

# Multi-soft gluon limits and extended current algebras at null-infinity

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**ABSTRACT:** In this note we consider aspects of the current algebra interpretation of multi-soft limits of tree-level gluon scattering amplitudes in four dimensions. Building on the relation between a positive helicity gluon soft-limit and the Ward identity for a level-zero Kac-Moody current, we use the double-soft limit to define the Sugawara energy-momentum tensor and, by using the triple- and quadruple-soft limits, show that it satisfies the correct OPEs for a CFT. We study the resulting Knizhnik-Zamolodchikov equations and show that they hold for positive helicity gluons in MHV amplitudes. Turning to the sub-leading soft-terms we define a one-parameter family of currents whose Ward identities corresponding to the universal tree-level sub-leading soft-behaviour. We compute the algebra of these currents formed with the leading currents and amongst themselves. Finally, by parameterising the ambiguity in the double-soft limit for mixed helicities, we introduce a non-trivial OPE between the holomorphic and anti-holomorphic currents and study some of its implications.

**KEYWORDS:** Scattering Amplitudes, Conformal and W Symmetry, Gauge Symmetry

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## 1 Introduction

The simple geometric fact that a null vector in Minkowski space defines a point on the sphere at null infinity naturally suggests interpreting scattering amplitudes for massless particles as two-dimensional correlation functions. An early and prominent example of such an interpretation was given by Nair in [1] where certain  $\mathcal{N} = 4$  super-Yang-Mills (SYM) amplitudes were constructed using the operator product expansion (OPE) of the Kac-Moody currents which define a two-dimensional Wess-Zumino-Witten model. This construction in part led to the development of the twistor string [2] which gives a complete description of tree-level  $\mathcal{N} = 4$  SYM in terms of a two-dimensional world-sheet theory.

Kac-Moody structures have also recently appeared in the study of the soft-limits of gluon amplitudes [3, 4]. Scattering amplitudes of massless particles such as gluons can be formally expanded in powers of a small scaling parameter  $\delta$  multiplying the momentum of a particle which is thus taken to be soft,  $p^\mu = \delta q^\mu$ , and which schematically gives

$$\lim_{\delta \rightarrow 0} \mathcal{A}_{n+1}(\delta q) = \left( \frac{1}{\delta} S^{(0)}(q) + S^{(1)}(q) \right) \mathcal{A}_n + \mathcal{O}(\delta), \quad (1.1)$$

where  $\mathcal{A}_n$  depends only on the remaining hard momenta. That scattering amplitudes of photons and gravitons have a universal, leading divergent behaviour as the external leg

becomes soft has been long understood [5–8]. It is also known that at tree-level the behaviour at the sub-leading order is universal, which is sometimes referred to as the Low theorem [5–7, 9–11], while for tree-level graviton amplitudes there is further universal behaviour at sub-sub-leading order [12]. Analogous sub-leading behaviour for gluon amplitudes has been recently studied [13–18]. In [4] He et al. showed that one could identify the tree-level leading soft theorem for positive helicity gluons with the Ward identity for holomorphic two-dimensional Kac-Moody currents. These currents were shown to be related to the asymptotic symmetries of Yang-Mills theory given by large CPT-invariant gauge transformations [3].<sup>1</sup> The OPE of the currents was extracted from the limit of two soft positive helicity gluons and it was shown that the level is zero in this identification, that is the algebra is a standard current algebra

$$J^a(z_1)J^b(z_2) \sim \frac{if^{ab}_c J^c(z_2)}{z_{12}}. \quad (1.2)$$

It was further noted that the limit where one positive and one negative helicity gluon are both taken soft is ambiguous with the result depending on the order in which the limit is taken. In [4] the authors gave the prescription of first taking all positive helicity gluons soft which resulted in one copy of the holomorphic Kac-Moody algebra but without a second anti-holomorphic copy arising from the negative helicity gluons.

Multi-soft limits of gluon amplitudes have been studied more generally in a number of recent works and universal behaviour in the limit where all gluons are taken soft simultaneous was found for tree-level amplitudes using the BCFW [33] and CHY [34, 35] formalisms in [36, 37]. These limits have also been studied using the CSW [38] formalism by [39] and by considering the field theory limit of string theory [40]. At leading order in the soft parameter this simultaneous double- and multi-soft limit differs from the consecutive case when there are particles of both positive and negative helicity. There is also universal behaviour in the sub-leading order of the multi-soft limit however the order in which the limit is taken is important even when all gluons have the same helicity.

We use these multi-soft limits to study further aspects of the current algebra interpretation in four dimensions. Starting with the leading soft-limit for positive helicity gluons we show that, by analogy with the Sugawara construction [41], one can use the double soft-limit to define a holomorphic energy-momentum tensor. Moreover we show, by analyzing the triple- and quadruple-soft limits of positive helicity gluons that this energy-momentum tensor has the standard OPE with both the holomorphic current and with itself. In a conformal field theory with a Kac-Moody symmetry the correlation functions of primary fields should satisfy certain differential equations, the Knizhnik-Zamolodchikov (KZ) equations [42]. We consider the KZ equation for the amplitudes and show that while they hold for positive helicity gluons in MHV amplitudes this is not the case for the negative helicity gluons or for non-MHV amplitudes. While this suggests that the holomorphic currents do

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<sup>1</sup>We focus entirely on gluon amplitudes but the relation between asymptotic symmetries and soft-behaviour has been the subject of significant recent interest particularly for gravitons and photons, see for example [19–32].

not provide a complete description of the putative CFT it motivates further analysing the sub-leading and negative helicity soft-limit.

We thus study the soft limit at sub-leading order and give these sub-leading soft-theorems an interpretation as Ward identities for two-dimensional currents which depend on a continuous parameter corresponding to the soft gluon energy,  $J_{\text{sub}}^a(z; \omega_z)$ . The OPE of these currents with themselves and with the leading order currents is then extracted from the sub-leading double-soft limits. As the algebra depends on exactly how the double soft limit is taken we introduce parameters to characterise this ambiguity. In the language of the two-dimensional field theory the leading terms of the OPEs are given by

$$J^a(z_1)J_{\text{sub}}^b(z_2; \omega_{z_2}) \sim if^{ab}_c \frac{J_{\text{sub}}^c(z_2; \omega_{z_2})}{z_{12}}, \tag{1.3}$$

$$J_{\text{sub}}^a(z_1; \omega_{z_1})J_{\text{sub}}^b(z_2; \omega_{z_2}) \sim \frac{if^{ab}_c J_{\text{sub}}^c(z_2; c_2^{ss}\omega_{z_1} + c_1^{ss}\omega_{z_2})}{z_{12}}. \tag{1.4}$$

As in [4] the anti-holomorphic currents are defined using the soft-limits of negative helicity gluons and to study the algebra with the holomorphic currents we consider double-soft limits involving both a positive and negative helicity gluon. However rather than picking a specific ordering of soft-limits we consider a linear combination and introduce variables with which we parameterise the ambiguity. The resulting algebra of currents is found to be

$$J^a(z_1)\bar{J}^b(z_2) \sim if^{ab}_c \left[ d_1 \frac{\bar{J}^c(z_2)}{z_{12}} - d_2 \frac{J^c(z_2)}{\bar{z}_{12}} - d_2 \frac{z_{12}}{\bar{z}_{12}} \partial J^c(z_2) \right]. \tag{1.5}$$

This OPE differs from the usual CFT chiral current algebra where one would find  $J^a(z_1)\bar{J}^b(z_2) \sim 0$ . However such OPEs can be considered in general [43] and bear resemblance to the non-chiral current algebras that appear in the study of non linear sigma-models on super-group manifolds [44]. In the super-group case it is still possible to construct a holomorphic stress-energy tensor as the terms which would spoil conformality are removed by the fact that the Killing form is zero. While these theories are conformal they are however non-unitary and lack many of the features familiar from the study of two-dimensional CFTs (see [45] for a review of super-group CFTs). Of course there are a number of very significant differences between the current algebra in the super-group non-linear sigma-models and in our considerations, not least the absence of logarithmic terms, at least at tree-level, and naturally the Killing form for the gauge group in Yang-Mills theory is not usually taken to vanish as in the super-group case. To complete the algebra of currents we calculate the OPEs of the sub-leading currents with the anti-holomorphic currents and sub-leading anti-holomorphic coordinates

$$\bar{J}^a(z_1)J_{\text{sub}}^b(z_2; \omega_{z_2}) \sim if^{ab}_c \frac{\bar{J}_{\text{sub}}^c(z_2; \omega_{z_2})}{\bar{z}_{12}}, \tag{1.6}$$

$$J_{\text{sub}}^a(z_1; \omega_{z_1})\bar{J}_{\text{sub}}^b(z_2; \omega_{z_2}) \sim if^{ab}_c \left[ \frac{J_{\text{sub}}^c(z_2; d_2^s \omega_{z_2})}{\bar{z}_{12}} - \frac{\bar{J}_{\text{sub}}^c(z_2; d_1^s \omega_{z_1})}{z_{12}} + \frac{\bar{z}_{12}}{z_{12}} \bar{\partial} \bar{J}_{\text{sub}}^c(z_2; d_1^s \omega_{z_1}) + 2 \frac{z_{12}}{\bar{z}_{12}} \partial J_{\text{sub}}^c(z_2; d_2^s \omega_{z_2}) \right]. \tag{1.7}$$

Using (1.5) one can compute the OPE of the anti-holomorphic currents with the holomorphic stress-energy tensor. We do this in section 5.2 and show that it is consistent with the triple-soft limit involving two positive helicity gluons and one negative. Finally, using the leading order holomorphic and anti-holomorphic currents, we define a conjugation operator  $C(z) \propto J^a \bar{J}^a(z)$  and compute its OPE with the currents.

While we don't attempt to give a more fundamental interpretation of these two-dimensional currents the renewed focus on understanding the asymptotic symmetries has lead to a number of interesting proposals for a holographic description of flat space scattering amplitudes. In [46], building on [47], a world-sheet theory with the null infinity boundary of flat space as its target space was proposed. An alternative approach based on the ambi-twistor string, [48, 49], was studied in [50, 51]. Recently, Cheung et al. [52] have proposed a holographic description which, following [53, 54], makes use of a slicing of flat space into a family of AdS<sub>3</sub> sub-spaces to interpret four-dimensional amplitudes in terms of a two-dimensional CFT. They identify the conserved currents and energy-momentum tensor with the asymptotic symmetries of gluon and graviton amplitudes. While there are a number of differences with our current work, for example the central charge of the current algebra, it would be very interesting to understand better if there is a holographic interpretation of the sub-leading and anti-holomorphic currents.

## 2 Soft limits for gluon amplitudes

The central objects of our study are the soft limits of gluon scattering amplitudes. While we will be interested in the full colour dressed amplitudes it is convenient to start with colour stripped amplitudes. Consider an  $n+1$ -leg amplitude where we take the first leg to be a soft particle of helicity  $h_q$  with momentum  $\delta q$ , or in terms of spinor variables,  $\{\sqrt{\delta}\lambda_q, \sqrt{\delta}\tilde{\lambda}_q\}$ .<sup>2</sup> The soft limit corresponds to expanding the amplitude in powers of  $\delta_1$  and we keep only those terms in the expansion which are universal. As it will make subsequent formulae simpler we will often simply denote the non-soft legs with momenta  $p_i$ ,  $i = 1, \dots, n$  by  $1, 2, \dots, n$ .

### 2.1 Single-soft limits

The single-soft limits for gluon amplitudes were found, up to the first sub-leading term, in [7, 9, 13]

$$A_{n+1}(\delta_1 q^{h_q}, 1, \dots, n) \rightarrow \left( \frac{1}{\delta_1} S_{n,1}^{(0)}(q^{h_q}) + S_{n,1}^{(1)}(q^{h_q}) \right) A_n(1, \dots, n) \quad (2.1)$$

where, for a positive helicity gluon,  $h_q = +$ , between neighboring particles  $n$  and  $1$ , the universal soft-factors can be written using the spinor-helicity formalism as

$$S_{n,1}^{(0)}(q^+) = \frac{\langle n1 \rangle}{\langle n q \rangle \langle q1 \rangle}, \quad S_{n,1}^{(1)}(q^+) = \frac{1}{\langle q1 \rangle} \tilde{\lambda}_q^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_1^{\dot{\alpha}}} + \frac{1}{\langle n q \rangle} \tilde{\lambda}_q^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_n^{\dot{\alpha}}}. \quad (2.2)$$

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<sup>2</sup>We will make extensive use of the spinor-helicity formalism. Useful reviews can be found in [55–57].

Here we use the conventions

$$\langle ab \rangle = \epsilon_{\alpha\beta} \lambda_a^\alpha \lambda_b^\beta = \lambda_{a\beta} \lambda_b^\beta = -\lambda_a^\alpha \lambda_{b\alpha} \quad (2.3)$$

and similarly for the dotted indices,  $[ab] = \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_a^{\dot{\alpha}} \tilde{\lambda}_b^{\dot{\beta}}$ . Correspondingly we will write the contraction of a spinor with a derivative as

$$\tilde{\lambda}_a^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_b^{\dot{\alpha}}} = -[a \bar{\partial}_b]. \quad (2.4)$$

For a negative helicity gluon the soft factors are given by conjugation of the spinor variables,  $\lambda_i \leftrightarrow \tilde{\lambda}_i$ .

## 2.2 Multi-soft limits

One can similarly consider the limit of gluon amplitudes with multiple soft gluons. However as mentioned there is in general an ambiguity in the result which depends on the order in which the gluons are taken to be soft. Starting with the case of two soft gluons we consider the  $n+2$ -leg amplitude where we will take the first and second legs to be soft particles with helicities  $h_{q_1}$  and  $h_{q_2}$  and momenta  $\{\sqrt{\delta_1} \lambda_{q_1}, \sqrt{\delta_1} \tilde{\lambda}_{q_1}\}$  and  $\{\sqrt{\delta_2} \lambda_{q_2}, \sqrt{\delta_2} \tilde{\lambda}_{q_2}\}$  respectively. In particular, if one takes the gluons soft sequentially we call this a consecutive soft limit in contradistinction to the simultaneous or double soft limit. These consecutive soft limits can be calculated straightforwardly from repeated action of the above single soft factors.

We define the consecutive soft factor,  $\text{CSL}_{n,1}(q_1^{h_{q_1}}, q_2^{h_{q_2}})$ , to be

$$\begin{aligned} \text{CSL}_{n,1}(q_1^{h_{q_1}}, q_2^{h_{q_2}}) A_n(1, \dots, n) &\equiv \lim_{\delta_2 \rightarrow 0} \lim_{\delta_1 \rightarrow 0} A_{n+2}(\delta_1 q_1^{h_{q_1}}, \delta_2 q_2^{h_{q_2}}, 1, \dots, n) \\ &= \left( \frac{1}{\delta_1} S_{n,q_2}^{(0)}(q_1^{h_{q_1}}) + S_{n,q_2}^{(1)}(q_1^{h_{q_1}}) \right) \left( \frac{1}{\delta_2} S_{n,1}^{(0)}(q_2^{h_{q_2}}) + S_{n,1}^{(1)}(q_2^{h_{q_2}}) \right) A_n(1, \dots, n), \end{aligned} \quad (2.5)$$

where we take the particle  $q_1$  soft first and then  $q_2$ . As it will be of interest later, let us give the explicit expression:

$$\text{CSL}_{n,1}(q_1^+, q_2^+) = \frac{1}{\delta_1 \delta_2} \frac{\langle n 1 \rangle}{\langle n q_1 \rangle \langle q_1 q_2 \rangle \langle q_2 1 \rangle} + \mathcal{O}(\delta_2^0 / \delta_1, \delta_1^0 / \delta_2). \quad (2.6)$$

If we take the reverse consecutive limit, i.e. take leg  $q_1$  soft and then leg  $q_2$ , the leading term in  $\text{CSL}(q_1^+, q_2^+)$  is unchanged. However when the particles have different helicity, or when we extract sub-leading multi-soft terms, the order of limits will be important and so we consider linear combinations by defining more general multi-soft limits

$$\lim_{\alpha} \equiv \left( \alpha_{12\dots m} \lim_{\delta_m \rightarrow 0} \dots \lim_{\delta_2 \rightarrow 0} \lim_{\delta_1 \rightarrow 0} + \alpha_{21\dots m} \lim_{\delta_m \rightarrow 0} \dots \lim_{\delta_1 \rightarrow 0} \lim_{\delta_2 \rightarrow 0} + \dots \right) \quad (2.7)$$

with the ellipses denoting all further permutations. Thus we can define a multi-parameter family of consecutive soft limits

$$\alpha \text{CSL}_{n,1} \left( 1^{h_{q_1}}, 2^{h_{q_2}}, \dots, m^{h_{q_m}} \right) A_n(1, \dots, n) = \lim_{\alpha} A_{n+m} \left( \delta_1 q_1^{h_{q_1}}, \dots, \delta_m q_m^{h_{q_m}}, 1, \dots, n \right). \quad (2.8)$$

The effect of this general limit can be seen in the sub-leading term in the expansion of two soft, positive helicity gluons which a short calculation shows is given by

$$\begin{aligned} \lim_{\alpha} A_{n+2}(\delta_1 q_1^+, \delta_2 q_2^+, \dots) \Big|_{\text{sub-leading}} &= \left[ \frac{1}{\delta_1} \left( \frac{\alpha_{12} \langle nq_2 \rangle \langle q_1 1 \rangle + \alpha_{21} \langle n1 \rangle \langle q_1 q_2 \rangle}{\langle q_1 q_2 \rangle \langle q_1 1 \rangle \langle nq_1 \rangle \langle q_2 1 \rangle} \right) [q_2 \partial_1] \right. \\ &\quad \left. + \frac{\alpha_{12}}{\delta_1} \frac{[q_2 \partial_n]}{\langle nq_1 \rangle \langle q_1 q_2 \rangle} + \frac{1}{\delta_2} \left( \frac{\alpha_{12} \langle n1 \rangle \langle q_1 q_2 \rangle + \alpha_{21} \langle q_1 1 \rangle \langle nq_2 \rangle}{\langle q_1 q_2 \rangle \langle nq_2 \rangle \langle nq_1 \rangle \langle q_2 1 \rangle} \right) [q_1 \partial_n] + \frac{\alpha_{21}}{\delta_2} \frac{[q_1 \partial_1]}{\langle q_2 1 \rangle \langle q_1 q_2 \rangle} \right] A_n. \end{aligned} \tag{2.9}$$

It is perhaps worthwhile to note that this expression is only valid for generic external momenta as we have neglected holomorphic anomaly terms that can arise when external legs are collinear with soft legs, however it will turn out that they are not necessary for our considerations.

For the case of mixed helicity the two orderings of limits already differ at leading order in the soft expansion and so we can define a two parameter double-soft limit

$$\alpha \text{CSL}_{n,1}^{(0)}(q_1^+, q_2^-) = \frac{1}{\langle nq_1 \rangle [q_2 1]} \left( \alpha_{21} \frac{\langle n1 \rangle [q_1 1]}{[q_1 q_2] \langle q_1 1 \rangle} + \alpha_{12} \frac{[n1] \langle q_2 n \rangle}{\langle q_1 q_2 \rangle [q_2 n]} \right), \tag{2.10}$$

where  $\alpha \text{CSL}^{(0)}$  has the overall  $(\delta_1 \delta_2)^{-1}$  dependence stripped off. As described previously, we can of course also consider the simultaneous double-soft limit where both particles are taken soft together. The case of two gluons of the same helicity is identical to the consecutive limit but the case of one negative helicity and one positive is different. In four space-time dimensions the leading order double-soft mixed helicity factor is given in the spinor helicity formalism by

$$\text{DSL}_{n,1}^{(0)}(q_1^+, q_2^-) = \frac{1}{\langle n | q_{12} | 1 \rangle} \left[ \frac{1}{2p_n \cdot q_{12}} \frac{[n1] \langle nq_2 \rangle^3}{\langle q_1 q_2 \rangle \langle nq_1 \rangle} - \frac{1}{2p_1 \cdot q_{12}} \frac{\langle n1 \rangle [1q_1]^3}{[q_1 q_2] [q_2 1]} \right], \tag{2.11}$$

where

$$q_{12} := q_1 + q_2. \tag{2.12}$$

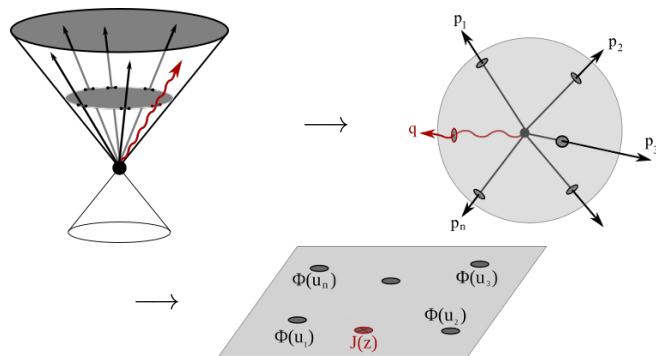
This expression differs from the consecutive limit due to the sum of soft momenta in the denominator but it is closest to the symmetric  $\alpha_{12} = \alpha_{21} = \frac{1}{2}$  case. A similar ambiguity appears at sub-leading order when we consider the double-soft limit in the mixed helicity case. Explicit formulae can be found in [36] but in this case we will focus on the consecutive limits which can be calculated by repeated use of the single-soft limits (2.2).

### 3 Current algebra interpretation of Yang-Mills soft limits

Let us consider the single and double soft limits of Yang-Mills amplitudes but now instead of just focussing on colour-ordered partial amplitudes we examine the full amplitude

$$\mathcal{A}_n(\{p_i, h_i, a_i\}) = g_{\text{YM}}^{n-2} \sum_{\sigma \in S_n / Z_n} A_n(\sigma_1, \dots, \sigma_n) \text{Tr}(T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}) \tag{3.1}$$

where the sum runs over all permutations,  $S_n$ , modulo those which are cyclic,  $Z_n$ , and  $T^a$  are the generators of the colour algebra which we will take to be  $\mathfrak{su}(N)$ . We will drop the



**Figure 1.** The gluons emerging from the amplitude travel on the light-cone and intersect with the sphere at null infinity, or any finite time representation, at points which can be given complex coordinates  $z$ , for the soft gluon, and  $u_i$  for the hard gluons. The soft gluon is then interpreted as the insertion of a current operator at the point  $z$  while the  $u_i$  denote the insertion points of additional fields.

factors of  $g_{\text{YM}}$  but as we are only concerned with tree-level amplitudes they can be trivially restored. If we take the soft-limit of the  $n + 1$ -particle amplitude we have at leading order in the soft-expansion

$$\lim_{\delta \rightarrow 0} \mathcal{A}_{n+1}(\delta q_1, h_{q_1}, a; \{p_i, h_i, a_i\}) \Big|_{\frac{1}{\delta}} = \sum_{\sigma \in S_n} \frac{1}{\delta} S_{\sigma_n, \sigma_1}^{(0)}(q_1^{h_{q_1}}) A_n(\sigma_1, \dots, \sigma_n) \text{Tr}(T^a T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}).$$

We can rewrite this soft limit in terms of two-dimensional position space variables by using the parameterisation for the soft-gluon momentum

$$\lambda_{q_1} = \sqrt{\omega}(1, z), \quad \tilde{\lambda}_{q_1} = \sqrt{\omega}(1, \bar{z}), \quad (3.2)$$

and for the hard momenta the parameterisation  $\lambda_i = \sqrt{\nu_i}(1, u_i)$  and  $\bar{\lambda}_i = \sqrt{\nu_i}(1, \bar{u}_i)$ . Essentially the complex  $z$ -variable describes the position of the intersection of the soft gluon's trajectory with the sphere at asymptotic infinity and the  $u_i$ 's are the corresponding intersection points for the hard gluons, see figure 1. We take all gluons to be outgoing and so this is in fact the anti-sky mapping. In these variables we have for the leading single soft factor (2.2) ,

$$(-\delta\omega) S_{i,j}^{(0)}(q_1^+) = \frac{1}{z - u_j} - \frac{1}{z - u_i}, \quad (3.3)$$

where we see that it has simple poles as the soft momenta approach the hard momentum in position space i.e.  $z \rightarrow u_i$  for each  $i$ . Now we define,

$$\mathcal{J}^a(z) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i, \nu_i\}) \equiv - \lim_{\delta \rightarrow 0} (\delta\omega) \mathcal{A}_{n+1}(\delta p, +, a; \{p_i, h_i, a_i\}), \quad (3.4)$$

and so

$$\begin{aligned} \mathcal{J}^a(z) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) &= \sum_{\ell=1}^n \left[ \frac{1}{z - u_\ell} \right] \sum_{\tilde{\sigma} \in S_{n-1}} A_n(\ell, \tilde{\sigma}_2, \dots, \tilde{\sigma}_n) \text{Tr}([T^a, T^{a_\ell}] T^{a_{\tilde{\sigma}_2}} \dots T^{a_{\tilde{\sigma}_n}}) \\ &= \sum_{\ell=1}^n \frac{T_\ell^a}{z - u_\ell} \mathcal{A}_n(\{u_i, \bar{u}_i\}) \end{aligned} \quad (3.5)$$



where in the last line  $T_\ell^a$  is understood to act (in the adjoint representation) only on the  $\ell$ -th particle colour factor in  $\mathcal{A}_n$ . This is the result of He et al. [4] who compared it to the OPE for a Kac-Moody current  $J^a$  and a primary field in a representation  $R$

$$J^a(z)\Phi_R^r(u) \sim \frac{(T_R^a)^r_s \Phi_R^s(u)}{z-u} \tag{3.6}$$

and the consequent Ward Identity

$$\langle J^a(z)\Phi_{R_1}(u_1, \bar{u}_1) \dots \Phi_{R_n}(u_n, \bar{u}_n) \rangle = \sum_{\ell=1}^n \frac{T_{R_\ell}^a}{z-u_\ell} \langle \Phi_{R_1}(u_1, \bar{u}_1) \dots \Phi_{R_n}(u_n, \bar{u}_n) \rangle. \tag{3.7}$$

Note that we use the notation  $\mathcal{J}^a$  in (3.5) to denote the expression derived from a four-dimensional scattering amplitude perspective and the  $J^a$  in (3.7) to be the corresponding quantity from the two dimensional CFT perspective. It is the formal similarity of these two expressions that suggests the interpretation of the soft-gluon limits as corresponding to the insertion of the current operator and it is interesting to ask to what extent this analogy can be extended.

### 3.1 An $\mathfrak{su}(N)$ current algebra

As a first step, it is natural to ask whether one can reproduce the current-current OPE

$$J^a(z_1)J^b(z_2) \sim \frac{k\delta^{ab}}{(z_1-z_2)^2} + \frac{if^{abc}J^c(z_2)}{z_1-z_2}. \tag{3.8}$$

At the level of correlation functions we would expect to find,

$$\langle J^a(z_1)J^b(z_2)\Phi(u_1) \dots \rangle \sim \frac{k\delta^{ab}}{(z_1-z_2)^2} \langle \Phi(u_1) \dots \rangle + \frac{if^{abc}}{z_1-z_2} \sum_{\ell} \frac{T_\ell^c}{z_2-u_\ell} \langle \Phi(u_1) \dots \rangle. \tag{3.9}$$

That the corresponding result can be found by analyzing the double-soft limits of amplitudes was shown in [4] and as it is useful for our later results we rederive this fact in our notations. To this end we consider the double soft limit which we can write as

$$\begin{aligned} \lim_{\delta \rightarrow 0} \mathcal{A}_{n+2}(\delta_1 q_1, h_{q_1}, a_{q_1}; \delta_2 q_2, h_{q_2}, a_{q_2}; \{p_i, h_i, a_i\}) = \\ \sum_{\sigma \in S_n} \left\{ (\text{CSL}_{\sigma_n, \sigma_1}(q_1^{h_{q_1}}, q_2^{h_{q_2}}) + \text{CSL}_{\sigma_n, \sigma_1}(q_2^{h_{q_2}}, q_1^{h_{q_1}})) \text{Tr}(q_1 q_2 \sigma_1 \dots \sigma_n) \right. \\ \left. \sum_{i=2}^n S_{\sigma_n, \sigma_1}(q_1^{h_{q_1}}) S_{\sigma_{i-1}, \sigma_i}(q_2^{h_{q_2}}) \text{Tr}(q_1 \sigma_1 \dots \sigma_{i-1} q_2 \sigma_i \dots \sigma_n) \right. \\ \left. i \text{CSL}_{\sigma_n, \sigma_1}(q_2^{h_{q_2}}, q_1^{h_{q_1}}) f^{a_{q_2} a_{q_1} c} \text{Tr}(c \sigma_1 \dots \sigma_n) \right\} A_n(\sigma_1, \dots, \sigma_n) \tag{3.10} \end{aligned}$$

in a slightly condensed notation with  $\text{Tr}(q_1 q_2 \sigma_1 \dots \sigma_n) = \text{Tr}(T^{a_{q_1}} T^{a_{q_2}} T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}})$  etc. As we first want to find the OPE of two holomorphic currents we consider the leading order double soft limit of two positive helicity gluons. As the gluons have the same helicity the

order of limits is not relevant and so we simply take the simultaneous limit with a single soft parameter  $\delta_1 = \delta_2 = \delta$ . We use the identity for consecutive soft-limits

$$\text{CSL}_{\sigma_n, \sigma_1}^{(0)}(q_1^{h_{q_1}}, q_2^{h_{q_2}}) + \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(q_2^{h_{q_2}}, q_1^{h_{q_1}}) = S_{\sigma_n, \sigma_1}^{(0)}(q_1^{h_{q_1}}) S_{\sigma_n, \sigma_1}^{(0)}(q_2^{h_{q_2}}) \quad (3.11)$$

here with both helicities positive  $h_{q_1} = h_{q_2} = +$ , and so we can combine the first two terms to find

$$\begin{aligned} \lim_{\delta \rightarrow 0} \mathcal{A}_{n+2}(\delta q_1, +, a_{q_1}; \delta q_2, +, a_{q_2}; \{p_i, h_i, a_i\}) = \\ \sum_{\sigma \in S_n} \left\{ \sum_{i=1}^n S_{\sigma_n, \sigma_1}^{(0)}(q_1^+) S_{\sigma_{i-1}, \sigma_i}^{(0)}(q_2^+) \text{Tr}(q_1 \sigma_1 \dots \sigma_{i-1} q_2 \sigma_i \dots \sigma_n) \right. \\ \left. + i \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(q_2^+, q_1^+) f^{a_{q_2} a_{q_1} c} \text{Tr}(c \sigma_1 \dots \sigma_n) \right\} A_n(\sigma_1, \dots, \sigma_n). \quad (3.12) \end{aligned}$$

Re-writing the soft-factors in terms of polarisation vectors one recovers the result quoted in [4].

The singularities of these double soft factors in the collinear limit, as the soft momenta approach each other, can be made clear by rewriting them in position space and examining the  $z_1 \rightarrow z_2$  limit. We can see that there will be no singularities from the terms involving the products of single soft factors; these terms instead give poles when the soft momenta approach the hard momenta. However the term involving the consecutive double soft limit does have such a singularity and gives rise to the non-trivial algebra between the holomorphic currents. More explicitly, defining

$$\mathcal{J}^a(z_1) \mathcal{J}^b(z_2) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) \equiv \lim_{\delta \rightarrow 0} (4 \delta^2 \omega_{z_1} \omega_{z_2}) \mathcal{A}_{n+2}(\delta q_1, +, a; \delta q_2, +, b; \{p_i, h_i, a_i\}) \quad (3.13)$$

we find

$$\begin{aligned} \mathcal{J}^a(z_1) \mathcal{J}^b(z_2) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) &\sim \frac{1}{z_1 - z_2} \sum_{\ell \neq m=1}^n \sum_{\sigma \in S_{n-2}} i f^{ba}{}_c \left( \frac{1}{z_2 - u_\ell} - \frac{1}{z_2 - u_m} \right) \\ &\quad \times \text{Tr}(\ell c m \sigma_1 \dots) A_n(\ell m \sigma_1 \dots) \\ &= \frac{i f^{ba}{}_c}{z_1 - z_2} \sum_{\ell=1}^n \frac{1}{z_2 - u_\ell} \sum_{\tilde{\sigma} \in S_{n-1}} \text{Tr}([\ell, c] \tilde{\sigma}_1 \dots) A_n(\ell \tilde{\sigma}_1 \dots) \\ &= \frac{i f^{ab}{}_c}{z_1 - z_2} \sum_{\ell=1}^n \frac{T_\ell^c}{z_2 - u_\ell} \mathcal{A}_n(\{u_i, \bar{u}_i\}). \quad (3.14) \end{aligned}$$

Hence comparing with (3.9) we see that the double soft-limit reproduces the structure of level-zero Kac-Moody or simply a current algebra as described in [4].

### 3.2 Sugawara construction for amplitudes

If this is a correct interpretation, i.e. there is a level-zero Kac-Moody symmetry acting on amplitudes, we may try to construct a holomorphic energy-momentum tensor via the Sugawara construction. Namely, we define

$$T^S(z_1) = \gamma : J^a J^a(z_1) : \equiv \gamma \lim_{z_2 \rightarrow z_1} J^a(z_1) J^a(z_2) \quad (3.15)$$

where  $\gamma$  is a constant related to the dual Coxeter number,  $h^\vee$ , by  $\gamma = \frac{1}{2h^\vee}$ , and the normal ordering is missing the usual singular term that occurs in the Kac-Moody case as the current algebra has no central extension. At the level of correlation functions this leads us to again consider double insertions of holomorphic currents and so, as the analogue for amplitudes is the double soft limit, we again consider (3.13) but now with contracted adjoint indices. That is we define

$$\mathcal{T}^S(z_1) = \frac{1}{2h^\vee} \lim_{z_2 \rightarrow z_1} \mathcal{J}^a \mathcal{J}^a(z_1). \quad (3.16)$$

We could in principle work with a general Lie group but for simplicity we focus on the case of  $\mathfrak{su}(N)$  colour where we can contract the indices  $a$  and  $b$  using the identity

$$(T^a)_i^j (T^a)_k^l = \delta_i^l \delta_k^j - \frac{1}{N} \delta_i^j \delta_k^l. \quad (3.17)$$

and in this case  $h^\vee = N$ .

To calculate the scattering amplitude analogue of the insertion of an energy-momentum tensor into a correlation function we consider the singularities as either of the soft momenta approach one of the hard momenta, say  $u_m$ , and then expand the  $z_2$  dependence near  $z_1$ . Using the notation  $z_{1m} = z_1 - u_m$  for  $m = 1, \dots, n$  we write the result

$$\begin{aligned} \mathcal{T}^S(z_1) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) &\equiv \frac{1}{2N} \mathcal{J}^a(z_1) \mathcal{J}^a(z_1) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) \\ &= \sum_{m=1}^n \sum_{\sigma \in S_{n-1}} \left\{ \frac{1}{2N} \frac{1}{z_{1m}} \sum_{i=1}^{n-1} \frac{1}{z_{1\sigma_i}} \text{Tr}([a, m] \sigma_1 \dots [a, \sigma_i] \dots) \right. \\ &\quad \left. + \frac{1}{z_{1m}^2} \text{Tr}(m \sigma_1 \dots) \right\} \mathcal{A}_n(m \sigma_1 \dots). \end{aligned} \quad (3.18)$$

While this definition seems sensible we should ask whether it satisfies the basic properties of a energy-momentum tensor for a conformal field theory.

### 3.3 Energy-momentum tensor/current OPE

In two-dimensional conformal field theory a defining characteristic of  $T^S(z_1)$  is its OPE with the holomorphic current operator

$$T^S(z_1) J^a(z_3) \sim \frac{J^a(z_3)}{z_{13}^2} + \frac{\partial J^a(z_3)}{z_{13}}. \quad (3.19)$$

The analogous object for the amplitude is a specific case of the triple soft-limit

$$\begin{aligned} &\lim_{\delta \rightarrow 0} \mathcal{A}_{n+3}(\delta q_1, h_{q_1}, a_{q_1}; \delta q_2, h_{q_2}, a_{q_2}; \delta q_3, h_{q_3}, a_{q_3}; \{p_i, h_i, a_i\}) \\ &= \sum_{\sigma \in S_n} \left\{ \sum_{s=1}^{n-1} \sum_{\substack{r=1 \\ r \neq s}}^n S_{\sigma_n, \sigma_1}^{(0)}(q_1^{h_{q_1}}) S_{\sigma_s, \sigma_{s+1}}^{(0)}(q_2^{h_{q_2}}) S_{\sigma_r, \sigma_{r+1}}^{(0)}(q_3^{h_{q_3}}) \text{Tr}(q_1 \dots \sigma_s q_2 \dots \sigma_r q_3 \dots) \right. \\ &\quad + \sum_{r=1}^{n-1} \left( (\text{CSL}_{\sigma_n, \sigma_1}^{(0)}(q_1^{h_{q_1}}, q_2^{h_{q_2}}, ) + \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(q_2^{h_{q_2}}, q_1^{h_{q_1}})) S_{\sigma_r, \sigma_{r+1}}^{(0)}(q_3^{h_{q_3}}) \text{Tr}(q_1 q_2 \dots \sigma_r q_3 \dots) \right. \\ &\quad \left. + i \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(q_2^{h_{q_2}}, q_1^{h_{q_1}}) S_{\sigma_r, \sigma_{r+1}}^{(0)}(q_3^{h_{q_3}}) f^{a_{q_2} a_{q_1} c} \text{Tr}(c \dots \sigma_r q_3 \dots) + (q_2 \leftrightarrow q_3) + (q_1 \leftrightarrow q_3) \right) \\ &\quad \left. + \left( \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(1^{h_1}, 2^{h_2}, 3^{h_3}) \text{Tr}(123 \sigma_1 \dots \sigma_n) + \text{perm}(1, 2, 3) \right) \right\} \mathcal{A}_n(\sigma_1, \dots, \sigma_n). \end{aligned} \quad (3.20)$$

We take all gluons to have positive helicity, such that the triple and double soft limits are the same regardless of whether we take the consecutive or simultaneous soft limit and so, as in the double-soft case, we set all soft parameters equal. The product of three holomorphic currents is then defined to be

$$\begin{aligned} \mathcal{J}^a(z_1)\mathcal{J}^b(z_2)\mathcal{J}^c(z_3) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) &\equiv \\ &- \lim_{\delta \rightarrow 0} (\delta^2 \omega_{z_1} \omega_{z_2} \omega_{z_3}) \mathcal{A}_{n+3}(\delta q_1, +, a, \delta q_2, +, b, \delta q_3, +, c; \{p_i, h_i, a_i\}). \end{aligned} \tag{3.21}$$

In the standard two-dimensional conformal field theory calculation, to compute the OPE of the energy-momentum tensor with the current operator we collect the terms which are singular as  $z_3$  approaches either  $z_1$  or  $z_2$ . In the soft-limit such terms correspond to the soft momentum  $q_3$  becoming collinear with either  $q_1$  or  $q_2$ . In particular the contributions from the triple soft terms split into two groups, those where the soft particles  $q_1$  and  $q_2$  are adjacent which come with a colour factor  $(N - \frac{1}{N})\text{Tr}(c\sigma_1 \dots \sigma_n)$  and those where  $q_1$  and  $q_2$  are split by particle  $q_3$  which have  $(-\frac{1}{N})\text{Tr}(c\sigma_1 \dots \sigma_n)$ . Combining these terms we find that the  $(-\frac{1}{N})$  factors cancel. There are further contributions from terms involving double soft limits times single soft factors which have colour structures  $f^{ca}_d \text{Tr}(a \dots \sigma_r d \dots)$  however these terms cancel between themselves. This makes use of the identity (3.11) as well as the fact that

$$\text{Tr}(T^a T^{a\sigma_1} \dots T^{a\sigma_r} [T^c, T^a] \dots) + \text{Tr}([T^c, T^a] T^{a\sigma_1} \dots T^{a\sigma_r} T^a \dots) = 0. \tag{3.22}$$

Following the conformal field theory calculation, we then expand this in powers of  $z_{21} = z_2 - z_1$  and keep the leading term (in principle there could be a more singular term however this term is absent which corresponds to the level of the current algebra being zero). We finally have

$$\begin{aligned} \mathcal{T}^S(z_1)\mathcal{J}^c(z_3) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) &\sim \frac{1}{z_{31}^2} \sum_{\sigma \in S_n} \left( \frac{1}{z_1 - u_{\sigma_1}} - \frac{1}{z_1 - u_{\sigma_n}} \right) \text{Tr}(c\sigma_1 \dots \sigma_n) A_n(\sigma_1 \dots \sigma_n) \\ &= \frac{1}{z_{31}^2} \sum_{\ell=1}^n \frac{1}{z_1 - u_\ell} \sum_{\tilde{\sigma} \in S_{n-1}} \text{Tr}([c, \ell] \tilde{\sigma}_1 \dots) A_n(\ell \tilde{\sigma}_1 \dots) \\ &= \frac{1}{z_{31}^2} \mathcal{J}^c(z_1) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) \end{aligned} \tag{3.23}$$

which after expanding the  $z_1$  dependence on the right hand side is the required result.

### 3.4 Energy-momentum OPE

Using the OPE of holomorphic currents it can be shown that the Sugawara energy-momentum tensor for a current algebra ( $k = 0$ ) satisfies the OPE

$$T(z_1)T(z_3) \sim \frac{2T(z_3)}{(z_1 - z_3)^2} + \frac{\partial T(z_3)}{(z_1 - z_3)} \tag{3.24}$$

and so defines a conformal field theory with vanishing central charge. Correspondingly, inserted into a correlator of primary fields  $\Phi(u_i)$ , we have that

$$\langle T(z_1)T(z_3)\Phi(u_1)\dots\Phi(u_n)\rangle = \left[ \frac{2}{(z_1-z_3)^2} + \frac{\partial_{z_3}}{(z_1-z_3)} + \sum_{i=1}^n \left\{ \frac{\Delta}{(z_1-u_i)^2} + \frac{\partial_{u_i}}{(z_1-u_i)} \right\} \right] \times \langle T(z_3)\Phi(u_1)\dots\Phi(u_n)\rangle. \tag{3.25}$$

In terms of the soft-limits of amplitudes this Ward identity should be reproduced by the collinear part of the quadruple soft limit for positive helicity gluons. Once again as we are only considering positive helicity gluons there is no ambiguity in the order of limits. In the two-dimensional conformal field theory construction one starts with pairs of currents,  $J^{a_1}(z_1)J^{a_2}(z_2)$  and  $J^{a_3}(z_3)J^{a_4}(z_4)$ , and to calculate the OPE one first extracts the terms which become singular as either  $z_1$  or  $z_2$  approaches either  $z_3$  or  $z_4$  and then takes the limits  $z_2 \rightarrow z_1$  and  $z_4 \rightarrow z_3$ . For the amplitude one similarly considers those terms which are singular as either  $q_1$  or  $q_2$  become collinear with  $q_3$  or  $q_4$ . The singular terms get contributions from quadruple soft terms, where all soft particles are adjacent, from triple-soft terms and from the “double” double-soft terms. Carefully combining all these terms we find, after taking the  $z_2 \rightarrow z_1$  limit, that the singular terms in the quadruple soft limit are

$$\begin{aligned} \mathcal{T}^S(z_1)\mathcal{T}^S(z_3)\cdot\mathcal{A} &\equiv \lim_{\delta\rightarrow 0} \frac{(\prod_{i=1}^4 \omega_{z_i})}{(2N)^2} \mathcal{A}_{n+4}(\delta q_1, +, a; \delta q_1, +, a; \delta q_3, +, b; \delta q_3, +, b; \{p_i, h_i, a_i\}) \\ &\sim \frac{2}{z_{13}^2(2N)} \sum_{\sigma\in S_n} \sum_{i=1}^n \left( \frac{u_{\sigma_n}-u_{\sigma_1}}{(z_3-u_{\sigma_n})(z_3-u_{\sigma_1})} \right) \left( \frac{u_{\sigma_{i-1}}-u_{\sigma_i}}{(z_1-u_{\sigma_{i-1}})(z_1-u_{\sigma_i})} \right) \text{Tr}(a\sigma_1\dots a\sigma_i\dots)A_n. \end{aligned}$$

This can be compared with the definition of a single insertion of the energy-momentum tensor in (3.18) and, after further expanding the  $z_1$  dependence near  $z_3$ , we see that we indeed reproduce the OPE in (3.24) as expected.

While we restrict our considerations to gluon amplitudes it is interesting to note that the sub-leading soft theorem for a graviton has been identified with the Ward identity for a two-dimensional energy-momentum tensor [58] which suggests interpreting the collinear limit of double-soft gluons as a graviton. This is reminiscent of recent work [59–62] showing that amplitudes describing the interactions of gravitons with gluons can be written as linear combinations of amplitudes in which the graviton is replaced by a pair of collinear gluons.

### 3.5 Comments on a Knizhnik-Zamolodchikov equation

Given the above construction for the energy-momentum tensor it is interesting to ask if we can derive conformal Ward identities by considering correlation functions with insertions of the energy-momentum tensor. However, *a priori*, even if there is a sensible conformal field theory interpretation of the asymptotic states the current algebra may only form part of it, which is to say the full energy-momentum tensor  $T(z)$  would be given by

$$T(z) = T'(z) + T^S(z). \tag{3.26}$$

If we consider the individual terms in the mode expansion

$$\begin{aligned} (\hat{L}_{-n}\Phi)(u) &= \oint \frac{dz}{2\pi i} \frac{1}{(z-u)^{n-1}} T(z)\Phi(u), \\ (\hat{J}_{-n}^a\Phi)(u) &= \oint \frac{dz}{2\pi i} \frac{1}{(z-u)^n} J^a(z)\Phi(u) \end{aligned} \tag{3.27}$$

we have that

$$\hat{L}_{-m} = \hat{L}'_{-m} + \gamma \sum_{l \leq -1} \hat{J}_l^a \hat{J}_{m-l}^a + \gamma \sum_{l > -1} \hat{J}_{m-l}^a \hat{J}_{-l}^a. \tag{3.28}$$

In conformal field theory we can insert this inside a correlation function of primary fields and if we consider the case where  $T'(z)$  is absent we can derive the Knizhnik-Zamolodchikov equation

$$\begin{aligned} \langle \Phi(u_1) \dots (\hat{L}_{-1}\Phi)(u_m) \dots \Phi(u_n) \rangle = \\ \langle \Phi(u_1) \dots \left( \gamma \left( \sum_{l \leq -1} \hat{J}_l^a \hat{J}_{-l-1}^a + \sum_{l > -1} \hat{J}_{-l-1}^a \hat{J}_{-l}^a \right) \Phi \right) (u_m) \dots \Phi(u_n) \rangle. \end{aligned}$$

Turning now to the scattering amplitudes, we consider the individual terms in the expression for an insertion of the energy-momentum tensor following from the double-soft limit, (3.18). We have at leading order

$$\oint_{\mathcal{C}_{u_m}} \frac{dz_1}{2\pi i} z_{1m} \mathcal{T}^S(z_1) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) = \mathcal{A}_n(\{u_i, \bar{u}_i\}) \tag{3.29}$$

where  $\mathcal{C}_{u_m}$  is a contour surrounding the point  $u_m$ . This has the expected form for an insertion of the energy-momentum tensor into a correlation function of primary fields with weight one. More non-trivially the sub-leading term is given by

$$\oint_{\mathcal{C}_{u_m}} \frac{dz_1}{2\pi i} \mathcal{T}^S(z_1) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) = \frac{1}{N} \sum_{\substack{\ell=1 \\ \ell \neq m}}^n \frac{T_m^a \otimes T_\ell^a}{u_m - u_\ell} \mathcal{A}_n(\{u_i, \bar{u}_i\}). \tag{3.30}$$

The usual conformal Ward identity would then imply

$$\partial_{u_m} \mathcal{A}_n(\{u_i, \bar{u}_i\}) = \mathcal{L}'_{-1} \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) + \frac{1}{N} \sum_{\substack{\ell=1 \\ \ell \neq m}}^n \frac{T_m^a \otimes T_\ell^a}{u_m - u_\ell} \mathcal{A}_n(\{u_i, \bar{u}_i\}) \tag{3.31}$$

where  $\mathcal{L}'_{-1}$  is the residue of  $T'(z_1)$  at  $z_1 = u_m$ . It is tempting to ask what would happen if the  $\mathcal{L}'_{-1}$  term would be absent, that is, if the current algebra described the complete CFT. As an example one can consider the MHV amplitudes. It is useful to define a rescaled quantity  $G_n$  as follows

$$G_n(\{u_i, \bar{u}_i, h_i\}) = \prod_{\ell=1}^n \nu_\ell^{h_\ell} \mathcal{A}_n(\{u_i, \bar{u}_i\}). \tag{3.32}$$

In the simplest three-particle case

$$G_3^{\text{MHV}}(1^-, 2^-, 3^+) = \frac{(u_1 - u_2)^3}{(u_2 - u_3)(u_3 - u_1)} (\text{Tr}(T^{a_1} T^{a_2} T^{a_3}) - \text{Tr}(T^{a_1} T^{a_3} T^{a_2})) \quad (3.33)$$

and it is immediately obvious that (3.31) with  $\mathcal{L}'_{-1}$  absent is satisfied for  $u_3$  but not  $u_1$  or  $u_2$ . In fact it is easy to check explicitly that the KZ-equation with  $\mathcal{L}'_{-1}$  absent continues to hold for the positive helicity gluons in higher point MHV amplitudes (currently done with Mathematica to seven-points). It is slightly non-trivial to check that the double trace terms, which arise when  $m$  and  $\ell$  in (3.31) are not adjacent in a specific colour-ordered term, vanish. This requires that the colour ordered amplitudes satisfy a number of identities. For example, for the  $m = 3$  KZ equation one such relation at four points is the vanishing of the  $\text{Tr}(T^{a_1} T^{a_4})\text{Tr}(T^{a_2} T^{a_3})$  double-trace terms, which requires that

$$\begin{aligned} & \frac{A(1, 2, 3, 4)}{u_3 - u_1} + \frac{A(1, 4, 3, 2)}{u_3 - u_1} - \frac{A(1, 2, 3, 4)}{u_3 - u_2} - \frac{A(1, 3, 2, 4)}{u_3 - u_2} \\ & - \frac{A(1, 4, 2, 3)}{u_3 - u_2} - \frac{A(1, 4, 3, 2)}{u_3 - u_2} + \frac{A(1, 3, 2, 4)}{u_3 - u_4} + \frac{A(1, 4, 2, 3)}{u_3 - u_4} = 0. \end{aligned} \quad (3.34)$$

More generally for the  $m$ -th KZ-equation the vanishing of the

$$\text{Tr}(T^{a_m} T^{a_{\sigma_1}} \dots T^{a_{\sigma_{\ell-1}}}) \text{Tr}(T^{a_{\sigma_\ell}} \dots T^{a_{\sigma_{n-1}}}) \quad (3.35)$$

term requires

$$\begin{aligned} \sum_{\text{cyc}\{\sigma_\ell, \dots, \sigma_{n-1}\}} & \left( \left[ \frac{1}{u_m - u_{\sigma_\ell}} - \frac{1}{u_m - u_{\sigma_{\ell-1}}} \right] A_n(m, \sigma_1 \dots \sigma_\ell \dots \sigma_{n-1}) \right. \\ & \left. - \left[ \frac{1}{u_m - u_{\sigma_1}} - \frac{1}{u_m - u_{\sigma_{n-1}}} \right] A_n(m, \sigma_\ell \dots \sigma_{n-1} \sigma_1 \dots \sigma_{\ell-1}) \right) = 0 \end{aligned} \quad (3.36)$$

where  $\text{cyc}\{\sigma_\ell, \dots, \sigma_{n-1}\}$  denotes the sum over all cyclic permutations. We checked all such identities hold up to seven points for MHV amplitudes. However, starting with the six-point NMHV amplitude it no longer appears that the KZ equations holds even for the positive helicity gluons. This suggests, unsurprisingly, that if there is a CFT interpretation there is additional structure and as there is further universal behaviour in the soft-limits we attempt to also interpret these in the context of a two-dimensional description.

Let us round off this section with an interesting connection of the above amplitude relations with the so called Bern-Carrasco-Johansson (BCJ) [63] relations resulting from the color-kinematic duality of gauge theory amplitudes. We can re-write the previous example of the four point amplitude relation (3.34) in a different way using Kleiss-Kuiff and the photon decoupling relations, which gives  $A(1, 2, 3, 4) = A(1, 4, 3, 2)$  and  $A(1, 3, 2, 4) = A(1, 4, 2, 3)$ . Using these relations we can rewrite the left-hand side of (3.34) as,

$$\begin{aligned} A(1, 2, 3, 4) \left( \frac{1}{u_{31}} - \frac{1}{u_{32}} \right) + A(1, 4, 2, 3) \left( \frac{1}{u_{34}} - \frac{1}{u_{32}} \right) &= 0 \\ A(1, 2, 3, 4) \frac{u_{12}}{u_{31}} + A(1, 4, 2, 3) \frac{u_{42}}{u_{34}} &= 0. \end{aligned} \quad (3.37)$$

Now using the representations of the spinor helicity variables  $\langle ij \rangle = \sqrt{\nu_i \nu_j} u_{ij}$  and  $[ij] = \sqrt{\nu_i \nu_j} \bar{u}_{ij}$  and four-point momentum conservation we get,

$$\frac{u_{12}u_{34}}{u_{31}u_{24}} = \frac{\langle 12 \rangle \langle 34 \rangle [12]}{\langle 24 \rangle \langle 31 \rangle [12]} = \frac{s_{12}}{s_{24}}, \quad (3.38)$$

where  $s_{ij} = (p_i + p_j)^2$  are the Mandelstam invariants. Hence, using these change of variables (3.37) and resultantly (3.34) becomes,

$$A(1, 2, 3, 4)_{s_{12}} = s_{13}A(1, 4, 2, 3), \quad (3.39)$$

which is remarkably the 4-point BCJ relation. Thus the MHV amplitude relation derived from demanding the vanishing of the double-trace terms in our KZ equations leads to the BCJ relation at least for 4 points. This connection is harder to see for higher number of points and we will report on this in a future publication.

#### 4 Sub-leading currents

So far we have only been considering the leading soft terms, however it is known that the sub-leading soft terms for Yang-Mills amplitudes are also universal. This behaviour, the gluon version of the Low theorem [5–7, 9–11] described in section 2.1, is understood to be valid only at tree-level and for generic configurations of the remaining external hard legs [17]. We will only consider such configurations and so neglect boundary terms in the kinematic space where additional particles become collinear. At loop level such corrections could no longer be avoided and so it would be interesting to carefully understand these terms, and more generally to make contact with the SCET [64–67] description used in [17]. We thus repeat the above analysis at the sub-leading order by defining the sub-leading currents,  $\mathcal{J}_{\text{sub}}^a$ , as

$$\mathcal{J}_{\text{sub}}^a(z; \omega_z) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i, \nu_i\}) \equiv - \lim_{\delta \rightarrow 0} (1 + \delta \partial_\delta) \omega_z \mathcal{A}_{n+1}(\delta q, +, a; \{p_i, h_i, a_i\}) \quad (4.1)$$

where the  $(1 + \delta \partial_\delta)$  factor picks out the sub-leading soft term. We have that

$$\mathcal{J}_{\text{sub}}^a(z; \omega_z) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i, \nu_i\}) = \sum_{\ell=1}^n \frac{[\omega_z \partial_{\nu_\ell} + \frac{\omega_z}{\nu_\ell} (\bar{z} - \bar{u}_\ell) \bar{\partial}_\ell] T_\ell^a}{z - u_\ell} \mathcal{A}_n(\{u_i, \bar{u}_i, \nu_i\}) \quad (4.2)$$

where  $\bar{\partial}_\ell = \frac{\partial}{\partial \bar{z}_\ell}$  and  $\partial_{\omega_\ell} = \frac{\partial}{\partial \omega_\ell}$ . If we again attempt to interpret this in terms of the OPE of a current with a primary field we must consider the primary fields as depending on an auxiliary variable  $\nu$  and we have

$$J_{\text{sub}}^a(z; \omega) \Phi_R^r(w; \nu) \sim \frac{(T_R^a)_s^r [\omega \partial_\nu + \frac{\omega}{\nu} (\bar{z} - \bar{u}) \bar{\partial}]}{z - u} \Phi_R^s(w; \nu). \quad (4.3)$$

The appearance of an auxiliary parameter and derivatives in the representation is (very vaguely) reminiscent of the Cartan element of the non-compact affine  $\widehat{\mathfrak{sl}}_2$  current



algebra in the principle continuous series representations,  $\Phi^{j,\ell}(w)$ , for which we introduce the complex parameter  $x$ , and the OPE is given by

$$J^3(z)\Phi^{j,\ell}(w;x) \sim \frac{t^3\Phi^{j,\ell}(w;x)}{z-u} + \frac{k\ell}{2}\Phi^{j,\ell}(w;x) \tag{4.4}$$

where  $t^3 = x \frac{\partial}{\partial x}$ . In the case at hand we see that, in addition to having a one-parameter family of such currents, there is no extension as  $k = 0$ . Adamo and Casali [46] have, in their world-sheet theory with null infinity as its target space, given the definition of a charge with reproduces the sub-leading soft factor. It would be interesting to understand if there is a relation between the currents introduced here and their charges which have the interpretation as rotating the space of null generators. It is also known that these sub-leading soft terms in four-dimensions are intimately connected to the conformal invariance of the theory [17].

### 4.1 Sub-leading algebra

To continue the two-dimensional interpretation of the collinear soft limits to sub-leading order and given these new currents we analyse the algebra that they form with the leading currents and amongst themselves.

**$J_{\text{sub}}J$ .** To find the OPE of the Kac-Moody current  $J^a(z)$  with  $J_{\text{sub}}^a(z)$  we must consider part of the double soft limit at sub-leading order. The order of limits in this case is relevant and a specific prescription must be given with the results dependent on the prescription. We start with the consecutive limit and take leg  $q_2$  soft before leg  $q_1$ :

$$\begin{aligned} & \lim_{\delta_1 \rightarrow 0} \lim_{\delta_2 \rightarrow 0} (1 + \delta_1 \partial_{\delta_1}) \delta_2 \mathcal{A}_{n+2}(\delta_1 q_1, h_{q_1}, a_{q_1}; \delta_2 q_2, h_{q_2}, a_{q_2}; \{p_i, h_i, a_i\}) = \\ & \sum_{\sigma \in S_n} \left\{ \left[ S_{q_1, \sigma_1}^{(0)}(q_2^{h_{q_2}}) S_{\sigma_n, \sigma_1}^{(1)}(q_1^{h_{q_1}}) + S_{\sigma_n, q_1}^{(0)}(q_2^{h_{q_2}}) S_{\sigma_n, \sigma_1}^{(1)}(q_1^{h_{q_1}}) \right] \text{Tr}(q_1 q_2 \sigma_1 \dots \sigma_n) \right. \\ & \quad + \sum_{i=2}^n S_{\sigma_{i-1}, \sigma_i}^{(0)}(q_2^{h_{q_2}}) S_{\sigma_n, \sigma_1}^{(1)}(q_1^{h_{q_1}}) \text{Tr}(q_1 \sigma_1 \dots \sigma_{i-1} q_2 \sigma_i \dots \sigma_n) \\ & \quad \left. + i S_{\sigma_n, q_1}^{(0)}(q_2^{h_{q_2}}) S_{\sigma_n, \sigma_1}^{(1)}(q_1^{h_{q_1}}) f^{a_{q_2} a_{q_1} c} \text{Tr}(c \sigma_1 \dots \sigma_n) \right\} A_n(\sigma_1, \dots, \sigma_n). \tag{4.5} \end{aligned}$$

An important feature of this formula is that, as previously, the singularities as  $q_1$  and  $q_2$  become collinear only occur in the third set of terms on the right-hand side as those that appear in the first set of terms cancel amongst themselves

$$\begin{aligned} & \lim_{\delta_1 \rightarrow 0} \lim_{\delta_2 \rightarrow 0} (1 + \delta_1 \partial_{\delta_1}) \delta_2 \mathcal{A}_{n+2}(\delta_1 q_1, h_{q_1}, a_{q_1}; \delta_2 q_2, h_{q_2}, a_{q_2}; \{p_i, h_i, a_i\}) = \\ & - \sum_{\sigma \in S_n} \left\{ \left[ \frac{\langle \sigma_n \sigma_1 \rangle}{\langle \sigma_n q_2 \rangle \langle q_2 \sigma_1 \rangle} \left( \frac{[q_1 \bar{\partial}_{\sigma_1}]}{\langle q_1 \sigma_1 \rangle} + \frac{[q_1 \bar{\partial}_{\sigma_n}]}{\langle \sigma_n q_1 \rangle} \right) \right] \text{Tr}(q_1 q_2 \sigma_1 \dots \sigma_n) \right. \\ & \quad + \dots \\ & \quad \left. + i \frac{\langle \sigma_n q_1 \rangle}{\langle \sigma_n q_2 \rangle \langle q_2 q_1 \rangle} \left( \frac{[q_1 \bar{\partial}_{\sigma_1}]}{\langle q_1 \sigma_1 \rangle} + \frac{[q_1 \bar{\partial}_{\sigma_n}]}{\langle \sigma_n q_1 \rangle} \right) f^{a_{q_2} a_{q_1} c} \text{Tr}(c \sigma_1 \dots \sigma_n) \right\} A_n(\sigma_1, \dots, \sigma_n). \tag{4.6} \end{aligned}$$

Using the coordinate representation (3.2) we can find the most singular part as  $z_1 \rightarrow z_2$

$$\lim_{\delta_1 \rightarrow 0} \lim_{\delta_2 \rightarrow 0} (1 + \delta_1 \partial_{\delta_1}) \delta_2 \omega_{z_1} \omega_{z_2} \mathcal{A}_{n+2} \sim \frac{i f^{a_{q_1} a_{q_2} c}}{z_{12}} \sum_{\ell=1}^n \left[ \frac{T_\ell^c \omega_{z_1} (\partial_{\nu_\ell} + \frac{\bar{z}_2 - \bar{u}_\ell}{\nu_\ell} \bar{\partial}_\ell)}{z_2 - u_\ell} \right] \mathcal{A}_n. \quad (4.7)$$

If we had instead taken  $\delta_2$  to zero after  $\delta_1$  we would find

$$\begin{aligned} & \lim_{\delta_2 \rightarrow 0} \lim_{\delta_1 \rightarrow 0} (1 + \delta_1 \partial_{\delta_1}) \delta_2 \mathcal{A}_{n+2}(\delta_1 q_1, h_{q_1}, a_{q_1}; \delta_2 q_2, h_{q_2}, a_{q_2}; \{p_i, h_i, a_i\}) = \\ & \sum_{\sigma \in S_n} \left\{ \left[ S_{\sigma_n, q_2}^{(1)}(q_1^{h_{q_1}}) S_{\sigma_n, \sigma_1}^{(0)}(q_2^{h_{q_2}}) + S_{q_2, \sigma_1}^{(1)}(q_1^{h_{q_1}}) S_{\sigma_n, \sigma_1}^{(0)}(q_2^{h_{q_2}}) \right] \text{Tr}(q_1 q_2 \sigma_1 \dots \sigma_n) \right. \\ & \quad \left. + \dots + i S_{q_2, \sigma_1}^{(1)}(q_1^{h_{q_1}}) S_{\sigma_n, \sigma_1}^{(0)}(q_2^{h_{q_2}}) f^{a_{q_2} a_{q_1} c} \text{Tr}(c \sigma_1 \dots \sigma_n) \right\} A_n(\sigma_1, \dots, \sigma_n). \quad (4.8) \end{aligned}$$

Again one can show that in the collinear limit the potentially singular terms only come from the last line above, however in this case there are in fact no non-vanishing singular terms in the collinear limit. Such terms potentially could have arisen from the holomorphic anomaly

$$\frac{\partial}{\partial \bar{\lambda}^{\dot{a}}} \frac{1}{\langle \lambda \mu \rangle} = \pi \epsilon_{\dot{a} b} \bar{\mu}^{\dot{b}} \delta^{(2)}(\langle \lambda \mu \rangle) \quad (4.9)$$

which gives

$$\begin{aligned} & \lim_{\delta_2 \rightarrow 0} \lim_{\delta_1 \rightarrow 0} (1 + \delta_1 \partial_{\delta_1}) \delta_2 \mathcal{A}_{n+2} = \\ & - \sum_{\sigma \in S_n} \left\{ \dots + i \pi f^{a_{q_2} a_{q_1} c} \frac{\langle \sigma_n \sigma_1 \rangle}{\langle q_2 q_1 \rangle} \left( - \frac{[q_1 \sigma_n]}{\langle q_2 \sigma_1 \rangle} \delta^{(2)}(\langle \sigma_n q_2 \rangle) \right. \right. \\ & \quad \left. \left. + \frac{[q_1 \sigma_1]}{\langle q_2 \sigma_n \rangle} \delta^{(2)}(\langle q_2 \sigma_1 \rangle) \right) \text{Tr}(c \sigma_1 \dots \sigma_n) \right\} A_n(\sigma_1, \dots, \sigma_n). \quad (4.10) \end{aligned}$$

However these terms are also of order one when  $q_1$  becomes collinear with  $q_2$ . Thus taking a combination with parameters  $\alpha_{12}$  and  $\alpha_{21}$  only the  $\alpha_{21}$  parameter contributes and from the definition

$$\begin{aligned} & \mathcal{J}_{\text{sub}}^a(z_1) \mathcal{J}^b(z_2) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) \equiv \\ & \lim_{\alpha} (1 + \delta_1 \partial_{\delta_1}) \delta_2 \omega_{z_1} \omega_{z_2} \mathcal{A}_{n+2}(\delta_1 q_1, +, a; \delta_2 q_2, +, b; \{p_i, h_i, a_i\}) \end{aligned} \quad (4.11)$$

we find

$$\mathcal{J}_{\text{sub}}^a(z_1) \mathcal{J}^b(z_2) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i, \omega_i\}) = \frac{i \alpha_{21} f^{ab} c}{z_{12}} \sum_{\ell=3}^{n+2} \left[ \frac{T_\ell^c \omega_{z_1} (\partial_{\nu_\ell} + \frac{\bar{z}_2 - \bar{u}_\ell}{\nu_\ell} \bar{\partial}_\ell)}{z_2 - u_\ell} \right] \mathcal{A}_n$$

or in the notation of the OPE, after a slight rearrangement involving exchanging the soft legs and renaming the parameter  $c_1^s$ ,

$$J^a(z_1) J_{\text{sub}}^b(z_2; \omega_{z_2}) \sim \frac{i c_1^s f^{ab} c J_{\text{sub}}^c(z_2; \omega_{z_2})}{z_{12}}. \quad (4.12)$$

Note that if we choose  $c_1^s$  to vanish then the OPE between the leading soft current and the sub-leading current vanishes and these sectors decouple as the parameter  $\alpha_{12}$  doesn't

appear. For brevity we will make the choice to set  $c_1^s = 1$  and  $c_2^s = 0$ , however it can always be reintroduced. Using this OPE, one can compute the OPE of the Sugawara energy-momentum tensor with  $J_{\text{sub}}^a(z; \omega_z)$

$$T^S(z_1)J_{\text{sub}}^a(z_2; \omega_{z_2}) \sim \frac{J_{\text{sub}}^a(z_2; \omega_{z_2})}{z_{12}^2} + \frac{\partial J_{\text{sub}}^a(z_2; \omega_{z_2})}{z_{12}}, \quad (4.13)$$

where we have not included the composite term  $f^{adc} : J_{\text{sub}}^a J_{\text{sub}}^d : (z_2, \omega_{z_2})$  which appears at  $1/z_{12}$ . This is what one would expect for a field of weight one.

**$J_{\text{sub}} J_{\text{sub}}$ .** To find the OPE of the sub-leading currents with themselves we must calculate the sub-leading behaviour of each soft particle in the limit of two soft gluons

$$\begin{aligned} & \lim_{\alpha} (1 + \delta_1 \partial_{\delta_1})(1 + \delta_2 \partial_{\delta_2}) \mathcal{A}_{n+2}(\delta_1 q_1, +, a; \delta_2 q_2, +, b; \{p_i, h_i, a_i\}) \\ &= \sum_{\sigma \in S_n} \left\{ \dots + i \frac{f^{bac}}{\langle q_2 q_1 \rangle} \left( \frac{\alpha_{21} [q_2 \bar{\partial}_{\sigma_n}]}{\langle \sigma_n q_1 \rangle} + \frac{\alpha_{21} [q_2 \bar{\partial}_{\sigma_1}]}{\langle q_1 \sigma_1 \rangle} \right. \right. \\ & \quad \left. \left. + \frac{\alpha_{12} [q_1 \bar{\partial}_{\sigma_n}]}{\langle \sigma_n q_2 \rangle} + \frac{\alpha_{12} [q_1 \bar{\partial}_{\sigma_1}]}{\langle q_2 \sigma_1 \rangle} \right) \text{Tr}(c \sigma_3 \dots \sigma_{n+2}) \right\} \mathcal{A}_n(\sigma_1, \dots, \sigma_n) \end{aligned}$$

where we have included contributions from both orderings and from which we have

$$\begin{aligned} \mathcal{J}_{\text{sub}}^a(z_1; \omega_{z_1}) \mathcal{J}_{\text{sub}}^b(z_2; \omega_{z_2}) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) = \\ \frac{if^{ab}_c}{z_{12}} \sum_{\ell=1}^n \left[ \frac{T_{\ell}^c(\alpha_{21} \omega_{z_1} + \alpha_{12} \omega_{z_2})(\partial_{\nu_{\ell}} + \frac{(\bar{z}_2 - \bar{u}_{\ell})}{\nu_{\ell}}) \bar{\partial}_{\ell}}{z_2 - u_{\ell}} \right] \mathcal{A}_n \end{aligned}$$

or in the notation of the OPE and writing the parameters as  $c_1^{ss}$  and  $c_2^{ss}$  we have

$$J_{\text{sub}}^a(z_1; \omega_{z_1}) J_{\text{sub}}^b(z_2; \omega_{z_2}) \sim \frac{if^{ab}_c J_{\text{sub}}^c(z_2; c_2^{ss} \omega_{z_1} + c_1^{ss} \omega_{z_2})}{z_{12}}. \quad (4.14)$$

Here we see an example where there doesn't appear to be a natural choice for the ordering of the soft limits and so we simply keep both and parameterise the ambiguity by  $c_1^{ss}$  and  $c_2^{ss}$ .

Given our previous considerations and the form of the OPE, one might attempt to repeat the Sugawara construction for these sub-leading currents. While we don't analyse the general case, at least in the symmetric case where  $c_1^{ss} = c_2^{ss} = 1$  using the OPE it is straightforward to show that one can define an operator for each value of  $\omega_{z_1}$

$$T_{\text{sub}}^S(z_1, \omega_{z_1}) = \frac{1}{2N} J_{\text{sub}}^a(z_1; \omega_{z_1}) J_{\text{sub}}^a(z_1; -\omega_{z_1}) \quad (4.15)$$

which acts like a sub-leading energy-momentum tensor in that it satisfies

$$T_{\text{sub}}^S(z_1, \omega_{z_1}) J_{\text{sub}}^a(z_2; \omega_{z_2}) \sim \frac{J_{\text{sub}}^a(z_2; \omega_{z_2})}{z_{12}^2} + \frac{\partial J_{\text{sub}}^a(z_2; \omega_{z_2})}{z_{12}}. \quad (4.16)$$

## 5 Anti-holomorphic currents

We can of course repeat the previous calculations for the negative helicity gluon and find the anti-holomorphic currents. We define

$$\bar{\mathcal{J}}^a(z) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i, \nu_i\}) \equiv \lim_{\delta \rightarrow 0} (\delta \omega) \mathcal{A}_{n+1}(\delta p, -, a; \{p_i, h_i, a_i\}), \quad (5.1)$$

so that the current still acts with the adjoint action but because of the missing minus sign it acts from the right rather than the left or, alternatively, with the complex conjugate generator

$$\bar{\mathcal{J}}^a(z) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) = \sum_{\ell=1}^n \frac{\bar{T}_\ell^a}{\bar{z} - \bar{u}_\ell} \mathcal{A}_n(\{u_i, \bar{u}_i\}). \quad (5.2)$$

### 5.1 OPE for anti-holomorphic currents

Slightly more non-trivially we can consider the mixed helicity double soft limit and attempt to reproduce the action of the holomorphic currents on the anti-holomorphic discussed in [4]. We can again start from (3.10) however now  $h_{q_1} = +$  while  $h_{q_2} = -$ . In this case there is an ambiguity in the double soft limit which we parameterise in case of consecutive limits by using a specific case of the general multi-parameter limit

$$\lim_{\alpha} \equiv \left( \alpha_{12} \lim_{\delta_2 \rightarrow 0} \lim_{\delta_1 \rightarrow 0} + \alpha_{21} \lim_{\delta_1 \rightarrow 0} \lim_{\delta_2 \rightarrow 0} \right). \quad (5.3)$$

The corresponding two-parameter family of consecutive double-soft factors,  $\alpha \text{CSL}_{\sigma_n, \sigma_1}^{(0)}$ , still satisfies the identity analogous to (3.11)

$$\alpha \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(1^+, 2^-) + \alpha \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(2^-, 1^+) = S_{\sigma_n, \sigma_3}^{(0)}(1^+) S_{\sigma_n, \sigma_3}^{(0)}(2^-). \quad (5.4)$$

We thus find that the terms singular in  $z_{12}$  arise from the  $\alpha \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(2^-, 1^+)$  term. More explicitly, if we start from

$$\mathcal{J}^a(z_1) \bar{\mathcal{J}}^b(z_2) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) \equiv - \lim_{\alpha} \delta_1 \delta_2 \omega_{z_1} \omega_{z_2} \mathcal{A}_{n+2}(\delta_1 q_1, +, a; \delta_2 q_2, -, b; \{p_i, h_i, a_i\})$$

then we find that

$$\begin{aligned} \mathcal{J}^a(z_1) \bar{\mathcal{J}}^b(z_2) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) = & i f^{ab}{}_c \sum_{\ell=1}^n \left[ \cdots + \alpha_{12} \frac{\bar{T}_\ell^c}{z_{12}(\bar{z}_2 - \bar{u}_\ell)} - \alpha_{21} \frac{T_\ell^c}{\bar{z}_{12}(z_2 - u_\ell)} \right. \\ & \left. + \alpha_{21} \frac{z_{12}}{\bar{z}_{12}} \frac{T_\ell^c}{(z_2 - z_\ell)^2} + \cdots \right] \mathcal{A}_n, \end{aligned}$$

where we have included the sub-leading term that has a non-trivial phase as  $z_1$  encircles  $z_2$ . This expression can be interpreted as a non-trivial OPE between the holomorphic and anti-holomorphic currents of the form

$$\mathcal{J}^a(z_1) \bar{\mathcal{J}}^b(z_2) \sim i f^{ab}{}_c \left[ d_1 \frac{\bar{J}^c(z_2)}{z_{12}} - d_2 \frac{J^c(z_2)}{\bar{z}_{12}} - d_2 \frac{z_{12}}{\bar{z}_{12}} \partial J^c(z_2) \right], \quad (5.5)$$

where the OPE parameters are related to the order of the soft limits by  $d_1 = \alpha_{12}$  and  $d_2 = \alpha_{21}$ . If we require that the current-current OPE be consistent with the complex conjugation  $(J^a(z_1))^* = \bar{J}^a(z_1)$  we find the constraint  $d_1^* = d_2$  and  $d_2^* = d_1$ . For the soft-limit which naturally implies real parameters this requires  $d_1 = d_2$  which is to say the symmetric choice of parameters. The structure for the OPE (5.5) between the holomorphic and anti-holomorphic currents appears similar to that found in the CFTs describing supergroup coset models considered in [44]. As mentioned previously, in the case of super-group non-linear sigma models the vanishing of the Killing form allows the construction of a holomorphic stress-energy tensor. Supergroup CFTs are generically non-unitary and logarithmic which suggests that in our case if one can construct a corresponding CFT it will potentially be non-unitary. However, without a better definition of such a putative CFT it is difficult to understand the meaning of such an interpretation.

It is interesting to compare this with what one finds in the simultaneous double-soft by computing

$$-\lim_{\delta \rightarrow 0} \delta^2 \omega_{z_1} \omega_{z_2} \mathcal{A}_{n+2}(\delta q_1, +, a, \delta q_2, -, b; \{p_i, h_i, a_i\}),$$

and making use of the results of [36, 37]. Focusing on the singular terms this gives

$$\frac{if^{ab}_c}{(\omega_{z_1} + \omega_{z_2})^2} \sum_{\ell=1}^n \left[ \dots + \frac{\bar{T}_\ell^c \omega_{z_2}^2}{z_{12}(\bar{z}_2 - \bar{u}_\ell)} - \frac{T_\ell^c \omega_{z_1}^2}{\bar{z}_{12}(z_2 - u_\ell)} + \dots \right] \mathcal{A}_n.$$

With the previous interpretation of the leading order soft limits in terms of currents  $J^a$  and  $\bar{J}^a$  this expression doesn't appear to make sense due to the appearance of the soft-particle energies  $\omega_{z_i}$  on the right-hand side. One could of course attempt to introduce a family of currents parametrized by  $\omega$  already at leading order; alternatively we can additionally demand that  $\omega_{z_1} = \omega_{z_2}$  in which case we reproduce the consecutive answer with a symmetric choice for the parameters  $d_1$  and  $d_2$ .

**$\bar{J}J_{\text{sub}}$ .** To complete the algebra we must compute the OPE of the anti-holomorphic current with the sub-leading current. The calculations are essentially the same as those above and we again start from

$$\begin{aligned} \bar{J}^a(z_1) \mathcal{J}_{\text{sub}}^b(z_2; \omega_{z_2}) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) \equiv \\ - \lim_{\alpha} \delta_1 (1 + \delta_2 \partial_{\delta_2}) \omega_{z_1} \omega_{z_2} \mathcal{A}_{n+2}(\delta_1 q_1, -, a, \delta_2 q_2, +, b; \{p_i, h_i, a_i\}) \end{aligned}$$

which implies

$$\begin{aligned} \bar{J}^a(z_1) \mathcal{J}_{\text{sub}}^b(z_2; \omega_{z_2}) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) = if^{ab}_c \sum_{\ell=1}^n \left[ \dots - 2 \frac{\alpha_{21}}{z_{12}} \frac{\omega_{z_2} \bar{T}_\ell^c}{(\bar{z}_2 - \bar{u}_\ell)} + 3 \alpha_{21} \frac{\bar{z}_{12}}{z_{12}} \frac{\omega_{z_2} \bar{T}_\ell^c}{(\bar{z}_2 - \bar{u}_\ell)^2} \right. \\ \left. - \frac{\alpha_{12}}{\bar{z}_{12}} \frac{T_\ell^c \omega_{z_2} [\partial_{\omega_\ell} + \frac{\bar{z}_2 - \bar{u}_\ell}{v_\ell} \bar{\partial}_\ell]}{(z_2 - u_\ell)} + \dots \right] \mathcal{A}_n. \end{aligned} \tag{5.6}$$

As it stands this doesn't appear to be interpretable as an OPE between an anti-holomorphic current and a sub-leading current depending on the parameter  $\omega_{z_2}$  due to the explicit

appearance of  $\omega_{z_1}$  on the right-hand side. Thus we are led to imposing a particular ordering for the soft limits where we take the particle corresponding to the sub-leading current to be soft after the leading order current, that is  $\alpha_{21} = 0$  and  $\alpha_{12} = 1$ . This, taking the sub-leading limit after the leading limit, is the same as in the holomorphic sector and as there seems a reasonable choice. This corresponds to an OPE between the non-holomorphic currents and the sub-leading holomorphic current

$$\bar{J}^a(z_1) J_{\text{sub}}^b(z_2; \omega_{z_2}) \sim -i f_c^{ab} \frac{J_{\text{sub}}^c(z_2; \omega_{z_2})}{\bar{z}_{12}}. \quad (5.7)$$

In some aspects this lack of choice is unappealing and it would be interesting to understand it better. As a small step in this direction one can again consider the simultaneous double soft limit at sub-leading order the expressions for which can be found in [36]. In this case the simultaneous limit is not the same as the consecutive limit and for the case of mixed helicity is not given by products of single soft limits. The simultaneous limit mixes the terms where particle  $q_1$  and  $q_2$  are sub-leading but by focussing on the terms with anti-holomorphic derivatives one can identify those terms corresponding to the sub-leading terms in the soft-expansion for the positive helicity gluon. Denoting the simultaneous double-soft factor  $\text{DSL}_{n,1}(q_1^{h_1}, q_2^{h_2})$  the relevant, singular, sub-leading terms are

$$\text{DSL}_{n,1}(q_1^-, q_2^+) \Big|_{\text{singular}} = \frac{[nq_2]^2 [q_2 \tilde{\partial}_1]}{[nq_1][q_1q_2] \langle 1|q_{12}|n \rangle} - \frac{[nq_2]^2 [q_2 \tilde{\partial}_n]}{[nq_1][q_1q_2] (2p_n \cdot q_{12})}. \quad (5.8)$$

This prescription does not include the contact terms, which involve no derivatives,

$$\text{DSL}_{n,1}^{(1)}(q_1^-, q_2^+) \Big|_{\text{contact}} = \frac{[nq_2]^2 \langle q_1 n \rangle}{[nq_1]} \frac{1}{(2p_n \cdot q_{12})^2} + \frac{\langle 1q_1 \rangle^2 [q_2 1]}{\langle 1q_2 \rangle} \frac{1}{(2p_1 \cdot q_{12})^2} \quad (5.9)$$

however as they are not singular in the collinear limit this is no loss. Expanding the collinear limit, gives the analogous result to (5.6)

$$i f_c^{ab} \sum_{\ell=1}^n \left[ \dots - \frac{1}{\bar{z}_{12}} \frac{T_\ell^c \frac{\omega_{z_2}^2}{\omega_{z_1} + \omega_{z_2}} [\partial_{\omega_\ell} + \frac{\bar{z}_2 - \bar{u}_\ell}{v_\ell} \bar{\partial}_\ell]}{(z_2 - u_\ell)} + \dots \right] \mathcal{A}_n.$$

Here we see that the troublesome non-derivative terms in (5.6) don't appear. Of course, as before, due to the non-local nature of the simultaneous limit we find the soft-particle energies entering as  $\frac{\omega_{z_2}^2}{\omega_{z_1} + \omega_{z_2}}$  and so the simultaneous limit, by itself, does not give a good prescription. As this limit involves the sub-leading soft-terms this behaviour may be related to the failure of the Low theorem in general. Understanding these terms better will be essential if there is to be any progress at loop-level where despite the breakdown of the Low theorem there is some evidence of universal behaviour [68] of a restricted type and in particular the SCET framework is useful in trying to better understand these terms [17].

**$J_{\text{sub}} \bar{J}_{\text{sub}}$ .** Finally we consider the OPE of a sub-leading current with its anti-holomorphic analogue. This is related to the sub-sub-leading double-soft limit. While such sub-sub-leading behaviour has not been studied in the simultaneous double-soft limit it is straightforward to define using consecutive single-soft limits though naturally this again leads to

the introduction of parameters. Starting from

$$\mathcal{J}_{\text{sub}}^a(z_1; \omega_{z_1}) \bar{\mathcal{J}}_{\text{sub}}^b(z_2; \omega_{z_2}) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) \equiv - \lim_{\alpha} (1 + \delta_2 \partial_{\delta_2})(1 + \delta_2 \bar{\partial}_{\delta_2}) \omega_{z_1} \omega_{z_2} \mathcal{A}_{n+2}(\delta_1 q_1, +, a; \delta_2 q_2, -, b; \{p_i, h_i, a_i\})$$

and so

$$\begin{aligned} \mathcal{J}_{\text{sub}}^a(z_1; \omega_{z_1}) \bar{\mathcal{J}}_{\text{sub}}^b(z_2; \omega_{z_2}) \cdot \mathcal{A}_n(\{u_i, \bar{u}_i\}) = & if^{ab}_c \sum_{\ell=1}^n \left[ \dots - \frac{\alpha_{12}}{z_{12}} \frac{\bar{T}_{\ell}^c \omega_{z_1} [\partial_{\omega_{\ell}} + \frac{z_2 - u_{\ell}}{\nu_{\ell}} \partial_{\ell}]}{(\bar{z}_2 - \bar{u}_{\ell})} \right. \\ & - \frac{\alpha_{12} \bar{z}_{12}}{z_{12}} \frac{\bar{T}_{\ell}^c \omega_{z_1} [\partial_{\omega_{\ell}} + \frac{z_2 - u_{\ell}}{\nu_{\ell}} \partial_{\ell}]}{(\bar{z}_2 - \bar{u}_{\ell})^2} + \frac{\alpha_{21}}{\bar{z}_{12}} \frac{T_{\ell}^c \omega_{z_2} [\partial_{\omega_{\ell}} + \frac{\bar{z}_2 - \bar{u}_{\ell}}{\nu_{\ell}} \bar{\partial}_{\ell}]}{(z_2 - u_{\ell})} \\ & \left. - 2 \frac{\alpha_{21} z_{12}}{\bar{z}_{12}} \frac{T_{\ell}^c \omega_{z_2} [\partial_{\omega_{\ell}} + \frac{\bar{z}_2 - \bar{u}_{\ell}}{\nu_{\ell}} \bar{\partial}_{\ell}]}{(z_2 - u_{\ell})^2} + \dots \right] \mathcal{A}_n \end{aligned}$$

which corresponds to the OPE

$$\begin{aligned} J_{\text{sub}}^a(z_1; \omega_{z_1}) \bar{J}_{\text{sub}}^b(z_2; \omega_{z_2}) \sim & if^{ab}_c \left[ \frac{J_{\text{sub}}^c(z_2; d_2^s \omega_{z_2})}{\bar{z}_{12}} - \frac{\bar{J}_{\text{sub}}^c(z_2; d_1^s \omega_{z_1})}{z_{12}} \right. \\ & \left. + \frac{\bar{z}_{12}}{z_{12}} \bar{\partial} \bar{J}_{\text{sub}}^c(z_2; d_1^s \omega_{z_1}) + 2 \frac{z_{12}}{\bar{z}_{12}} \partial J_{\text{sub}}^c(z_2; d_2^s \omega_{z_2}) \right]. \end{aligned} \quad (5.10)$$

## 5.2 CFT interpretation of anti-holomorphic currents

It is interesting to compute the OPE of the anti-holomorphic current with the holomorphic energy-momentum tensor found via the Sugawara construction. This can be done directly by using the formulae above to compute the OPE of two holomorphic currents with an anti-holomorphic current and then extracting the singular terms as the two holomorphic currents approach each other. At leading order in  $z_{13}$  we find

$$J^a(z_1) J^a(z_2) \bar{J}^c(z_3) \sim f^{ca} d f^{ad}_e \left[ c_1 \frac{\bar{J}^c(z_3)}{z_{13}^2} + \left[ c'_2 \frac{z_{21}}{\bar{z}_{21}} \frac{1}{z_{13}^2} + c'_3 \frac{\bar{z}_{21}}{z_{21}} \frac{1}{\bar{z}_{13}^2} \right] J^e(z_3) \right], \quad (5.11)$$

where the constants  $c_1$  and  $c'_2, c'_3$  are related to the parameters in the OPEs of  $J^a$  with  $J^a$  (1.2) and  $J^a$  with  $\bar{J}^a$  (1.5). The terms less singular in  $1/z_{13}$  can be computed and are more complicated but one can already see at this leading order the unwanted appearance of  $J^a(z_3)$  in the OPE; however all such terms, while non-vanishing as  $z_2 \rightarrow z_1$ , can be identified by their phase. If we define the energy-momentum tensor by the contour integral

$$T(z_1) = \frac{1}{4\pi i h^{\vee}} \oint \frac{dz_2}{z_{21}} J^a(z_1) J^a(z_2) \quad (5.12)$$

where  $h^{\vee}$  is again the dual Coxeter number then we can drop all unwanted terms depending on  $z_{21}$  and  $\bar{z}_{21}$  so that we have the result

$$T(z_1) \bar{J}^c(z_3) \sim c_1 \frac{\bar{J}^c(z_3)}{z_{13}^2} + c_1 \frac{\bar{z}_{13}}{z_{13}^2} \bar{\partial} \bar{J}^c(z_3) + \dots \quad (5.13)$$

where we have now included some of the sub-leading terms and the coefficient  $c_1$  is given by  $c_1 = d_1^2$ . The remaining sub-leading terms are either composite operators of the form

$f^{ca}_d : J^a \bar{J}^d : (z_3)$  or of the form  $\bar{\partial} J^c, \partial \bar{J}^c$ . These later terms would be related to composite terms of the former type if we imposed the Maurer-Cartan

$$\bar{\partial} J^c - \partial \bar{J}^c - i f^c_{ad} J^a \bar{J}^d = 0 \quad (5.14)$$

and current conservation

$$\bar{\partial} J^c + \partial \bar{J}^c = 0 \quad (5.15)$$

equations. However we restrict our attention to the leading singularity terms for the present.

One can alternatively start from the triple soft-limit of (3.20) but where we now take the third helicity to be negative,  $h_{q_3} = -$ . We are interested in the case where we trace over the colour indices  $a_{q_1}$  and  $a_{q_2}$ , extracting the terms that are singular as  $z_3$  approaches  $z_1$  or  $z_2$  and then taking the limit  $z_2 \rightarrow z_1$ . For example, and again taking the gauge group to be  $\mathfrak{su}(N)$ , the terms that potentially contribute are

$$\begin{aligned} & \lim_{\delta_1 \rightarrow 0} \lim_{\delta_2 \rightarrow 0} \lim_{\delta_3 \rightarrow 0} \sum_a \mathcal{A}_{n+3}(\delta_1 q_1, +, a; \delta_2 q_2, +, a; \delta_3 q_3, -, b; \{p_i, h_i, a_i\}) \\ &= \sum_{\sigma \in S_n} \left\{ \cdots + i \sum_{r=1}^{n-1} \left( \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(q_3^-, q_1^+) S_{\sigma_r, \sigma_{r+1}}^{(0)}(q_2^+) \right. \right. \\ & \quad - \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(q_3^-, q_2^+) S_{\sigma_r, \sigma_{r+1}}^{(0)}(q_1^+) \Big) f^{ba}_c \text{Tr}(a \dots \sigma_r c \dots) \\ & \quad + \left( N - \frac{1}{N} \right) \left( \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(q_1^+, q_2^+, q_3^-) + \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(q_2^+, q_1^+, q_3^-) \right. \\ & \quad + \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(q_3^-, q_1^+, q_2^+) + \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(q_3^-, q_2^+, q_1^+) \Big) \text{Tr}(b \sigma_1 \dots) \\ & \quad \left. - \frac{1}{N} \left( \text{CSL}_{\sigma_n, \sigma_1}^{(0)}(q_1^+, q_3^-, q_2^+) + \text{CSL}_{\sigma_n, \sigma_4}^{(0)}(q_2^+, q_3^-, q_1^+) \right) \text{Tr}(b \sigma_1 \dots) \right\} \\ & \quad \times A_n(\sigma_1, \dots, \sigma_n). \end{aligned} \quad (5.16)$$

However it remains to specify exactly how to take the multi-soft limit and the result depends heavily on the prescription. We could use the general multi-parameter soft limit

$$\lim_{\alpha} = \alpha_{123} \lim_{\delta_3 \rightarrow 0} \lim_{\delta_2 \rightarrow 0} \lim_{\delta_1 \rightarrow 0} + \alpha_{321} \lim_{\delta_1 \rightarrow 0} \lim_{\delta_2 \rightarrow 0} \lim_{\delta_3 \rightarrow 0} \dots \quad (5.17)$$

however it is simplified by symmetrizing the order in which the positive helicity gluons are taken soft i.e. we take  $\alpha_{123} = \alpha_{213}$ ,  $\alpha_{132} = \alpha_{231}$  and  $\alpha_{312} = \alpha_{321}$ . In the above expression one can see two different colour structures: those with  $\text{Tr}(b \sigma_1 \dots)$  and those with  $f^{ba}_c \text{Tr}(a \dots \sigma_r c \dots)$ . Focusing on the former we can show that the terms with coefficient  $\frac{1}{N}$  cancel while the remaining terms give, here only including terms of order  $\frac{1}{z_{13}^2}$  and  $\frac{1}{\bar{z}_{13}^2}$ ,

$$\begin{aligned} N \sum_{\sigma \in S_n} \left\{ \cdots - \left[ 4\alpha_{123} \frac{1}{z_{13}^2} \frac{1}{\bar{z}_3 - \bar{u}_{\sigma_1}} + 2\alpha_{132} \frac{z_{21}}{z_{13}^2 \bar{z}_{21}} \frac{1}{z_3 - u_{\sigma_1}} \right. \right. \\ \left. \left. + 2(2\alpha_{312} + \alpha_{132}) \frac{\bar{z}_{21}}{\bar{z}_{13}^2 z_{21}} \frac{1}{z_3 - u_{\sigma_1}} \right] \text{Tr}([b, \sigma_1] \sigma_2 \dots) + \dots \right\} A_n(\sigma_1 \dots). \end{aligned} \quad (5.18)$$



Using the prescription to drop terms with non-trivial monodromy as  $z_2$  circles  $z_1$  this multi-soft limit can be written as

$$\mathcal{T}(z_1)\bar{\mathcal{J}}^a(z_3) \sim 2\alpha_{123} \frac{\bar{\mathcal{J}}^a(z_3)}{z_{13}^2} + 2\alpha_{123} \frac{\bar{z}_{13}}{z_{13}^2} \bar{\partial} \bar{\mathcal{J}}^a(z_3) + \dots \quad (5.19)$$

where we have here included the local term at the sub-leading  $\frac{\bar{z}_{13}}{z_{13}^2}$  order. We can see that we reproduce at  $\frac{1}{z_{13}^2}$ -order the structure following from the OPE calculation if we make the appropriate choice for the ordering of the consecutive soft-limits. There are additionally terms at order  $\frac{1}{z_{13}}$  which have a bi-local structure and in this case the terms with colour structure  $f^{ba}{}_c \text{Tr}(a \dots \sigma_r c \dots)$  in (5.16) do not vanish but instead also have a bi-local form. We have not carefully matched these terms with those appearing in the OPE.

To compare this result to the one obtained from the simultaneous triple soft limit we use the above formula (5.16) and specifically focus on the third and fourth lines since those are the only singular contributions in the collinear limit. In [37] the simultaneous triple-soft formulae with three adjacent soft gluons of mixed helicities were derived for the following cases

$$\begin{aligned} A_{n+3}(\delta_1 q_1^+, \delta_2 q_2^-, \delta_3 q_3^-, 1, \dots, n) \Big|_{\delta_1 \sim \delta_2 \sim \delta_3 \rightarrow 0} &\rightarrow \mathcal{S}_{n,1}^{+-} A_n \\ A_{n+3}(\delta_1 q_1^+, \delta_2 q_2^-, \delta_3 q_3^+, 1, \dots, n) \Big|_{\delta_1 \sim \delta_2 \sim \delta_3 \rightarrow 0} &\rightarrow \mathcal{S}_{n,1}^{++} A_n. \end{aligned} \quad (5.20)$$

We can obtain the other possible triple-soft terms needed for the different permutations in (5.16) from (5.20) by conjugation, i.e. flipping bra to ket and vice-versa., hence

$$\mathcal{S}_{n,1}^{-+-} = \mathcal{S}_{n,1}^{+-+} |^* \quad \text{and} \quad \mathcal{S}_{n,1}^{-++} = \mathcal{S}_{n,1}^{+--} |^*, \quad (5.21)$$

and, to get the remaining configurations, by exchanging the neighbouring labels  $n$  and  $1$ , i.e.

$$\mathcal{S}_{n,1}^{+-+} = \mathcal{S}_{1,n}^{-++} \quad \text{and} \quad \mathcal{S}_{n,1}^{+--} = \mathcal{S}_{1,n}^{-+-}. \quad (5.22)$$

Using these expressions we again get a cancelation of all the sub-leading color terms of order  $\mathcal{O}(-\frac{1}{N})$  and we see the same structure as in (5.19) but with specific coefficients. That is the triple-soft limit can be written as

$$\mathcal{T}(z_1)\bar{\mathcal{J}}^a(z_3) \sim \frac{2}{9} \frac{\bar{\mathcal{J}}^a(z_3)}{z_{13}^2} + \frac{2}{9} \frac{\bar{z}_{13}}{z_{13}^2} \bar{\partial} \bar{\mathcal{J}}^a(z_3). \quad (5.23)$$

**A conjugation operator.** Due to the non-trivial OPE between  $J^a$  and  $\bar{J}^b$  it is interesting to define the operator

$$C(z_1) = \frac{1}{4\pi i h^\vee} \oint \frac{dz_2}{z_{21}} J^a(z_1) \bar{J}^a(z_2) \quad (5.24)$$

which satisfies the OPE with current  $J^a$

$$C(z_1) J^a(z_3) \sim \left[ d_1^2 \frac{\bar{J}^a(z_3)}{z_{13}^2} + d_1^2 \frac{\bar{z}_{13}}{z_{13}^2} \bar{\partial} \bar{J}^a(z_3) \right] + \dots, \quad (5.25)$$

where the constants  $d_1$  are those appearing in the  $J\bar{J}$ -OPE (1.5) and we see that  $C$  essentially acts as a charge conjugation operator.

Just as for the energy-momentum tensor we can analyze the same OPE by examining the triple soft-limit of two positive helicity gluons and one negative but now  $h_{q_2} = -$  rather than  $h_{q_3}$ . The calculation is identical to that above and the final result can be written as

$$\mathcal{C}(z_1)\mathcal{J}^a(z_3) \sim 2\alpha_{123}\frac{\bar{\mathcal{J}}^a(z_3)}{z_{13}^2} + 2\alpha_{123}\frac{\bar{z}_{13}}{z_{13}^2}\bar{\partial}\bar{\mathcal{J}}^a(z_3) + \dots \quad (5.26)$$

where we again see that with the appropriate choice of parameter in the multi-soft limit that we find the expression calculated directly from the  $J\bar{J}$  OPE.

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