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Universal corner entanglement from twist operators

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Abstract: The entanglement entropy in three-dimensional conformal field theories (CFTs) receives a logarithmic contribution characterized by a regulator-independent function $a(\theta)$ when the entangling surface contains a sharp corner with opening angle θ. In the limit of a smooth surface ($\theta \to \pi$), this corner contribution vanishes as $a(\theta) = \sigma\,(\theta - \pi)^2$. In arXiv:1505.04804, we provided evidence for the conjecture that for any $d = 3$ CFT, this corner coefficient σ is determined by C_T, the coefficient appearing in the two-point function of the stress tensor. Here, we argue that this is an instance of a much more general relation connecting the analogous corner coefficient σ_n appearing in the nth Rényi entropy and the scaling dimension h_n of the corresponding twist operator. In particular, we find the simple relation $h_n/\sigma_n = (n-1)\pi$. We show how it reduces to our previous result as $n \to 1$, and explicitly check its validity for free scalars and fermions. With this new relation, we show that as $n \to 0$, σ_n yields the coefficient of the thermal entropy, c_S. We also reveal a surprising duality relating the corner coefficients of the scalar and the fermion. Further, we use our result to predict σ_n for holographic CFTs dual to four-dimensional Einstein gravity. Our findings generalize to other dimensions, and we emphasize the connection to the interval Rényi entropies of $d = 2$ CFTs.

Keywords: Field Theories in Lower Dimensions, Conformal and W Symmetry, Holography and condensed matter physics (AdS/CMT)

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1 Introduction & main results

Understanding the structure of quantum entanglement in complex systems has become an active area of study in a variety of areas of physics, including condensed matter physics, e.g., [1–5]; quantum field theory, e.g., [6–10]; and quantum gravity, e.g., [11–21]. For these investigations, the entanglement entropy $S_{\rm EE}$ and Rényi entropies S_n [22, 23] have proven to be two particularly useful measures of the relevant degrees of freedom. In the context of quantum field theory, these are defined for a spatial region V with:

$$S_n(V) = \frac{1}{1-n} \log \operatorname{Tr} \rho_V^n \,, \qquad S_{\rm EE}(V) = \lim_{n\to 1} S_n(V) = -\operatorname{Tr}\left(\rho_V \log \rho_V\right), \tag{1.1}$$

where ρ_V is the reduced density matrix computed by integrating out the degrees of freedom in the complementary region $\overline{V}$.

In the present paper, we will focus on the Rényi entropy in three-dimensional conformal field theories (CFTs), which takes the form

$$S_n = B_n \frac{\ell}{\delta} - a_n(\theta) \log(\,\ell/\delta\,) + c_n + \mathcal{O}\big(\delta/\ell\big)\,, \tag{1.2}$$

where δ is a UV cut-off and ℓ is a length scale characteristic of the size of the region V — see figure 1a. The result is dominated by the first contribution, the celebrated 'area law' term, but the corresponding coefficient B_n depends on the details of the UV regulator. The subleading logarithmic contribution appears when the entangling surface (i.e., the boundary of V) contains a sharp corner of opening angle θ, as in figure 1a. The corresponding corner function $a_n(\theta)$ is regulator independent and hence it is a useful quantity to characterize the underlying CFT. For instance, several groups have numerically studied the corner function using lattice Hamiltonians [24–33], and obtained results independent of the lattice details which probe the low energy degrees of freedom.

As the corner function $a_n(\theta)$ is central to our investigation, let us summarize a few of its key properties: if eq. (1.2) is evaluated for the vacuum state, reflection positivity [34, 35] constrains $a_n(\theta)$ to be a positive convex function of θ, i.e.,

$$a_n(\theta) \geq 0\,, \qquad \partial_\theta a_n(\theta) \leq 0\,, \qquad \partial_\theta^2 a_n(\theta) \geq 0\,, \tag{1.3}$$

in the range $0 \leq \theta \leq \pi$. In fact, reflection positivity gives rise to an infinite tower of nonlinear higher-derivative constraints as well [34].[1] Further, this function satisfies $a_n(\theta) = a_n(2\pi - \theta)$ if eq. (1.2) is evaluated for a pure state, e.g., in the vacuum state of the CFT. The form of $a_n(\theta)$ is also constrained on general grounds in the limits where the corner becomes very sharp ($\theta \to 0$) and where it becomes almost smooth ($\theta \to \pi$, figure 1b) [8–10]:

$$a_n(\theta \to 0) = \frac{\kappa_n}{\theta}\,, \qquad a_n(\theta \to \pi) = \sigma_n\,(\pi - \theta)^2\,. \tag{1.4}$$

This behaviour is schematically illustrated in figure 2a. Hence these limits define two regulator-independent coefficients, κ_n and σ_n, which are representative of the CFT.

[1] We thank Horacio Casini for explaining these points. We discuss the nonlinear constraints further in section 6.

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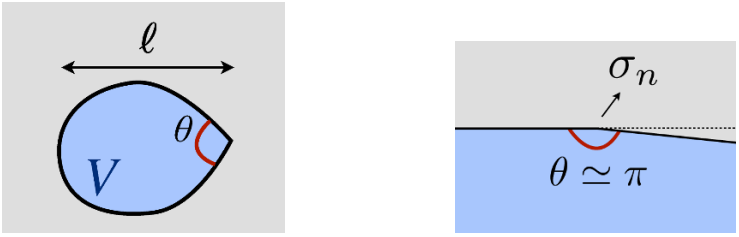

(a) (b)

Figure 1. a) Region V whose boundary contains a sharp corner with opening angle θ. b) The contribution to the Rényi entropy S_n from a corner in the almost smooth limit yields a great deal of insight into the degrees of freedom of the CFT via the coefficient σ_n.

In studying the corner contribution in the entanglement entropy [36, 37], we recently conjectured that the smooth-corner coefficient σ_1 has a universal form in general three-dimensional CFTs,

$$\sigma_1 = \frac{\pi^2}{24} C_T \,, \tag{1.5}$$

where C_T is the central charge appearing in the two-point function of the stress tensor — see eq. (1.12). We have verified that this relation holds for a free conformally coupled scalar and a free massless fermion, as well as for an eight-parameter family of strongly coupled holographic CFTs [36, 37]. A more general holographic proof appears in [38, 39]. Our primary result here is the generalization of eq. (1.5) to arbitrary values of the Rényi index $n > 0$:

$$\sigma_n = \frac{1}{\pi} \frac{h_n}{n-1} \,, \tag{1.6}$$

where h_n is the scaling dimension of the twist operator τ_n appearing in calculations of $S_n(V)$. It is defined in the n-fold replicated theory as the surface operator on the boundary of V which acts to permute the n copies of the original QFT — see section 2 for more details.

A first test of eq. (1.6) is to verify that we recover eq. (1.5) from this new relation in the limit $n \to 1$. To do so, we make two observations: first, the twist operator becomes trivial at $n = 1$ and hence the scaling dimension h_n vanishes in this limit. Second, at $n = 1$ the first derivative of h_n with respect to n is proportional to the central charge C_T in any d-dimensional CFT [40, 41]. In $d = 3$, the precise relation is

$$\partial_n h_n|_{n=1} = \frac{\pi^3}{24} C_T \,. \tag{1.7}$$

Therefore in the vicinity of $n = 1$, the scaling dimension can be expanded as

$$h_n \overset{n\to 1}{=} \frac{\pi^3}{24} C_T \,(n-1) + \mathcal{O}\big((n-1)^2\big) \,. \tag{1.8}$$

Now it is straightforward to see that substituting this expression into eq. (1.6) and taking the limit $n \to 1$ precisely reproduces the original relation (1.5).

Further, we have been able to verify eq. (1.6) for a free conformally coupled scalar field and for a free massless fermion. In particular, as we discuss in the following, we can evaluate

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the corner coefficient σ_n using the results of [8–10, 42] while the scaling dimension h_n can be determined with the results of [40, 41]. We demonstrate that these two independent calculations yield complete agreement with eq. (1.6) for any integer values of $n > 1$. Since the methods involved in these two calculations are so completely disparate, we find this agreement to be very strong evidence for the new conjecture.

Before moving to detailed discussions, let us draw an interesting parallel with two-dimensional CFTs where twist operators are well understood. In the case of $d = 2$, the twist operator is a local primary operator with scaling dimension [6, 7, 43]:

$$h_n^{(2)} = \frac{c}{12}\left(n - \frac{1}{n}\right) , \tag{1.9}$$

where c is the Virasoro central charge of the theory. The Rényi entropy of a single interval is calculated by first evaluating the correlator of two twist operators inserted at each of the endpoints and the final result can be written as

$$S_n^{d=2} = 2\,\sigma_n^{(2)}\,\log(\,\ell/\delta\,) + \cdots , \qquad \text{where} \qquad \sigma_n^{(2)} \equiv \frac{h_n^{(2)}}{n-1} . \tag{1.10}$$

In this expression, ℓ is the length of the interval and δ, the UV cut-off. The factor of two here comes from having two endpoints [43]. Hence, there is a striking similarity between the coefficient of the logarithmic contribution in $d = 2$ and in $d = 3$ — in the limit $\theta \to \pi$ for the latter. The parallel extends to the $n \to 1$ limit, where one recovers the well-known result $S_{\rm EE}^{d=2} = (c/3)\,\log(\ell/\delta)$ by applying the two-dimensional analog[2] of eq. (1.7): $\partial_n h_n|_{n=1} = c/6$. While it calls for some deeper physical insight, this interesting connection serves as a stepping stone from the well-known results for $d = 2$ to new considerations of Rényi entropies in higher dimensions — see further discussion in sections 4 and 7.

1.1 Main results

Eq. (1.6) is our main result. In particular, we conjecture this to be a relation valid for general CFTs in $d = 3$. We thus predict that the corner coefficient of the Rényi entropies in the smooth limit σ_n is proportional to the scaling dimension h_n of the twist operator. Most of the remainder of the paper is devoted to supporting this new conjecture, and to extract various consequences from it.

Eq. (1.6) incorporates our previous conjecture (1.5) when $n = 1$, as we discussed above. Using independent computations of σ_n [8–10, 42] and h_n [40, 41], we have verified that (1.6) holds exactly for a free scalar and for a free massless Dirac fermion at integer values of n (up to $n = 500$) as well as in the $n \to \infty$ limit. Further, since we have a simple expression for h_n that holds for any real n, we can predict the behavior of σ_n for non-integer n. This is plotted in figure 3 for the free complex scalar and Dirac fermion. Figure 3 also includes σ_n for a holographic CFT dual to Einstein gravity. This is a prediction of our conjecture as h_n is known for these holographic theories, but direct access to σ_n is currently limited

[2] Eq. (2.5) provides the general equation relating $\partial_n h_n|_{n=1}$ and C_T in d dimensions. The above result follows from simply substituting $d = 2$ and also $C_T = c/(2\pi^2)$.

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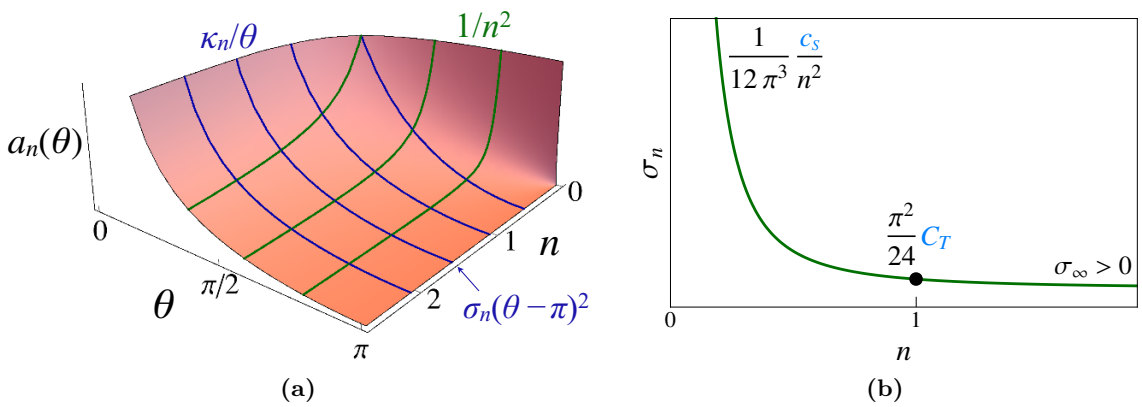

Figure 2. Schematics for the corner entanglement function in general CFTs in $d = 3$ spacetime dimensions. a) Rényi corner contribution $a_n(\theta)$ versus the Rényi index n and opening angle θ, with the asymptotics for $\theta \to 0, \pi$ and $n \to 0$. b) Smooth-corner coefficient σ_n versus n. The divergence at small n is determined by the flat-space thermal entropy density: $s_{\text{flat}} = c_S\, T^2$.

to $n = 1$. Indeed, the celebrated Ryu-Takayanagi prescription [18–21] only provides a holographic description of entanglement entropy.

A general formula for the scaling dimensions h_n in dimensions $d \geq 2$ was obtained in [41] — see (2.4) below. As we explain in section 3, h_n is expressed in terms of the thermal energy density $\mathcal{E}(T)$ of the CFT on the hyperbolic space $\mathbb{H}^{d-1}$ (the simplest geometry with constant negative curvature). Using eq. (1.6), we find a new formula for the Rényi corner coefficient:

$$\sigma_n = \frac{n}{n-1} R^3 \left(\mathcal{E}(T_0) - \mathcal{E}(T_0/n)\right) , \tag{1.11}$$

where R is the curvature radius of the hyperbolic plane $\mathbb{H}^2$, which also determines the temperature $T_0 = 1/(2\pi R)$. Note that neither σ_n nor h_n depend on R, and that this expression is always positive since the energy density $\mathcal{E}(T)$ increases with temperature. Eq. (1.11) allows for explicit calculations of the Rényi corner coefficient σ_n for any $n > 0$, including non-integer values of n. It also sheds light on the physical content of σ_n. For example, following the analysis of [41], it yields our earlier conjecture (1.5) where σ_1 is proportional to C_T, the central charge appearing in the vacuum two-point function of the stress tensor:

$$\langle T_{\mu\nu}(x)\, T_{\eta\kappa}(0)\rangle = \frac{C_T}{x^{2d}}\, \mathcal{I}_{\mu\nu,\eta\kappa}(x) , \tag{1.12}$$

where $\mathcal{I}_{\mu\nu,\eta\kappa}(x)$ is a dimensionless tensor, whose structure is fixed by conformal symmetry [44]. As noted above, the desired result follows from expanding h_n around $n = 1$ in eq. (1.6). Further, in the limit $n \to 0$, we find that the corner coefficient yields another important physical constant, the coefficient c_S which controls the thermal entropy density of the CFT (in flat space), i.e., $s_{\text{flat}} = c_S\, T^{d-1}$. In particular, we show that eq. (1.11) yields

$$\lim_{n\to 0} \sigma_n = \frac{c_S}{12\pi^3} \frac{1}{n^2} . \tag{1.13}$$

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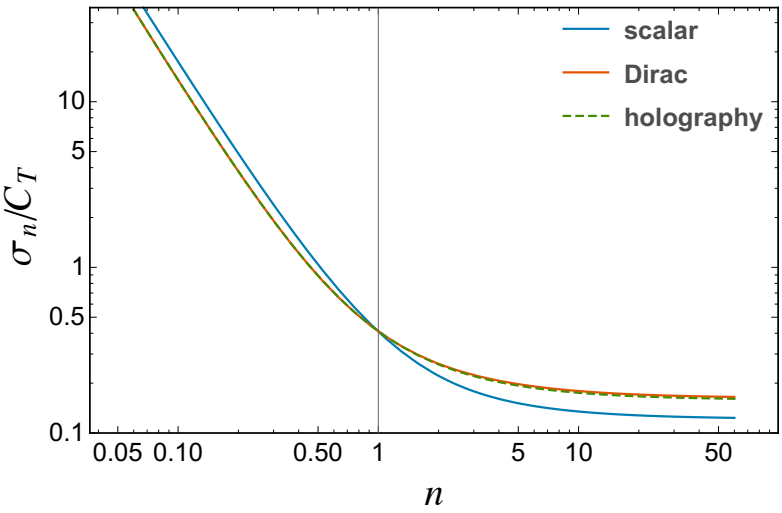

Figure 3. Log-log plot of σ_n/C_T versus the Rényi index n for the free complex scalar, free Dirac fermion, and holographic theories. Strikingly, the latter two are almost equal. We note that all three lines must cross at $n = 1$, where one recovers the universal ratio, $\sigma_1/C_T = \pi^2/24 \simeq 0.411$.

In section 6 we extend this result by establishing that the corner function $a_n(\theta)$ diverges as $1/n^2$ for any opening angle. In the opposite limit, i.e., as $n \to \infty$, eq. (1.11) becomes

$$\sigma_\infty = R^3 \left(\mathcal{E}(T_0) - \mathcal{E}(0)\right) > 0\,. \tag{1.14}$$

This quantity does not seem to lend itself to an interpretation in terms of flat space observables, in contrast to the $n \to 0, 1$ limits. In section 6, we show that this asymptotic behavior extends to finite angles, i.e., $a_\infty(\theta)$ is finite for all θ. In figure 2b, we show the generic dependence of the σ_n coefficient on the Rényi index n as deduced from eq. (1.6).

In figure 3, we have normalized the σ_n by the corresponding central charges C_T, which makes them cross at $n = 1$ by eq. (1.5). The three curves all exhibit a similar dependence on n. In particular, they diverge as $1/n^2$ for $n \to 0$, and asymptote to constant values for $n \to \infty$. A salient feature of figure 3 is that the curves *only* coincide at $n = 1$, meaning that the ratio σ_n/C_T is only universal for the entanglement entropy, as was independently observed in [42].

Interestingly, the free fermion and holographic curves in figure 3 are hardly distinguishable in the whole range of values, with the agreement becoming even better for values $0 < n \leq 1$. In particular, this shows that the ratios c_S/C_T are very close to each other in both theories. The exact values are

$$\frac{c_S^{\rm f}}{C_T^{\rm f}} = 12\pi\zeta(3) \simeq 45.3165\,, \qquad \frac{c_S^{\rm hol}}{C_T^{\rm hol}} = \frac{4\pi^5}{27} \simeq 45.3362\,. \tag{1.15}$$

While the irrational numbers involved in both cases are different, the final results differ by only approximately 0.04%.

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Our results for σ_n also reveal a surprising duality relating the corner coefficients of the complex scalar $\sigma_n^{\rm cs}$ and fermion $\sigma_n^{\rm f}$:

$$n^2\,\sigma_n^{\rm cs,\,f} = \sigma_{1/n}^{\rm f,\,cs}\,, \tag{1.16}$$

which is valid for any real positive n. As we show in proving this result, it requires that the corresponding thermal partition functions (on $S^1 \times \mathbb{H}^2$) are related in a relatively simple way. This surprising relation (1.16) also implies further connections between the two free theories, e.g., the above expression can be rewritten in terms of the conformal dimensions of the corresponding twist operators, yielding $n\,h_n^{\rm cs,\,f} = -h_{1/n}^{\rm f,\,cs}$.

The remainder of the paper is organized as follows: in section 2, we review some results concerning twist operators and their conformal dimensions h_n. In section 3, we compute h_n for a free conformally coupled complex scalar and a massless Dirac fermion in three dimensions. We compare these results with those obtained previously for the Rényi corner coefficient σ_n, finding perfect agreement with our conjecture (1.6). In this section, we also compute the coefficients κ_n, arising in the sharp corner limit ($\theta \to 0$), for the free fields. These coefficients play a central role in section 6 — see below. Given our results for σ_n, in section 4, we discuss a surprising duality for the corner coefficients of the scalar and the fermion theories. In section 5, we use our conjecture to produce a new expression for σ_n in holographic CFTs dual to Einstein gravity. In section 6, we study the n dependence for corners of arbitrary opening angles. Then, for a given index n, we propose a method for estimating the curve $a_n(\theta)$ using the coefficients κ_n and σ_n alone. We apply this approach to construct interpolating functions for the scalar and the fermion for $n = 1, 2, 3$ and in the limit $n \to \infty$, and we show that they accurately fit the lattice values in the cases in which these are available. A number of technical details are provided in the appendices: in appendix A, we evaluate certain integral expressions for the scaling dimensions in the free field theories to express h_n in terms of finite closed-form sums, valid for odd n. Using the integral expressions for h_n, we also provide explicit values of σ_n for certain rational values of the Rényi index, which provides an interesting demonstration of the duality between the coefficients for the scalar and fermion theories. In appendix B, we describe the details of calculations for the κ_n presented in section 3. Appendix C explains the origin of a relation observed in [42] between the limit of the corner coefficient at large n and the thermal partition function on $S^1 \times \mathbb{H}^2$ in the same limit. In appendix D, we provide a general proof of the duality between the corner coefficients of the free scalar and fermion theories. In appendix E, we connect the Rényi corner coefficients to the Rényi entropies S_n for a circle, using the relation [41] between these entropies and the twist scaling dimensions h_n.

2 Twist operators

As noted in the introduction, twist operators were originally defined in discussing Rényi entropies in two-dimensional CFTs [6, 7]. Twist operators are well understood in this context since they are local primary operators. They can be formally defined for quantum field theories (QFTs) in any number of dimensions, with the replica method, e.g., see [40, 41, 45]. However, in higher dimensions, they become nonlocal surface operators and their

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properties are less well understood. The replica method begins by evaluating the reduced density matrix ρ_V as a Euclidean path integral where independent boundary conditions are fixed on the region V as it is approached from above and below in Euclidean time, i.e., with $t_{\rm E} \to 0^{\pm}$. This path integral for the partition function Z_n is then extended to the Euclidean path integral on a n-sheeted geometry [6, 7], where the consecutive sheets are sewn together on cuts running over V, to represent

$$\mathrm{Tr}[\,\rho_V^n\,] = \frac{Z_n}{Z_1^n}\,. \tag{2.1}$$

The denominator is introduced here to ensure the correct normalization, i.e., $\mathrm{Tr}[\rho_V] = 1$. In defining the twist operator $\tau_n[V]$, the above construction is replaced by a path integral over n copies of the underlying QFT on a single copy of the geometry. The twist operator is then defined as the codimension-two surface operator extending over the entangling surface, i.e., the boundary of the region V, whose expectation value yields

$$\langle\,\tau_n\,\rangle_n = \mathrm{Tr}[\,\rho_V^n\,]\,, \tag{2.2}$$

where the subscript n on the expectation value on the left-hand side indicates that it is taken in the n-fold replicated QFT. Further, here and in the following, we omit the V dependence of τ_n to alleviate the notation. Hence eq. (2.2) implies that τ_n opens a branch cut over the region V which connects consecutive copies of the QFT in the n-fold replicated theory. Closely related to the twist operator, the so-called swap operator was introduced to compute Rényi entropies on the lattice [46].

In the case of a CFT, the conformal scaling dimension h_n of the twist operator is defined by the coefficient of the leading power-law divergence in the correlator $\langle T_{\mu\nu}\,\tau_n\rangle_n$ as the location of $T_{\mu\nu}$ approaches that of τ_n [40, 41]. In the case of a twist operator on an infinite (hyper)plane, as is illustrated in figure 4, this correlator reads

$$\begin{aligned}
\langle T_{ab}\,\tau_n\rangle_n &= -\frac{h_n}{2\pi}\frac{\delta_{ab}}{y^d}\,, \qquad \langle T_{ai}\,\tau_n\rangle_n = 0\,,\\
\langle T_{ij}\,\tau_n\rangle_n &= \frac{h_n}{2\pi}\frac{(d-1)\delta_{ij} - d\,\hat{n}_i\hat{n}_j}{y^d}\,,
\end{aligned} \tag{2.3}$$

where the indices i, j and a, b denote the two transverse directions and the d-2 parallel directions to the twist operator.[3] Further, y is the perpendicular distance from the stress tensor insertion to the twist operator and $\hat{n}_i$ is the unit vector orthogonally directed from τ_n to the stress tensor. Note that $T_{\mu\nu}$ here denotes the stress tensor for the entire n-fold replicated CFT.

While the above expressions are only valid for a twist operator on a (hyper)plane, we stress that in general the leading singularity takes this form whenever $y \ll \ell$, where ℓ is any scale entering in the description of the geometry of the entangling surface. Hence the scaling dimension h_n is a fixed coefficient which is characteristic of all twist operators τ_n (in a given CFT), independent of the details of the geometry of the corresponding entangling surface. Finally, let us add that $h_1 = 0$ since the twist operator τ_n becomes trivial for $n = 1$.

[3] Let us add that implicitly the above expressions are normalized by dividing by $\langle\tau_n\rangle_n$ but we left this normalization implicit to avoid the clutter that would otherwise be created.

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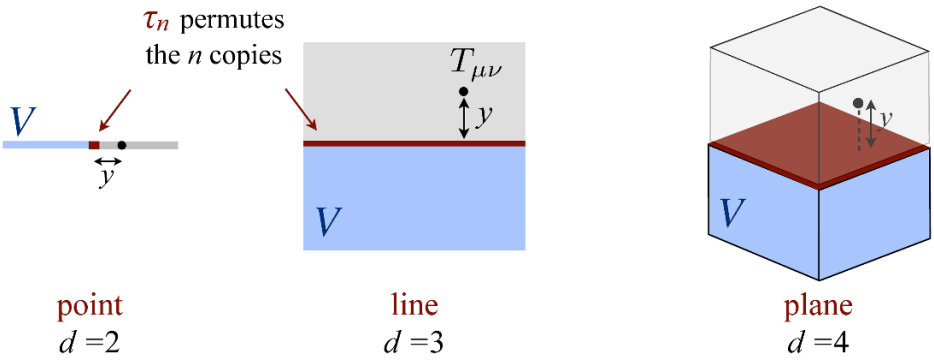

Figure 4. Planar twist operators τ_n in $d = 2, 3, 4$ used in the definition of the twist scaling dimension h_n, via eq. (2.3). They have support on the boundary (in red) of the entangling region V, which is at time $t_{\rm E} = 0$. The stress tensor $T_{\mu\nu}$ insertions lie a distance y from the twist operators. In $d > 2$, the use of planar boundaries here is a matter of convenience, as h_n can be extracted from any entangling surface ∂V by taking $y \to 0$.

2.1 Insights from hyperbolic space

In refs. [40, 47], the entanglement and Rényi entropies for a spherical entangling region $V = S^{d-1}$ in general CFTs were studied using a conformal mapping from the conformal vacuum in flat space to a thermal ensemble on the hyperbolic cylinder. The Euclidean version of this transformation maps a n-fold cover of $\mathbb{R}^d$ to the thermal spacetime $S^1 \times \mathbb{H}^{d-1}$. The flat space geometry has branch cuts beginning at a sphere of radius R, and in the hyperbolic geometry, the radius of curvature of $\mathbb{H}^{d-1}$ is R and the period of S^1 is $2\pi R\, n$. Using this construction, the conformal dimension h_n can be expressed in terms of the thermal energy density $\mathcal{E}(T)$ of the (non-replicated) CFT on the hyperbolic space $\mathbb{H}^{d-1}$ [40, 41]:

$$h_n = \frac{2\pi\, n}{d-1}\, R^d\, \left(\mathcal{E}(T_0) - \mathcal{E}(T_0/n)\right), \tag{2.4}$$

where $T_0 = 1/(2\pi R)$. Note that the conformal dimension is independent of R since this radius is the only scale in the above expression, i.e., both of the energy densities above are proportional to $1/R^d$. Of course, the R independence is required since h_n is a dimensionless coefficient and as described above, it is independent of the geometry of the entangling surface.

With the above definition (2.4), it is clear that setting $n = 1$ yields $h_1 = 0$. However, as noted in the introduction, studying the limit $n \to 1$ of this expression also allows one to prove [41]

$$\partial_n h_n|_{n=1} = 2\pi^{\frac{d}{2}+1} \frac{\Gamma(d/2)}{\Gamma(d+2)}\, C_T\,, \tag{2.5}$$

where C_T is the central charged appearing in the two-point correlator (1.12) of the stress tensor.

In the $n \to 0$ limit, the temperature diverges for the second energy density in eq. (2.4) and this contribution dominates the result for the conformal dimension. Further, at large temperatures, the curvature of the hyperbolic space becomes unimportant and to leading

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order this energy density will match that in flat space [48]. In particular, for a d-dimensional CFT, the thermal energy density in flat space directly relates to the entropy density, $s_{\rm flat} = c_S\, T^{d-1}$, which defines a useful coefficient c_S that characterizes the number of degrees of freedom in the CFT. The corresponding thermal energy density in flat space is then given by $\mathcal{E}_{\rm flat}(T) = \frac{d-1}{d}\, c_S\, T^d$ and hence, with $\mathcal{E}(T) \simeq \mathcal{E}_{\rm flat}(T)\,[1 + \mathcal{O}(1/(RT)^2)]$ at high temperatures, eq. (2.4) yields

$$\lim_{n\to 0} h_n = -\frac{c_S}{d}\left(\frac{1}{2\pi\, n}\right)^{d-1} . \tag{2.6}$$

The negative sign arises here because $n < 1$. Thus, in this limit, the conformal dimension of the twist operator diverges with (d-1) power of the Rényi index, and is proportional to the flat-space thermal entropy coefficient. In particular, we can see this behaviour arises in two dimensions from eq. (1.9). The small-n behaviour of the twist dimension is then $h^{(2)}_{n\to 0} = -c/(12n)$ and comparing with eq. (2.6), we can extract $c_S = \pi\, c/3$, which precisely agrees with the well-known result for this coefficient in $d = 2$ [49].

In the opposite limit, $n \to \infty$, the temperature vanishes in the second term in eq. (2.4), leaving

$$\lim_{n\to\infty} h_n = \frac{2\pi\, n}{d-1}\, R^d\, \left(\mathcal{E}(T_0) - \mathcal{E}(0)\right), \tag{2.7}$$

which shows that h_n grows linearly with n when the Rényi index is large, independent of the dimension d. Let us comment here that the physical interpretation of the factor $\mathcal{E}(T_0) - \mathcal{E}(0)$ in eq. (2.7) depends on the details of the renormalization scheme used in defining the energy density $\mathcal{E}(T)$. For example, using a heat kernel regularization with free fields in the next section, we will find $\mathcal{E}(0) = 0$, and hence eq. (2.7) yields $h_\infty \propto \mathcal{E}(T_0)$. However, with the holographic calculations described in section 5, one finds $\mathcal{E}(T_0) = 0$ and hence in eq. (2.7), one is left with $h_\infty \propto -\mathcal{E}(0)$. In this case, $\mathcal{E}(0) < 0$ is interpreted as the Casimir energy density for the boundary CFT on the hyperbolic space. In any event, these ambiguities for the values of the individual energy densities in eqs. (2.4) and (2.7) are eliminated in the difference $\mathcal{E}(T_0) - \mathcal{E}(T_0/n)$ and so they do not affect the value of the scaling dimension of the twist operator.

3 Free fields

In this section, we present explicit the free field calculations which verify our new conjecture (1.6) relating σ_n to h_n. Hence in the following, we restrict the discussion to three spacetime dimensions, in which case, τ_n is a line operator as shown in figure 4. We begin by computing the scaling dimensions of the twist operators h_n corresponding to a free complex[4] scalar and a free Dirac fermion in three dimensions. This allows us to simply compute the smooth corner coefficient σ_n for any real $n > 0$. We use this to confirm our conjecture in the case of free CFTs, and examine some physical limits of σ_n. In this section, we also compute the coefficients κ_n appearing in the opposite limit, i.e., that corresponding

[4]Note that our discussion in [36] involved a real scalar, for which the values of h_n and σ_n are half those given here. We consider a complex scalar here to facilitate the comparison with the free fermion.

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to a very sharp corner with $\theta \to 0$. We will use the latter in section 6 to estimate the value of the corner functions $a_n(\theta)$ for arbitrary values of the opening angle.

3.1 Twist operators

Recall that the scaling dimension of the twist operator τ_n in a general CFT can be evaluated with the expression in eq. (2.4), which in three dimensions reduces to

$$h_n = \pi\, n\, R^3\, \left(\mathcal{E}(T_0) - \mathcal{E}(T_0/n)\right), \tag{3.1}$$

where $\mathcal{E}(T)$ is the thermal energy density of the CFT on the hyperbolic cylinder $S^1 \times \mathbb{H}^2$. Recall that in this curved space geometry, R is the radius of curvature of $\mathbb{H}^2$, and the period of S^1 is set to the inverse of corresponding temperature, with $T_0 = 1/(2\pi R)$. Now, our first step will be to determine the partition function of the theory on the hyperbolic cylinder and then the desired thermal energy densities can be evaluated as

$$\mathcal{E}(T) = \frac{T^2}{V_{\mathbb{H}^2}} \frac{\partial}{\partial T} \log Z(T), \tag{3.2}$$

where $V_{\mathbb{H}^2}$ is the regulated area of the hyperbolic plane $\mathbb{H}^2$. To be precise, we have [40, 50]

$$V_{\mathbb{H}^2} = 2\pi R^2 \left(\frac{R}{\delta} - 1 + \mathcal{O}(\delta/R)\right). \tag{3.3}$$

Through the conformal transformation discussed in section 2.1, UV divergences in the entanglement entropy or the Rényi entropy in $\mathbb{R}^3$ become IR divergences of the corresponding (total) thermal entropy on the hyperbolic plane $\mathbb{H}^2$. Therefore, in order to identify quantities in a meaningful way, the UV cut-off in the flat space calculations must be mapped to the IR cut-off in the calculations on the hyperbolic cylinder. Hence, we see the short distance cut-off δ (in flat space) appearing in the regulated area above.

The partition functions for free fields on the (Euclidean) manifold $S^1 \times \mathbb{H}^2$ are readily calculated using heat kernel techniques and the results for the conformally coupled complex scalar and a massless Dirac fermion are given respectively by [41][5]

$$\begin{aligned} \log Z^{\rm cs}(T) &= \frac{V_{\mathbb{H}^2} T}{2R} \int_0^\infty \frac{du}{u^3 \sinh^2 u} \frac{u^2 + \sinh^2 u\,(u\,\coth u - 2)}{\sinh(u/(2\pi R T))}, \\ \log Z^{\rm f}(T) &= \frac{V_{\mathbb{H}^2} T}{2R} \int_0^\infty \frac{du}{u^3 \sinh u} \frac{2\sinh u - u\,(u \coth u + 1)}{\tanh(u/(2\pi R T))}, \end{aligned} \tag{3.4}$$

where again R is the radius of curvature of the hyperbolic plane and the temperature T corresponds to the inverse period of the circle S^1. It is now straightforward to evaluate the scaling dimension (2.4) of the corresponding twist operators using eq. (3.2) and the final expressions can be written as

$$\begin{aligned} h_n^{\rm cs} &= \frac{1}{4\pi n^2} \int_0^\infty \frac{du}{\sinh u} \left[\frac{\cosh(u/n)}{\sinh^3(u/n)} - n^3 \frac{\cosh u}{\sinh^3 u}\right], \\ h_n^{\rm f} &= \frac{1}{8\pi n^2} \int_0^\infty \frac{du}{\tanh u} \left[n^3 \frac{2 + \sinh^2 u}{\sinh^3 u} - \frac{2 + \sinh^2(u/n)}{\sinh^3(u/n)}\right]. \end{aligned} \tag{3.5}$$

[5] We emphasize that both of these expressions are UV finite (i.e., the potential singularities as $x \to 0$ are removed). We also note that both yield $\mathcal{E}(0) = 0$ and a nonzero value for $\mathcal{E}(T_0)$ — see comments below eq. (2.7).

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n	$n \to 0$	$1+\epsilon$	2	3	4	5	6	$n \to \infty$
$h_n^{\rm cs}$	$-\frac{\zeta(3)}{4\pi^3}\frac{1}{n^2}$	$\frac{\pi}{128}\epsilon$	$\frac{1}{24\pi}$	$\frac{1}{27\sqrt{3}}$	$\frac{3\pi+8}{192\pi}$	$\frac{\sqrt{25-2\sqrt{5}}}{125}$	$\frac{81+34\sqrt{3}\pi}{1944\pi}$	$\frac{3\zeta(3)}{16\pi^3}n$
$h_n^{\rm f}$	$-\frac{3\zeta(3)}{16\pi^3}\frac{1}{n^2}$	$\frac{\pi}{128}\epsilon$	$\frac{1}{64}$	$\frac{5}{108\sqrt{3}}$	$\frac{1+6\sqrt{2}}{256}$	$\frac{\sqrt{425+58\sqrt{5}}}{500}$	$\frac{261+20\sqrt{3}}{5184}$	$\frac{\zeta(3)}{4\pi^3}n$

Table 1. Scaling dimensions h_n of the twist operator at various values of n for a massless complex scalar and a massless Dirac fermion. In the third column $|\epsilon| \ll 1$. For non-integer n, see table 5.

In appendix A, we show that these integrals can be evaluated for odd values of n, and expressed in terms of sums involving trigonometric functions. In any event, the above expressions can be easily evaluated for different values of n, and we will use this to test the conjecture (1.6) relating h_n to the corner coefficient σ_n in the next subsection. Some explicit results are collected in table 1.

Following our general discussion above, let us comment further on some of the results in the table. First, we consider the limit $n \to 0$ for which eq. (2.6) becomes

$$\lim_{n\to 0} h_n = -\frac{c_S}{12\pi^2}\frac{1}{n^2}\,. \tag{3.6}$$

By evaluating the integrals for small n, we find that our free field results have the desired form, with both of the scaling dimensions diverging as $1/n^2$ — see table 1. Further, examining the overall coefficient of these divergences, eq. (3.6) indicates that

$$c_S^{\rm cs} = \frac{3\zeta(3)}{\pi} \qquad \text{and} \qquad c_S^{\rm f} = \frac{9\zeta(3)}{4\pi}\,, \tag{3.7}$$

for the thermal entropy coefficient of the complex scalar and the Dirac fermion, respectively. One can easily verify that these values precisely match the expected thermal entropy coefficients for both fields, e.g., see [51]. Another interesting limit to consider is $n \to 1$. While the scaling dimension vanishes at precisely $n = 1$, evaluating eq. (3.5) in the vicinity of this point, we find

$$\begin{aligned} h_n^{\rm cs}|_{n=1+\epsilon} &= \frac{\pi}{128}\,\epsilon - \frac{17\pi}{1920}\,\epsilon^2 + \mathcal{O}(\epsilon^3)\,, \\ h_n^{\rm f}\Big|_{n=1+\epsilon} &= \frac{\pi}{128}\,\epsilon - \frac{13\pi}{1920}\,\epsilon^2 + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{3.8}$$

Now recall that eq. (2.5) determined the leading coefficient in terms of the central charge C_T. In particular, we have $\partial_n h_n|_{n=1} = \frac{\pi^3}{24}C_T$ for $d = 3$. Now given $C_T^{\rm cs,\,f} = 3/(16\pi^2)$ [44], this latter result yields $\partial_n h_n|_{n=1} = \pi/128$ for both fields, in agreement with the above expansions. Finally, the scaling dimension exhibits the expected linear growth with the Rényi index as $n \to \infty$. In particular, we find:

$$\lim_{n\to\infty} h_n^{\rm cs} = \frac{3\zeta(3)}{16\pi^3}\,n \qquad \text{and} \qquad \lim_{n\to\infty} h_n^{\rm f} = \frac{\zeta(3)}{4\pi^3}\,n\,. \tag{3.9}$$

3.2 Corner Rényi entropies

The regulator independent contributions to the entanglement and Rényi entropies produced by a sharp corner in the entangling surface for $d = 3$ free conformally coupled scalars

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and free Dirac fermions were studied in a series of papers by Casini and Huerta [8–10]. However, the expressions for, e.g., the Rényi coefficients σ_n were left in a very complicated and implicit form. Motivated by our original conjecture (1.5) relating σ_1 and C_T, and the possibility that a similar universal relation could also exist for σ_n, the results for $\sigma_n^{\text{cs, f}}$ were recently reduced to simple and closed-form expressions at integer n [42]. These new expressions were used to easily evaluate the first coefficients with $n \geq 1$, as well as determine the $n \to \infty$ behaviour. In this section, we use these results to confirm our new conjecture, verifying that eq. (1.6) holds for both the scalar and the fermion.

In the second part of this section, we consider the coefficients $\kappa_n^{\text{cs, f}}$, which control the corner contribution of the Rényi entropy in the limit $\theta \to 0$. In particular, we evaluate the corresponding expressions for $n = 2, 3, 4$, and at large n — see also appendix B. In section 6, we will use these coefficients, along with the $\sigma_n^{\text{cs, f}}$, to construct simple interpolating functions to approximate the corner function $a_n^{\text{cs, f}}(\theta)$ for all angles.

3.2.1 Smooth surface limit

The expressions for $\sigma_n^{\text{cs, f}}$ appearing in [8–10] were recently shown to reduce to the following simple sums (for integer n) [42]:

$$\sigma_n^{\text{cs}} = \sum_{k=1}^{n-1} \frac{k(n-k)(n-2k)\tan\left(\frac{\pi k}{n}\right)}{12\pi n^3(n-1)}\,, \qquad \sigma_n^{\text{f}} = \sum_{k=-(n-1)/2}^{(n-1)/2} \frac{k(n^2-4k^2)\tan\left(\frac{\pi k}{n}\right)}{24\pi n^3(n-1)}\,. \tag{3.10}$$

Now combining eq. (3.1) for the conformal dimension h_n with our new conjecture (1.6), we find a new general formula for the corner coefficient in terms of the thermal energy density on the hyperbolic cylinder:

$$\sigma_n = \frac{n}{n-1} R^3 \left(\mathcal{E}(T_0) - \mathcal{E}(T_0/n)\right). \tag{3.11}$$

In particular, focusing on the free fields, we can use the expressions (3.5) for h_n to produce the following prediction for the corner coefficients in these theories

$$\begin{aligned} \sigma_n^{\text{cs}} &= \frac{1}{4\pi^2 n^2(n-1)} \int_0^\infty \frac{du}{\sinh u} \left[\frac{\cosh(u/n)}{\sinh^3(u/n)} - n^3 \frac{\cosh u}{\sinh^3 u}\right], \\ \sigma_n^{\text{f}} &= \frac{1}{8\pi^2 n^2(n-1)} \int_0^\infty \frac{du}{\tanh u} \left[n^3 \frac{2+\sinh^2 u}{\sinh^3 u} - \frac{2+\sinh^2(u/n)}{\sinh^3(u/n)}\right]. \end{aligned} \tag{3.12}$$

We have explicitly evaluated these integrals for a few values of n in table 1, and also numerically plotted the coefficients for continuous n in figure 3. In appendix A, we also evaluate the integrals analytically for odd $n > 1$ to express these corner coefficients as certain closed sums — see eqs. (A.4) and (A.8). Our new closed-form sums agree with eq. (3.10) for all odd integer values that we have checked, $1 < n \leq 500$, but take a very different form from the latter. It would be desirable to explicitly prove that both are equal for all odd integers. In any event, we have also verified that the above integrals (3.12) exactly reproduce the values obtained from eq. (3.10) for $n = 1, 2, \cdots, 500$, as well as in the limit $n \to \infty$. In that limit, we have also verified that the subleading terms agree, as we discuss below. We find

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n	$n \to 0$	1	2	3	4	5	6	$n \to \infty$
$\sigma_n^{\rm cs}$	$\frac{\zeta(3)}{4\pi^4}\,\frac{1}{n^2}$	$\frac{1}{128}$	$\frac{1}{24\pi^2}$	$\frac{1}{54\sqrt{3}\pi}$	$\frac{3\pi+8}{576\pi^2}$	$\frac{\sqrt{25-2\sqrt{5}}}{500\pi}$	$\frac{81+34\sqrt{3}\pi}{9720\pi^2}$	$\frac{3\zeta(3)}{16\pi^4}$
$\sigma_n^{\rm f}$	$\frac{3\zeta(3)}{16\pi^4}\,\frac{1}{n^2}$	$\frac{1}{128}$	$\frac{1}{64\pi}$	$\frac{5}{216\sqrt{3}\pi}$	$\frac{1+6\sqrt{2}}{768\pi}$	$\frac{\sqrt{425+58\sqrt{5}}}{2000\pi}$	$\frac{261+20\sqrt{3}}{25920\pi}$	$\frac{\zeta(3)}{4\pi^4}$

Table 2. Corner coefficient σ_n for various Rényi indices for the complex scalar and Dirac CFTs in $d = 3$.

that this agreement provides strong evidence for our conjecture (1.6) relating the corner coefficient σ_n and the scaling dimension h_n of the corresponding twist operators.

Of course, we can also use this agreement to combine the relation $h_n = \pi(n-1)\sigma_n$ with eq. (3.10) to express the scaling dimensions $h_n^{\rm cs,f}$ in terms of sums:

$$h_n^{\rm cs} = \sum_{k=1}^{n-1} \frac{k(n-k)(n-2k)\tan\left(\frac{\pi k}{n}\right)}{12n^3}\,, \qquad h_n^{\rm f} = \sum_{k=-(n-1)/2}^{(n-1)/2} \frac{k(n^2-4k^2)\tan\left(\frac{\pi k}{n}\right)}{24n^3}\,, \tag{3.13}$$

which is valid for all integer values of $n > 1$.

One advantage of the integral expressions (3.12) for $\sigma_n^{\rm cs,f}$ over the sums in eq. (3.10) is that they allow us to evaluate the corner coefficients for non-integer values of n. For example, we can examine how these coefficients vary in the vicinity of $n = 1$, i.e.,

$$\begin{aligned} \sigma_n^{\rm cs}\big|_{n=1+\epsilon} &= \frac{1}{128} - \frac{17}{1920}\,\epsilon + \mathcal{O}(\epsilon^2)\,, \\ \sigma_n^{\rm f}\big|_{n=1+\epsilon} &= \frac{1}{128} - \frac{13}{1920}\,\epsilon + \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{3.14}$$

Hence we see that the agreement between the two coefficients at $n = 1$ does not extend beyond this precise point, as expected from eq. (3.8). We can also consider the limit $n \to 0$, which yields

$$\lim_{n\to 0} \sigma_n^{\rm cs} \simeq \frac{\zeta(3)}{4\pi^4}\,\frac{1}{n^2}\,, \qquad \text{and} \qquad \lim_{n\to 0} \sigma_n^{\rm f} \simeq \frac{3\zeta(3)}{16\pi^4}\,\frac{1}{n^2}\,. \tag{3.15}$$

Of course, these results are simply related to the corresponding limit of the scaling dimensions in eq. (3.6), and hence the corner coefficients are proportional to c_S and diverge as $1/n^2$ in this limit.

Next, we note that while the h_n diverge linearly with n as $n \to \infty$ in eq. (3.9), the σ_n asymptote to the constant values $\sigma_\infty^{\rm cs,f}$ given in table 2. The detailed large-n asymptotic behavior of both corner coefficients can be determined from eq. (3.12) and reads:[6]

$$\lim_{n\to\infty} \sigma_n = \sigma_\infty \left(1 + \frac{1}{n}\right)\left[1 + \frac{b_2}{n^2} - \frac{b_4}{n^4} + \mathcal{O}\left(\frac{1}{n^6}\right)\right]\,, \tag{3.16}$$

[6] We note that for the scalar, the large-n expansion can be performed directly on the corresponding integral in eq. (3.12). However, the analogous calculation for the fermion does not produce a consistent expansion. Rather in this case, we took advantage of an alternative representation of $\sigma_n^{\rm f}$ which makes use of the expression for $\log Z^{\rm f}(T)$ given in [52] — see eq. (7.5) and appendix D.

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where

$$
\begin{aligned}
b_2^{\rm cs} &= 1\,, \qquad b_4^{\rm cs} = -1 + \frac{\pi^4}{45\zeta(3)} \simeq 0.8008\,,\\
b_2^{\rm f} &= 1 - \frac{\pi^2}{12\zeta(3)} \simeq 0.3158\,,\\
b_4^{\rm f} &= -1 + \frac{\pi^2(120+7\pi^2)}{1440\zeta(3)} \simeq 0.0781\,.
\end{aligned}
\tag{3.17}
$$

We further find that the series in brackets only contains even powers of $1/n$, with terms that alternate in sign. These analytical results for the asymptotics agree with the numerical analysis presented in [42] of the sums in eq. (3.10). This analysis yielded precise estimates for $b_2^{\rm cs,\,f}$, which coincide with the values given above. Based on their numerical fits, the authors of [42] conjectured that the terms in the large-n expansions appear in pairs with identical coefficients, which leads to the overall factor of $1 + 1/n$ in eq. (3.16). Here the large-n expansion of our integral expressions allow us to confirm the appearance of this factor analytically. As an aside, we observe that the same factor $1 + 1/n$ fully determines the n dependence of the universal coefficient of the $d = 2$ Rényi entropy in eq. (1.10). Overall, the large-n asymptotic analysis discussed above provides further support for our main conjecture (1.6) relating h_n and σ_n.

Ref. [42] further observed that for both of the free fields

$$
\lim_{n\to\infty} \sigma_n = \frac{1}{4\pi^2}\,\mathcal{F}_\infty\,,
\tag{3.18}
$$

where the coefficient $\mathcal{F}_\infty$ is related to the logarithm of the partition function on $S^1 \times \mathbb{H}^2$, as calculated in [52]. Given the connection (1.6) between the corner coefficient and the scaling dimension, the origin of the above relation can be traced to the facts that the scaling dimension can be expressed in terms of the energy density on the hyperbolic geometry $\mathbb{H}^2$, as in eq. (3.1), and that the energy density is determined by a derivative of $\log Z(T)$ with respect to the temperature, as in eq. (3.2). We defer the precise details to appendix C because producing the exact result in eq. (3.18) requires taking into account differences between our present conventions and those in [52]. Our analysis in the appendix establishes that eq. (3.18) is a general relation that holds for any three-dimensional CFT.

3.2.2 Sharp corner limit

The main goal of our paper is to establish the relation (1.6) between the scaling dimensions of twist operators and the corner coefficients σ_n, arising for almost smooth corners. However, it is also interesting to study the opposite limit of the corner function $a_n(\theta)$, corresponding to a very small opening angle, i.e., $\theta \to 0$. In this limit, the corner contributions are characterized by the coefficients κ_n defined by

$$
a_n(\theta \to 0) = \kappa_n/\theta\,.
\tag{3.19}
$$

We will evaluate the first few $\kappa_n^{\rm cs,\,f}$ here and use the results to produce simple functions to approximate $a_n^{\rm cs,\,f}(\theta)$ for all angles, in section 6. Further, we will explicitly show that these simple functions provide an excellent approximation to the exact $a_n(\theta)$ for $n = 1, 2, 3$.

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n	1	2	3	4	$n \to \infty$
$\kappa_n^{\rm cs}$	0.0794	0.0455996(1)	0.037339(1)	0.033798(1)	0.0262(2)
$\kappa_n^{\rm f}$	0.0722	0.0472338(1)	0.040662(1)	0.0376674(1)	0.030(4)

Table 3. Sharp-corner coefficient for various Rényi indices for the massless complex scalar and massless Dirac fermion in $d = 3$.

Interestingly, in the case of free fields, the κ_n coefficients are related to the so-called "entropic c-functions" for a family of corresponding massive free fields in $d = 2$ [8–10]. In particular, they can be evaluated as

$$\kappa_n^{\rm cs,\,f} = \frac{1}{\pi}\int_0^\infty dt\, c_n^{\rm cs,\,f}(t)\,, \tag{3.20}$$

where the $c_n^{\rm cs,\,f}(t)$ are the universal functions

$$c_n^{\rm cs,\,f}(m\ell) = \ell\,\frac{dS_n(m\ell)}{d\ell}\,, \tag{3.21}$$

where $S_n(m\ell)$ are the Rényi entropies of $d = 2$ free fields with mass m for a single interval of length ℓ. We leave the details of the calculations of the $c_n^{\rm cs,\,f}(t)$ to appendix B. The final expressions for the coefficients $\kappa_n^{\rm cs,\,f}$ can be written as

$$\kappa_n^{\rm cs} = \frac{1}{\pi(n-1)}\sum_{k=1}^{n-1}\int_0^\infty dt\,\frac{t^2 u'^2_{k/n,\,\rm cs} - u^2_{k/n,\,\rm cs}((1-2k/n)^2 + t^2(1+u^2_{k/n,\,\rm cs}))}{2(1+u^2_{k/n,\,\rm cs})}\,, \tag{3.22}$$

$$\kappa_n^{\rm f} = \frac{2}{\pi(n-1)}\sum_{k>0}^{(n-1)/2}\int_0^\infty dt\,\frac{t^2 u'^2_{k/n,\,\rm f} - u^2_{k/n,\,\rm f}(4(k/n)^2 + t^2(1-u^2_{k/n,\,\rm f}))}{2(1-u^2_{k/n,\,\rm f})}\,, \tag{3.23}$$

which applies for integer $n > 1$. The functions $u_{k/n,{\rm cs,\,f}}(t)$ are solutions to the differential equations (B.4) and (B.6) given in the appendix, subject to the boundary conditions (B.5) and (B.7) in each case. Evaluating the expressions in eqs. (3.22) and (3.23) for $n = 2, 3, 4, \infty$, we obtain the numerical results shown in table 3.

As also described in the appendix, $\kappa^{\rm cs,\,f}$ corresponding to the special case $n = 1$ are evaluated using a similar approach using two-dimensional entropic c-functions. The values of these coefficients were evaluated by [8–10] and also appear in table 3 under $n = 1$.

3.3 n and θ factorization?

One can ask whether the Rényi index n and θ dependences of the corner function $a_n(\theta)$ factorize, namely,

$$a_n(\theta) \stackrel{?}{=} f(n)\,a_1(\theta)\,, \tag{3.24}$$

where $f(n{=}1) = 1$ and $a_1(\theta)$ is the corner function appearing at $n = 1$ (i.e., in the entanglement entropy). Such factorization occurs for the corner contribution in the extensive mutual information model [45, 53, 54], which can be written as

$$a_n^{\rm Ext}(\theta) = \frac{1}{\partial_n h_n|_{n=1}}\,\frac{h_n}{n-1}\,a_1^{\rm Ext}(\theta)\,, \qquad \text{where}\quad a_1^{\rm Ext}(\theta) = \frac{\pi^2}{8}\,C_T\,[1 + (\pi-\theta)\cot\theta]\,. \tag{3.25}$$

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This result follows from a straightforward generalization of our calculations for $n=1$ in [36]. We note that the precise form of h_n is undetermined for this model, apart from the general constraints valid for any CFT. Factorization also occurs in a more concrete setting in a class of quantum critical Lifshitz theories with dynamical exponent $z = 2$, which have a corner function characterizing their corner entanglement just like CFTs. Moore and Fradkin [55] indeed found that $f(n) \equiv 1$, i.e., the corner function $a_n(\theta)$ is independent of n for all angles.[7,8]

Using our above results, we find that the factorization (3.24) does not hold for the free complex scalar or for the Dirac fermion. Indeed, if eq. (3.24) were to hold, we would have the equality $\kappa_n/\kappa_1 = \sigma_n/\sigma_1$. It is sufficient to examine the $n = 2$ case to find a contradiction:

$$\begin{aligned} \kappa_2^{\rm cs}/\kappa_1^{\rm cs} &= 0.574\,, \qquad & \sigma_2^{\rm cs}/\sigma_1^{\rm cs} &= 0.540\,, \\ \kappa_2^{\rm f}/\kappa_1^{\rm f} &= 0.654\,, \qquad & \sigma_2^{\rm f}/\sigma_1^{\rm f} &= 0.637\,. \end{aligned} \tag{3.26}$$

The deviations from factorization here are relatively small, i.e., about 6% for the complex scalar and 3% for the fermion. However, these deviations are much greater than the numerical uncertainties in our evaluation of the corresponding κ_n coefficients — recall that the σ_n are exact.

The failure of the factorization here is reminiscent of a similar non-factorization found in [40] for the Rényi entropies of spherical regions in higher dimensional CFTs. There the authors considered the question of whether or not the coefficient of the universal contribution in these Rényi entropies took the form $f_d(n)\, a_d^*$, where a_d^* is a constant determined by the specific data of the underlying CFT but $f_d(n)$ is some function of the Rényi index which is the same for any CFT in d dimensions. This kind of factorization is found in the Rényi entropy of a single interval in two dimensions, as can be inferred from eqs. (1.9) and (1.10). However, by examining a variety of holographic models, ref. [40] found that this factorization fails in any number of spacetime dimensions beyond two.

4 Bose-Fermi duality

In this section, we study duality relations between the universal entanglement coefficients at Rényi indices n and $1/n$, in both $d = 2$ and 3, the latter being our main focus. First, let us make some observations comparing the corner coefficients σ_n of the complex scalar and the Dirac fermion in $d = 3$. We note that the ratio $\sigma_n^{\rm cs}/\sigma_n^{\rm f}$ monotonically decreases as n runs from 0 to ∞, i.e.,

$$\frac{\sigma_{n\to 0}^{\rm cs}}{\sigma_{n\to 0}^{\rm f}} = \frac{4}{3}\,, \qquad \frac{\sigma_1^{\rm cs}}{\sigma_1^{\rm f}} = 1\,, \qquad \frac{\sigma_\infty^{\rm cs}}{\sigma_\infty^{\rm f}} = \frac{3}{4}\,. \tag{4.1}$$

[7] We thank S. Furukawa for pointing this out.

[8] It has been numerically shown that non-analycities can arise in the n-dependence of the corner function of certain Lifshitz QCPs when these are studied using lattice realizations [56, 57]. However, there is no evidence that such singular effects arise in the ideal (fixed point) field theories describing those QCPs [55]. We thank Joel Moore for bringing this point to our attention.

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Interestingly, the values corresponding to the complex scalar and the Dirac fermion are closely related in the opposite limits of small and large n:

$$n^2\,\sigma_n^{\rm cs}\Big|_{n\to 0} = \sigma_\infty^{\rm f}\,, \qquad n^2\,\sigma_n^{\rm f}\Big|_{n\to 0} = \sigma_\infty^{\rm cs}\,. \tag{4.2}$$

This is not coincidental as these relations constitute special cases of a new duality between the corner coefficients of the two free CFTs:

$$n^2\,\sigma_n^{\rm cs,\,f} = \sigma_{1/n}^{\rm f,\,cs}\,, \qquad n>0,\, n\in\mathbb{R}. \tag{4.3}$$

The duality interchanges the boson and fermion coefficients, together with $n \leftrightarrow 1/n$. In particular $\sigma_n^{\rm cs}$ is fully determined by $\sigma_n^{\rm f}$, and vice versa. In appendix D, we prove this relation for any positive real Rényi index, assuming our conjecture (1.6). As an immediate consequence of this duality, we can use our large-n expansion eq. (3.16) to get the following expansion in the opposite limit:

$$\sigma_{n\to 0}^{\rm cs,\,f} = \frac{\sigma_\infty^{\rm f,\,cs}}{n^2}\,(1+n)\left[1 + b_2^{\rm f,\,cs}\,n^2 - b_4^{\rm f,\,cs}\,n^4 + \mathcal{O}\big(n^6\big)\right]\,, \tag{4.4}$$

with the same coefficients $b_k^{\rm f,\,cs}$ as in eq. (3.17). Of course, combined with eq. (3.6), these results imply that the coefficient σ_∞ of one theory is determined by the thermal entropy coefficient c_S of the "dual theory".

This surprising duality between the corner coefficients in the two free theories has numerous other consequences. For example, we can rewrite it in terms of the conformal dimensions of the corresponding twist operators as

$$n\,h_n^{\rm cs,\,f} = -h_{1/n}^{\rm f,\,cs}\,. \tag{4.5}$$

We can go further and use eqs. (1.13) and (1.14) to establish relations between the thermal entropy coefficient of each field with the thermal energy density on the hyperbolic plane $\mathbb{H}^2$ at T_0 of the other:[9]

$$\mathcal{E}^{\rm cs,\,f}(T_0) = \frac{c_S^{\rm f,\,cs}}{12\pi^3 R^3}\,, \tag{4.6}$$

where again $T_0 = 1/(2\pi R)$. In fact, combining eqs. (3.1) and (4.5), we find a more general relation between the energy densities of the two theories at generic temperatures on the hyperbolic cylinder,

$$\mathcal{E}^{\rm cs,\,f}(T_0^2/T) - \mathcal{E}^{\rm cs,\,f}(T_0) = -\frac{T^3}{T_0^3}\,\Big(\mathcal{E}^{\rm f,\,cs}(T) - \mathcal{E}^{\rm f,\,cs}(T_0)\Big)\,. \tag{4.7}$$

In appendix D, we establish a general relation for the logarithm of the partition functions of the two theories on $S^1\times\mathbb{H}^2$ and so there may be interesting relations, similar to eq. (4.7), for other thermal quantities.[10] These relations are relating the properties of one theory at

[9] The form of this result relies on $\mathcal{E}^{\rm cs,\,f}(T=0)=0$ in our conventions. In appendix D, we have adopted the conventions of [52] in which $\mathcal{E}^{\rm cs,\,f}(T_0)=0$ instead. As explained below eq. (2.7) and in the appendix C, the differences arise from the choice of the renormalization scheme for the partition function on $S^1\times\mathbb{H}^2$.

[10] There may also be interesting relations for the circle Rényi entropies using the results in appendix E — a topic we leave for future work.

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temperature T to the other at T_0^2/T. Hence, in this regard, our duality is somewhat analogous to the Kramers-Wannier duality [58], which relates quantities in the two-dimensional Ising model at low and high temperatures.

We point out that a duality closer in spirit to Kramers-Wannier can be obtained by considering the free CFT containing both the complex scalar and the Dirac fermion. This CFT has $\mathcal{N}=2$ supersymmetry, as it consists of a free chiral multiplet. Now, the corner function $a_n(\theta)$ is additive in that case as free fields factorize, implying $\sigma_n^{\text{free}\,\mathcal{N}=2} = \sigma_n^{\text{cs}} + \sigma_n^{\text{f}}$. The above Bose-Fermi duality (4.3) thus leads to

$$n^2\,\sigma_n^{\text{free}\,\mathcal{N}=2} = \sigma_{1/n}^{\text{free}\,\mathcal{N}=2}\,, \qquad n>0,\, n\in\mathbb{R}\,, \tag{4.8}$$

which is a self-duality connecting $\sigma_q^{\text{free}\,\mathcal{N}=2}$ at $q=n$ and $1/n$. In particular, in this theory, σ_∞ acquires a concrete physical meaning as it is determined by the flat-space thermal entropy coefficient c_S of the same theory. It would be interesting to investigate the fate of this self-duality in the interacting $\mathcal{N}=2$ supersymmetric CFT, i.e., the chiral Wess-Zumino model, which can be obtained by perturbing the theory of the free chiral multiplet.

It is further natural to ask: do similar entanglement dualities exist in dimensions other than $d=3$? Below we point out that this indeed occurs in $d=2$.

4.1 Entanglement self-duality in $d=2$

For any CFT in $d=2$, we have the following self-duality relation for the twist dimension:

$$h_n^{(2)} = -h_{1/n}^{(2)}\,, \tag{4.9}$$

as follows from eq. (1.9). This can then be rewritten in terms of the coefficient $\sigma_n^{(2)}$ of the logarithmic contribution to the Rényi entropy of a single interval as

$$n\,\sigma_n^{(2)} = \sigma_{1/n}^{(2)}\,, \tag{4.10}$$

where we recall from eq. (1.10) that

$$\sigma_n^{(2)} = \frac{h_n^{(2)}}{n-1} = \frac{c}{12}\left(1+\frac{1}{n}\right). \tag{4.11}$$

The relation (4.10) is clearly reminiscent of the $d=3$ Bose-Fermi duality presented above, and in particular of its manifestation as the self-duality in the free supersymmetric theory in eq. (4.8). As a special case of (4.10), we have a Bose-Fermi duality between a free Dirac fermion and a real scalar (compact or not): $n\,\sigma_n^{(2),\,\text{s}} = \sigma_{1/n}^{(2),\,\text{f}}$, as both have $c=1$.

Given the above results in $d=2,3$, we are led to propose that relations of the form $n^{d-1}\sigma_n^{(d)} \sim \sigma_{1/n}^{(d)}$ or $n^{d-2}h_n^{(d)} \sim -h_{1/n}^{(d)}$ might occur as well in other higher-dimensional theories. The dualities may connect n to $1/n$ in two different theories, as in eq. (4.3), or within the same theory, as in eqs. (4.8) and (4.10).

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5 Holography

Let us now consider strongly coupled holographic CFTs dual to Einstein gravity. We will use known results for the twist dimension $h_n^{\rm hol}$ to obtain the n-dependence of the corner coefficient $\sigma_n^{\rm hol}$. We begin with the AdS/CFT correspondence in its simplest setting, where it describes a given d-dimensional boundary CFT in terms of $(d+1)$-dimensional gravity in the bulk with the action

$$I = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{g}\left[\frac{d(d-1)}{L^2}+\mathcal{R}\right], \tag{5.1}$$

where G is the $(d+1)$-dimensional Newton's constant, L is the AdS$_{d+1}$ radius, and $\mathcal{R}$ is the Ricci scalar. In order to obtain $h_n^{\rm hol}$, we need to consider the thermal ensemble of the CFT on the hyperbolic geometry appearing in the construction of [47], which is then equivalent to a topological black hole [59–64] with a hyperbolic horizon. Using this connection together with eq. (2.4), the scaling dimension of twist operators in the corresponding holographic CFTs was evaluated as [40]

$$h_n^{\rm hol} = \frac{L^{d-1}}{8G}\,n\left(\frac{\sqrt{d^2n^2-2d\,n^2+1}+1}{d\,n}\right)^{d-2}\left(1-\frac{\left(\sqrt{d^2n^2-2d\,n^2+1}+1\right)^2}{d^2\,n^2}\right). \tag{5.2}$$

In the following, we will focus our attention on the case of a three-dimensional boundary theory. Of course, one particular example of such a boundary CFT would be the supersymmetric gauge theory constructed in [65], in an appropriate large N and strong coupling limit.

Setting $d=3$ in eq. (5.2) and using the conjectured relation (1.6), we can predict the following form for the Rényi corner coefficient in these holographic CFTs:

$$\sigma_n^{\rm hol} = \frac{L^2}{108\pi G}\,\frac{(3n^2-2)\sqrt{1+3n^2}-2}{n^2(n-1)}\,. \tag{5.3}$$

Of course, at present, there is no method available by which we may evaluate these coefficients directly in a holographic framework. This expression can be easily used to compute the corner coefficient for arbitrary values of n, and some results are presented in table 4. In the table, we normalized the corner coefficient by dividing by the central charge $C_T^{\rm hol} = 3L^2/(\pi^3 G)$ [66, 67]. We recover the expected result for the corner coefficient in the entanglement entropy (i.e., with $n=1$). That is, $\sigma_1^{\rm hol}/C_T^{\rm hol} = \pi^2/24$ [36, 37]. Let us note at this point that the numerical values for the σ_n/C_T in the holographic CFT are remarkably similar those for the free Dirac fermion, as shown in figure 3. The relative error is no more than 2.6% for $n \geq 1$ and 0.2% in the range $0 \leq n \leq 1$.

Turning now to the limits $n \to 0$ and $n \to 1$, we find

$$\begin{aligned}\sigma_{n\to 0}^{\rm hol} &= \frac{L^2}{27\pi G}\left[\frac{1}{n^2}+\frac{1}{n}+\mathcal{O}\left(n^0\right)\right],\\ \sigma_n^{\rm hol}\big|_{n=1+\epsilon} &= \frac{L^2}{8\pi G}\left[1-\frac{7}{8}\epsilon+\mathcal{O}(\epsilon^2)\right],\end{aligned} \tag{5.4}$$

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n	$n \to 0$	1	2	3	4	5	6	$n \to \infty$
$\sigma_n^{\rm hol}/C_T^{\rm hol}$	$\frac{\pi^2}{81\,n^2}$	$\frac{\pi^2}{24}$	$\frac{\pi^2(5\sqrt{13}-1)}{648}$	$\frac{\pi^2(25\sqrt{7}-1)}{2916}$	$\frac{5\pi^2}{243}$	$\frac{\pi^2(73\sqrt{19}-1)}{16200}$	$\frac{\pi^2(53\sqrt{109}-1)}{29160}$	$\frac{\pi^2}{36\sqrt{3}}$

Table 4. Corner coefficient for various Rényi indices in holographic CFTs dual to Einstein gravity.

Comparing the $n \to 0$ limit above with eq. (1.13), we find

$$c_S^{\rm hol} = \frac{4\pi^2 L^2}{9\,G}\,. \tag{5.5}$$

This result exactly matches the thermal entropy coefficient for the holographic CFT calculated from a planar AdS_4 black hole in the bulk, e.g., see [37].

In the large-n limit, we find:

$$\sigma_n^{\rm hol} = \sigma_\infty^{\rm hol}\left(1+\frac{1}{n}\right)\left[1+\frac{b_2}{n^2}-\frac{b_4}{n^4}+\mathcal{O}\left(\frac{1}{n^6}\right)\right]-\frac{2\sigma_\infty^{\rm hol}}{3\sqrt{3}n^3}\,, \tag{5.6}$$

where

$$b_2=\frac{1}{2}\,,\quad b_4=\frac{2}{3\sqrt{3}}-\frac{3}{8}\simeq 0.01 \quad \text{and} \quad \sigma_\infty^{\rm hol}=\frac{L^2}{12\sqrt{3}\pi G}=\frac{\pi^2}{36\sqrt{3}}\,C_T^{\rm hol}\,. \tag{5.7}$$

We emphasize that we have shown analytically that the overall factor of $(1+1/n)$ appears in the first term of eq. (5.6). Of course, this was the structure conjectured for the large-n expansions in eq. (3.16) for the free scalar and fermion CFTs. However in contrast to the free CFTs, the second term proportional to $1/n^3$ spoils this simple n-dependence for the holographic CFTs. As explained in eq. (1.14), the coefficient σ_∞ is proportional to $\mathcal{E}(T_0)-\mathcal{E}(0)$, a difference of thermal energy densities on the hyperbolic cylinder. We note that the holographic calculations yield $\mathcal{E}(T_0)=0$ and $\mathcal{E}(0)=-L^2/(12\sqrt{3}\pi G R^3)$ here [68]. These results contrast to those for the corresponding free field energy densities in section 3, where $\mathcal{E}(0)$ vanishes while $\mathcal{E}(T_0)$ does not.

Let us close this section by mentioning that our results here should be easily generalizable to other holographic theories. In particular, it would be interesting to study these corner contributions for holographic theories whose bulk action contains higher-curvature terms, e.g., using the results of [40].

6 Corners with arbitrary opening angles

In sections 3 and 5, we saw that the corner coefficient σ_n has an interesting systematic dependence on the Rényi index. Below, we argue that this dependence is a general property of the entire corner function $a_n(\theta)$, at all opening angles θ. Further, in this section, we provide an efficient approach to approximating the angular dependence of the corner function $a_n(\theta)$, using knowledge of the coefficients κ_n and σ_n, which control the asymptotic behaviour in the limits $\theta \to 0$ and π, respectively.

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6.1 General n dependence

We now go beyond the nearly smooth limit and establish some general properties of the Rényi index dependence of the corner function $a_n(\theta)$, valid for any opening angle θ. Without loss of generality, we consider a region V containing a single corner with opening angle $0 < \theta < \pi$ but that is otherwise smooth, as in figure 1a. As it will be useful below, we recall that the Rényi entropy associated with V is then given by an expression of the form shown in eq. (1.2), i.e.,

$$S_n(V) = \widetilde{B}_n \frac{\mathcal{A}(\partial V)}{\delta} - a_n(\theta) \log\left(\mathcal{A}(\partial V)/\delta\right) + \cdots , \tag{6.1}$$

where we have written the leading term in a geometric fashion, with $\mathcal{A}(\partial V)$ being the total length of the entangling surface, and δ a UV cutoff.

Let us begin by considering the $n \to 0$ limit of the corner function. Here we begin by adapting the arguments that were encountered in section 2.1 for the present discussion. First, we step back to consider a circular region with an entirely smooth boundary. The problem of evaluating the Rényi entropy for such a region in the vacuum of a general CFT can be mapped to one of examining a thermal ensemble on hyperbolic space $\mathbb{H}^2$. However, it was observed by [48] that in the $n \to 0$ limit the result is dominated by the high temperature behaviour, which in fact matches that in flat space. Hence we have that the Rényi entropy for a circular entangling curve in three dimensions diverges as $S_n \sim 1/n^2$. In particular then, this argument indicates that as $n \to 0$, the coefficient $\widetilde{B}_n$ in eq. (6.1) diverges as $1/n^2$ for a circular entangling surface. However, this coefficient is independent of the details of the geometry and so the same divergence appears in the leading area law contribution to the Rényi entropy for any entangling surface. Now note that the corner makes a negative contribution to S_n with the second term in eq. (6.1) but the total Rényi entropy must always be positive. Hence we conclude that as $n \to 0$, $a_n(\theta)$ can diverge at most as fast as $1/n^p$, with $p \leq 2$. Indeed, as $\theta \to \pi$, we have $a_n(\theta) \simeq \sigma_n\,(\theta - \pi)^2$ where σ_n diverges precisely as $1/n^2$, eq. (1.13). Next, we can use the fact that $a_n(\theta)$ decreases monotonically for $0 < \theta < \pi$, eq. (1.3), to establish that for all angles $a_n(\theta)$ diverges at least as fast as $1/n^p$ with $p \geq 2$. Combining these two inequalities we are lead to $p = 2$, i.e., the corner function diverges as $1/n^2$ for any θ as $n \to 0$.

Next we consider the opposite limit, $n \to \infty$. In this case, we know that σ_n asymptotes to a finite value $\sigma_\infty > 0$. Hence at this point, we again invoke the monotonically decreasing behaviour of the corner function to infer that $a_{n\to\infty}(\theta)$ must either tend to a finite value or diverge, i.e., it will behave as n^q with $q \geq 0$ in the large-n limit. However, if $a_{n\to\infty}(\theta)$ was growing (with $q > 0$), the positivity of S_n would require that the coefficient $\widetilde{B}_n$ of the area law term and hence the entire Rényi entropy are also growing with n at least as fast as the corner function. However, it follows from considerations of information theory that S_n must satisfy various inequalities [69, 70] and in particular, one finds that $\partial_n S_n \leq 0$. That is, the Rényi entropy can not grow with increasing n (at any point). Therefore, we are led to the conclusion that the corner function remains finite as $n \to \infty$ (i.e., $q = 0$), again for any opening angle. Our calculations for the free CFTs confirm this result since the sharp corner coefficients κ_n of the complex scalar and the Dirac fermion approach finite values as $n \to \infty$, see section 3.2.2.

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The general asymptotics of the corner function found above for the limits $n \to 0$ and ∞, as well as the behaviour for large and small angles, are schematically illustrated in figure 2a.

6.2 Simple approximations for $a_n(\theta)$

In this subsection we address the following question: for a given Rényi index n, can one efficiently approximate the angle dependence of the corner function $a_n(\theta)$? We will show that by knowing the asymptotic coefficients κ_n and σ_n alone, one can construct a simple closed-form expression for $a_n(\theta)$ that is not only very accurate for all θ, but also exact as $\theta \to 0, \pi$. We will use two physically motivated functions that obey the correct asymptotics as our building blocks:

$$\tilde{a}^{(1)}(\theta) = \frac{(\theta-\pi)^2}{\theta(2\pi-\theta)}\,, \tag{6.2}$$

$$\tilde{a}^{(2)}(\theta) = 1 + (\pi-\theta)\cot\theta\,. \tag{6.3}$$

The first one is the simplest algebraic function with a quadratic zero at $\theta = \pi$ and with the pole at $\theta = 0$, as $a_n(\theta)$ is expected to have on general grounds — see eq. (1.4). The additional factor of $(2\pi - \theta)$ in the denominator ensures the function satisfies the constraint $\tilde{a}^{(1)}(\theta) = \tilde{a}^{(1)}(2\pi - \theta)$, as required for the Rényi entropy of pure states. Interestingly, precisely the same function yields the corner function $a_n(\theta)$ of certain Lifshitz quantum critical points [55]. The second function $\tilde{a}^{(2)}$ corresponds to the functional form of the corner contribution in the extensive mutual information model [45, 53, 54], as described above at eq. (3.25). Both functions satisfy the expected constraints in eq. (1.3). However, we note that they also satisfy a stronger constraint obtained for the entanglement entropy coefficient $a_1(\theta)$ using strong subadditivity and Lorentz invariance [8–10, 71], i.e., $\partial_\theta^2 \tilde{a}^{(i)}(\theta) \geq |\partial_\theta \tilde{a}^{(i)}(\theta)|/\sin\theta$ for $0 \leq \theta \leq \pi$. Further, as mentioned in the introduction, reflection positivity also imposes a infinite tower of nonlinear constraints on the corner functions $a_n(\theta)$ [34]. These constraints can be schematically written as $\det(\{\partial_\theta^{j+k+2} a_n(\theta)\}_{j,k=0}^{M-1}) \geq 0$, with $M \geq 1$ and $\theta \in [0, \pi]$. For example, with $M = 1$, this reduces to the linear constraint $\partial_\theta^2 a_n \geq 0$, but the second inequality with $M = 2$ reads:

$$\partial_\theta^2 a_n\, \partial_\theta^4 a_n - (\partial_\theta^3 a_n)^2 \geq 0\,. \tag{6.4}$$

We have managed to explicitly verify that our basis functions $\tilde{a}^{(1)}(\theta)$ and $\tilde{a}^{(2)}(\theta)$ in eq. (6.2) satisfy these non-linear constraints for $M = 2, \ldots, 6$.

We now construct an approximation $\tilde{a}_n(\theta)$ for the corner functions with a simple linear combination of $\tilde{a}^{(1)}$ and $\tilde{a}^{(2)}$:

$$a_n(\theta) \simeq \tilde{a}_n(\theta) \equiv \lambda_n^{(1)} \tilde{a}^{(1)}(\theta) + \lambda_n^{(2)} \tilde{a}^{(2)}(\theta)\,, \tag{6.5}$$

where the constants $\lambda_n^{(i)}$ are fixed by requiring the right-hand side of eq. (6.5) to have the same asymptotics as $a_n(\theta)$ in the limits $\theta \to 0$ and π. Taylor expanding $\tilde{a}^{(1)}, \tilde{a}^{(2)}$ in the these limits, we obtain:

$$\begin{aligned} \theta \to 0 \;&:\quad \frac{\pi}{2}\lambda_n^{(1)} + \pi\lambda_n^{(2)} = \kappa_n\,, \\ \theta \to \pi \;&:\quad \frac{1}{\pi^2}\lambda_n^{(1)} + \frac{1}{3}\lambda_n^{(2)} = \sigma_n\,. \end{aligned} \tag{6.6}$$

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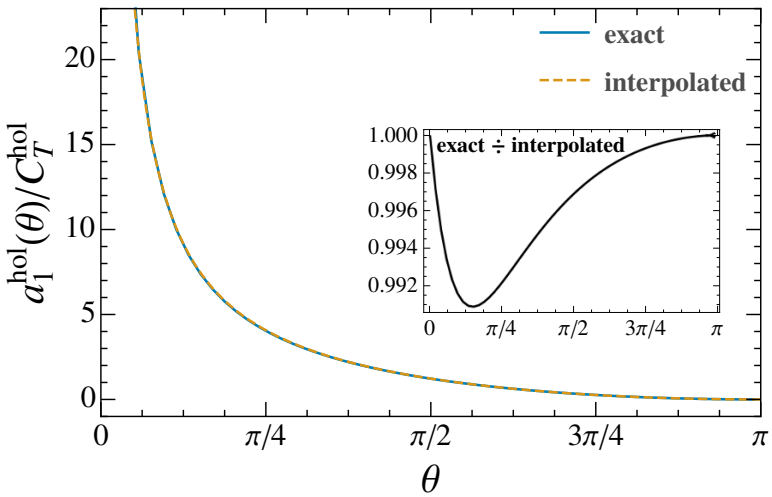

Figure 5. Holographic corner function $a_1^{\rm hol}(\theta)$ [72] normalized by $C_T^{\rm hol}$, and the approximate form obtained by interpolating the asymptotics with eq. (6.5). The two lines can hardly be distinguished. The inset shows the ratio of the exact and extrapolated functions.

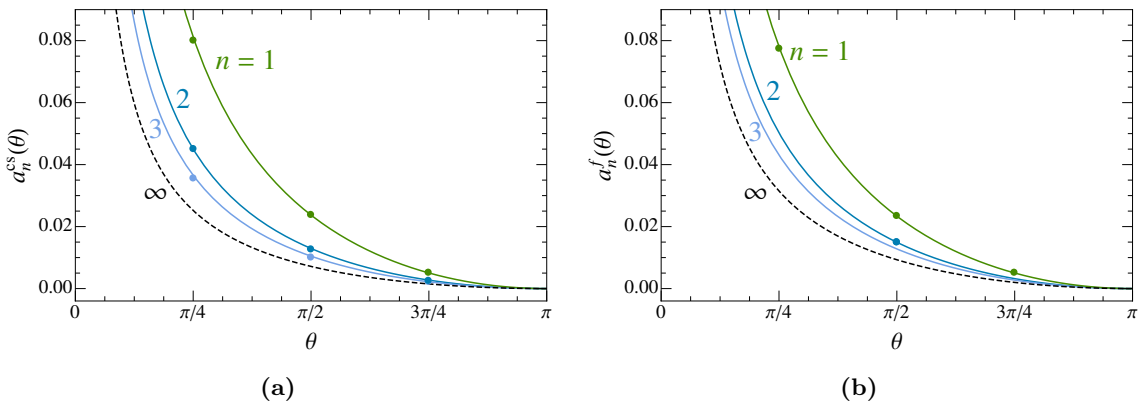

Figure 6. Closed-form approximations for the corner function $a_n(\theta)$ at $n = 1, 2, 3$ and $n = \infty$ for (a) the free complex scalar and (b) the free Dirac fermion. The dots were obtained from numerical lattice calculations [8–10]. The approximate functions obey the *exact* asymptotics as $\theta \to 0, \pi$.

The unique solution of eq. (6.6) for the two unknown coefficients is then:

$$\lambda_n^{(1)} = 2\pi\,\frac{\kappa_n - 3\pi\sigma_n}{\pi^2 - 6}\,, \qquad \lambda_n^{(2)} = -\frac{3}{\pi}\,\frac{2\kappa_n - \pi^3\sigma_n}{\pi^2 - 6}\,. \tag{6.7}$$

The holographic CFT dual to Einstein gravity provides a simple test of this approach (6.5), at least for $n = 1$. For this theory, we have an integral expression to evaluate $a_1^{\rm hol}(\theta)$ for any angle [72], and the necessary coefficients are $\kappa_1^{\rm hol} = \Gamma(3/4)^4\, L^2/(2\pi G)$ and $\sigma_1^{\rm hol} = L^2/(8\pi G)$, e.g., see [37]. In figure 5, we show the curves for the exact result for $a_1^{\rm hol}(\theta)/C_T^{\rm hol}$ and the approximate function also normalized by $C_T^{\rm hol} = 3L^2/(\pi^3 G)$. The agreement is seen to be excellent, with the relative error being less than 1% over the entire range of angles.

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We can also use the values of σ_n and κ_n computed in the section 3.2 to construct approximations of $a_n(\theta)$ for the free complex scalar and the free Dirac fermion. Now in principle, the $a_n(\theta)$ functions can also be determined exactly for these free fields, as described in [8–10]. Unfortunately, their results are written implicitly in terms of several functions which must be determined through a complicated system of coupled non-linear algebraic and differential equations. We explicitly used this approach to test our approximation for $a_2^{\rm cs}(\theta)$ for the complex scalar, and we found that the difference between our approximation and the 'exact' curve was comparable to our numerical errors in evaluating $a_n(\theta)$ with the method of [8–10]. The numerical errors were of the order of 0.20% while the maximum difference between the two curves was approximately 0.33%. Clearly, these results suggest that our approach produces a faithful approximation to $a_n(\theta)$ for the free fields. In figure 6, we show our approximate corner functions for $n = 1, 2, 3$ and ∞, both for the free scalar and fermion. The figure also includes the results of lattice simulations for $\theta = \pi/4,\ \pi/2$ and $3\pi/4$, in the cases where these are known,[11] and again, we see that our interpolating curves agree very well with these lattice points. As an additional check, $a_4^{\rm cs}(\pi/2)$ was recently estimated to be 0.0086(2) by numerically studying the Goldstone phase of antiferromagnets on the square lattice [33]. Using a different system, a numerical linked cluster expansion [73] yielded 0.0094(2). These values are in excellent agreement with our estimate 0.0094, obtained with the simple interpolating method described in this section.

Finally, we comment on the reasons behind the excellent accuracy of our interpolations. As noted in eq. (1.3), $a_n(\theta)$ is a monotonic convex function of θ. Thus we may expect that once the behavior in the vicinity of the endpoints $\theta = 0$ and π is fixed (by our knowledge of κ_n and σ_n), little freedom remains at intermediate angles. Our findings confirm this suspicion. Besides, we have checked that all the interpolating functions shown in figures 5 and 6 satisfy the first six inequalities arising from reflection positivity mentioned above.

7 Discussion

In this paper, we have introduced a generalized version of the conjecture presented in [36, 37] for the coefficient appearing in the universal contribution in the entanglement entropy for an almost smooth corner,

$$\sigma_1 = \frac{\pi^2}{24}\, C_T\,. \tag{7.1}$$

In particular, we proposed that the analogous Rényi corner coefficients σ_n — see eq. (1.4) — are related to the scaling dimension h_n of the corresponding twist operators τ_n by

$$\sigma_n = \frac{1}{\pi}\,\frac{h_n}{n-1}\,. \tag{7.2}$$

This expression reduces to our previous conjecture in the limit $n \to 1$, and is therefore supported in this limit by the extensive results presented in [36–39, 42], corresponding to

[11] We thank Horacio Casini for providing these lattice results, as well as his assistance in directly evaluating $a_2^{\rm cs}(\theta)$ using the method of [8–10]. We obtained $a_2^{\rm cs}(\theta)$ for $0.6 < \theta < 3.0$, but do not show it in figure 6 since given the uncertainties described in the main text, the result would be indistinguishable from the approximate interpolation in the plot.

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free scalars and fermions, as well as a general proof for holographic theories. Here we have checked the generalized version of the conjecture for a free conformally coupled complex scalar and a free massless Dirac fermion for integer Rényi indices $1 \leq n \leq 500$, as well as in the limit $n \to \infty$. The calculations involved in the evaluation of σ_n and h_n for the free fields are very different, both conceptually and computationally, which motivates us to conjecture that eq. (7.2) is a universal expression for general three-dimensional CFTs. It would also be interesting to explore how these expressions get modified in non-conformal theories, see e.g., [55, 74, 75].

Further, based on the properties of h_n at small n, we have obtained that

$$\lim_{n\to 0} \sigma_n = \frac{1}{12\pi^3}\frac{c_S}{n^2}\,, \tag{7.3}$$

which yields a key observable, namely the thermal entropy density coefficient c_S of the original CFT. As an interesting application of this result, we find that σ_n for the O(N) Wilson-Fisher IR fixed point in the $N \to \infty$ limit differs from its value at the UV Gaussian fixed point. Indeed, the ratio of the entropy coefficients at $N \to \infty$ is $c_S^{\rm IR}/c_S^{\rm UV} = 4/5$ [76]. Hence σ_n already distinguishes the entanglement structure of these two CFTs even in the absence of finite N corrections, in contrast to the RG monotone F [77, 78].[12] This suggests that the full corner function $a_n(\theta)$ differs at other values of n and θ between the $N \to \infty$ IR and UV fixed points. It will be interesting to verify this claim.

Our original motivation for the new conjecture (1.6) was as follows: essentially, the corner coefficient σ_n describes the universal response of the Rényi entropy to a small deformation of the (originally smooth) entangling surface, as shown in figure 1b. This small displacement of the twist operator can be accomplished by making appropriate insertions of the stress tensor, following the approach of [79–81] — see also [82–84]. Hence in the case of the entanglement entropy, the response is determined by correlators of the stress tensor and the entanglement Hamiltonian [79–81, 85]. Since the latter is given by an integral of the stress tensor over the region V [47], the calculation actually involves correlators of the stress tensor with itself and hence it is natural that the response is controlled by the central charge C_T as in eq. (1.5). In the case of the Rényi entropy (with $n \neq 1$), the corresponding calculation instead involves correlators of the stress tensor and the twist operator and so this reasoning naturally suggests that the response would be determined by the scaling dimension h_n as in eq. (1.6). This reasoning also outlines a path potentially leading to a proof of our new conjecture, however, the detailed calculations are subtle and we must leave this topic for future work [86].

In eq. (1.10), we observed that the universal contribution in the Rényi entropy of an interval in a two-dimensional CFT also contains a factor of $h_n/(n-1)$. This suggests that the form given in eq. (7.2) generalizes to higher dimension as well. In particular, using the results of [83, 84, 87], we have shown that the higher dimensional analogs of σ_1 describing the universal contribution in the entanglement entropy for an almost smooth cone is proportional to the central charge C_T for general holographic CFTs in any number

[12] Of course, the scaling dimension h_n of the twist operator serves the same purpose.

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of dimensions [39].[13] Further, as shown in eq. (2.5), $\partial_n h_n|_{n=1}$ is also related to C_T in general dimensions. Hence, it is natural to suggest that eq. (7.2) will have a higher dimensional counterpart for the coefficients $\sigma_n^{(d)}$ controlling the response of the Rényi entropy to an almost smooth conical defect in the entangling surface of the form

$$\sigma_n^{(d)} = g(d)\,\frac{h_n^{(d)}}{n-1}\,, \tag{7.4}$$

where $g(d)$ is a geometric factor that depends on the spacetime dimension, and $h_n^{(d)}$ is the scaling dimension of the corresponding twist operators in d dimensions. Proving this conjecture would give a new perspective on the geometric character of Rényi entropies in higher dimensions [82–84, 88].

In section 4, we observed an intriguing duality between the corner coefficients of the free scalar and the free fermion. In particular, this relates the corner coefficients for inverse values of the Rényi index through the simple expression $n^2\sigma_n^{\mathrm{cs,f}} = \sigma_{1/n}^{\mathrm{f,cs}}$, which we proved in appendix D. Further implications of this duality include that the energy density of one theory on the hyperbolic cylinder at temperature T is related to the corresponding energy density of the dual theory at T_0^2/T, as in eq. (4.7). It would be interesting to explore the physical implications of this new duality in more depth. Similarly, it would be worthwhile to search for similar relations for other theories in other spacetime dimensions.

Our proof of the Bose-Fermi duality makes use of the representation of $\log Z^{\mathrm{cs,f}}$ given in [52] — see appendices C and D. This allows us to give the following alternative representation of $\sigma_n^{\mathrm{cs,f}}$

$$\begin{aligned}\sigma_n^{\mathrm{cs}} &= \frac{n}{4\pi^2(n-1)}\left[\frac{3\zeta(3)}{4\pi^2} - \frac{2}{\pi^2}\int_0^\infty d\lambda\;\lambda^2\,\tanh\lambda\,\left(\coth(n\lambda)-1\right)\right],\\ \sigma_n^{\mathrm{f}} &= \frac{n}{4\pi^2(n-1)}\left[\frac{\zeta(3)}{\pi^2} + \frac{2}{\pi^2}\int_0^\infty d\lambda\;\lambda^2\,\coth\lambda\,\left(\tanh(n\lambda)-1\right)\right].\end{aligned} \tag{7.5}$$

It would be interesting prove the equivalence of these integral expressions with those in eq. (3.12), which were used in the main text. Alternatively, it also appears that these expressions may be somewhat simpler to relate to the sums in eq. (3.10), than those in eq. (3.12).

In this section 6, we devised an approach to constructing a simple analytic function which gives a (very) good approximation to the entire corner function $a_n(\theta)$ for all angles. These functions make an informed interpolation between the asymptotic limits (1.4) at $\theta \to 0$ and π. Hence the only data necessary to construct these approximations is the almost smooth corner coefficient σ_n and the small angle coefficient κ_n in eq. (1.4). Of course, our conjecture (7.2) indicates that the coefficient σ_n can be accessed by a different calculation. In particular, the scaling dimension h_n is determined by the leading singularity in correlation function of the stress tensor with a twist operator, as shown in eq. (2.3). Similarly, the small angle coefficient κ_n can be accessed by an alternative calculation. In particular, as described in [8–10, 37], the same coefficient controls the universal contribution in the Rényi entropy of an narrow strip. It would be interesting to investigate situations

[13] Similar observations were recently made in [38, 74].

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where these two alternate calculations would allow the corner contribution to be determined more simply than by evaluating the Rényi entropy for entangling surfaces with corners of varying angles.

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A Evaluation of h_n for the free CFTs

In this appendix, we evaluate the integral expressions in eq. (3.5) for the scaling dimension of the twist operator with odd Rényi index $n > 1$, in the free conformal field theories. Using the same integrals along with eq. (1.6), we also evaluate σ_n in closed-form for various rational *non-integer* values of n. These results provide an interesting explicit check of the Bose-Fermi duality discussed in section 4, and proven in appendix D.

A.1 Complex scalar

We begin with the conformal dimension h_n in the theory of a free complex scalar, which may be written as:

$$h_n^{\rm cs} = \frac{1}{8\pi n^2} \int_{-\infty}^{\infty} \frac{du}{\sinh u} \left[\frac{\cosh(u/n)}{\sinh^3(u/n)} - n^3 \frac{\cosh u}{\sinh^3 u} \right]. \tag{A.1}$$

Here we have used the fact that the integrand is even to extend the range of integration over the entire real line. In passing, we also note that the integrand is finite at the origin, where it becomes $(n^4 - 1)/(15n) + O(u^2)$. Further we note that the integrand decays exponentially fast as $u \to \pm\infty$. Next we observe that if we shift the contour in the complex u plane from the real axis to $\Im(u) = i\pi n$, then the integrand is invariant up to an overall phase of $e^{in\pi} = (-)^n$. Hence for odd n, we may re-express the conformal weight in terms of complex contour integral which runs along the contour C, illustrated in figure 7:

$$h_n^{\rm cs} = \frac{1}{16\pi n^2} \oint_C \frac{du}{\sinh u} \left[\frac{\cosh(u/n)}{\sinh^3(u/n)} - n^3 \frac{\cosh u}{\sinh^3 u} \right]. \tag{A.2}$$

Thus the result will be expressed in terms of the residues of the poles found to lie within the contour C. The integrand within the contour is only singular along the imaginary axis where $\sinh u$ vanishes, i.e., at $u = i\pi k$ with $k = 1, 2, \cdots, n-1$. The second term in the

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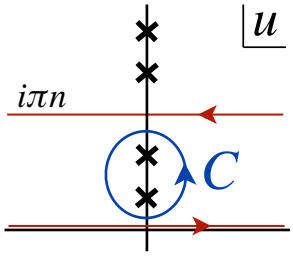

Figure 7. The integration contour C in the complex u plane. Poles occur at $u = i\pi k$, for all integers k that are not multiples of n. Here we illustrate the case $n = 3$, with two poles lying inside C.

integrand actually has a quartic and quadratic pole at each of these points but neither of these contribute to the closed contour integral. Hence only the first term contributes with the result

$$h_n^{\rm cs} = \frac{1}{8n^2}\sum_{k=1}^{n-1}(-)^{k+1}\frac{\cos(\pi k/n)}{\sin^3(\pi k/n)}\,. \tag{A.3}$$

We note again that this result only applies for odd values of n. One can confirm that this sum vanishes for n even, as expected. We have also explicitly verified that the above expression reproduces the value of $h_n^{\rm cs}$ given by directly evaluating the integral in eq. (3.5) for odd values of n up to $n = 500$. Combined with the new conjecture (1.6), the above expression yields the following sum for the corresponding corner coefficient (for odd $n > 1$):

$$\sigma_n^{\rm cs} = \frac{1}{8\pi n^2(n-1)}\sum_{k=1}^{n-1}(-)^{k+1}\frac{\cos(\pi k/n)}{\sin^3(\pi k/n)}\,. \tag{A.4}$$

Unfortunately, the sum appearing here does not seem to be simply related to that appearing in eq. (3.10).

A.2 Dirac fermion

Now we consider the conformal dimension of the twist operator in the theory of a free Dirac fermion. Again the integrand in eq. (3.5) is even and so we extend the integral to the entire real line:

$$h_n^{\rm f} = \frac{1}{16\pi n^2}\int_{-\infty}^{\infty}\frac{du}{\tanh u}\left[n^3\,\frac{2+\sinh^2 u}{\sinh^3 u} - \frac{2+\sinh^2(u/n)}{\sinh^3(u/n)}\right]. \tag{A.5}$$

Again, we note that the integrand is finite at the origin, where it becomes $7(n^4-1)/(30n)+O(u^2)$. Further we note that the integrand decays exponentially fast as $u \to \pm\infty$. Next we observe that if we shift the contour in the complex u plane from the real axis to $\Im(u) = i\pi n$, then the integrand is invariant up to an overall change of sign, but only for odd n. Hence we may re-express the conformal weight in terms of complex contour integral which runs along the contour C, illustrated in figure 7:

$$h_n^{\rm f} = \frac{1}{32\pi n^2}\oint_C\frac{du}{\tanh u}\left[n^3\,\frac{2+\sinh^2 u}{\sinh^3 u} - \frac{2+\sinh^2(u/n)}{\sinh^3(u/n)}\right]. \tag{A.6}$$

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n	$1/6$	$1/5$	$1/4$	$1/3$	$2/3$	$1/2$	$3/2$
$\sigma_n^{\rm cs}$	$\frac{261+20\sqrt{3}}{720\pi}$	$\frac{\sqrt{425+58\sqrt{5}}}{80\pi}$	$\frac{1+6\sqrt{2}}{48\pi}$	$\frac{5}{24\pi\sqrt{3}}$	$\frac{20\sqrt{3}\pi-81}{216\pi^2}$	$\frac{1}{16\pi}$	$\frac{243-128\sqrt{3}}{1296\pi}$
$\sigma_n^{\rm f}$	$\frac{81+34\sqrt{3}\pi}{270\pi^2}$	$\frac{\sqrt{25-2\sqrt{5}}}{20\pi}$	$\frac{8+3\pi}{36\pi^2}$	$\frac{1}{6\pi\sqrt{3}}$	$\frac{243-128\sqrt{3}}{576\pi}$	$\frac{1}{6\pi^2}$	$\frac{20\sqrt{3}\pi-81}{486\pi^2}$

Table 5. Corner coefficient σ_n for various rational values of the Rényi index for the complex scalar and Dirac CFTs in $d = 3$.

Hence the result can again be expressed in terms of the residues of the poles found to lie within the contour C. The integrand in C is only singular along the imaginary axis where $\sinh(u)$ vanishes, i.e., at $u = i\pi k$ with $k = 1, 2, \cdots, n-1$. Again, the first term in the integrand actually has a quartic and quadratic pole at each of these points and neither of these contribute to the closed contour integral. Hence only the second term contributes with the result

$$h_n^{\rm f} = \frac{1}{16n^2}\sum_{k=1}^{n-1}\left[\frac{2}{\sin^3(\pi k/n)} - \frac{1}{\sin(\pi k/n)}\right] . \tag{A.7}$$

Again, this result only applies for n odd. In this case, the sum does not vanish for n even, but the result should not be taken as the correct value of $h_n^{\rm f}$. We have again explicitly verified that this sum reproduces the value of $h_n^{\rm f}$ found by directly evaluating the integral in eq. (3.5) for odd values of n up to $n = 500$. Combining the above sum with the new conjecture (1.6), we find the following expression for the corresponding corner coefficient (for odd $n > 1$):

$$\sigma_n^{\rm f} = \frac{1}{16\pi n^2(n-1)}\sum_{k=1}^{n-1}\left[\frac{2}{\sin^3(\pi k/n)} - \frac{1}{\sin(\pi k/n)}\right] . \tag{A.8}$$

Here again, this new sum does not seem to be simply related to the sum in eq. (3.10) for the same coefficients.

A.3 Rational values of n

Using our integrals (3.5) for the scaling dimensions h_n of the free scalar and fermion, we can evaluate σ_n in closed-form at various rational *non-integer* values of n. We list select results in table 5. Comparing the results for $n = 2/3$ and $3/2$, we see that they explicitly satisfy the Bose-Fermi duality $n^2\sigma_n^{\rm sc} = \sigma_{1/n}^{\rm f}$ in eq. (4.3). Similarly, comparing $n = 1/6$, $1/5$, $1/4$, $1/3$ and $1/2$ with the dual results in table 2, we see that they obey the same duality relation. We provide a general proof of the duality eq. (4.3) in appendix D.

B Sharp corner coefficient calculation

In section 3.2.2, we computed the sharp corner coefficients κ_n, $n = 2, 3, 4$ and $n = \infty$ for the complex scalar and the Dirac fermion. Here we explain how these calculations are performed following the approach of [8–10]. In particular, we exploit the relation between

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these coefficients and the so-called 'entropic c-functions' for a family of corresponding massive free fields in $d = 2$. As was noted in section 3.2.2, the $\kappa_n^{\mathrm{cs,f}}$ coefficients are given by

$$\kappa_n^{\mathrm{cs,f}} = \int_0^\infty \frac{dt}{\pi}\, c_n^{\mathrm{cs,f}}(t)\,, \qquad \text{where} \qquad c_n^{\mathrm{cs,f}}(m\ell) = \ell\, \frac{dS_n^{\mathrm{cs,f}}(m\ell)}{d\ell}\,, \tag{B.1}$$

where $S_n^{\mathrm{cs,f}}(m\ell)$ are the Rényi entropies of $d = 2$ free fields with mass m for a single interval of length ℓ. The $c_n(t)$ for the complex scalar and the Dirac fermion can be evaluated as

$$c_n^{\mathrm{cs}} = \frac{1}{1-n}\sum_{k=1}^{n-1} \omega_{k/n}^{\mathrm{cs}}(t)\,, \qquad c_n^{\mathrm{f}} = \frac{2}{1-n}\sum_{k>0}^{(n-1)/2} \omega_{k/n}^{\mathrm{f}}(t)\,, \tag{B.2}$$

respectively, where the $\omega_a^{\mathrm{cs,f}}$ are given by

$$\omega_a(t) = -\int_t^\infty dy\, y\, u_a^2(y)\,, \tag{B.3}$$

and the functions $u_a(y)$ satisfy, respectively, the differential equations

$$u''_{a,\,\mathrm{cs}}(t) + \frac{u'_{a,\,\mathrm{cs}}(t)}{t} - \frac{u_{a,\,\mathrm{cs}}(t)}{1+u_{a,\,\mathrm{cs}}^2(t)} u'^2_{a,\,\mathrm{cs}}(t) - u_{a,\,\mathrm{sc}}(t)(1+u_{a,\,\mathrm{cs}}^2(t)) - \frac{(2a-1)^2}{t^2}\,\frac{u_{a,\,\mathrm{cs}}(t)}{1+u_{a,\,\mathrm{cs}}^2(t)} = 0, \tag{B.4}$$

$$u_{a,\,\mathrm{cs}}(t\to 0) = \frac{-1}{t(\log(t/2) + 2\gamma_{\mathrm{E}} + (\psi[a] + \psi[1-a])/2)}\,, \tag{B.5}$$

$$u_{a,\,\mathrm{cs}}(t\to\infty) = \frac{2}{\pi}\sin(a\pi) K_{1-2a}(t)\,,$$

and

$$u''_{a,\,\mathrm{f}}(t) + \frac{u'_{a,\,\mathrm{f}}(t)}{t} + \frac{u_{a,\,\mathrm{f}}(t)}{1-u_{a,\,\mathrm{f}}^2(t)} u'^2_{a,\,\mathrm{f}}(t) - u_{a,\,\mathrm{f}}(t)(1-u_{a,\,\mathrm{f}}^2(t)) - \frac{4a^2}{t^2}\,\frac{u_{a,\,\mathrm{f}}(t)}{1-u_{a,\,\mathrm{f}}^2(t)} = 0\,, \tag{B.6}$$

$$u_{a,\,\mathrm{f}}(t\to 0) = -2a(\log(t/2) + 2\gamma_{\mathrm{E}} + (\psi[a] + \psi[-a])/2)\,, \tag{B.7}$$

$$u_{a,\,\mathrm{f}}(t\to\infty) = \frac{2}{\pi}\sin(a\pi) K_{2a}(t)\,.$$

The differential equations can also be used to eliminate the integral in eq. (B.3) producing

$$\omega_a^{\mathrm{cs}}(t) = -\frac{t^2 u'^2_{a,\,\mathrm{cs}} - u_{a,\,\mathrm{cs}}^2((1-2a)^2 + t^2(1+u_{a,\,\mathrm{cs}}^2))}{2(1+u_{a,\,\mathrm{cs}}^2)}\,, \tag{B.8}$$

$$\omega_a^{\mathrm{f}}(t) = -\frac{t^2 u'^2_{a,\,\mathrm{f}} - u_{a,\,\mathrm{f}}^2(4a^2 + t^2(1-u_{a,\,\mathrm{f}}^2))}{2(1-u_{a,\,\mathrm{f}}^2)}\,. \tag{B.9}$$

The final expressions for the $\kappa_n^{\mathrm{cs,f}}$ are then given in eq. (3.22). Hence, our approach is to numerically solve eqs. (B.4) and (B.6) subject to the boundary conditions (B.5) and (B.7), respectively, and then use the results to evaluate eq. (3.22).

Let us add here that we can evaluate $\kappa^{\mathrm{cs,f}}$ corresponding to the special case $n = 1$ with a similar approach using two-dimensional entropic c-functions. In particular, we have

$$\kappa^{\mathrm{cs,f}} = \int_0^\infty \frac{dt}{\pi}\, c^{\mathrm{cs,f}}(t)\,, \tag{B.10}$$

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with

$$c^{\rm cs}(t) = -\pi \int_0^\infty \frac{db}{\cosh^2(\pi b)}\, \omega^{\rm cs}_{1/2-ib}(t)\,, \tag{B.11}$$

$$c^{\rm f}(t) = 2\pi \int_0^\infty \frac{db}{\sinh^2(\pi b)}\, \omega^{\rm f}_{-ib}(t)\,. \tag{B.12}$$

The values of these coefficients were evaluated by [8–10] and appear in table 3 under $n = 1$.

C Relation between σ_∞ and $\mathcal{F}_\infty$

In comparing the behaviour of eq. (3.10) for large n, the authors of [42] observed a relation (3.18) between σ_∞ for the free fields and corresponding results for the logarithm of the partition function on the hyperbolic cylinder $S^1 \times \mathbb{H}^2$, given in [52]. We will provide an explanation of this relation in this appendix and at the same time, we show that the same result holds for any three-dimensional CFT.

Following [52], we begin by defining the logarithm of the thermal partition function evaluated at $T = T_0/n$

$$\widehat{\mathcal{F}}_n \equiv -\log Z(T_0/n)\,, \tag{C.1}$$

which is now readily connected to the corner coefficient σ_n through eq. (3.11). In particular, it is straightforward to show that eq. (3.2) for the desired energy densities can be re-expressed as

$$\mathcal{E}(T_0/n) = \frac{1}{2\pi R\, V_{\mathbb{H}^2}}\, \partial_n \widehat{\mathcal{F}}_n\,, \tag{C.2}$$

and hence eq. (3.11) can be written as

$$\sigma_n = -\frac{R^2}{2\pi\, V_{\mathbb{H}^2}}\, \frac{n}{n-1}\, \left(\partial_n \widehat{\mathcal{F}}_n - \partial_n \widehat{\mathcal{F}}_n\big|_{n=1}\right) \tag{C.3}$$

Now, of course, the quantity $\widehat{\mathcal{F}}_n$ is infrared divergent because it involves an integration over the infinite volume of the hyperbolic plane $\mathbb{H}^2$. However, both of the previous two formulae compensate for this divergence by dividing by $V_{\mathbb{H}^2}$ to produce a finite quantity. One can produce expressions involving only finite quantities by focusing on the universal contribution to $\widehat{\mathcal{F}}_n$, which amounts here to only retaining the finite regulator-independent contribution. The IR regulator also renders the volume finite and one only keeps the regulator-independent term in $V_{\mathbb{H}^2}$. In particular, examining eq. (3.3), we have $V^{\rm univ}_{\mathbb{H}^2} = -2\pi R^2$. Hence eq. (C.3) can be written as

$$\sigma_n = \frac{1}{4\pi^2}\, \frac{n}{n-1}\, \left(\partial_n \widehat{\mathcal{F}}^{\rm univ}_n - \partial_n \widehat{\mathcal{F}}^{\rm univ}_n\big|_{n=1}\right). \tag{C.4}$$

However, before proceeding, we must first consider a difference in the conventions in the present paper and [52], which was alluded to in the discussion after eq. (2.7). With the heat kernel regularization used to produce the partition functions (3.4) for the free fields [41], one finds a nonvanishing energy density at $T = T_0$. On the other hand, with the approach used in [52], $\mathcal{E}(T_0) = 0$ and hence $\partial_n \widehat{\mathcal{F}}^{\rm univ}_n\big|_{n=1} = 0$ as a consequence of eq. (C.2).

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Since our present goal is to explain the result in eq. (3.18), we adopt the latter convention for the moment. Of course, this produces a simplification in eq. (C.4) and we are left with the following expression for the corner coefficient,

$$\sigma_n = \frac{1}{4\pi^2}\,\frac{n}{n-1}\,\partial_n \widehat{\mathcal{F}}_n^{\text{univ}}\,. \tag{C.5}$$

Finally, we consider the limit $n \to \infty$ for which one finds a linear growth in $\widehat{\mathcal{F}}_n^{\text{univ}}$. Hence defining $\mathcal{F}_\infty$ as the coefficient of the leading term, i.e., $\widehat{\mathcal{F}}_n^{\text{univ}} \sim n\,\mathcal{F}_\infty$, the above expression reproduces the desired relation (3.18)

$$\sigma_\infty = \frac{1}{4\pi^2}\,\mathcal{F}_\infty\,, \tag{C.6}$$

which was observed in [42] for the free CFTs. Of course, our analysis here made no reference to the free field theories discussed there and so we have established that eq. (3.18) holds generally for any three-dimensional CFT — assuming that the renormalization scheme is chosen such that $\mathcal{E}(T_0) = 0$.

To close, let us return to the difference in the conventions adopted here and in [52]. As noted above and discussed below eq. (2.7), differences in the renormalization of the Euclidean partition function produce a shift in $\widehat{\mathcal{F}}_n$:

$$\left[\widehat{\mathcal{F}}_n\right]_{\text{here}} = \left[\widehat{\mathcal{F}}_n - n\,\left(\partial_n \widehat{\mathcal{F}}_n\right)_{n=\infty}\right]_{[52]}\,. \tag{C.7}$$

Hence let us consider the form of eq. (3.18) if we instead adopt the conventions of the present paper. First, we note that with eq. (C.7), $\partial_n \widehat{\mathcal{F}}_n^{\text{univ}}$ vanishes in the limit $n \to \infty$. Therefore eq. (C.4) reduces to the following expression

$$\sigma_\infty = -\frac{1}{4\pi^2}\,\partial_n \widehat{\mathcal{F}}_n^{\text{univ}}\Big|_{n=1} = R^3\,\mathcal{E}(T_0)\,. \tag{C.8}$$

Again, we emphasize that although eqs. (C.6) and (C.8) are dissimilar in appearance, they both yield the same numerical value for σ_∞.

D Proof of the Bose-Fermi duality

In this appendix, we prove the intriguing duality relation (4.3) relating the Rényi corner coefficients of the free scalar and the free Dirac fermion through

$$n^2\,\sigma_n^{\text{cs}} = \sigma_{1/n}^{\text{f}}\,, \qquad \text{for } n > 0,\ n \in \mathbb{R}. \tag{D.1}$$

The proof is most simply presented by adopting the conventions and notation of [52], as described in the previous appendix. In this case, we can use eq. (C.5) to rewrite the duality relation (D.1) as

$$n^3 \partial_n \widehat{\mathcal{F}}_n^{\text{univ, cs}} + \partial_q \widehat{\mathcal{F}}_q^{\text{univ, f}}\Big|_{q=1/n} = 0\,. \tag{D.2}$$

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Now, ref. [52] provides the following simple expressions for $\widehat{\mathcal{F}}_n^{\text{univ, cs}}$ and $\widehat{\mathcal{F}}_n^{\text{univ, f}}$

$$\widehat{\mathcal{F}}_n^{\text{univ, cs}} = n\frac{3\zeta(3)}{4\pi^2} - \frac{2}{\pi^2}\int_0^\infty d\lambda\,\lambda\tanh(\lambda)\log\left(1-e^{-2n\lambda}\right), \tag{D.3}$$

$$\widehat{\mathcal{F}}_n^{\text{univ, f}} = n\frac{\zeta(3)}{\pi^2} + \frac{2}{\pi^2}\int_0^\infty d\lambda\,\lambda\coth(\lambda)\log\left(1+e^{-2n\lambda}\right). \tag{D.4}$$

Taking derivatives with respect to n of the expressions above then yields

$$\partial_n\widehat{\mathcal{F}}_n^{\text{univ, cs}} = \frac{3\zeta(3)}{4\pi^2} - \frac{2}{\pi^2}\int_0^\infty d\lambda\,\lambda^2\tanh(\lambda)\left(\coth(n\lambda)-1\right), \tag{D.5}$$

$$\partial_n\widehat{\mathcal{F}}_n^{\text{univ, f}} = \frac{\zeta(3)}{\pi^2} + \frac{2}{\pi^2}\int_0^\infty d\lambda\,\lambda^2\coth(\lambda)\left(\tanh(n\lambda)-1\right). \tag{D.6}$$

We note in passing that these can be used to produce yet another integral representation of $\sigma_n^{\text{cs,f}}$ through eq. (C.5) — see eq. (7.5). Now, after some manipulations, including a change of variables $\lambda \to \lambda/n$ in the scalar integral, we find

$$n^3\partial_n\widehat{\mathcal{F}}_n^{\text{univ, cs}} = \frac{3\zeta(3)}{4\pi^2}n^3 - \frac{2}{\pi^2}\int_0^\infty d\lambda\,\lambda^2\tanh(\lambda/n)\left(\coth(\lambda)-1\right), \tag{D.7}$$

$$\partial_q\widehat{\mathcal{F}}_q^{\text{univ, f}}\Big|_{q=1/n} = \frac{\zeta(3)}{\pi^2} + \frac{2}{\pi^2}\int_0^\infty d\lambda\,\lambda^2\coth(\lambda)(\tanh(\lambda/n)-1). \tag{D.8}$$

Hence the right-hand side of eq. (D.2) becomes

$$n^3\partial_n\widehat{\mathcal{F}}_n^{\text{univ, cs}} + \partial_q\widehat{\mathcal{F}}_q^{\text{univ, f}}\Big|_{q=1/n} = \frac{\zeta(3)}{4\pi^2}(3n^3+4) + \frac{2}{\pi^2}\int_0^\infty d\lambda\,\lambda^2\left[\tanh(\lambda/n)-\coth\lambda\right]. \tag{D.9}$$

Finally, this expression vanishes identically by virtue of the exact result for the integral

$$\int_0^\infty d\lambda\,\lambda^2\left[\tanh(\lambda/n)-\coth\lambda\right] = -\frac{\zeta(3)}{8}(3n^3+4), \quad \forall\, n>0. \tag{D.10}$$

This completes the proof of eq. (D.2), and hence of the duality eq. (D.1).

If we apply the simple identity $\partial_q f(q)|_{q=1/n} = -n^2\partial_n f(1/n)$, we can express eq. (D.2) as

$$\partial_n\widehat{\mathcal{F}}_{1/n}^{\text{univ, f}} = n\,\partial_n\widehat{\mathcal{F}}_n^{\text{univ, cs}}, \tag{D.11}$$

which can be integrated to yield

$$\widehat{\mathcal{F}}_{1/n}^{\text{univ, f}} = n\,\widehat{\mathcal{F}}_n^{\text{univ, cs}} - \int_1^n dq\,\widehat{\mathcal{F}}_q^{\text{univ, cs}} - \left(\widehat{\mathcal{F}}_1^{\text{univ, cs}} - \widehat{\mathcal{F}}_1^{\text{univ, f}}\right). \tag{D.12}$$

Of course, $\widehat{\mathcal{F}}_1^{\text{univ}}$ is equivalent to the universal coefficient in the sphere partition function which plays a role in the F-theorem [77, 78, 89–91]. Eq. (D.12) shows that the thermal partition functions of the two free theories are not simply exchanged under $n \leftrightarrow 1/n$, but they are still related in a relatively simple manner.

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E Relation to circular region Rényi entropies

Let us first recall eq. (1.7):

$$\partial_n h_n\big|_{n=1} = 2\pi^{\frac{d}{2}+1}\frac{\Gamma(d/2)}{\Gamma(d+2)}C_T\,. \tag{E.1}$$

This relation was first observed to hold for certain holographic theories in [40] and later proven for general CFTs in [41]. This result was also connected to a similar relation found in [92] for derivatives of the Rényi entropy S_n of a spherical entangling region

$$\partial_n S_n\big|_{n=1} = -\pi^{\frac{d}{2}+1}\frac{(d-1)\Gamma(d/2)}{\Gamma(d+2)}\,\frac{V_{\mathbb{H}^{d-1}}}{R^{d-1}}\,C_T\,. \tag{E.2}$$

Here again, R is the radius of the sphere and $V_{\mathbb{H}^{d-1}}$ is the regulated (dimensionful) volume of the hyperbolic geometry $\mathbb{H}^{d-1}$, which appears in the construction of [47]. In particular, the following identity holds [41]

$$\partial_n S_n\big|_{n=1} = -\frac{d-1}{2}\,\frac{V_{\mathbb{H}^{d-1}}}{R^{d-1}}\,\partial_n h_n\big|_{n=1}\,, \tag{E.3}$$

and more generally,[14] for $k \geq 1$

$$\partial_n^k S_n\big|_{n=1} = -\frac{d-1}{k+1}\,\frac{V_{\mathbb{H}^{d-1}}}{R^{d-1}}\sum_{m=0}^{k-1}\frac{(-)^m k!}{(k-m)!}\,\partial_n^{k-m} h_n\big|_{n=1}\,. \tag{E.4}$$

Note that the above sum does not extend to $m = k$ because $h_1 = 0$. The relations between these derivatives can be written in a more compact form as

$$\partial_n^k h_n\big|_{n=1} = -\frac{R^{d-1}}{(d-1)\,V_{\mathbb{H}^{d-1}}}\left[(k+1)\,\partial_n^k S_n\big|_{n=1} + k^2\,\partial_n^{k-1} S_n\big|_{n=1}\right], \tag{E.5}$$

where we drop the second term for $k = 1$.

Note that the scaling dimension on the left-hand side is a finite quantity of this last identity (E.5), while the Rényi entropies on the right-hand side are UV divergent. Of course, this divergence is compensated by dividing by $V_{\mathbb{H}^2}$ to produce a finite quantity. One can produce an identity relating finite quantities by focusing on the universal contribution to the Rényi entropy, which is either the coefficient of the logarithmically divergent term in an even number of dimensions or the finite contribution in an odd number of dimensions. For example, let us focus on $d = 3$ where we can use eq. (3.3) to extract the finite term from $V_{\mathbb{H}^2}$. Then eq. (E.5) yields

$$\partial_n^k h_n\big|_{n=1} = \frac{1}{4\pi}\left[(k+1)\,\partial_n^k S_n^{\text{univ}}\big|_{n=1} + k^2\,\partial_n^{k-1} S_n^{\text{univ}}\big|_{n=1}\right]. \tag{E.6}$$

It is straightforward to explicitly verify this relation in, e.g., the holographic model in section 5.

[14] Please note that the original expression for $k > 2$ in eq. (2.53) of [41] was incomplete.

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Now, expanding our new conjecture (1.6) around $n = 1$ yields

$$\partial_n^k h_n|_{n=1} = \pi\, k\, \partial_n^{k-1} \sigma_n|_{n=1}\,, \tag{E.7}$$

for $k \geq 1$. Hence we can combine this result with eq. (E.6) as

$$\partial_n^k \sigma_n\big|_{n=1} = \frac{1}{4\pi^2} \left[\frac{k+2}{k+1}\, \partial_n^{k+1} S_n^{\text{univ}}\big|_{n=1} + (k+1)\, \partial_n^k S_n^{\text{univ}}\big|_{n=1} \right]\,, \tag{E.8}$$

which is valid for $k \geq 1$. However, eqs. (E.6) and (E.7) yield an additional relation for $k = 0$, which can be seen as another realization of our original conjecture (1.5):

$$\partial_n S_n^{\text{univ}}\Big|_{n=1} = 2\,\pi^2\,\sigma_1\,. \tag{E.9}$$

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