

Diagonal form factors and heavy-heavy-light three-point functions at weak coupling

Laszlo Hollo,^a Yunfeng Jiang^b and Andrei Petrovskii^b

^a*MTA Lendület Holographic QFT Group, Wigner Research Centre for Physics, H-1525 Budapest 114, P.O.B. 49, Hungary*

^b*Institut de Physique Théorique, DSM, CEA, URA2306 CNRS, Saclay, F-91191 Gif-sur-Yvette, France*

E-mail: hollo.laszlo@wigner.mta.hu, jinagyf2008@gmail.com, andrey Petrovskij@gmail.com

ABSTRACT: In this paper we consider a special kind of three-point functions of HHL type at weak coupling in $\mathcal{N} = 4$ SYM theory and analyze its volume dependence. At strong coupling this kind of three-point functions were studied recently by Bajnok, Janik and Wereszczynski [1]. The authors considered some cases of HHL correlator in the $\mathfrak{su}(2)$ sector and, relying on their explicit results, formulated a conjecture about the form of the volume dependence of the symmetric HHL structure constant to be valid at any coupling up to wrapping corrections. In order to test this hypothesis we considered the HHL correlator in $\mathfrak{su}(2)$ sector at weak coupling and directly showed that, up to one loop, the finite volume dependence has exactly the form proposed in [1]. Another side of the conjecture suggests that computation of the symmetric structure constant is equivalent to computing the corresponding set of infinite volume form factors, which can be extracted as the coefficients of finite volume expansion. In this sense, extracting appropriate coefficients from our result gives a prediction for the corresponding infinite volume form factors.

KEYWORDS: Extended Supersymmetry, AdS-CFT Correspondence, Bethe Ansatz, Integrable Field Theories

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1 Introduction

It's strongly believed that $\mathcal{N} = 4$ SYM theory is integrable. There are a huge amount of supportive evidences in the literature starting from [2], where the relation between spin chain Hamiltonian and the dilatation operator of the theory was established, for detailed review, see [3]. Among one of the most exciting achievements revealing the power of the integrability is the so-called Quantum Spectral Curve (QSC) approach [4, 5] which provides efficient technique for computing anomalous dimension of a gauge invariant operators of the theory.

Due to conformal symmetry, the computation of general correlation functions of $\mathcal{N} = 4$ SYM can be reduced to computation of two- and three-point functions by means of the operator product expansion (OPE). In this sense these two quantities are the fundamental blocks of the theory. Two-point functions of gauge invariant operators, again due to the conformal symmetry, are completely defined by the value of the corresponding anomalous dimension, and thus can be computed in terms of QSC technique. However for the moment there is no such an efficient analogue for computing three-point functions. Nevertheless significant progress have been done in this direction starting from [6, 7] and later developed systematically in [8], where the three-point functions of the $\mathfrak{su}(2)$ sector at tree level were considered and, using the mapping between spin chains and gauge invariant operators, the computation of the structure constants was reduced to computation of the scalar products between two off-shell Bethe-states. This method was improved in [9] for some special configuration, which allows to express the structure constant in terms of on-shell/off-shell scalar products and can be computed by Slavnov determinants. Later this result was extended to one loop [10, 11] and other rank-one sectors also were investigated [12, 13]. The great advancement was made very recently in [14], where the authors proposed all-loop procedure for the structure constant in $\mathcal{N} = 4$ SYM. Another interesting direction inspired by the light-cone string field theory was initiated in [15], where the authors interpreted the OPE coefficients as “generalized Neumann coefficients” and proposed a set of bootstrap axioms for these generalized Neumann coefficients. In parallel, the spin vertex approach was developed in [16–18] which can be seen as a weak coupling counterpart of the string vertex.

Apart from direct approach, there is another method for the computation of three-point functions by means of their relations with the form factors. This approach was initiated in [19, 20], where the set of the axioms for the world-sheet form factor of the light-cone gauge fixed $AdS_5 \times S^5$ string theory was proposed. Recently in [1] the symmetric HHL (heavy-heavy-light) correlator at strong coupling of the $\mathfrak{su}(2)$ sector was considered. The proposals for computing the HHL three-point functions were first formulated in [21, 22]. The authors of [1] showed that the prescription in [21] was inadequate and proposed an improved prescription. Using this new prescription the authors computed several examples of the three-point correlators and showed that their volume dependence exactly coincide with the finite volume structure of the appropriate form factor. Guided by this result they proposed a conjecture that this finite volume dependence should hold at any coupling of the theory.

The conjecture consists two parts. First, the finite volume dependence (neglecting wrapping) of a symmetric HHL structure constant is completely encoded into diagonal

minors of the Gaudin determinant of a heavy state and have the following form dictated by results of form factor theory ([23]):

$$C_{HHL} = \frac{f^{\mathcal{O}} + \sum_i \rho_N(\{i\}) f^{\mathcal{O}}(1, \dots, \hat{i}, \dots, N) + \sum_{i,j} \rho_N(\{i, j\}) f^{\mathcal{O}}(1, \dots, \hat{i}, \dots, \hat{j}, \dots, N) + \dots}{\rho_N(\{1, \dots, N\})}, \tag{1.1}$$

and $\rho_N(\{i, j, \dots\})$'s are diagonal minors of the determinant:

$$\rho_N(\{1, \dots, N\}) = \det_{j,k} \left(\frac{\partial}{\partial u_j} \left(L p_k(u) + \sum_{l \neq k} \frac{1}{l} \log S(u_l, u_k) \right) \right). \tag{1.2}$$

Second part of the conjecture, also coming from the form factor approach, is that the coefficients $f_k^{\mathcal{O}}$ in the expansion (1.1) are diagonal infinite volume form factors of a light operator \mathcal{O} . In this paper we test the first part of the conjecture at weak coupling limit and show that the form (1.1) of the finite volume dependence holds at tree level as well as at one loop. At the same time the second part of the conjecture allowed us to predict expressions for diagonal infinite volume form factors by extracting corresponding coefficients from the expansion (1.1).

The structure of the paper is the following. In section 2 we set up the problem. In section 3, we will describe in detail the map between the field theory operator \mathcal{O}_α and the spin chain operator \hat{O}_α . In section 4, we review the integrability tools that we need, which includes algebraic Bethe ansatz (ABA), the scalar products between Bethe states and the solution of quantum inverse scattering problem (QISP). In section 5, we review the form factor formalism and the finite volume corrections. In section 6 and 7, we give a proof of the finite volume structure of the three-point function conjectured in [1] at tree level. In section 8 we compute some examples of the symmetric HHL correlator and list expressions for the corresponding infinite volume form factors. In section 9 we give general arguments proving that the same finite volume structure holds as well at one loop. Finally we conclude and discuss future directions to pursue in section 10.

2 Set-up

The central object of our paper is the symmetric HHL three-point correlation function of gauge invariant operators, by which we mean that two heavy¹ (the length $L \gg 1$) operators are conjugated to each other and the third operator has a few number of sites. The technique, which we are going to use, is due to the relation between the gauge invariant operators and the spin chains. Following [24], the symmetric structure constant can be mapped to diagonal matrix element of a spin chain operator. We take the heavy operators to be the eigenstates of the $\mathfrak{su}(2)$ sector dilatation operator and conjugated to each other. They are constructed from the following scalar fields

$$\mathcal{O}_1 : \{Z, X\}, \quad \mathcal{O}_{\bar{1}} : \{\bar{Z}, \bar{X}\}. \tag{2.1}$$

¹Although we call it “heavy” the results obtained in this paper are valid for small values of L as well.

At tree-level, we consider operators with definite one-loop anomalous dimension [8]. The one-loop dilatation operator in the $\mathfrak{su}(2)$ sector is the Heisenberg $XXX_{1/2}$ spin chain Hamiltonian whose eigenstates can be constructed by Bethe Ansatz techniques. Since we are considering diagonal matrix elements, the wave functions of the two operators are conjugate to each other. The third operator will be denoted hereafter by \mathcal{O}_α . In this paper, we consider \mathcal{O}_α in the compact sector. Only the zero R -charge terms give non-zero contribution. Therefore, we are interested in the operators of the following form

$$\text{Tr } Z\bar{Z}, \quad \text{Tr } X\bar{X}, \quad \text{Tr } XZ\bar{Z}\bar{X}, \quad \dots \tag{2.2}$$

This kind of operators are in the $\mathfrak{so}(4)$ sector of $\mathcal{N} = 4$ SYM theory. In addition, we require \mathcal{O}_α have definite anomalous dimension. The three-point function is fixed by conformal symmetry up to the structure constant C_α

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_\alpha(x_2)\mathcal{O}_{\bar{1}}(x_3) \rangle = \frac{L^2 L_\alpha}{N_c} \frac{\mathcal{N}_1 \sqrt{\mathcal{N}_\alpha} C_\alpha}{|x_{12}|^{\Delta_{12}} |x_{13}|^{\Delta_{13}} |x_{23}|^{\Delta_{23}}} \tag{2.3}$$

where

$$x_{ij}^\mu = x_i^\mu - x_j^\mu, \quad \Delta_{ij} = \frac{1}{2}(\Delta_i + \Delta_j - \Delta_k), \tag{2.4}$$

and L is the length of \mathcal{O}_1 while L_α is the length of the operator \mathcal{O}_α . The two-point functions are normalized as

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_{\bar{1}}(x_2) \rangle = \frac{L\mathcal{N}_1}{|x_{12}|^{2\Delta_1}}, \quad \langle \mathcal{O}_\alpha(x_1)\bar{\mathcal{O}}_\alpha(x_2) \rangle = \frac{L_\alpha \mathcal{N}_\alpha}{|x_{12}|^{2\Delta_\alpha}} \tag{2.5}$$

The structure constant C_α can be expressed in terms of correlation functions of the Heisenberg spin chain

$$C_\alpha = \frac{\langle \mathbf{u} | \hat{\mathcal{O}}_\alpha | \mathbf{u} \rangle}{\langle \mathbf{u} | \mathbf{u} \rangle} \tag{2.6}$$

where $|\mathbf{u}\rangle \equiv |\{u_1, \dots, u_N\}\rangle$ denotes an on-shell Bethe state, corresponding to the heavy operator, and $\hat{\mathcal{O}}_\alpha = \hat{\mathcal{O}}_\alpha(\sigma_i^\pm, \sigma_i^z)$ is an operator made of local spin operators. In this way, the computation of three-point function in planar $\mathcal{N} = 4$ SYM theory is recast into the calculation of correlation functions in the Heisenberg spin chain.

3 From field theory correlation functions to spin chain matrix elements

In this section, we summarize how to write the field theoretic operators \mathcal{O}_α in terms of spin chain operators $\hat{\mathcal{O}}_\alpha$. Let us introduce the following notation

$$Z \equiv \phi^0, \quad X \equiv \phi^1, \quad \bar{Z} \equiv \bar{\phi}^0, \quad \bar{X} \equiv \bar{\phi}^1. \tag{3.1}$$

The light operator is the linear combination of the single trace operators $\text{Tr } \phi^{i_1} \bar{\phi}^{i_2} \bar{\phi}^{i_3} \dots$. By planarity, only operators of the following form will contribute to the three-point function

$$\text{Tr } \phi_1^{i_1} \dots \phi_l^{i_l} \bar{\phi}_{l+1}^{j_1} \dots \bar{\phi}_{2l}^{j_l}, \quad i_k, j_k = 0, 1. \tag{3.2}$$

where the indices $1, \dots, 2l$ denotes the position on the third spin chain and $2l = L_\alpha$ is the length of the third operator. The zero R -charge condition is given by

$$\sum_{n=1}^l (i_n - j_n) = 0. \tag{3.3}$$

It is not hard to see that the operator (3.2) can be mapped to the following spin operator [24]

$$\text{Tr} \phi_1^{i_1} \cdots \phi_l^{i_l} \bar{\phi}_{l+1}^{j_1} \cdots \bar{\phi}_{2l}^{j_l} \longrightarrow \text{E}_{n+l-1}^{i_1+1, j_1+1} \cdots \text{E}_n^{i_l+1, j_l+1}. \tag{3.4}$$

where the indices $n, \dots, n-l+1$ denote the positions on the long spin chain, and the operators E_n^{ab} are the basis 2×2 matrices $(\text{E}_n^{ab})_{ij} = \delta_{ai} \delta_{bj}$ in the local quantum space $\mathcal{H}_n = \mathbb{C}^2$. The operators E_n^{ab} are related to the local spin operators as follows

$$\text{E}_n^{11} \equiv \frac{1}{2}(\mathbb{I} + \sigma_n^z), \quad \text{E}_n^{12} \equiv \sigma_n^+, \quad \text{E}_n^{21} \equiv \sigma_n^-, \quad \text{E}_n^{22} \equiv \frac{1}{2}(\mathbb{I} - \sigma_n^z). \tag{3.5}$$

Here σ_n^\pm, σ_n^z are the usual Pauli matrices acting on the space spanned by $|\uparrow\rangle$ and $|\downarrow\rangle$. By the mapping (3.4), we can translate the field theory operators into the spin operators. As an example, we consider the Konishi operator

$$\mathcal{O}_K = \text{Tr} X \bar{X} + \text{Tr} Y \bar{Y} + \text{Tr} Z \bar{Z}. \tag{3.6}$$

Since the heavy operators are in the $\mathfrak{su}(2)$ sector, the contraction with Y and \bar{Y} are zero and can be neglected. The Konishi operator can be mapped to the following spin operator

$$\mathcal{O}_K \longrightarrow \hat{\mathcal{O}}_K = \text{E}_n^{11} + \text{E}_n^{22} = \mathbb{I}. \tag{3.7}$$

Therefore we see that at tree level the structure constant, with the light operator being the Konishi operator, is trivial

$$C_K = \frac{\langle \mathbf{u} | \hat{\mathcal{O}}_K | \mathbf{u} \rangle}{\langle \mathbf{u} | \mathbf{u} \rangle} = 1. \tag{3.8}$$

In order to obtain non-trivial structure constant, the light operator must have at least $L_\alpha = 4$. This corresponds to the insertion of two spin operators between the Bethe states. An example for length-4 operator is given in figure 1.

The main focus of the current paper is the length-4 field theory operators, which corresponds to operators in the spin chain which acts on two neighboring sites. These are the simplest non-trivial cases which can be studied thoroughly. We use L_α and l_s to denote the lengths of field theory operators and spin chain operators, it is obvious that $L_\alpha = 2l_s$.

In order to have non-zero diagonal matrix element in (2.6), the spin chain operator should not alter the total spin of the state it acts on. At $l_s = 1$, there are two independent operator, satisfying this condition, the identity and

$$o_1(n) = \text{E}_n^{11}. \tag{3.9}$$

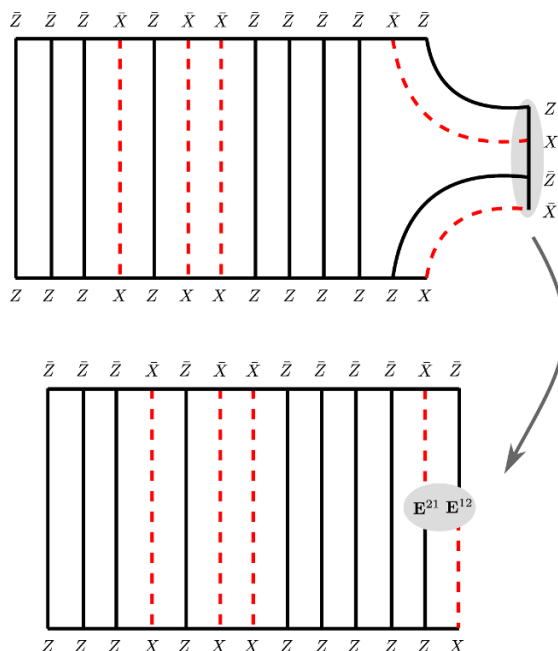


Figure 1. An example of the mapping between field operators to spin chain operators. The operator in the field theory $\mathcal{O} = \text{Tr} ZX\bar{Z}\bar{X}$ is mapped to the spin chain operator $\hat{O} = E_n^{21} E_{n+1}^{12}$.

For $l_s = 2$, there are six independent operators: the unity, $o_1(n)$, $o_1(n+1)$ ² and the other three operators are the following

$$o_2^1(n) = E_n^{11} E_{n+1}^{11}, \quad o_2^2(n) = E_n^{12} E_{n+1}^{21}, \quad o_2^3(n) = E_n^{21} E_{n+1}^{12}. \quad (3.10)$$

For later convenience we also introduce the following operator

$$o_2^4 = E_n^{22} E_{n+1}^{22} = (\mathbb{I} - E_n^{11})(\mathbb{I} - E_{n+1}^{11}) = \mathbb{I} - o_1(n) - o_1(n+1) + o_2^1(n). \quad (3.11)$$

In what follows, we will study in detail the diagonal matrix elements of the spin operators o_1 and o_2^i ($i = 1, 2, 3, 4$).

4 The algebraic Bethe ansatz approach

In this section, we review the algebraic Bethe Ansatz (ABA) approach to the correlation functions in the $XXX_{1/2}$ Heisenberg spin chain, see [25–27] and references therein. This approach is based on two main elements: the Slavnov determinant formula and the solution of the Quantum Inverse Scattering Problem (QISP). The Slavnov formula states that the scalar product of any Bethe state with an on-shell one can be written in terms of a determinant. The solution of the QISP enable us to express the local spin operators in terms of the elements of the monodromy matrix, which are non-local operators acting on the spin chain.

²Due to translation invariance of the spin chain at the level of correlation functions these two operators obviously coincide.

4.1 Algebraic Bethe ansatz

In this subsection we give a brief review of the algebraic Bethe Ansatz method which also serves to fix our conventions. For more detailed and pedagogical discussions we refer the readers to [28]. The central object in ABA is the quantum R -matrix. For the Heisenberg $XXX_{1/2}$ spin chain, the R -matrix is given by

$$R_{ab}(u) = u I_{ab} + i P_{ab} = \begin{pmatrix} u+i & 0 & 0 & 0 \\ 0 & u & i & 0 \\ 0 & i & u & 0 \\ 0 & 0 & 0 & u+i \end{pmatrix}_{ab} \quad (4.1)$$

where I_{ab} and P_{ab} are identity and permutation operators acting on the space $\mathbb{C}_a^2 \otimes \mathbb{C}_b^2$, respectively. It has a complex parameter which is usually referred to as the spectral parameter. Consider a spin chain of length L . At each site of the spin chain we define the Lax matrix

$$L_{an}(u - \theta_n) = R_{an}(u - \theta_n - i/2), \quad n = 1, \dots, L \quad (4.2)$$

where n and a denote the quantum space and auxiliary space, respectively. The parameter θ_n is called the inhomogeneity associated to the site n . In the case of the homogeneous Heisenberg spin chain all the inhomogeneities are set to be zero, $\theta_n = 0$. The Lax operator obeys an important property that it satisfies the following braiding relation, called the RLL relation

$$R_{ab}(u - v)L_{an}(u)L_{bn}(v) = L_{bn}(v)L_{an}(u)R_{ab}(u - v). \quad (4.3)$$

Taking the ordered product of Lax operators at all sites, we obtain the monodromy matrix

$$T_a(u) = \prod_{n=1}^L L_{an}(u - \theta_n). \quad (4.4)$$

It can be represented in the auxiliary space as a 2×2 matrix

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \quad (4.5)$$

By construction, the monodromy matrix is a non-local object and its elements are operators which act on the whole spin chain. From the RLL relation (4.3) and the definition of the monodromy matrix (4.4), one can show that the monodromy matrix satisfies a similar relation, called the RTT relation

$$R_{ab}(u - v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u - v). \quad (4.6)$$

Once written in terms of the components, the RTT relation leads to a quadratic algebra, between the ABA operators A , B , C , D . The most relevant relations for us are

$$\begin{aligned} A(v)B(u) &= f(u - v)B(u)A(v) + g(v - u)B(v)A(u), \\ D(v)B(u) &= f(v - u)B(u)D(v) + g(u - v)B(v)D(u), \end{aligned} \quad (4.7)$$

$$[C(v), B(u)] = g(u - v)[A(v)D(u) - A(u)D(v)]$$

where the functions $f(u)$ and $g(u)$ are

$$f(u) = \frac{u+i}{u}, \quad g(u) = \frac{i}{u}. \tag{4.8}$$

All the conserved charges of the system are encoded into the transfer matrix, which is defined to be the trace of monodromy matrix in the auxiliary space

$$\mathcal{T}(u) = \text{Tr}_a T(u) = A(u) + D(u). \tag{4.9}$$

The eigenstates of the transfer matrix simultaneously diagonalize all the conserved charges, including the Hamiltonian. In order to diagonalize it, one starts with a ferromagnetic reference state, called the pseudovacuum $|\Omega\rangle = |\uparrow^L\rangle$, such that

$$A(u)|\Omega\rangle = a(u)|\Omega\rangle, \quad D(u)|\Omega\rangle = d(u)|\Omega\rangle, \quad C(u)|\Omega\rangle = 0. \tag{4.10}$$

For later convenience, we define the Baxter polynomials

$$Q_{\theta}(u) = \prod_{n=1}^L (u - \theta_n), \quad Q_{\theta}^{\pm} = Q_{\theta}(u \pm i/2). \tag{4.11}$$

Then we have

$$a(u) = Q_{\theta}^{+}(u), \quad d(u) = Q_{\theta}^{-}(u). \tag{4.12}$$

The eigenstates of the transfer matrix are generated by the action of a sequence of B -operators on the pseudovacuum

$$|\mathbf{u}\rangle = B(u_1) \cdots B(u_N)|\Omega\rangle \tag{4.13}$$

provided that the rapidities $\mathbf{u} = \{u_1, \dots, u_N\}$ satisfy the Bethe Ansatz equation (BAE)

$$\frac{a(u_k)}{d(u_k)} = \prod_{\substack{j=1 \\ j \neq k}}^N \frac{u_k - u_j + i}{u_k - u_j - i}, \quad k = 1, \dots, N. \tag{4.14}$$

These elementary excitations are commonly referred to as magnons. If the rapidities satisfy (4.14), the corresponding Bethe state $|\mathbf{u}\rangle$ is called *on-shell*, otherwise it is called *off-shell*. The on-shell Bethe states diagonalize the transfer matrix

$$\mathcal{T}(u)|\mathbf{u}\rangle = t_{\mathbf{u}}(u)|\mathbf{u}\rangle \tag{4.15}$$

with the eigenvalue

$$t_{\mathbf{u}}(u) = a(u) \frac{Q_{\mathbf{u}}(u-i)}{Q_{\mathbf{u}}(u)} + d(u) \frac{Q_{\mathbf{u}}(u+i)}{Q_{\mathbf{u}}(u)}. \tag{4.16}$$

Here, similarly we have defined the Baxter polynomial for the rapidities

$$Q_{\mathbf{u}}(u) = \prod_{k=1}^N (u - u_k). \tag{4.17}$$

In this paper, we will compute the matrix elements of spin chain operators between two Bethe states. To this end, it is important to have a manageable expression for the scalar products between Bethe states.

4.2 Slavnov determinant and Gaudin norm

The scalar product of Bethe states is defined as

$$\langle \mathbf{v} | \mathbf{u} \rangle = \langle \Omega | \prod_{j=1}^N C(v_j) \prod_{k=1}^N B(u_k) | \Omega \rangle. \quad (4.18)$$

The Slavnov theorem [29] states that, if one of the Bethe states, say $|\mathbf{u}\rangle$ is on-shell, the scalar product (4.18) is given by

$$\langle \mathbf{v} | \mathbf{u} \rangle = \prod_{j=1}^N a(v_j) d(u_j) S_{\mathbf{u}, \mathbf{v}}, \quad (4.19)$$

where

$$S_{\mathbf{u}, \mathbf{v}} = \frac{\det_{jk} \Omega(u_j, v_k)}{\det_{jk} \frac{1}{u_j - v_k + i}} \quad (4.20)$$

The matrix element $\Omega(u_j, v_k)$ is given by

$$\begin{aligned} \Omega(u_j, v_k) &= t(u_j - v_k) - e^{2ip_{\mathbf{u}}(v_k)} t(v_k - u_j) \\ &= i \frac{(u_j - v_k - i) - (u_j - v_k + i) e^{2ip_{\mathbf{u}}(v_k)}}{(u_j - v_k)[(u_j - v_k)^2 + 1]} \end{aligned} \quad (4.21)$$

where $p_{\mathbf{u}}(v)$ is the pseudomomentum

$$e^{2ip_{\mathbf{u}}(u)} \equiv \frac{d(u) Q_{\mathbf{u}}(u+i)}{a(u) Q_{\mathbf{u}}(u-i)}. \quad (4.22)$$

and $t(u)$ is

$$t(u) = \frac{1}{u} - \frac{1}{u+i}. \quad (4.23)$$

In terms of the pseudomomentum, the Bethe Ansatz equation takes a particularly simple form

$$e^{2ip_{\mathbf{u}}(u_k)} = -1. \quad (4.24)$$

In the Slavnov formula, the rapidities $\{\mathbf{v}\}$ can be any set of complex numbers. Let us now consider the case when some of the v_k 's coincide with the Bethe roots, namely $v_k = u_k$ for some k . The matrix element reads as

$$\Omega(u_j, u_k) = \frac{2i}{(u_j - u_k)^2 + 1} - i\delta_{jk} \left(\sum_{l=1}^N \frac{2}{(u_j - u_l)^2 + 1} - \sum_{m=1}^L \frac{1}{(u_j - \theta_m)^2 + 1/4} \right) \quad (4.25)$$

When all $\{\mathbf{v}\}$ coincide with $\{\mathbf{u}\}$, the scalar product corresponds to the norm of the Bethe state. This norm is given by the Gaudin formula which takes the form of a Jacobian determinant. Let us define the following quantity

$$\Phi_k(\mathbf{u}) = p(u_k)L - i \sum_{j \neq k}^N \log S(u_k - u_j) \quad (4.26)$$

where the momentum and the two-body scattering matrix of the Heisenberg spin chain are

$$p(u) = -\frac{i}{L} \log \frac{a(u)}{d(u)}, \quad S(u-v) = \frac{u-v-i}{u-v+i}. \quad (4.27)$$

In terms of the function $\Phi_k(\mathbf{u})$, the Bethe Ansatz equation (4.14) reads as

$$\Phi_k(\mathbf{u}) = 2\pi I_k, \quad k = 1, \dots, N \quad (4.28)$$

where $I_k \in \mathbb{Z}$ are the Bethe quantum numbers. The Jacobian for the change of variables between u_k and I_k is given by

$$\rho_N(\{1, \dots, N\}) = \left| \frac{\partial \Phi_k(\mathbf{u})}{\partial u_j} \right| = \det_{jk} i \Omega(u_j, u_k). \quad (4.29)$$

The norm of an on-shell N -magnon Bethe states is proportional to $\rho_N(\{1, \dots, N\})$, explicitly

$$\langle \mathbf{u} | \mathbf{u} \rangle = \left(\prod_{j=1}^N a(u_j) d(u_j) \prod_{j < k}^N \frac{1 + (u_j - u_k)^2}{(u_j - u_k)^2} \right) \rho_N(\{1, \dots, N\}) \quad (4.30)$$

We will see below that the Jacobian $\rho_N(\{1, \dots, N\})$ and its sub-diagonal counterparts take into account all the finite volume corrections of the diagonal matrix elements.

4.3 The solution of quantum inverse scattering problem

In the ABA approach, the idea to compute the diagonal matrix element $\langle \mathbf{u} | \hat{O}_n | \mathbf{u} \rangle$ is to act the spin chain operator \hat{O}_n on the bra state, so that the ket Bethe state $|\mathbf{u}\rangle$ is left on-shell. The operator \hat{O}_n is typically a multilocal operator in the sense that it acts on a finite interval of the spin chain. In order to apply the Slavnov formula, the bra state also needs to be a Bethe state, although we do not require it to be on-shell. The solution of the quantum inverse scattering problem (QISP) relates the local spin operators to the matrix elements of the monodromy matrix. These latter are non-local operators. In addition, from the Yangian algebra (4.7), it is clear that the action of A, B, C, D operators on a Bethe state always gives a sum over Bethe states. Therefore, using the solution of QISP we can translate any local operator \hat{O}_n into a sequence of ABA operators, and the state $\langle \mathbf{u} | \hat{O}_n$ can be written as a sum of dual Bethe states. This enable us to apply the Slavnov formula discussed in the last subsection. We present the main statement of the solution of QISP in this subsection and refer to [26] for the proof and details.

Let E_n^{ab} ($a, b = 1, 2$) be the operators defined in (3.5) which act on the local quantum space $\mathcal{H}_n = \mathbb{C}^2$. They can be represented by the elements of monodromy matrix as

$$E_n^{ab} = \left\{ \prod_{k=1}^{n-1} \mathcal{T}(\theta_k + i/2) \right\} T^{ab}(\theta_n + i/2) \left\{ \prod_{k=1}^n \mathcal{T}(\theta_k + i/2) \right\}^{-1} \quad (4.31)$$

where

$$T^{11}(u) = A(u), \quad T^{12}(u) = B(u), \quad T^{21}(u) = C(u), \quad T^{22}(u) = D(u). \quad (4.32)$$

Once inserted inside a correlator, the transfer matrices in (4.31) act on a Bethe state and can be replaced by their eigenvalues

$$\langle \mathbf{u} | E_n^{ab} | \mathbf{u} \rangle = \frac{\langle \mathbf{u} | T^{ab}(\theta_n + i/2) | \mathbf{u} \rangle}{t_{\mathbf{u}}(\theta_n + i/2)} \quad (4.33)$$

where $t_{\mathbf{u}}(u)$ is given by (4.16). At $u = \theta_n + i/2$, we have

$$t_{\mathbf{u}}(\theta_n + i/2) = Q_{\theta}(\theta_n + i) \frac{Q_{\mathbf{u}}^{-}(\theta_n)}{Q_{\mathbf{u}}^{+}(\theta_n)}. \quad (4.34)$$

The generalization of (4.33) to a string of $l + 1$ operators is straightforward

$$\langle \mathbf{u} | E_n^{a_0 b_0} \dots E_{n+l}^{a_l b_l} | \mathbf{u} \rangle = \left(\prod_{k=0}^l \frac{Q_{\mathbf{u}}^{+}(\theta_{n+k})}{Q_{\theta}(\theta_{n+k} + i) Q_{\mathbf{u}}^{-}(\theta_{n+k})} \right) \langle \mathbf{u} | \prod_{k=0}^l T_{n+k}^{a_k b_k}(\theta_{n+k} + i/2) | \mathbf{u} \rangle \quad (4.35)$$

We can now compute the r.h.s. of (4.35) by using the Yangian algebra and the Slavnov formula.

5 Finite volume diagonal form factors

The interest in the volume dependence of diagonal matrix elements of local operators is due to their appearance in different physical quantities. They are central objects in the calculation of finite temperature one-point functions [30, 31], as well as they play an essential role in the form factor perturbation theory [32]. The short distance behavior of correlation functions also involves the vacuum expectation values as basic ingredients [33]. Recently, it was conjectured [1] that the Heavy-Heavy-Light symmetric structure constants of the AdS/CFT correspondence (2.6) are also related to these quantities. This latter case is of the central interest of the current paper. To better understand this conjecture, in this section we summarize the theory of form factors in integrable models, starting from the infinite volume description and then take into account the finite volume corrections, up to wrapping.

5.1 Form factors in infinite volume

We consider a $1 + 1$ dimensional integrable quantum field theory defined by its S -matrix. For simplicity we restrict ourselves to diagonally scattering theories with a self-conjugated particle, the generalization to any diagonally scattering theory is straightforward. For a detailed review of the general case see [34].

The infinite volume states can be characterized by the set of momenta of particles. In $1 + 1$ dimension it is convenient to label the particles by their rapidities u_i , the energy $\varepsilon(u)$ and momentum $p(u)$ being a single valued functions. In the remote past, an *in* state consists of particles of ordered rapidities: the fastest one is the leftmost while the slowest is the rightmost. Contrarily, the particles in an *out* state are reversely ordered,

$$|u_1, \dots, u_N\rangle = \begin{cases} |u_1, \dots, u_N\rangle^{\text{in}} & u_1 > \dots > u_N \\ |u_1, \dots, u_N\rangle^{\text{out}} & u_1 < \dots < u_N \end{cases} \quad (5.1)$$

The infinite volume states that differ only in the order of rapidities are related by the two particle S -matrix³

$$|u_1, \dots, u_i, u_{i+1}, \dots, u_N\rangle = S(u_i, u_{i+1}) |u_1, \dots, u_{i+1}, u_i, \dots, u_N\rangle \quad (5.2)$$

The energy of a multiparticle state is the sum of the one particle energies

$$E(u_1, \dots, u_N) = \sum_{i=1}^N \varepsilon(u_i). \quad (5.3)$$

In infinite volume we normalize the in states as

$${}^{in}\langle u'_1, \dots, u'_M | u_1, \dots, u_N \rangle^{in} = (2\pi)^N \delta_{NM} \delta(u_1 - u'_1) \cdots \delta(u_N - u'_N), \quad (5.4)$$

and the norm of a general state can be determined from (5.4) by (5.2).

Let us consider the matrix elements of a *local* operator $\mathcal{O}(t, x)$ between asymptotic states. The space-time dependence can be easily factored out

$${}^{out}\langle u'_1, \dots, u'_M | \mathcal{O}(x, t) | u_1, \dots, u_N \rangle^{in} = e^{it\Delta E - ix\Delta P} {}^{out}\langle u'_1, \dots, u'_M | \mathcal{O}(0, 0) | u_1, \dots, u_N \rangle^{in},$$

where

$$\begin{aligned} \Delta E &= \sum_{j=1}^M \varepsilon(u'_j) - \sum_{k=1}^N \varepsilon(u_k), \\ \Delta P &= \sum_{j=1}^M p(u'_j) - \sum_{k=1}^N p(u_k), \end{aligned} \quad (5.5)$$

and we define the *form factor* of operator \mathcal{O} as

$$F_{M,N}^{\mathcal{O}}(u'_1, \dots, u'_M | u_1, \dots, u_N) = {}^{out}\langle u'_1, \dots, u'_M | \mathcal{O}(0, 0) | u_1, \dots, u_N \rangle^{in} \quad (5.6)$$

The form factors are *a priori* defined for ordered set of incoming and outgoing rapidities but can be analytically continued by (5.2). A form factor is a meromorphic function in all variables and each pole has a physical origin [34].

Suppose that the theory possesses crossing symmetry, i.e. a transformation which maps an outgoing particle with rapidity u to an incoming anti-particle with rapidity \bar{u} . The crossing symmetry implies the crossing equation for the form factors which, in case of a single self-conjugated particle, reads as

$$\begin{aligned} F_{M,N}^{\mathcal{O}}(u'_1, \dots, u'_M | u_1, \dots, u_N) &= F_{M-1, N+1}^{\mathcal{O}}(u'_1, \dots, u'_{M-1} | \bar{u}_M, u_1, \dots, u_N) \\ &+ \sum_{k=1}^N \langle u'_M | u_k \rangle \prod_{l=1}^{k-1} S(u_l, u_k) F_{M-1, N-1}^{\mathcal{O}}(u'_1, \dots, u'_{M-1} | u_1, \dots, \hat{u}_k, \dots, u_N) \end{aligned} \quad (5.7)$$

where the terms on the second line of (5.7) describe disconnected processes that occur if one of the incoming and outgoing particle has the same rapidity. The hat \hat{u}_k denotes that

³Although the states with non-ordered rapidities are not physical.

u_k is missing from the list of rapidities. By using the crossing relation all form factors can be expressed in terms of *elementary* form factors

$$F_N^{\mathcal{O}}(u_1, \dots, u_N) = \langle 0 | \mathcal{O}(0, 0) | u_1, \dots, u_N \rangle. \tag{5.8}$$

These elementary form factors satisfy several functional relations, called the form factor axioms. As these axioms are not relevant for the aims of this paper we will not list them here but rather refer to [34] for relativistic models and [19] for the AdS/CFT case.

5.1.1 Diagonal form factors

The diagonal form factor of the local operator \mathcal{O} , defined as

$$\text{in} \langle u_1, \dots, u_N | \mathcal{O}(0, 0) | u_1, \dots, u_N \rangle^{\text{in}}, \tag{5.9}$$

is singular due to the disconnected terms in the crossing relation (5.7). To avoid the singularities we regularize it by slightly shifting the outgoing rapidities. After crossing we get

$$F_{2N}^{\mathcal{O}}(\bar{u}_1 + \epsilon_1, \dots, \bar{u}_N + \epsilon_N, u_N, \dots, u_1) = \langle 0 | \mathcal{O} | \bar{u}_1 + \epsilon_1, \dots, \bar{u}_N + \epsilon_N, u_1, \dots, u_N \rangle^{\text{in}} \tag{5.10}$$

The diagonal limit, $\epsilon_i \rightarrow 0$, of (5.10) is not well-defined. It was first noticed in [32] that the singular parts vanish in the limit when all $\epsilon_i \rightarrow 0$, but the result depends on the direction of the limit. Its general structure can be written as

$$\begin{aligned} F_{2N}^{\mathcal{O}}(\bar{u}_1 + \epsilon_1, \dots, \bar{u}_N + \epsilon_N, u_N, \dots, u_1) & \tag{5.11} \\ &= \prod_{i=1}^N \frac{1}{\epsilon_i} \cdot \sum_{i_1=1}^N \sum_{i_2=1}^N \dots \sum_{i_N=1}^N a_{i_1 i_2 \dots i_N}(u_1, \dots, u_N) \epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_N} + \dots \end{aligned}$$

where $a_{i_1 i_2 \dots i_N}$ is a completely symmetric tensor of rank N . The ellipsis denote terms which vanish in the $\epsilon_i \rightarrow 0$ limit.

There are two generally used regularization scheme in the literature. The first is the so-called *symmetric* evaluation when we set all ϵ_i to be the same

$$F_{2N}^{\mathcal{O},s}(u_1, \dots, u_N) = \lim_{\epsilon \rightarrow 0} F_{2N}^{\mathcal{O}}(\bar{u}_1 + \epsilon, \dots, \bar{u}_N + \epsilon, u_N, \dots, u_1). \tag{5.12}$$

The second scheme is called *connected*, in which the diagonal form factors are defined as the finite part of (5.11), i.e. the ϵ -independent term,

$$F_{2N}^{\mathcal{O},c}(u_1, \dots, u_N) = N! a_{12 \dots N}. \tag{5.13}$$

Both the symmetric and the connected diagonal form factors are completely symmetric in the rapidity variables u_1, \dots, u_N . Of course these two quantities are not independent and each can be expressed with use of the other [23].

5.2 Diagonal form factors in finite volume

In this section we will summarize the results about the volume dependence of the diagonal form factors in all polynomial orders in the inverse of the volume, neglecting the exponentially small wrapping corrections, following [23, 35].

In finite volume L , the rapidities are quantized and a generic multiparticle state can be labeled by the Bethe quantum numbers $|\{I_1, \dots, I_N\}\rangle_L$. In finite volume we cannot order the particles by spatial separation in the remote past or future, as we did in the infinite volume case (5.2). In finite volume the states are completely symmetric under the exchange of particles and can be characterized by the *set* of quantum numbers. We adapt our notation to the conventions used in [23, 35] and order the quantum numbers in a monotonically decreasing sequence, $I_1 \geq \dots \geq I_N$.⁴ The quantized rapidities with the quantum numbers $\{I_1, \dots, I_N\}$ are solutions of the corresponding Bethe Ansatz equations. Similarly to (4.26) we define

$$\Phi_j(\{u_1, \dots, u_N\}) = p(u_j)L - i \sum_{\substack{k=1 \\ k \neq j}} \log S(u_k, u_j), \quad (5.14)$$

and the quantization condition reads as

$$\Phi_j(\{u_1, \dots, u_N\}) = 2\pi I_j, \quad j = 1, \dots, N. \quad (5.15)$$

These finite volume states are orthogonal to each other

$${}_L\langle \{J_1, \dots, J_M\} | \{I_1, \dots, I_N\} \rangle_L \propto \delta_{NM} \delta_{I_1, J_1} \dots \delta_{I_N, J_N} \quad (5.16)$$

and their normalization is a question of convention.

One can change from the quantum number representation of states to the rapidity representation which gives the direct connection between the finite and infinite volume states [35]. This change of variables involves the Jacobian, which is the density of N -particle states, defined as

$$\begin{aligned} \varrho_N(u_1, \dots, u_N)_L &= \det \mathcal{J}^{(N)}(u_1, \dots, u_N)_L \\ \mathcal{J}_{k,l}^{(N)}(u_1, \dots, u_N)_L &= \frac{\partial \Phi_k(u_1, \dots, u_N)}{\partial u_l}, \quad k, l = 1, \dots, N. \end{aligned} \quad (5.17)$$

where we explicitly indicated the volume dependence of these quantities. The determinant (5.17) is closely related to the Gaudin norm of Bethe states (4.30).⁵ Then the relation between the infinite and finite volume states reads as

$$|\{I_1, \dots, I_N\}\rangle_L = \frac{1}{\sqrt{\varrho_N(u_1, \dots, u_N)_L \prod_{i < j} S(u_i, u_j)}} |u_1, \dots, u_N\rangle \quad (5.18)$$

⁴Apart from the free boson case all known S -matrix obey the property $S(u, u) = -1$ which is an effective Pauli exclusion. In this cases we have $I_1 > \dots > I_N$.

⁵The Gaudin norm itself is not physical as it depends on the conventions. However, in any convention, it is proportional to the Jacobian (5.17).

where the rapidities $\{u_i\}$ are the solutions of the Bethe Ansatz equations (5.14) corresponding to the quantum numbers $\{I_1, \dots, I_N\}$. This identification holds up to exponential corrections. The product of S -matrices in the denominator ensures that the finite volume state is indeed symmetric under the exchange of particles.

Defining the system in finite volume regularizes all the divergences appearing in the diagonal limit of form factors (5.11), thus the normalized finite volume diagonal matrix element

$$\frac{{}_L\langle\{I_1, \dots, I_N\}|\mathcal{O}(0,0)|\{I_1, \dots, I_N\}\rangle_L}{{}_L\langle\{I_1, \dots, I_N\}|\{I_1, \dots, I_N\}\rangle_L} \quad (5.19)$$

is finite, completely well defined and does not depend on the normalization of states. However, it is a challenging task to relate them to the infinite volume ones in the general case [30, 31]. The problem become considerably simpler if we neglect the exponentially small wrapping corrections.

Up to wrapping, the finite volume N -particle diagonal form factor (5.19) can be expressed as a sum over the bipartite partitions of the full set $\{1, 2, \dots, N\}$, involving the infinite volume form factors and some kind of densities of states. As the diagonal form factors in infinite volume depend on the regularization scheme, this series is also scheme dependent. In case of the connected evaluation the relation reads as [23, 36]

$$\frac{{}_L\langle\{I_1, \dots, I_N\}|\mathcal{O}(0,0)|\{I_1, \dots, I_N\}\rangle_L}{{}_L\langle\{I_1, \dots, I_N\}|\{I_1, \dots, I_N\}\rangle_L} = \frac{1}{\rho_N(\{1, \dots, N\})} \sum_{\alpha \subseteq \{1, \dots, N\}} f^{\mathcal{O}}(\{u_k\}_{k \in \bar{\alpha}}) \rho_N(\alpha) \quad (5.20)$$

where $\bar{\alpha}$ denotes the complement of α in the full set. The functions appearing on the right hand side are exactly the connected diagonal form factors

$$f^{\mathcal{O}}(u_1, \dots, u_l) = F_{2l}^{\mathcal{O},c}(u_1, \dots, u_l) \quad (5.21)$$

The functions ρ_N are defined as the diagonal minor determinants of the N -particle Jacobian (5.17),

$$\rho_N(\alpha) = \det_{k,l \in \alpha} \mathcal{J}_{k,l}^{(N)}(u_1, \dots, u_N)_L, \quad \alpha \subseteq \{1, \dots, N\}. \quad (5.22)$$

They can also be referred to as partial Gaudin norms. As special cases we have

$$\rho_N(\{1, \dots, N\}) = \varrho_N(u_1, \dots, u_N)_L; \quad \rho_N(\emptyset) = 1. \quad (5.23)$$

We want to emphasize that the function $\rho_N(\alpha)$ depend on *all* the N rapidities. The set of rapidities $\{u_i\}$ in the right hand side of (5.20) is the solution of the Bethe Ansatz equations (5.14) corresponding to the quantum numbers $\{I_1, \dots, I_N\}$. Thus, the *explicit* volume dependence is encoded *only* into the factors ρ_N , the connected form factors $f^{\mathcal{O}}$ depend on the volume only *implicitly* via the Bethe Ansatz equations.

As the connected and symmetric diagonal form factors are not independent, we can express the finite volume matrix element in the symmetric regularization scheme. In this case the series take the form [35]

$$\frac{{}_L\langle\{I_1, \dots, I_N\}|\mathcal{O}(0,0)|\{I_1, \dots, I_N\}\rangle_L}{{}_L\langle\{I_1, \dots, I_N\}|\{I_1, \dots, I_N\}\rangle_L} = \frac{1}{\rho_N(\{1, \dots, N\})} \sum_{\alpha \subseteq \{1, \dots, n\}} F_{2|\bar{\alpha}|}^s(\{u_k\}_{k \in \bar{\alpha}}) \rho_{|\alpha|}(\alpha). \quad (5.24)$$

Here again, the rapidities $\{u_i\}$ are the solutions of the Bethe Ansatz equations (5.14) with the quantum numbers $\{I_1, \dots, I_N\}$. The $\rho_{|\alpha|}$ functions appearing in the sum are the $|\alpha|$ -particle densities of state (5.23), (5.17) evaluated at the rapidities $\{u_i\}_{i \in \alpha}$. Note that, contrary to the connected expansion, they depend *only* on the rapidities labeled by the set α . The explicit volume dependence is carried only by the ρ functions.

5.2.1 Form factor of densities of conserved charges

An important special case of local operators is the density of a conserved quantity,

$$Q = \int_0^L J(x, t) dx$$

where Q acts diagonally and additively on the multiparticle states. Its density therefore satisfies

$$\frac{{}_L \langle \{I_1, \dots, I_N\} | J(0, 0) | \{I_1, \dots, I_N\} \rangle_L}{{}_L \langle \{I_1, \dots, I_N\} | \{I_1, \dots, I_N\} \rangle_L} = \frac{1}{L} \sum_{j=1}^N q(u_j),$$

where $\{u_i\}$ are the solutions of the Bethe Ansatz equations (5.14) corresponding to the quantum numbers $\{I_1, \dots, I_N\}$, and $q(u)$ is the one-particle eigenvalue of the operator Q .

A compact expression for the connected diagonal form factors of these densities was presented in [30, 36], however the proof was found recently [37]. The connected form factors can be cast into the form

$$F_{2N}^{J,c}(u_1 \dots, u_N) = \sum_{\sigma \in S_N} p'(u_{\sigma(1)}) \varphi(u_{\sigma(1)}, u_{\sigma(2)}) \varphi(u_{\sigma(2)}, u_{\sigma(3)}) \dots \varphi(u_{\sigma(N-1)}, u_{\sigma(N)}) q(u_{\sigma(N)}) \tag{5.25}$$

where the summation runs over all the permutation of the set $\{1, \dots, N\}$. Here, p' denotes the derivative of the momentum w.r.t. the rapidity,

$$p'(u) = \frac{\partial}{\partial u} p(u).$$

For massive relativistic models we have $p(u) = m \sinh u$ and $\varepsilon(u) = m \cosh u$, so that $p'(u) = \varepsilon(u)$ and (5.25) reduces to the expression presented in [30, 36]. However, in the case of the Heisenberg $XXX_{1/2}$ spin chain an extra sign appears, $p'(u) = -\varepsilon(u)$.

5.3 Conjecture for the symmetric structure constants

Based on explicit calculations the authors of [1] conjectured that the Heavy-Heavy-Light symmetric structure constant in the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM is equal to a finite volume form factor and its volume dependence (up to wrapping corrections) at any coupling has the form (1.1). Let us suppose that the two, conjugated heavy operators correspond to a multiparticle state, labeled by the rapidities $\{u_i\}$, in finite, but large volume L , such that the exponential corrections are negligible. Then the symmetric structure constant are conjectured to be the finite volume diagonal form factor of the vertex operator \mathcal{O} of the light operator,

$$C_\alpha = {}_L \langle u_1, \dots, u_N | \mathcal{O} | u_1, \dots, u_N \rangle_L \tag{5.26}$$

that can be expressed in terms of the infinite volume quantities (5.20) and (5.24).

6 Matrix elements of spin operators

In this section we study the diagonal matrix elements of spin operators of the Heisenberg spin chain using the ABA and the solution of QISP discussed in section 4. We show that, in general, the matrix elements can be written as linear combinations of a special kind of Slavnov determinant. In section 7, we show that this determinant has the structure conjectured in [1], namely it can be written as linear combinations of diagonal minors of Gaudin determinants (5.20). We call the procedure of expanding quantities in terms of diagonal minors of Gaudin determinants the *finite volume expansion*, as it captures all the finite volume dependence. We will discuss the case $l_s = 1, 2$ in detail and comment on the general $l_s > 2$ case.

6.1 Form factors of length-2 operators

We have shown in section 3 that all the length-2 diagonal matrix elements can be written as linear combinations of the following building blocks

$$\mathcal{F}^{o_1} = \langle \mathbf{u} | o_1(n) | \mathbf{u} \rangle, \quad \mathcal{F}^{o_2^i} = \langle \mathbf{u} | o_2^i(n) | \mathbf{u} \rangle, \quad i = 1, \dots, 3 \quad (6.1)$$

where the local operators $o_1(n)$ and $o_2^i(n)$ are given in (3.9) and (3.10). According to (4.35), these matrix elements are proportional to the following quantities

$$\mathcal{F}^{o_1} \propto F^A \equiv \langle \mathbf{u} | A(\theta_n + i/2) | \mathbf{u} \rangle. \quad (6.2)$$

$$\mathcal{F}^{o_2^1} \propto F^{AA} \equiv \langle \mathbf{u} | A(\theta_n + i/2) A(\theta_{n+1} + i/2) | \mathbf{u} \rangle, \quad (6.3)$$

$$\mathcal{F}^{o_2^2} \propto F^{BC} \equiv \langle \mathbf{u} | B(\theta_n + i/2) C(\theta_{n+1} + i/2) | \mathbf{u} \rangle,$$

$$\mathcal{F}^{o_2^3} \propto F^{CB} \equiv \langle \mathbf{u} | C(\theta_n + i/2) B(\theta_{n+1} + i/2) | \mathbf{u} \rangle,$$

$$\mathcal{F}^{o_2^4} \propto F^{DD} \equiv \langle \mathbf{u} | D(\theta_n + i/2) D(\theta_{n+1} + i/2) | \mathbf{u} \rangle.$$

In order to compute the building blocks (6.2) and (6.2), we act all the operators on the ket state $|\mathbf{u}\rangle$. The action of A and D on a Bethe state is

$$A(v) |\mathbf{u}\rangle = a(v) \frac{Q_{\mathbf{u}}(v-i)}{Q_{\mathbf{u}}(v)} |\mathbf{u}\rangle + \sum_{n=1}^N M_n(v) |\{\mathbf{u}, v\} \setminus \{u_n\}\rangle, \quad (6.4)$$

$$D(v) |\mathbf{u}\rangle = d(v) \frac{Q_{\mathbf{u}}(v+i)}{Q_{\mathbf{u}}(v)} |\mathbf{u}\rangle + \sum_{n=1}^N N_n(v) |\{\mathbf{u}, v\} \setminus \{u_n\}\rangle$$

where $M_k(v)$, $N_k(v)$ are given by

$$M_k(v) = \frac{ia(u_n)}{v-u_n} \prod_{j \neq n}^N \frac{u_n - u_j - i}{u_n - u_j}, \quad (6.5)$$

$$N_k(v) = \frac{id(u_n)}{u_n - v} \prod_{j \neq n}^N \frac{u_n - u_j + i}{u_n - u_j}.$$

These relations can be derived from the Yangian algebra (4.7). From (6.4) we see that the action of the operators A and D on a Bethe state preserve the number of magnons. In addition to the original Bethe state $|\mathbf{u}\rangle$, there is a sum of Bethe states $|\{\mathbf{u}, v\} \setminus \{u_k\}\rangle$ where one of the rapidities u_k is replaced by the spectral parameter v of the operator. These are called the *unwanted terms* and are off-shell for generic v . On the other hand, they are not too far from the on-shell Bethe state $|\mathbf{u}\rangle$ since most of the rapidities remain unchanged.

The action of the C operator on the Bethe state is more involved

$$C(v)|\mathbf{u}\rangle = \sum_{n=1}^N K_n |\{\mathbf{u}\} \setminus \{u_n\}\rangle + \sum_{k>n} K_{kn} |\{\mathbf{u}, v\} \setminus \{u_k, u_n\}\rangle \quad (6.6)$$

where

$$\begin{aligned} K_n &= \frac{ia(v)d(u_n)}{u_n - v} \prod_{j \neq n}^N \frac{u_j - u_n - i}{u_j - u_n} \cdot \frac{u_j - v + i}{u_j - v} + \\ &+ \frac{ia(u_n)d(v)}{v - u_n} \prod_{j \neq n}^N \frac{u_j - u_n + i}{u_j - u_n} \cdot \frac{u_j - v - i}{u_j - v} \\ K_{kn} &= \frac{d(u_k)a(u_n)}{(u_k - v)(u_n - v)} \frac{u_k - u_n + i}{u_k - u_n} \prod_{j \neq k, n} \frac{u_j - u_k - i}{u_j - u_k} \cdot \frac{u_j - u_n + i}{u_j - u_n} + \\ &+ \frac{d(u_n)a(u_k)}{(u_n - v)(u_k - v)} \frac{u_k - u_n - i}{u_k - u_n} \prod_{j \neq k, n} \frac{u_j - u_k + i}{u_j - u_k} \cdot \frac{u_j - u_n - i}{u_j - u_n} \end{aligned} \quad (6.7)$$

The coefficients K_n and K_{kn} can be expressed in terms of M_n and N_n

$$\begin{aligned} K_n(v) &= M_n(v)N_0(v) \frac{v - u_n}{v - u_n + i} + M_0(v)N_n(v) \frac{v - u_n}{v - u_n - i}, \\ K_{nk}(v) &= M_k(v)N_n(v) \frac{u_n - u_k}{u_n - u_k + i} + M_n(v)N_k(v) \frac{u_n - u_k}{u_n - u_k - i} \end{aligned} \quad (6.8)$$

if we define

$$M_0 \equiv a(v) \frac{Q_{\mathbf{u}}(v - i)}{Q_{\mathbf{u}}(v)}, \quad N_0 \equiv d(v) \frac{Q_{\mathbf{u}}(v + i)}{Q_{\mathbf{u}}(v)}. \quad (6.9)$$

From (6.6) it is clear that C reduces the number of magnons by one. For diagonal matrix elements, any C operator has to be accompanied by a B operator in order to preserve S^z , and obtain non-vanishing results. For length-2 operators the only possibilities are $B(u)C(v)$ and $C(u)B(v)$. Both combinations preserve the number of magnons but will lead to a sum of unwanted terms with one or two magnons replaced by the spectral parameters of the operators.

6.2 Form factors of length l_s operators

In general, the action of l_s ABA operators on a Bethe state generates unwanted terms with at most l_s rapidities replaced by the spectral parameters of the operators. In particular, it is clear now that all the building blocks (6.2) and (6.3) can be written as the scalar products of the following three types

$$\langle \mathbf{u} | \mathbf{u} \rangle, \quad \langle \mathbf{u} | \{\mathbf{u}, \theta_n^+\} \setminus \{u_k\} \rangle, \quad \langle \mathbf{u} | \{\mathbf{u}, \theta_n^+, \theta_{n+1}^+\} \setminus \{u_j, u_k\} \rangle. \quad (6.10)$$

where we have used the notation $\theta_n^+ = \theta_n + i/2$. The first scalar product is the Gaudin norm (4.30). The finite volume dependence of other two determinants is the subject of discussion in section 7.

The discussion of the previous subsection can be generalized to the diagonal matrix elements of the operators with $l_s > 2$. As before, any such matrix element can be spanned by some building blocks such as $F^{A\dots A}$, $F^{BCA\dots A}$, $F^{D\dots D}$. Of course, the number of the building blocks grows with the length of the operator.

We act all the ABA operators on the ket state which give rise to the unwanted terms

$$|\{\mathbf{u}, \theta_n^+, \dots, \theta_{n+M}^+\} \setminus \{u_{k_1}, \dots, u_{k_M}\}\rangle, \quad M \leq l_s. \quad (6.11)$$

Thus the diagonal matrix element of any length- l_s operator can be written as a linear combination of the following scalar products

$$\langle \mathbf{u} | \{\mathbf{u}, \theta_n^+, \dots, \theta_{n+M}^+\} \setminus \{u_{k_1}, \dots, u_{k_M}\} \rangle, \quad M \leq l_s. \quad (6.12)$$

The number of terms and the complexity of the coefficients will grow quickly with the increase of number of magnons and length of the operators, nevertheless the structure is robust.

7 Finite volume expansion

In this section, we analyze the structure of the scalar products (6.12) and show that any of them can be expanded in terms of diagonal minors of Gaudin norms. We call this procedure the *finite volume expansion*.

Above mentioned scalar products can be computed by the Slavnov determinant formula. In the Slavnov determinant (4.20), the denominator is a simple Cauchy determinant and can be computed readily. We therefore focus on the non-trivial numerator $\det_{jk} \Omega(u_j, v_k)$. Let us first consider the scalar product for the case of length-2 operators, $\langle \mathbf{u} | \{\mathbf{u}, \theta_n, \theta_{n+1}\} \setminus \{u_j, u_k\} \rangle$. The determinant takes the following form

$$\det \Omega = \begin{vmatrix} i\phi_{11} & \cdots & \Omega_{1j} & \cdots & \Omega_{1k} & \cdots & i\phi_{1N} \\ i\phi_{21} & \cdots & \Omega_{2j} & \cdots & \Omega_{2k} & \cdots & i\phi_{2N} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ i\phi_{N1} & \cdots & \Omega_{Nj} & \cdots & \Omega_{Nk} & \cdots & i\phi_{NN} \end{vmatrix}. \quad (7.1)$$

where we have defined $i\phi_{jk} = \Omega(u_j, u_k)$ and $\Omega_{ik} = \Omega_{ik}(u_i, \theta_n + i/2)$. The procedure is straightforward: *perform Laplace expansion with respect to the column or row that does not have any element of the form ϕ_{nn} repeatedly, until one can not do it further*. Note that after one Laplace expansion, we will obtain sub-determinants. We shall also perform the same procedure for all the sub-determinants until it terminates. This procedure will terminate when all the determinants in the expression take the form of diagonal minors (5.22) of

Gaudin norm (5.17)

$$\rho_N(\{i_1, \dots, i_m\}) = (-1)^m \begin{vmatrix} \phi_{i_1 i_1} & \cdots & \cdots & \cdots \\ \cdots & \phi_{i_2 i_2} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \phi_{i_m i_m} \end{vmatrix} \quad (7.2)$$

Therefore, the following expansion holds

$$\langle \mathbf{u} | \{ \mathbf{u}, \theta_n, \theta_{n+1} \} \setminus \{ u_j, u_k \} \rangle = \sum_{\alpha \subseteq A} \mathcal{F}(\bar{\alpha}) \rho_N(\alpha), \quad (7.3)$$

where $A = \{1, \dots, \hat{j}, \dots, \hat{k}, \dots, N\}$ ⁶ and the summation runs over all possible subsets α of A . Here $\bar{\alpha}$ is the complement of α in A . For an explicit and simple example, see appendix A.

Finally we need to justify why we call this procedure “finite volume expansion”. From a simple analysis below, it is clear that all the explicit L dependence are contained in the diagonal minors of the Gaudin norm. In the ABA approach, the diagonal matrix elements are given in terms of the following functions: the eigenvalue of the diagonal elements of the transfer matrix, $a(u)$ and $d(u)$, the products of functions $f(u-v)$ and $g(u-v)$ (6.4), (6.6), and the matrix elements in the Slavnov determinant formula Ω_{jk} and ϕ_{jk} . Under proper normalization, the functions $a(u)$ and $d(u)$ always appear in the expression as the ratio $a(u)/d(u) = e^{ipL}$. In fact, this kind of phase factor is either canceled by the same factors from the norm, or be replaced by products of scattering matrices using the Bethe Ansatz equations and they do not appear in the final expression. The products of $f(u-v)$ and $g(u-v)$ functions do not depend on L . The matrix element Ω_{jk} defined in (4.21) also has no dependence on L . Finally, ϕ_{jk} with $j \neq k$ reads

$$\phi_{jk} = \phi(u_j, u_k) = \frac{2}{(u_j - u_k)^2 + 1}, \quad j \neq k, \quad (7.4)$$

again, do not depend on L . The only dependence on L is hidden in the diagonal element ϕ_{nn} . Recall that we have

$$\phi_{nn} = \sum_{m=1}^L \frac{1}{(u_n - \theta_m)^2 + 1/4} - \sum_{\substack{l=1 \\ l \neq n}}^N \phi_{nl} \quad (7.5)$$

In the homogeneous limit, where $\theta_m = 0$ ($m = 1, \dots, L$), the first term of (7.5) becomes $L/(u_n^2 + 1/4)$ which depends linearly on L . When we perform the Laplace expansion, we carefully avoid expansion with this kind of terms and they only appear in the diagonal minor $\rho_N(\alpha)$. Therefore, the finite volume corrections are all contained in $\rho_N(\alpha)$. This is one part of the conjecture in [1].⁷

We have shown in section 6 that any diagonal matrix element can be written as a linear combination of specific determinant with coefficients that do not depend explicitly on L . As

⁶Here \hat{j} and \hat{k} mean these two indices are absent.

⁷There the authors used an equivalent description of the diagonal matrix element (5.26) in terms of the symmetric expansion (5.24), instead the connected one (5.20) that we used here.

we showed above, these determinants allow finite volume expansion, thus we can perform the finite volume expansion of any diagonal matrix element in the Heisenberg spin chain. As was shown in section 3, the diagonal matrix elements correspond to three-point functions of HHL type. Therefore we have shown that the structure of finite volume dependence of three-point functions conjectured in [1] is also valid at weak coupling at the leading order in the $\mathfrak{su}(2)$ sector. In section 9, we will show that the structure also holds at one-loop level.

This is only half of the story. In the conjecture [1], each coefficient $\mathcal{F}(\bar{\alpha})$ of $\rho(\alpha)$ is identified with the form factor of the same operator in *infinite volume*. In order to check this statement, it is desirable to have a formulation of the diagonal matrix elements of the Heisenberg spin chain directly in infinite volume. However, we are not aware of such a formulation, although it seems possible to do it in the framework of coordinate Bethe Ansatz. In principle, the infinite volume form factors for our case can also be obtained by first solving the Klose-McLoughlin axioms [19] and then take the weak coupling limit. However, no solution has been found up to now. Because of these reasons, we are not able to confirm that the coefficients we obtain from finite volume expansion are indeed the infinite volume form factors.

It is still of interest to know the explicit form of coefficients from our finite volume expansion. These will be our predictions for the diagonal form factors in the infinite volume theory. We perform the finite volume expansion for all the diagonal matrix elements of length-1 and length-2 operators and extract the coefficients. The results exhibit a nice structure and will be presented in section 8.

8 Infinite volume form factors

First, let us comment on the identity operator. As any multi-magnon diagonal matrix element of the identity operator equals to 1, matching it with the series (5.20), one can easily derive that all infinite volume connected form factors vanish except from the vacuum expectation value,

$$f^{\mathbb{I}}(\emptyset) = 1; \quad f^{\mathbb{I}}(u_1, \dots, u_N) = 0, \quad N \geq 1. \quad (8.1)$$

We should also discuss separately the simple case of the vacuum expectation values of spin chain operators. In the series (5.20), the zero magnon diagonal matrix element only contains the vacuum expectation value of the given operator in the infinite volume theory. So that, for the length-1 and length-2 operators one can easily find

$$f^{o_1}(\emptyset) = f^{o_2^1}(\emptyset) = 1, \quad f^{o_2^2}(\emptyset) = f^{o_2^3}(\emptyset) = f^{o_2^4}(\emptyset) = 0. \quad (8.2)$$

It holds also in the general case. Let us take the operators E_n^{ab} , ($a, b = 1, 2$) as a basis on the local quantum space, and linearly extend it to l_s neighboring site. Then only one, among this 4^{l_s} basis element, has non-vanishing vacuum expectation value, namely the one containing E^{11} at each site.

In the rest of this section, we will perform the finite volume expansion for the diagonal matrix elements of length-1 and length-2 operators. We will discuss a simple example, namely the case of length-1 operator with 2 magnons in detail and present the results for more complicated form factors.

8.1 An example: length-1 operator with -2 magnons

We consider the finite volume diagonal matrix element for the operator $o_1(n) = E_n^{11}$ with two magnons

$$\mathcal{F}_L^{o_1}(u_1, u_2) = \frac{\langle u_1, u_2 | o_1(n) | u_1, u_2 \rangle}{\langle u_1, u_2 | u_1, u_2 \rangle}. \quad (8.3)$$

It has the following structure in finite volume

$$\mathcal{F}_L^{o_1}(u_1, u_2) = \frac{1}{\rho_2(\{1, 2\})} (\rho_2(\{1, 2\}) + f^{o_1}(u_2) \rho_2(\{1\}) + f^{o_1}(u_1) \rho_2(\{2\}) + f^{o_1}(u_1, u_2)) \quad (8.4)$$

where $f^{o_1}(\mathbf{u})$ is to be identified with the connected diagonal form factor of o_1 in the infinite volume theory (5.13).

We proceed as described in the previous sections. Using the solution of QISP, we have

$$\mathcal{F}_L^{o_1}(u_1, u_2) = \frac{1}{t_{\mathbf{u}}(\theta_n^+)} \frac{\langle u_1, u_2 | A(\theta_n^+) | u_1, u_2 \rangle}{\langle u_1, u_2 | u_1, u_2 \rangle}, \quad (8.5)$$

where the denominator is the Gaudin norm (4.30),

$$\langle u_1, u_2 | u_1, u_2 \rangle = \left(\prod_{j=1}^2 a(u_j) d(u_j) \right) \frac{1 + (u_1 - u_2)^2}{(u_1 - u_2)^2} \rho_2(\{1, 2\}). \quad (8.6)$$

From (6.4),

$$\begin{aligned} \langle u_1, u_2 | A(\theta_n^+) | u_1, u_2 \rangle &= M_0(\theta_n^+) \langle u_1, u_2 | u_1, u_2 \rangle \\ &+ M_1(\theta_n^+) \langle u_1, u_2 | u_2, \theta_n^+ \rangle + M_2(\theta_n^+) \langle u_1, u_2 | u_1, \theta_n^+ \rangle. \end{aligned} \quad (8.7)$$

We introduce some notations in order to simplify the expressions. Let us define

$$\mathcal{C}_{\mathbf{u}, \mathbf{v}} = \frac{\prod_{j=1}^N a(v_j) d(u_j)}{\det_{jk} \frac{1}{u_j - v_k + i}}, \quad (8.8)$$

so that

$$\langle \mathbf{v} | \mathbf{u} \rangle = \mathcal{C}_{\mathbf{u}, \mathbf{v}} \det_{jk} \Omega(u_j, v_k). \quad (8.9)$$

By perform the finite volume expansion for the three scalar products, we obtain

$$\begin{aligned} \langle u_1, u_2 | A(\theta_n^+) | u_1, u_2 \rangle &= -\mathcal{C}_{\{u_1, u_2\}, \{u_1, u_2\}} M_0(\theta_n^+) \rho_2(\{1, 2\}) \\ &- i \mathcal{C}_{\{u_1, u_2\}, \{u_1, \theta_n^+\}} M_2(\theta_n^+) \Omega(u_2, \theta_n^+) \rho_2(\{1\}) \\ &- i \mathcal{C}_{\{u_1, u_2\}, \{u_2, \theta_n^+\}} M_1(\theta_n^+) \Omega(u_1, \theta_n^+) \rho_2(\{2\}) \\ &- \phi_{12} \left(\mathcal{C}_{\{u_1, u_2\}, \{u_2, \theta_n^+\}} \Omega(u_2, \theta_n^+) M_1(\theta_n^+) + \mathcal{C}_{\{u_1, u_2\}, \{u_1, \theta_n^+\}} \Omega(u_1, \theta_n^+) M_2(\theta_n^+) \right) \end{aligned} \quad (8.10)$$

Plugging (8.10) into (8.5) and comparing to the expansion (8.4), we obtain the expression for the various form factors in infinite volume

$$f^{o_1}(u_1) = i \frac{\mathcal{C}_{\{u_1, u_2\}, \{u_2, \theta_n^+\}} M_1(\theta_n^+)}{\mathcal{C}_{\{u_1, u_2\}, \{u_1, u_2\}} M_0(\theta_n^+)} \Omega(u_1, \theta_n^+) \quad (8.11)$$

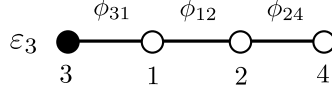


Figure 2. Diagrammatic representation of one term in (8.13) with 4 magnons.

$$f^{o_1}(u_2) = i \frac{\mathcal{C}_{\{u_1, u_2\}, \{u_1, \theta_n^+\}} M_2(\theta_n^+)}{\mathcal{C}_{\{u_1, u_2\}, \{u_1, u_2\}} M_0(\theta_n^+)} \Omega(u_2, \theta_n^+)$$

$$f^{o_1}(u_1, u_2) = \frac{\phi_{12}}{\mathcal{C}_{\{u_1, u_2\}, \{u_1, u_2\}} M_0(\theta_n^+)} \left(\mathcal{C}_{\{u_1, u_2\}, \{u_2, \theta_n^+\}} \Omega(u_2, \theta_n^+) M_1(\theta_n^+) \right. \\ \left. + \mathcal{C}_{\{u_1, u_2\}, \{u_1, \theta_n^+\}} \Omega(u_1, \theta_n^+) M_2(\theta_n^+) \right)$$

Substituting the explicit expressions in (8.11) and, at the end, taking the homogeneous limit $\theta_n \rightarrow 0$, we obtain very compact results for the infinite volume connected form factors,

$$f^{o_1}(u_k) = \frac{1}{u_k^2 + 1/4}, \quad k = 1, 2 \quad (8.12)$$

$$f^{o_1}(u_1, u_2) = \left(\frac{1}{u_1^2 + 1/4} + \frac{1}{u_2^2 + 1/4} \right) \frac{2}{1 + (u_1 - u_2)^2}.$$

8.2 Length-1 operator with N magnons

We can perform the same calculation as in the previous subsection and extract the form factors with more magnons. The process becomes cumbersome for higher number of particles. However, from the first few magnon cases, we are able to observe a nice pattern of the connected form factors. The N -magnon connected diagonal form factor for $o_1(n)$ is given as

$$f^{o_1}(u_1, \dots, u_N) = \varepsilon_1 \phi_{12} \phi_{23} \dots \phi_{N-1, N} + \text{permutations} \quad (8.13)$$

where ε_k is the energy of the magnon with rapidity u_k and ϕ_{jk} can be seen as some “propagator” defined as

$$\varepsilon_k = \varepsilon(u_k) = \frac{1}{u_k^2 + 1/4}, \quad \phi_{jk} = \frac{2}{1 + (u_j - u_k)^2}, \quad j \neq k. \quad (8.14)$$

The expression (8.13) can be represented by the diagrams in figure 2. Each node is labeled by a number from 1 to N . The leftmost node is associated with the energy of its label. The lines between two neighboring nodes are associated with a propagator. Multiplying the factors we obtain the value of the diagram. Summing over all the permutations of the labeling gives the result for infinite volume form factor f^{o_1} . The result for an N magnon state is thus a sum over $N!$ terms.

The structure of f^{o_1} is exactly the structure of the connected form factors of conserved charge densities (5.25). This is not surprising, since $o_1(n) = \frac{1}{2}(\mathbb{I} + \sigma_n^z)$ is indeed a length-1 conserved charge density of the Heisenberg spin chain. The nice feature is that once we know the one particle eigenvalue $q(u)$ of the charge, we can immediately write down the expression for the corresponding infinite volume form factors. We remark here that our result (8.13) is consistent with the determinant formula of [25].

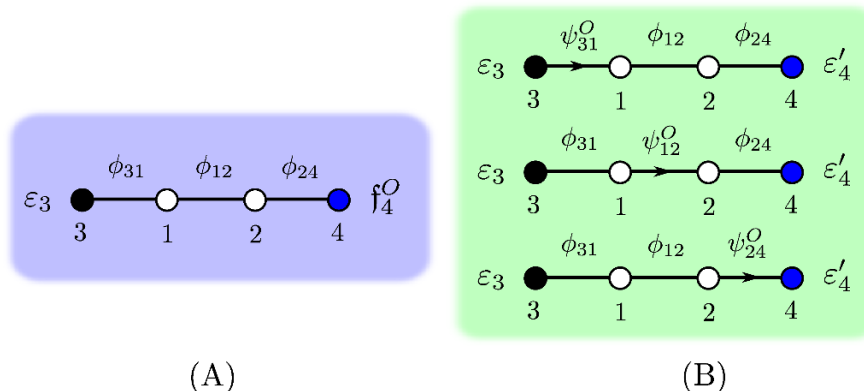


Figure 3. Different kinds of diagrams encoding length-2 form factors.

8.3 Length-2 form factors of N magnons

The calculation of infinite volume matrix elements can be performed following the same line as in section 8.1 but the process is more involved. Nevertheless, we again found some patterns for the various matrix elements which we present below. The structure for the length-2 operators can be encoded into diagrams similar to the one in figure 2. However, in this case we have two types of them, as are shown in figure 3.

Each diagonal matrix element is given by two kinds of diagrams. This structure was tested for all length-2 operators up to 5 magnons and we conjecture that it holds for any number of magnons. The first kind is depicted in the blue region. We label the nodes by number from 1 to N . The leftmost node is associated with $\varepsilon(u_j)$ while the rightmost node is associated with a function denoted by $f^O(u_k)$, and it depends on the operator. The other kind, depicted in the green region, is more interesting. The leftmost and rightmost nodes are associated with ε_j and ε'_k where $\varepsilon'(u) = \frac{\partial}{\partial u}\varepsilon(u)$. In addition, for a given label of the nodes, one needs to sum over the diagrams which replaces one of the propagators by a “directed propagator”, $\psi^O_{ij} = \psi^O(u_i, u_j)$. The directed propagator is antisymmetric with respect to its arguments $\psi^O_{ij} = -\psi^O_{ji}$ and its explicit form depends on the operator under consideration.

In summarizing, the infinite volume diagonal matrix element of a length-2 operator, $f^O(u_1, \dots, u_N)$, is characterized by two functions $f^O(u)$ and $\psi^O(u, v)$. The result for N -magnon is given by

$$f^O(u_1, \dots, u_N) = (\varepsilon_1 \phi_{12} \dots \phi_{N-1,N} f^O_N + \text{permutations}) + \left(\sum_{i=1}^{N-1} \varepsilon_1 \phi_{12} \dots \psi^O_{i,i+1} \dots \phi_{N-1,N} \varepsilon'_N + \text{permutations} \right) \quad (8.15)$$

We list the data for o_2^i (3.9), (3.10) is the following:

$$\begin{aligned} f^{o_2^1}(u) &= 2 & \psi^{o_2^1}(u, v) &= -(u-v)(uv-1/4)\phi(u, v) \\ f^{o_2^2}(u) &= -\frac{u-i/2}{u+i/2} & \psi^{o_2^2}(u, v) &= (u-v)(u-i/2)(v-i/2)\phi(u, v) \end{aligned}$$

$$\begin{aligned}
 f^{o_2^3}(u) &= -\frac{u+i/2}{u-i/2} & \psi^{o_2^3}(u,v) &= (u-v)(u+i/2)(v+i/2)\phi(u,v) \\
 f^{o_2^4}(u) &= 0 & \psi^{o_2^4}(u,v) &= -(u-v)(uv-1/4)\phi(u,v)
 \end{aligned}
 \tag{8.16}$$

Let us comment on this results (8.16). These data for the operators can be read off simply from the computation of 2 magnon case. Therefore, one should simply compute the 2-magnon matrix elements and perform the finite volume expansion to extract the data. Once the functions in (8.16) are known, we can write down any diagonal form factor of length-2 operators in the infinite volume. Any length-2 operator is a linear combination of the identity operator, o_1 and o_2^i ,

$$\mathcal{O}_n = b\mathbb{I} + c_0 o_1(n) + \tilde{c}_0 o_1(n+1) + \sum_{i=1}^3 c_i o_2^i(n)
 \tag{8.17}$$

where b , c_0 and c_i ($i = 1, 2, 3$) are some numbers. Then the data of \mathcal{O} is simply given by

$$f^{\mathcal{O}}(u) = c_0 + \tilde{c}_0 + \sum_{i=1}^3 c_i f^{o_2^i}(u), \quad \psi^{\mathcal{O}}(u,v) = \sum_{i=1}^3 c_i \psi^{o_2^i}(u,v),
 \tag{8.18}$$

and its vacuum expectation value is

$$f^{\mathcal{O}}(\emptyset) = b + c_0 + \tilde{c}_0 + c_1.
 \tag{8.19}$$

For example, the operator o_2^4 is not independent

$$o_2^4(n) = \mathbb{I} - o_1(n) - o_1(n+1) + o_2^1(n).$$

Note that by translational invariance, $o_1(n)$ gives the same result as $o_1(n+1)$ when computing the form factors. It is easy to check that this resolution is consistent with (8.2), (8.16). The diagonal matrix elements of the operators o_2^2 and o_2^3 are related by complex conjugation which is also manifest in (8.16).

8.4 Examples of length-2 operators

We compute two examples below. First, the matrix elements of the length-2 conserve charge density, which is the permutation operator $P_{k,k+1}$, or equivalently the Hamiltonian density $H_{k,k+1} = I_{k,k+1} - P_{k,k+1}$. We will see that the data for the permutation operator simplifies and the final result takes exactly the form predicted in (5.25). This is a non-trivial check of our functions (8.16). Another example is an HHL three-point function, with the light operator being the rotated BMN vacuum of length 4.

Permutation operator. The permutation operator $P_{k,k+1}$ is a length-2 operator of the Heisenberg spin chain. It can be written in terms of the operators o_2^i with equal weights

$$P_{k,k+1} = \sum_{i,j=1}^2 E_k^{ij} E_{k+1}^{ji} = \sum_{i=1}^4 o_2^i.
 \tag{8.20}$$

According to (8.18), the data of permutation operator is given by

$$f^P(u) = \varepsilon(u) = \frac{1}{u^2 + 1/4}, \quad \psi^P(u, v) = 0, \quad f^P(\emptyset) = 1. \quad (8.21)$$

The infinite volume form factor (8.15) with the entire (8.21) has the structure as a conserved charge should have (5.25), with the one particle eigenvalue of the corresponding charge being $-\varepsilon(u)$.

A three-point function with $\mathcal{O}_\alpha = \text{Tr } \tilde{Z}\tilde{Z}\tilde{Z}\tilde{Z}$. As another example, we compute a HHL three-point function with the light operator being $\mathcal{O}_\alpha = \tilde{\mathcal{O}} = \text{Tr } \tilde{Z}\tilde{Z}\tilde{Z}\tilde{Z}$, which is the rotated BMN vacuum. The scalar field \tilde{Z} is defined as

$$\tilde{Z} = Z + \bar{Z} + i(X + \bar{X}). \quad (8.22)$$

Following our strategy, we first map the field theory operator to the spin chain operator $\tilde{\mathcal{O}}$ using (3.4). Then we write the operator $\tilde{\mathcal{O}}$ in terms of linear combinations of the basis operators o_1, o_2^i ($i = 1, \dots, 3$). This enable us to write down the data for the operator $\tilde{\mathcal{O}}$ and thus the infinite volume form factor. We have⁸

$$\tilde{\mathcal{O}}_n = 4(\mathbb{I} - 2o_1(n) - 2o_1(n+1) + 4o_2^1(n) - o_2^2(n) - o_2^3(n)). \quad (8.23)$$

Using the representation (8.23) and also the data (8.16), we get for the operator $\tilde{\mathcal{O}}$

$$f^{\tilde{\mathcal{O}}}(u) = \frac{96u^2 + 8}{4u^2 + 1}, \quad \psi^{\tilde{\mathcal{O}}}(u, v) = -6(u - v)(4uv - 1)\phi(u, v). \quad (8.24)$$

The corresponding infinite volume form factors can be obtained from the general prescription (8.15) for the length-2 form factors.

9 Matrix elements at one loop

In this section, we generalize the above considerations from tree level to one loop. We show that the form factors at one loop can again be written in terms of a finite number of “building blocks”. These building blocks are matrix elements of the inhomogeneous Heisenberg $XXX_{1/2}$ spin chain with the inhomogeneities fixed to some specific values, called the BDS values, and can be written in terms of scalar products (6.12) for which one can perform the finite volume expansion, as at the tree level.

There are several new features for three-point functions at higher loops. The dilatation operator in the $\mathfrak{su}(2)$ sector is no longer the Hamiltonian of the Heisenberg $XXX_{1/2}$ spin chain, but becomes long-range interacting, called the BDS spin chain [38]. Therefore the two large operators correspond to the eigenvectors of the BDS spin chain. The BDS spin chain is related to a special inhomogeneous Heisenberg $XXX_{1/2}$ spin chain by a unitary transformation [11, 39]. Therefore its eigenstates can be obtained from the eigenstates

⁸Factor of 4 appeared due to the symmetry of the $\tilde{\mathcal{O}}$.

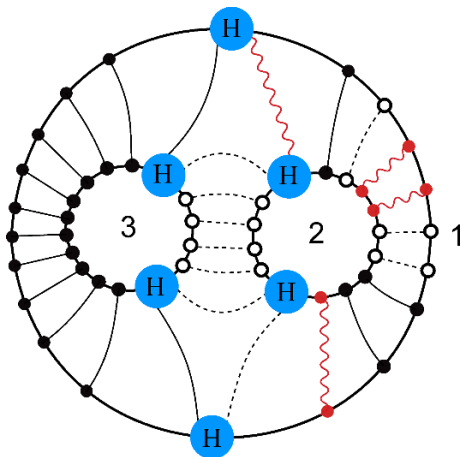


Figure 4. The quantum corrections are taken into account by operator insertions at the splitting points. At one-loop in the $\mathfrak{so}(6)$ sector, the insertion takes the form of one-loop Hamiltonian density.

of the inhomogeneous Heisenberg spin chain by performing the unitary transformation, namely $|\mathbf{u}\rangle_{\text{BDS}} = \text{S}|\mathbf{u}; \boldsymbol{\theta}^{\text{BDS}}\rangle$. The BDS values of the inhomogeneities are given by

$$\theta_k^{\text{BDS}} = 2g \sin \frac{2\pi k}{L}, \quad k = 1, \dots, L \quad (9.1)$$

where g is the coupling constant. The unitary operator S has been worked out in [11] up to g^2 order

$$\text{S} = \exp i \sum_{k=1}^L \left[\nu_k \text{H}_k + \frac{i}{2} \rho_k [\text{H}]_k \right], \quad (9.2)$$

where $\text{H}_k \equiv \text{I}_{k,k+1} - \text{P}_{k,k+1}$ and $[\text{H}]_k \equiv [\text{H}_k, \text{H}_{k+1}]$. The parameters ν_k and ρ_k are related to the inhomogeneities by

$$\nu_k = - \sum_{j=1}^k \theta_j, \quad \rho_k = 2g^2 k - \theta_k \nu_k - \sum_{j=1}^k \theta_j^2, \quad k = 1, \dots, L. \quad (9.3)$$

It is obvious that $\nu_k \sim g$ and $\rho_k \sim g^2$.

Another new feature is that the quantum corrections manifest themselves as operator insertions at the splitting points [6, 7], as is shown in figure 4. The operator insertions take the form of the Hamiltonian density of the spin chain. In our set up we need insertions for the $\mathfrak{so}(6)$ sector, since the light operator belongs to $\mathfrak{so}(4)$, which is not closed at one loop.

At one loop level, we need to take into account the aforementioned new features. In what follows, we first discuss the effect of the operator insertions and show that the three-point function under consideration can be reduced into the calculation of correlation functions of the BDS spin chain. Then we consider the effect of the S operator on the spin operators and reduce the correlation functions of BDS spin chain into the correlation functions of the inhomogeneous Heisenberg $\text{XXX}_{1/2}$ spin chain. As we discussed in the tree level case, the matrix elements of the inhomogeneous Heisenberg spin chain can be written in terms of scalar products (6.12), and we can perform the finite volume expansion.

9.1 The effects of one-loop operator insertions

For each spin chain state, there are two operator insertions at the two splitting points. We first discuss the effects of insertions for the “light” operator \mathcal{O}_α . The one-loop insertion takes the form of the Hamiltonian density

$$H_l^{\text{so}(6)} = K_{l,l+1} + 2I_{l,l+1} - 2P_{l,l+1} \tag{9.4}$$

where $I_{l,l+1}$, $P_{l,l+1}$ and $K_{l,l+1}$ are the identity, permutation and trace operators. They act on the $\mathfrak{so}(6)$ spin chain states as

$$\begin{aligned} I_{l,l+1}|\dots\phi_l^i\phi_{l+1}^j\dots\rangle &= |\dots\phi_l^i\phi_{l+1}^j\dots\rangle, \\ P_{l,l+1}|\dots\phi_l^i\phi_{l+1}^j\dots\rangle &= |\dots\phi_l^j\phi_{l+1}^i\dots\rangle, \\ K_{l,l+1}|\dots\phi_l^i\phi_{l+1}^j\dots\rangle &= \delta_{ij} \sum_{k=1}^6 |\dots\phi_l^k\phi_{l+1}^k\dots\rangle. \end{aligned} \tag{9.5}$$

At one loop level, the light operator should be an eigenvalue of the two-loop dilatation operator. The $\mathfrak{so}(6)$ sector is closed only at one-loop so in principle one needs fields outside the $\mathfrak{so}(6)$ sector, like fermionic fields, to construct the eigenstates of the two loop dilatation operator. However, when computing the three-point functions, the Feymann diagrams, involving fields apart from $\{X, Z, Y, \bar{X}, \bar{Z}, \bar{Y}\}$, will not contribute at one loop order. Thus, the only new fields which we have to take into account at one loop are Y and \bar{Y} . It's easy to see from figure 5 that the light operator, in the presence of one-loop insertions, can again be written in terms of local spin chain operators σ^\pm, σ^z , due to the fact that the heavy operators are still can be expressed as $\mathfrak{su}(2)$ spin chain states. Therefore, our considerations before can be generalized here. To summarize, the one-loop structure constants can be recast to the calculation of matrix elements of the BDS spin chain ${}_{\text{BDS}}\langle \mathbf{u} | \hat{O}(\sigma^\pm, \sigma^z; g^2) | \mathbf{u} \rangle_{\text{BDS}}$.

Now we consider the operator insertions for the heavy states. The effect of these insertions is increasing the length of the spin chain operator, as is shown in (6).

This can be seen easily by noticing that

$$P_{k,k+1} = \sum_{i,j=1}^2 E_m^{ij} \otimes E_{m+1}^{ji} \tag{9.6}$$

For example, we have the following

$$\star E_m^{11} H_m \star = (\star E_m^{11} \star) - (\star E_m^{11} E_{m+1}^{11} \star) - (\star E_m^{11} E_{m+1}^{12} \star) \tag{9.7}$$

where the star stands for some strings of operators.

9.2 The effects of the unitary S operator

In this subsection, we discuss the action of unitary operator S on the spin operators. We are interested in the following quantity

$$\langle \mathbf{u}; \boldsymbol{\theta}^{\text{BDS}} | S^{-1} \hat{O}_{l+1}(\sigma^\pm, \sigma^z) S | \mathbf{u}; \boldsymbol{\theta}^{\text{BDS}} \rangle. \tag{9.8}$$

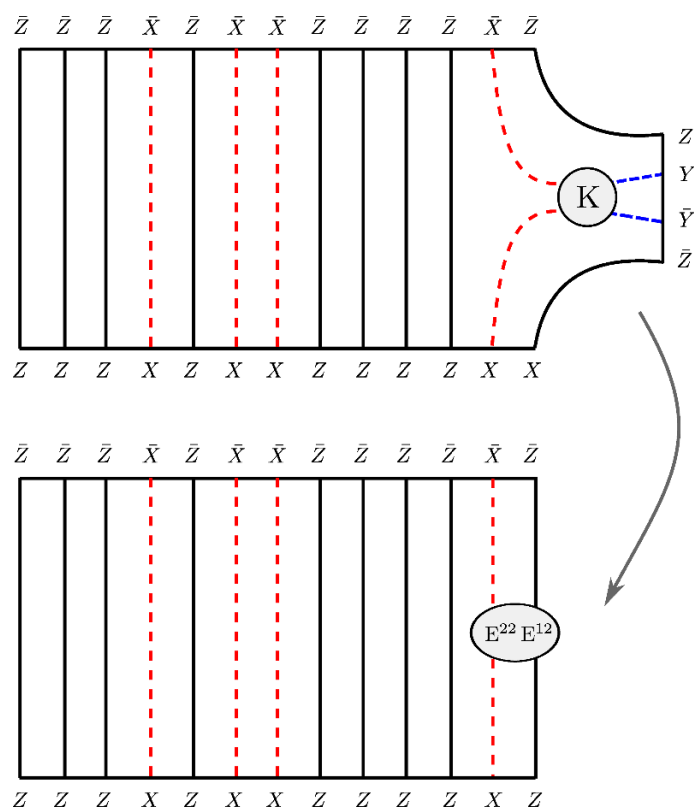


Figure 5. An example of mapping the light operator to the spin chain operator at one-loop in the presence of one-loop insertion.

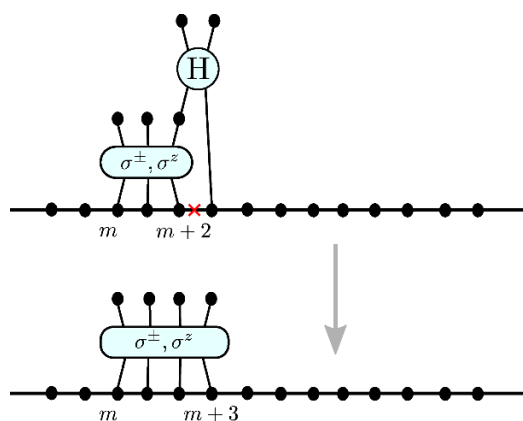


Figure 6. The effect of operator insertions for the heavy operators. They increase the length of the spin operator by 1. The red cross denotes the splitting point.

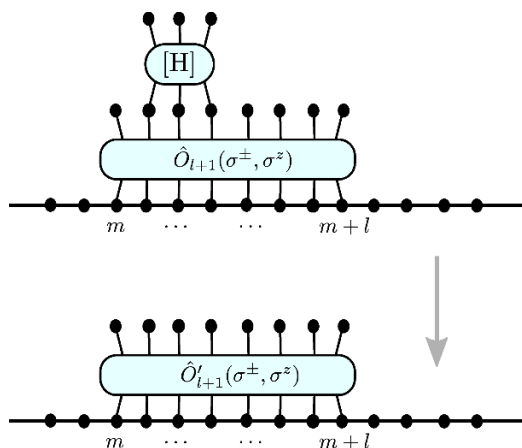


Figure 7. The length preserving action generated by $[H]_k$ on the spin chain vertex.

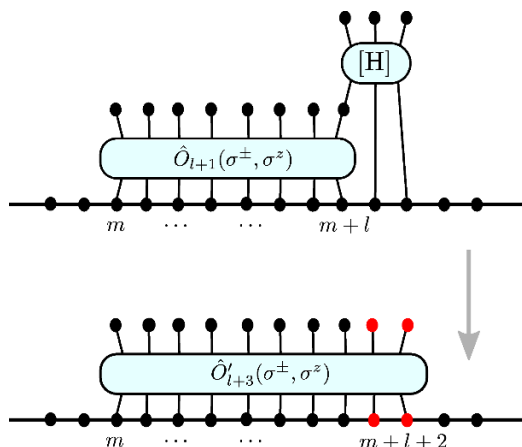


Figure 8. The length changing action generated by $[H]_k$ on the spin chain vertex. In this example, it increases the length of the spin chain operator by 2.

The S operator takes an exponential form $S = \exp \hat{F}$, thus we have

$$S^{-1} \hat{O}_{l+1}(\sigma^\pm, \sigma^z) S = \hat{O}_{l+1}(\sigma^\pm, \sigma^z) - [\hat{F}, \hat{O}_{l+1}(\sigma^\pm, \sigma^z)] + \frac{1}{2} [\hat{F}, [\hat{F}, \hat{O}_{l+1}(\sigma^\pm, \sigma^z)]] + \mathcal{O}(g^3) \tag{9.9}$$

where we have truncated up to $\mathcal{O}(g^2)$ order. The action of S operator on the spin chain operator can be divided into two types. The first type is length preserving, it originates from the operators H_k and $[H]_k$ that act within the range of the spin chain operator \hat{O}_{l+1} , which gives rise to an operator with the same length, this is depicted in figure 7.

The other type of the action increases the length of the operator by 1 or 2, which are generated from the operators at the boundary of the spin chain operator. There are two kinds of length changing processes at one loop. One process is generated by a single H_k or $[H]_k$, which is given in figure 8.

The other process is generated by two H_k 's, one example of which is given in figure 9.

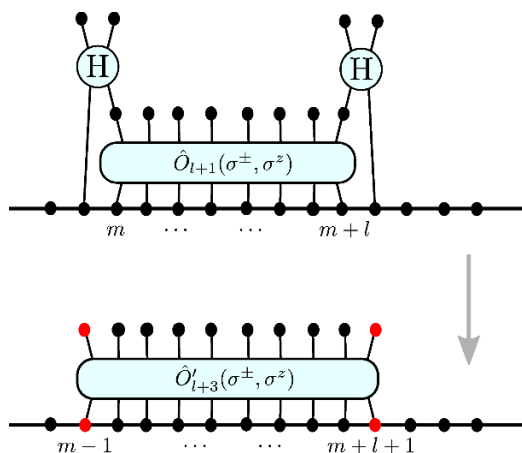


Figure 9. The length changing action generated by two H_k 's on both ends of the spin chain vertex. The length of the spin operator also increases by 2 in this example.

From our analysis we see that the action of the S operator on the spin chain operators, in general, increases the length of the spin chain operator. Up to $\mathcal{O}(g^2)$ order, the length of the operator increases at most by 2.

$$S^{-1} \hat{O}_l(\sigma^\pm, \sigma^z) S = \hat{O}'_l(\sigma^\pm, \sigma^z) + \hat{O}'_{l+1}(\sigma^\pm, \sigma^z) + \hat{O}'_{l+2}(\sigma^\pm, \sigma^z) + \mathcal{O}(g^3) \quad (9.10)$$

This implies that in order to compute the form factor of length l operator for BDS spin chain, we need to compute the form factors of length $l + 2$, $l + 1$ and l operators in the inhomogeneous Heisenberg $XXX_{1/2}$ spin chain.

Once we write the three-point function in terms of matrix elements of the inhomogeneous $XXX_{1/2}$ spin chain, we can perform the finite volume expansion and organize the results in the form conjectured in [1]. At one loop, the matrix element of Gaudin norm is modified. The equations (4.26), (4.27) and (4.29) are still valid, but the eigenvalues $a(u)$ and $d(u)$ are corrected

$$a(u) = \prod_{k=1}^L (u - \theta_k^{\text{BDS}} + i/2) = x(u + i/2)^L + \mathcal{O}(g^{2L}), \quad (9.11)$$

$$d(u) = \prod_{k=1}^L (u - \theta_k^{\text{BDS}} - i/2) = x(u - i/2)^L + \mathcal{O}(g^{2L}),$$

where $x(u)$ is the Zhukowsky map given by

$$x(u) + \frac{g^2}{x(u)} = u. \quad (9.12)$$

By replacing

$$p(u_k) = \frac{u_k + i/2}{u_k - i/2} \longrightarrow \frac{x(u_k + i/2)}{x(u_k - i/2)} \quad (9.13)$$

in (4.26) and expanding the result up to $\mathcal{O}(g^2)$ order, we obtain the Gaudin norm at one-loop, $\rho_n^{1\text{-loop}}$. In fact, the replacement (9.13) gives the correct Gaudin norm up to wrapping

orders [10, 11]. Our conclusion is that the structure conjectured in [1] is also valid at one loop level with respect to the one-loop Gaudin norm.

Of course the coefficients or the infinite volume form factors at one-loop will be more complicated. It is an interesting question to see how the infinite volume form factors are deformed at one loop and whether it is possible to bootstrap to all loops.

10 Conclusion

In this paper we considered symmetric HHL correlators at weak coupling of the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM theory. Using the formalism of computing matrix element of $XXX_{1/2}$ spin chain (see e.g. [25]), we showed that at tree level, as well as at one loop, the finite volume dependence (up to the wrapping corrections) of this kind of correlator is given by the expression conjectured in [1]. Apart from giving the general arguments of the proof we computed the simplest non-trivial correlators at tree level, which correspond to the case when the length of the light operator is equal to 4. The structure of the coefficients of the finite volume expansion allowed us to conjecture their form for any number of excitations of the heavy operator. We showed that these coefficients can be expressed in terms of several functions of the rapidities defined by the light operator. Another aspect of the conjecture proposed in [1] suggests that the coefficients of the above mentioned finite volume expansion can be interpreted as appropriate infinite volume form factors. Therefore our explicit calculation provides us with concrete proposal for the infinite volume form factor.

As a continuation of our work it would be interesting to see how the coefficients in the finite volume expansion are deformed at one loop. As an obstacle, one should mention that, as it follows from section 9, the complexity of the simplest one-loop computation is equivalent to a tree level computation with a light operator of the length equal to 6, and the calculation becomes much more involved. On the other hand, the recent result of [14] suggests an all-loop method for computing three-point correlators. And it would be interesting to merge their method with the approach developing in this paper and check the conjecture of [1] at all loop. In addition, it would give us all-loop prediction for the diagonal infinite volume form factors.

Another interesting direction to pursue is to try to see whether the same structure of the finite volume dependence holds for symmetric HHL correlators of the other sectors of $\mathcal{N}=4$ SYM.

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A An example of finite volume expansion

In this appendix, we give an explicit example in order to illustrate how to perform the finite volume expansion of the special scalar products defined in section 7. The scalar product under consideration is $\langle \{u_1, u_2, u_3, u_4\} | \{u_1, \theta, u_3, u_4\} \rangle$, where $\{u_1, u_2, u_3, u_4\}$ is a set of Bethe roots. Consider the numerator of the Slavnov determinant formula (4.20),

$$\langle \{u_1, u_2, u_3, u_4\} | \{u_1, \theta, u_3, u_4\} \rangle \propto \begin{vmatrix} \phi_{11} & \Omega_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \Omega_{22} & \phi_{23} & \phi_{24} \\ \phi_{31} & \Omega_{32} & \phi_{33} & \phi_{34} \\ \phi_{41} & \Omega_{42} & \phi_{43} & \phi_{44} \end{vmatrix} \quad (\text{A.1})$$

We first perform the Laplace expansion for the second column, which gives

$$\begin{aligned} & -\Omega_{12} \begin{vmatrix} \phi_{21} & \phi_{23} & \phi_{24} \\ \phi_{31} & \phi_{33} & \phi_{34} \\ \phi_{41} & \phi_{43} & \phi_{44} \end{vmatrix} + \Omega_{22} \begin{vmatrix} \phi_{11} & \phi_{13} & \phi_{14} \\ \phi_{31} & \phi_{33} & \phi_{34} \\ \phi_{41} & \phi_{43} & \phi_{44} \end{vmatrix} \\ & -\Omega_{32} \begin{vmatrix} \phi_{11} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{23} & \phi_{24} \\ \phi_{41} & \phi_{43} & \phi_{44} \end{vmatrix} + \Omega_{42} \begin{vmatrix} \phi_{11} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{23} & \phi_{24} \\ \phi_{31} & \phi_{33} & \phi_{34} \end{vmatrix} \end{aligned} \quad (\text{A.2})$$

The Laplace expansion gives rise to 4 terms, which we shall denote T_i , $i = 1, \dots, 4$. For T_1 , we do Laplace expansion by the first column

$$\begin{aligned} T_1 &= -\Omega_{12} \{ \phi_{21} \rho_4(\{3, 4\}) - \phi_{31}(\phi_{23} \rho_4(\{4\}) - \phi_{24} \phi_{43}) + \phi_{41}(\phi_{23} \phi_{34} - \phi_{24} \rho_4(\{3\})) \} \\ &= -\Omega_{12} \phi_{21} \rho_4(\{3, 4\}) + \Omega_{12} \phi_{41} \phi_{23} \rho_4(\{3\}) + \Omega_{12} \phi_{31} \phi_{23} \rho_4(\{4\}) \\ &\quad - \Omega_{12}(\phi_{31} \phi_{24} \phi_{43} + \phi_{41} \phi_{24} \phi_{34}) \end{aligned}$$

The second term already takes the form of diagonal minor of the Gaudin norm

$$T_2 = \Omega_{22} \rho_4(\{1, 3, 4\}) \quad (\text{A.3})$$

For the third term, we perform Laplace expansion with respect to the second column

$$\begin{aligned} T_3 &= -\Omega_{32} \phi_{23} \rho_4(\{1, 4\}) + \Omega_{32} \phi_{43} \phi_{24} \rho_4(\{1\}) + \Omega_{32} \phi_{13} \phi_{21} \rho_4(\{4\}) \\ &\quad - \Omega_{32}(\phi_{13} \phi_{24} \phi_{41} + \phi_{43} \phi_{14} \phi_{21}) \end{aligned}$$

For the last term, we perform Laplace expansion with respect to the last column

$$\begin{aligned} T_4 &= -\Omega_{42} \phi_{24} \rho_4(\{1, 3\}) + \Omega_{42} \phi_{34} \phi_{23} \rho_4(\{1\}) + \Omega_{42} \phi_{14} \phi_{21} \rho_4(\{3\}) \\ &\quad - \Omega_{42}(\phi_{14} \phi_{23} \phi_{31} + \phi_{34} \phi_{13} \phi_{21}) \end{aligned}$$

Collecting terms from the above calculation, we obtain the finite volume expansion of the scalar product

$$\begin{aligned}
 & \langle \{u_1, u_2, u_3, u_4\} | \{u_1, \theta, u_3, u_4\} \rangle \propto \tag{A.4} \\
 & \Omega_{22} \rho_4(\{1, 3, 4\}) - \Omega_{42} \phi_{24} \rho_4(\{1, 3\}) - \Omega_{32} \phi_{23} \rho_4(\{1, 4\}) - \Omega_{12} \phi_{21} \rho_4(\{3, 4\}) \\
 & + (\Omega_{32} \phi_{43} \phi_{24} + \Omega_{42} \phi_{34} \phi_{23}) \rho_4(\{1\}) + (\Omega_{42} \phi_{14} \phi_{21} + \Omega_{12} \phi_{41} \phi_{24}) \rho_4(\{3\}) \\
 & + (\Omega_{12} \phi_{31} \phi_{23} + \Omega_{32} \phi_{13} \phi_{21}) \rho_4(\{4\}) - \Omega_{12} (\phi_{31} \phi_{24} \phi_{43} + \phi_{41} \phi_{23} \phi_{34}) \\
 & - \Omega_{32} (\phi_{13} \phi_{24} \phi_{41} + \phi_{43} \phi_{14} \phi_{21}) - \Omega_{42} (\phi_{14} \phi_{23} \phi_{31} + \phi_{34} \phi_{13} \phi_{21})
 \end{aligned}$$

In fact, it is not hard to convince ourselves that the similar expansion can be performed for general scalar products defined in section 7. For length-2 operators, we have the following finite volume expansion

$$\langle \mathbf{u} | \{\mathbf{u}, \theta_n, \theta_{n+1}\} \setminus \{u_j, u_k\} \rangle = \sum_{\alpha \subseteq A} F_{\bar{\alpha}} \rho(\alpha), \quad A = \{1, 2, \dots, \hat{j}, \dots, \hat{k}, \dots, N\} \tag{A.5}$$

In general, the terms of the expansion grows quickly with the number of excitations and the expansion coefficients $F_{\bar{\alpha}}$ might get quite involved.

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