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A proposal on culling & filtering a coxeter group for 4D, $\mathcal{N} = 1$ spacetime SUSY representations: revised

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Abstract: We present an expanded and detailed discussion of the mathematical tools required to cull and filter representations of the Coxeter Group BC_4 into providing bases for the construction of minimal off-shell representations of the 4D, $\mathcal{N} = 1$ spacetime supersymmetry algebra.

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Contents

1 Introduction

This paper is a revised version of a previous work [1] and as such possesses a considerable overlap with this earlier effort. However, this paper is a major revision of [1] which, due to its brevity, left a number of statements implicit or unsaid. We have received a critique of this earlier work in [1] which asserts it did not succeed in giving enough detail so as to provide convincing arguments on the basis of the preponderance of evidence that was presented. The aim of the current work is to "pull back the curtain" on such information and give a much more expansive presentation.

After a presentation at the 2015 Miami Topical Physics Conference and during a question-and-answer session afterward there arose a query from Prof. J. Lukierski about what he perceived as the non-obvious relationship between the Euclidean SO(4) symmetry manifest in the use of adinkras [2] and their adjacency-like matrices [3–5] versus the Lorentzian structure required to describe theories of interest realizing the 4D, $\mathcal{N} = 1$ spacetime supersymmetry algebra.

This exchange motivated us to review [1] tools developed in prior works to show evidence that although adinkras with four colors, four open nodes, and four closed nodes manifestly realize a Euclidean SO(4) symmetry there is a "hidden path" that also relates each of them to a set of SO(1, 3) Dirac γ-matrices.

In the second section, we review results that are standard to the usual Dirac matrices appropriate for a four dimensional Minkowski space. We also show how the generators of SO(4) possess an apparently often overlooked relationship to the Dirac matrices appropriate for a four dimensional Minkowski space. The main purpose of this discussion is to proffer a direct response to the query raised at the Miami Topical Physics Conference. We also discuss the issue of the inherent ambiguities that arise in connecting matrices describing the generators of SO(4) to the SO(1, 3) gamma-matrices appropriate for a four dimensional Minkowski space.

In the third section, there follows a discussion of the L-matrices and R-matrices [3, 4] (obtainable by a modification of the standard notion of an adjacency matrix [5]) associated with every adinkra graph [2]. Forming the absolute values of the entries in the L-matrices and R-matrices leads to the conventional definitions of adjacency matrices for bipartitte graphs. The commutator algebra of the L-matrices and R-matrices define the 'holoraumy' [6–10] associated with the graph. Since this class of adinkra graphs possess an obvious SO(4) symmetry, by exploiting the results in section 3, the spinorial nodes of these adinkra graphs can provide a realization upon which the SO(1, 3) Dirac $\boldsymbol{\gamma}$-matrices can act. By this means it is shown that one of the 'holoraumy' matrices associated with any of these adinkras can be written as a linear combination of matrices in the enveloping algebra of the SO(1, 3) Dirac $\boldsymbol{\gamma}$-matrices. These linear combinations are expressed in terms of a set of parameters denoted by ℓ and $\widetilde{\ell}$ which provide a characterization of each adinkra graph relative to its relation to the SO(1, 3) Dirac $\boldsymbol{\gamma}$-matrices.

In the fourth section, we discuss a starting point of adinkras with four colors, four open nodes and four closed nodes within the context of a Coxeter group because we constructed all possible representations of this kind in a previous analysis [11]. Using a computer software program, it was found one can start with the Coxeter Group BC_4 [12, 13] and associate with every one of the 384 elements of this group to an adinkra with links of a single color. In particular, quartets of elements of BC_4 form representations of the adjacency-like matrices associated with this class of adinkras. This previous work showed there are 96 such quartets possible and built strictly from the elements of a faithful BC_4 representation. By taking the elements of the BC_4 Coxeter Group as our starting point, we have a rigorous mathematically well-defined beginning for our analysis. However, when considering quartets of elements of the Coxeter group we find that "twists" arise and this leads to the appearance of 672 additional adinkras, after "moding out" by adinkras that are related by an overall change of signs of all links.

In the fifth and sixth sections, we introduce two quadratic forms on the space of the ℓ and $\widetilde{\ell}$ parameters. These quadratic forms are then used to define two scalar functions over the space of the parameters and it is noted zeros of one of the quadratic form are directly tied to the value of the other quadratic form. The seventh section consists of a listing of the explicit values of the ℓ and $\widetilde{\ell}$ parameters over the 96 adinkra formed from quartets of BC_4 elements without including twists. The eighth section is a departure from the graph theoretic structure of the work. In this section, we show the three minimal off-shell supermultiplets of 4D $\mathcal{N} = 1$ SUSY possess a set of numerical quantities which can be calculated directly within the context of four dimensional field theory. When these three supermultiplets are considered as elements of a representation space indexed by the symbol $(\widehat{\mathcal{R}})$, it is shown how to define two functions over this representation space.

The ninth section is the one where all the previous results are collected together to describe a procedure by which some of the 768 adinkras can be used to provide a description of the supermultiplets described in section 8. The essential key is that the scalar functions defined on the basis of the ℓ and $\widetilde{\ell}$ parameters (related to the adinkra graphs) can be matched to the two scalar functions define on the basis of the 4D $\mathcal{N}=1$ supermultiplets. This is the realization of concept of "SUSY holography".[1] This presents a criteria by which a subset of the elements of Coxeter Group BC_4 can be consistently interpreted as a projection of the 4D, $\mathcal{N}=1$ fundamental irreducible supermultiplet representations. Thus we also identify obstructions that prevent such identifications for all the elements of BC_4.

There is a final section that includes our summary comments and perspective.

2 Connecting Dirac SO(1, 3) γ-matrices to SO(4) rotation matrices

A set of Dirac γ-matrices is provided by $\boldsymbol{\gamma}^\mu = (\boldsymbol{\gamma}^0, \boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \boldsymbol{\gamma}^3)$ and must satisfy the usual condition

$$\boldsymbol{\gamma}^\mu \boldsymbol{\gamma}^\nu + \boldsymbol{\gamma}^\nu \boldsymbol{\gamma}^\mu = 2\eta^{\mu\nu}\mathbf{I}_4 , \tag{2.1}$$

where the 4×4 identity matrix is denoted by $\mathbf{I}_4$ and the Minkowski metric $\eta^{\mu\nu}$ in (2.1) has non-vanishing diagonal entries $(-1, +1, +1, +1)$.

Given a set of Dirac gamma matrices $\boldsymbol{\gamma}^\mu$, we define $\boldsymbol{\gamma}^5$ via the usual definition

$$\boldsymbol{\gamma}^5 = i\boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 , \tag{2.2}$$

and we also define a representation of the generators of spatial rotations provided by the set containing the three elements $\{\boldsymbol{\sigma}^{12}, \boldsymbol{\sigma}^{23}, \boldsymbol{\sigma}^{31}\}$ where

$$\boldsymbol{\sigma}^{12} = -i\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2 , \qquad \boldsymbol{\sigma}^{23} = -i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 , \qquad \boldsymbol{\sigma}^{31} = -i\boldsymbol{\gamma}^3\boldsymbol{\gamma}^1 . \tag{2.3}$$

The commutator algebra of these takes the usual form

$$[\boldsymbol{\sigma}^{12}, \boldsymbol{\sigma}^{23}] = i2\boldsymbol{\sigma}^{31} , \qquad [\boldsymbol{\sigma}^{23}, \boldsymbol{\sigma}^{31}] = i2\boldsymbol{\sigma}^{12} , \qquad [\boldsymbol{\sigma}^{31}, \boldsymbol{\sigma}^{12}] = i2\boldsymbol{\sigma}^{23} , \tag{2.4}$$

with all other commutators vanishing.

We now introduce another set of matrices containing three elements $\{i\boldsymbol{\gamma}^0, \boldsymbol{\gamma}^5, \boldsymbol{\gamma}^0\boldsymbol{\gamma}^5\}$. The commutator algebra of these elements is

$$[i\boldsymbol{\gamma}^0, \boldsymbol{\gamma}^5] = i2\boldsymbol{\gamma}^0\boldsymbol{\gamma}^5 , \qquad [\boldsymbol{\gamma}^5, \boldsymbol{\gamma}^0\boldsymbol{\gamma}^5] = i2(i\boldsymbol{\gamma}^0) , \qquad [\boldsymbol{\gamma}^0\boldsymbol{\gamma}^5, i\boldsymbol{\gamma}^0] = i2(\boldsymbol{\gamma}^5) , \tag{2.5}$$

with all other commutators vanishing. The form of this commutator algebra (2.5) is identical to the one in (2.4) and both are recognizable as SU(2) algebras. Furthermore, it is easy to show

$$[i\boldsymbol{\gamma}^0, \boldsymbol{\sigma}^{ij}] = 0 , \qquad [\boldsymbol{\gamma}^5, \boldsymbol{\sigma}^{ij}] = 0 , \qquad [\boldsymbol{\gamma}^0\boldsymbol{\gamma}^5, \boldsymbol{\sigma}^{ij}] = 0 . \tag{2.6}$$

[1]This concept was initially simply called "holography" in the work of [14].

This all implies given a set of Dirac $\boldsymbol{\gamma}$-matrices together with a $\boldsymbol{\gamma}^5$-matrix it is possible to construct the six matrices above (2.3) and (2.5), each forming a representation of an SU(2) algebra, and each SU(2) algebra commutes with the other.

A standard result for $\boldsymbol{\gamma}$-matrices implies

$$\boldsymbol{\gamma}^1 = \boldsymbol{\gamma}^0\boldsymbol{\gamma}^5\boldsymbol{\sigma}^{23}, \qquad \boldsymbol{\gamma}^2 = \boldsymbol{\gamma}^0\boldsymbol{\gamma}^5\boldsymbol{\sigma}^{31}, \qquad \boldsymbol{\gamma}^3 = \boldsymbol{\gamma}^0\boldsymbol{\gamma}^5\boldsymbol{\sigma}^{12}, \tag{2.7}$$

which demonstrates that given the data of the two distinct SU(2) matrices, the three spatial $\boldsymbol{\gamma}$-matrices can be reconstructed. Actually, the information in both commuting SU(2)'s is over-complete as it is only the "third" component of the "non-orbital" SU(2) along the three components of the "orbital" SU(2) that are required.

At this point, a different set of 4×4 matrices can be introduced via the definitions

$$\begin{aligned} \boldsymbol{\alpha}^1 &= \boldsymbol{\sigma}^2 \otimes \boldsymbol{\sigma}^1, \qquad & \boldsymbol{\beta}^1 &= \boldsymbol{\sigma}^1 \otimes \boldsymbol{\sigma}^2, \\ \boldsymbol{\alpha}^2 &= \mathbf{I} \otimes \boldsymbol{\sigma}^2, \qquad & \boldsymbol{\beta}^2 &= \boldsymbol{\sigma}^2 \otimes \mathbf{I}, \\ \boldsymbol{\alpha}^3 &= \boldsymbol{\sigma}^2 \otimes \boldsymbol{\sigma}^3, \qquad & \boldsymbol{\beta}^3 &= \boldsymbol{\sigma}^3 \otimes \boldsymbol{\sigma}^2, \end{aligned} \tag{2.8}$$

where these matrices satisfy the identities

$$\begin{aligned} \boldsymbol{\alpha}^{\widehat{I}}\boldsymbol{\alpha}^{\widehat{K}} &= \delta^{\widehat{I}\widehat{K}}\mathbf{I}_4 + i\,\epsilon^{\widehat{I}\widehat{K}\widehat{L}}\boldsymbol{\alpha}^{\widehat{L}}, \\ \boldsymbol{\beta}^{\widehat{I}}\boldsymbol{\beta}^{\widehat{K}} &= \delta^{\widehat{I}\widehat{K}}\mathbf{I}_4 + i\,\epsilon^{\widehat{I}\widehat{K}\widehat{L}}\boldsymbol{\beta}^{\widehat{L}}, \\ [\boldsymbol{\alpha}^{\widehat{I}}, \boldsymbol{\beta}^{\widehat{J}}] &= 0. \end{aligned} \tag{2.9}$$

The commutator algebra derivable from (2.9) allows us to identify the six matrices (2.8) as the hermitian 4×4 matrix generators of SO(4). We also have

$$\begin{aligned} \mathrm{Tr}\big(\boldsymbol{\alpha}^{\widehat{I}}\boldsymbol{\alpha}^{\widehat{J}}\big) &= \mathrm{Tr}\big(\boldsymbol{\beta}^{\widehat{I}}\boldsymbol{\beta}^{\widehat{J}}\big) = 4\,\delta^{\widehat{I}\widehat{J}}, \qquad \mathrm{Tr}\big(\boldsymbol{\alpha}^{\widehat{I}}\boldsymbol{\beta}^{\widehat{J}}\big) = 0, \\ \mathrm{Tr}\big(\boldsymbol{\alpha}^{\widehat{I}}\big) &= \mathrm{Tr}\big(\boldsymbol{\beta}^{\widehat{I}}\big) = 0. \end{aligned} \tag{2.10}$$

However, the commutator algebra defined by (2.4), (2.5), and (2.6) is isomorphic to that which is derivable from (2.9). Hence both are representations of SO(4). Therefore, the Dirac gamma matrices can be expressed using the '$\boldsymbol{\alpha}$-set' and '$\boldsymbol{\beta}$-set'. One such set of definitions are

$$\boldsymbol{\gamma}^0 = i\,\boldsymbol{\beta}^3, \qquad \boldsymbol{\gamma}^1 = \boldsymbol{\alpha}^1\boldsymbol{\beta}^2, \qquad \boldsymbol{\gamma}^2 = \boldsymbol{\alpha}^2\boldsymbol{\beta}^2, \qquad \boldsymbol{\gamma}^3 = \boldsymbol{\alpha}^3\boldsymbol{\beta}^2, \tag{2.11}$$

which imply

$$\boldsymbol{\sigma}^{12} = \boldsymbol{\alpha}^3, \qquad \boldsymbol{\sigma}^{23} = \boldsymbol{\alpha}^1, \qquad \boldsymbol{\sigma}^{31} = \boldsymbol{\alpha}^2. \tag{2.12}$$

These equations make manifest that the $\boldsymbol{\sigma}^{ij}$-matrices are the generators of SU(2). A definition of the $\boldsymbol{\gamma}^5$ matrix following from (2.2) takes the form

$$\boldsymbol{\gamma}^5 = -\,\boldsymbol{\beta}^1, \tag{2.13}$$

together with the definition of $\boldsymbol{\gamma}^0$ shown above implies

$$\boldsymbol{\gamma}^0\boldsymbol{\gamma}^5 = \boldsymbol{\beta}^2. \tag{2.14}$$

It is not commonly noted that given a set of four dimensional Lorentzian gamma matrices, it is possible to use them to define *two mutually commuting* SU(2)'s and this is valid independent of the representation chosen for the gamma matrices.

> *Let us also note the identifications between the $\boldsymbol{\gamma}$-matrices on the one side of the equations (2.11) compared to the $\boldsymbol{\alpha}$-matrices and the $\boldsymbol{\beta}$-matrices on the other are not unique.*

One can cyclically permute, independently, the $\boldsymbol{\alpha}$-matrices and the $\boldsymbol{\beta}$-matrices simultaneously in all of these equations and this still leads to a properly defined set of $\boldsymbol{\gamma}$-matrices. Similarly, one can perform the exchanges of the form $\boldsymbol{\alpha}^{\widehat{I}} \leftrightarrow \boldsymbol{\beta}^{\widehat{I}}$ simultaneously in all of the equations (2.11) and this also leads to a properly defined set of $\boldsymbol{\gamma}$-matrices. Finally, one can introduce $(\pm)$ factors on the r.h.s. of each of the equations in (2.11) and this also leads to proper definitions of a set of four dimensional gamma matrices.

Thus one can ask a question, "what is the group of transformations acting on the $\boldsymbol{\alpha}$-matrices and the $\boldsymbol{\beta}$-matrices in (2.11), (2.13), and (2.14) such that the so defined $\boldsymbol{\gamma}$-matrices satisfy (2.1)?". The answer we find is $(\boldsymbol{Z}_2)^4 \times \boldsymbol{S}_{\boldsymbol{\mathcal{P}}_2} \times (\boldsymbol{S}_{\boldsymbol{\mathcal{P}}_3^+})^2$ and thus the number of ambiguities in making such identifications is $16 \times 2 \times 9 = 288$. The bottom line is there is *not* a unique way to construct a set of Dirac $\boldsymbol{\gamma}$-matrices for a four dimensional Minkowski space from the two commuting SU(2) groups contained in SO(4).

However, the fact that SO(4) can 'secretly' carry information about SO(1, 3) spinors and $\boldsymbol{\gamma}$-matrices is one of the important mechanisms for use of adinkras with four colors to describe 4D, $\mathcal{N} = 1$ SUSY theories that utilize Minkowski space spinors.

3 From L-matrices, R-matrices to Dirac $\boldsymbol{\gamma}$-matrices

In this section, we will show how the L-matrices and R-matrices [3, 4] that occur in the description of any adinkra graph [2, 5] with four colors ($\mathrm{I} = 1, \ldots, 4$) four open nodes ($i = 1, \ldots, 4$), and four closed nodes ($\hat{k} = 1, \ldots, 4$) are related to a set of SO(4) rotation matrices. From the last section, we showed there exist a possibility of linking the $\boldsymbol{\gamma}$-matrices of SO(1, 3) to the SO(4) rotation matrices. Combining these two results, we thus find a pathway that connects all adinkras with four colors, four open nodes, and four closed nodes to the representations of SO(1, 3) $\boldsymbol{\gamma}$-matrices.

Every adinkra representation $(\mathcal{R})$ of this type leads to a set of four adjacency-like matrices denoted by $\mathrm{L}_{\mathrm{I}}^{(\mathcal{R})}$ and $\mathrm{R}_{\mathrm{I}}^{(\mathcal{R})}$ which satisfy the conditions

$$
\begin{aligned}
(\mathrm{L}_{\mathrm{I}}^{(\mathcal{R})})_i{}^{\hat{j}}\,(\mathrm{R}_{\mathrm{J}}^{(\mathcal{R})})_{\hat{j}}{}^{k} \;+\; (\mathrm{L}_{\mathrm{J}}^{(\mathcal{R})})_i{}^{\hat{j}}\,(\mathrm{R}_{\mathrm{I}}^{(\mathcal{R})})_{\hat{j}}{}^{k} &= 2\,\delta_{\mathrm{IJ}}\,\delta_i{}^{k}\,, \\
(\mathrm{R}_{\mathrm{J}}^{(\mathcal{R})})_{\hat{i}}{}^{j}\,(\mathrm{L}_{\mathrm{I}}^{(\mathcal{R})})_{j}{}^{\hat{k}} \;+\; (\mathrm{R}_{\mathrm{I}}^{(\mathcal{R})})_{\hat{i}}{}^{j}\,(\mathrm{L}_{\mathrm{J}}^{(\mathcal{R})})_{j}{}^{\hat{k}} &= 2\,\delta_{\mathrm{IJ}}\,\delta_{\hat{i}}{}^{\hat{k}}\,, \\
(\mathrm{R}_{\mathrm{I}}^{(\mathcal{R})})_{\hat{j}}{}^{k}\,\delta_{ik} &= (\mathrm{L}_{\mathrm{I}}^{(\mathcal{R})})_i{}^{\hat{k}}\,\delta_{\hat{j}\hat{k}}\,.
\end{aligned}
\tag{3.1}
$$

This we call the "Garden Algebra". Given a set of L-matrices and R-matrices for a specified adinkra representation $(\mathcal{R})$, we can define two additional matrix sets denoted by $\boldsymbol{V}_{\mathbf{IJ}}^{(\boldsymbol{\mathcal{R}})}$ and

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$\widetilde{\boldsymbol{V}}_{\mathbf{IJ}}^{(\boldsymbol{\mathcal{R}})}$ [6–10] via the equations

$$
\begin{aligned}
(\mathrm{L}_{\mathrm{I}}^{(\mathcal{R})})_i{}^{\hat{\jmath}}\,(\mathrm{R}_{\mathrm{J}}^{(\mathcal{R})})_{\hat{\jmath}}{}^{k} \;-\; (\mathrm{L}_{\mathrm{J}}^{(\mathcal{R})})_i{}^{\hat{\jmath}}\,(\mathrm{R}_{\mathrm{I}}^{(\mathcal{R})})_{\hat{\jmath}}{}^{k} &= i\,2\,(V_{\mathrm{IJ}}^{(\mathcal{R})})_i{}^{k}\,,\\
(\mathrm{R}_{\mathrm{I}}^{(\mathcal{R})})_{\hat{\imath}}{}^{j}\,(\mathrm{L}_{\mathrm{J}}^{(\mathcal{R})})_{j}{}^{\hat{k}} \;-\; (\mathrm{R}_{\mathrm{J}}^{(\mathcal{R})})_{\hat{\imath}}{}^{j}\,(\mathrm{L}_{\mathrm{I}}^{(\mathcal{R})})_{j}{}^{\hat{k}} &= i\,2\,(\widetilde{V}_{\mathrm{IJ}}^{(\mathcal{R})})_{\hat{\imath}}{}^{\hat{k}}\,.
\end{aligned} \tag{3.2}
$$

We have given the name of "bosonic holoraumy matrices" to the quantities $\boldsymbol{V}_{\mathbf{IJ}}^{(\boldsymbol{\mathcal{R}})}$ and "fermionic holoraumy matrices" to the quantities $\widetilde{\boldsymbol{V}}_{\mathbf{IJ}}^{(\boldsymbol{\mathcal{R}})}$ defined here. Due to the definitions in (3.1), it follows that both sets of holoraumy matrices satisfy the commutator algebra that describes SO(4). Since the $\widetilde{\boldsymbol{V}}_{\mathbf{IJ}}^{(\boldsymbol{\mathcal{R}})}$ matrices act in the spinor space of the adinkras, we concentrate upon it. This means we can write an equation of the form

$$
\widetilde{\boldsymbol{V}}_{\mathrm{IJ}}^{(\mathcal{R})} \;=\; \Big[\,\ell_{\mathrm{IJ}}^{(\mathcal{R})1}\,\boldsymbol{\alpha}^{\mathbf{1}} + \ell_{\mathrm{IJ}}^{(\mathcal{R})2}\,\boldsymbol{\alpha}^{\mathbf{2}} + \ell_{\mathrm{IJ}}^{(\mathcal{R})3}\,\boldsymbol{\alpha}^{\mathbf{3}}\,\Big] \;+\; \Big[\,\widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})1}\,\boldsymbol{\beta}^{\mathbf{1}} + \widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})2}\,\boldsymbol{\beta}^{\mathbf{2}} + \widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})3}\,\boldsymbol{\beta}^{\mathbf{3}}\,\Big]\,, \tag{3.3}
$$

for some set of coefficients $\ell_{\mathrm{IJ}}^{(\mathcal{R})1}$, $\ell_{\mathrm{IJ}}^{(\mathcal{R})2}$, $\ell_{\mathrm{IJ}}^{(\mathcal{R})3}$, $\widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})1}$, $\widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})2}$, and $\widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})3}$. Using the results of the last section, this becomes

$$
\widetilde{\boldsymbol{V}}_{\mathrm{IJ}}^{(\mathcal{R})} \;=\; \Big[\,\ell_{\mathrm{IJ}}^{(\mathcal{R})1}\,\boldsymbol{\Sigma}^{\mathbf{23}} + \ell_{\mathrm{IJ}}^{(\mathcal{R})2}\,\boldsymbol{\Sigma}^{\mathbf{31}} + \ell_{\mathrm{IJ}}^{(\mathcal{R})3}\,\boldsymbol{\Sigma}^{\mathbf{12}}\,\Big] \;+\; \Big[\,-\,\widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})1}\,\boldsymbol{\gamma}^{\mathbf{5}} + \widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})2}\,\boldsymbol{\gamma}^{\mathbf{0}}\boldsymbol{\gamma}^{\mathbf{5}} - i\,\widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})3}\,\boldsymbol{\gamma}^{\mathbf{0}}\,\Big]\,. \tag{3.4}
$$

We have referred to (3.4) in the past [6, 7] as the "Adinkra/$\boldsymbol{\gamma}$-matrix Holography Equation".

The importance of (3.4), when combined with (2.7), is it implies for any four color, four open-node, four-closed node adinkra along with the introduction of the complete specification of two distinct commuting SU(2) algebras, $\{\Sigma^{ij}\}$ and $\{i\,\gamma^0,\,\gamma^5\,,\gamma^0\gamma^5\}$, derivable from adinkras, it is possible to find a set of three spatial γ-matrices and connect to the Lorentz symmetries. The link between any specific adinkra, of the type under consideration, to the representations of the Minkowski space SU(2) algebras, $\{\Sigma^{ij}\}$ and $\{i\,\gamma^0,\,\gamma^5\,,\gamma^0\gamma^5\}$, is specified by the constants $\ell_{\mathrm{IJ}}^{(\mathcal{R})1}$, $\ell_{\mathrm{IJ}}^{(\mathcal{R})2}$, $\ell_{\mathrm{IJ}}^{(\mathcal{R})3}$, $\widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})1}$, $\widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})2}$, and $\widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})3}$.

4 The Coxeter group BC_4 embedding starting point

For our purposes, we can define the elements of BC_4 [12, 13] in the following manner. Consider the set of all real 4×4 matrices that can be formally written as [11]

$$
\mathbf{L} \;=\; \boldsymbol{\mathcal{S}}\cdot\boldsymbol{\mathcal{P}}\,. \tag{4.1}
$$

We call the matrix $\boldsymbol{\mathcal{S}}$ the "Boolean Factor" [11] as it is a real diagonal 4×4 matrix that squares to the identity. The matrix $\boldsymbol{\mathcal{P}}$ is a matrix representation of a permutation of 4 objects. There are $2^d\,d! \,=\, 2^4 \times 4! \,=\, 384$ ways to choose the Boolean Factor *and* the Permutation matrix. This is the dimension of the Coxeter group BC_4.

By embedding the L-matrices as the elements in the entirety of BC_4 we know that for each one we can write the equation

$$
(\mathrm{L}_{\mathrm{I}}{}^{(\mathcal{R})})_i{}^{\hat{k}} \;=\; [\mathcal{S}^{(\mathrm{I})(\mathcal{R})}]_i{}^{\hat{\ell}}\,[\mathcal{P}_{(\mathrm{I})}{}^{(\mathcal{R})}]_{\hat{\ell}}{}^{\hat{k}}\,, \qquad \text{for each fixed} \quad \mathrm{I} = 1,2,3,4 \text{ on the l.h.s.} \tag{4.2}
$$

This notation anticipates that there are distinct adinkra representations denoted by the label $(\mathcal{R})$ and each adinkra leads to four matrices labeled by the index I. In other words,

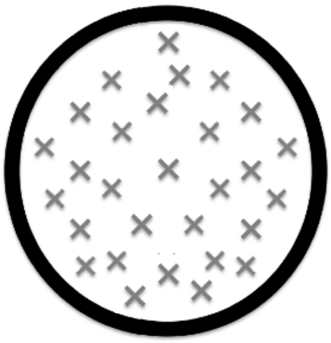

Figure 1. Set of all elements of BC_4.

the L-matrix for a single fixed value of I can be chosen to be any element in the Coxeter group BC_4.

We can illustrate some of these 384 elements in the context of a Venn diagram where each element is represented by the symbol ✕ in the image of figure 1. and where we only include a small sample of all the BC_4 elements.

If we were simply picking quartets of distinct elements of the Coxeter group BC_4 in an arbitrary manner there would be n_4 where

$$n_4 \;=\; \frac{384 \cdot 383 \cdot 382 \cdot 381}{4!} \;=\; 891,881,376 \tag{4.3}$$

ways to pick the quartets. However, we wish to pick the distinct quartet elements of the BC_4 Coexeter Group so that they satisfy the "Garden Algebra". This requirement is so severe there are only 1,536 ways in which four elements of the BC_4 Coexeter Group can be chosen to form a supersymmetry quartet. This was discovered by utilizing a code [11] to exhaustively construct all possible quartets starting from the BC_4 Coexeter Group elements. The label $(\mathcal{R})$ written in (4.2) takes its values over these representations and a more detailed description is given later.

This startlingly smaller number is mostly determined by the permutation elements from which any L-matrix is constructed. It turns out only particular choices of the permutation elements can appear within any given quartet. This is shown in the following collections of sets

$$\begin{aligned}
\{\mathcal{P}_1\} &= \{(243),\ (123),\ (134),\ (142)\} = (123)\,\{\mathcal{V}\}\,,\\
\{\mathcal{P}_2\} &= \{(234),\ (124),\ (132),\ (143)\} = (124)\,\{\mathcal{V}\}\,,\\
\{\mathcal{P}_3\} &= \{(1243),\ (23),\ (14),\ (1342)\} = (14)\,\{\mathcal{V}\}\,,\\
\{\mathcal{P}_4\} &= \{(24),\ (1234),\ (13),\ (1432)\} = (13)\,\{\mathcal{V}\}\,,\\
\{\mathcal{P}_5\} &= \{(34),\ (12),\ (1324),\ (1432)\} = (12)\,\{\mathcal{V}\}\,,\\
\{\mathcal{P}_6\} &= \{(),\ (12)(34),\ (13)(24),\ (14)(23)\} = ()\,\{\mathcal{V}\}\,,
\end{aligned} \tag{4.4}$$

where we use cycle notation to indicate the distinct permutations and relate all the permutations to the Vierergruppe[2] denoted by $\{\mathcal{V}\}$ [15, 16] thus making manifest its critical role.

[2]We thank our colleague K. Iga for this observation.

While the terminology "Vierergruppe" or "Klein Group" is used in the mathematical literature, for physicists this structure is recognizable as $\{\boldsymbol{Z}_2 \times \boldsymbol{Z}_2\}$. This is the group of symmetries of the regular rhombus and is therefore isomorphic to $\boldsymbol{D}_2$ which is described in the 2-plane by the generators $\boldsymbol{\mathcal{I}}$, $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, and $\boldsymbol{\mathcal{AB}}$,

$$\boldsymbol{\mathcal{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{\mathcal{A}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{4.5}$$

$$\boldsymbol{\mathcal{B}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{\mathcal{AB}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{4.6}$$

when expressed as 2×2 matrices.

At first, the six cosets written explicitly in (4.4) appear not to exhaust all possible such cosets. Of the cosets that use 2-cycles, the ones including $(23)\,\{\mathcal{V}\}$, $(24)\,\{\mathcal{V}\}$, and $(34)\,\{\mathcal{V}\}$, do not appear. In a similar manner, of the cosets that use 3-cycles, the ones including $(132)\,\{\mathcal{V}\}$, $(142)\,\{\mathcal{V}\}$, $(134)\,\{\mathcal{V}\}$, $(143)\,\{\mathcal{V}\}$, $(234)\,\{\mathcal{V}\}$, and $(243)\,\{\mathcal{V}\}$, do not appear. The reasons for the missing 2-cycle cosets in the list can be understood since

$$(12)\,\{\mathcal{V}\} = (34)\,\{\mathcal{V}\}\,, \qquad (13)\,\{\mathcal{V}\} = (24)\,\{\mathcal{V}\}\,, \qquad (14)\,\{\mathcal{V}\} = (23)\,\{\mathcal{V}\}\,, \tag{4.7}$$

and similar results hold for the missing 3-cycle cosets from the list since

$$\begin{aligned} &(142)\,\{\mathcal{V}\} = (134)\,\{\mathcal{V}\} = (234)\,\{\mathcal{V}\} = (243)\,\{\mathcal{V}\} = (123)\,\{\mathcal{V}\}\,, \\ &(132)\,\{\mathcal{V}\} = (143)\,\{\mathcal{V}\} = (124)\,\{\mathcal{V}\}\,, \end{aligned} \tag{4.8}$$

are also valid results. Thus, the listing of cosets as given in (4.4) is in actually exhaustive in accounting for the existence of 6 sets with each set containing four elements and all twenty-four element of the permutation group appear within one of the listed cosets.

We collectively express the permutations subsets as $\{\boldsymbol{\mathcal{P}}_{\boldsymbol{\Lambda}}\}$, with the index Λ taking on values 1 thru 6, as cosets involving the Vierergruppe and this allows a partitioning of BC_4 (since it contains S_4) into six distinct subsets as shown in figure 2. All of the 384 elements associated with figure 1. now reside inside 96 quartets which are equally distributed among all six partitions (i.e. 16 quartets per section).

The action of transposition (denoted by the symbol *) on these sets is straightforward to calculate and we find

$$\begin{aligned} &{}^*\{\mathcal{P}_1\} = \{\mathcal{P}_2\}\,, \qquad {}^*\{\mathcal{P}_3\} = \{\mathcal{P}_3\}\,, \qquad {}^*\{\mathcal{P}_5\} = \{\mathcal{P}_5\}\,, \\ &{}^*\{\mathcal{P}_2\} = \{\mathcal{P}_1\}\,, \qquad {}^*\{\mathcal{P}_4\} = \{\mathcal{P}_4\}\,, \qquad {}^*\{\mathcal{P}_6\} = \{\mathcal{P}_6\}\,, \end{aligned} \tag{4.9}$$

and, we define two sets to be equal if they contain the same elements, independent of order. In figure 3 these subsets of permutations together with the action of the $*$ map are shown.

We now turn to the assignments of the Boolean factors to the permutation elements. In order to do this, we first observe there exits 16 sets of Boolean factors that can be assigned to each of the permutation partition factors and faithfully represent BC_4. It suffices to

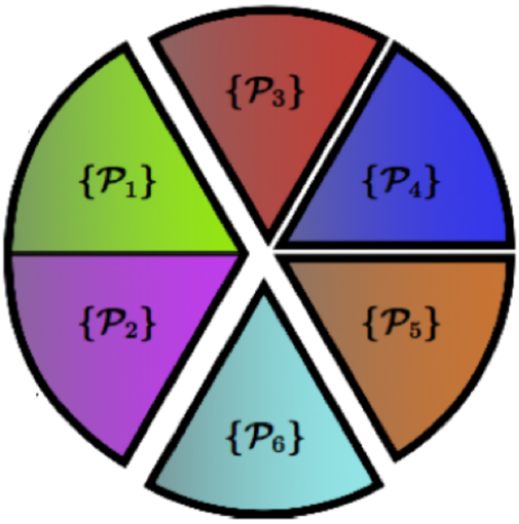

Figure 2. $\{\boldsymbol{Z}_2 \times \boldsymbol{Z}_2\}$ Partitioning of $\boldsymbol{S}_4$.

specify the Boolean factors in the same order as the permutation quartet factors appear in (4.4). Thus for each of the six sectors we find

$$\begin{aligned}
\boldsymbol{S}_{\boldsymbol{\mathcal{P}}1}[\alpha] :& \{(0)_b,(6)_b,(12)_b,(10)_b\},\{(0)_b,(12)_b,(10)_b,(6)_b\},\{(2)_b,(4)_b,(14)_b,(8)_b\},\\
&\{(2)_b,(14)_b,(8)_b,(4)_b\},\{(4)_b,(2)_b,(8)_b,(14)_b\},\{(4)_b,(8)_b,(14)_b,(2)_b\},\\
&\{(6)_b,(0)_b,(10)_b,(12)_b\},\{(6)_b,(10)_b,(12)_b,(0)_b\},\{(8)_b,(4)_b,(2)_b,(14)_b\},\\
&\{(8)_b,(14)_b,(4)_b,(2)_b\},\{(10)_b,(6)_b,(0)_b,(12)_b\},\{(10)_b,(12)_b,(6)_b,(0)_b\},\\
&\{(12)_b,(0)_b,(6)_b,(10)_b\},\{(12)_b,(10)_b,(0)_b,(6)_b\},\{(14)_b,(2)_b,(4)_b,(8)_b\},\\
&\{(14)_b,(8)_b,(2)_b,(4)_b\}\,,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
\boldsymbol{S}_{\boldsymbol{\mathcal{P}}2}[\alpha] :& \{(0)_b,(10)_b,(6)_b,(12)_b\},\{(0)_b,(12)_b,(10)_b,(6)_b\},\{(2)_b,(8)_b,(4)_b,(14)_b\},\\
&\{(2)_b,(14)_b,(8)_b,(4)_b\},\{(4)_b,(8)_b,(14)_b,(2)_b\},\{(4)_b,(14)_b,(2)_b,(8)_b\},\\
&\{(6)_b,(10)_b,(12)_b,(0)_b\},\{(6)_b,(12)_b,(0)_b,(10)_b\},\{(8)_b,(2)_b,(14)_b,(4)_b\},\\
&\{(8)_b,(4)_b,(2)_b,(14)_b\},\{(10)_b,(0)_b,(12)_b,(6)_b\},\{(10)_b,(6)_b,(0)_b,(12)_b\},\\
&\{(12)_b,(0)_b,(6)_b,(10)_b\},\{(12)_b,(6)_b,(10)_b,(0)_b\},\{(14)_b,(2)_b,(4)_b,(8)_b\},\\
&\{(14)_b,(4)_b,(8)_b,(2)_b\}\,,
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\boldsymbol{S}_{\boldsymbol{\mathcal{P}}3}[\alpha] :& \{(0)_b,(6)_b,(10)_b,(12)_b\},\{(0)_b,(12)_b,(6)_b,(10)_b\},\{(2)_b,(4)_b,(8)_b,(14)_b\},\\
&\{(2)_b,(14)_b,(4)_b,(8)_b\},\{(4)_b,(2)_b,(14)_b,(8)_b\},\{(4)_b,(8)_b,(2)_b,(14)_b\},\\
&\{(6)_b,(0)_b,(12)_b,(10)_b\},\{(6)_b,(10)_b,(0)_b,(12)_b\},\{(8)_b,(4)_b,(14)_b,(2)_b\},\\
&\{(8)_b,(14)_b,(2)_b,(4)_b\},\{(10)_b,(6)_b,(12)_b,(0)_b\},\{(10)_b,(12)_b,(0)_b,(6)_b\},\\
&\{(12)_b,(0)_b,(10)_b,(6)_b\},\{(12)_b,(10)_b,(6)_b,(0)_b\},\{(14)_b,(2)_b,(8)_b,(4)_b\},\\
&\{(14)_b,(8)_b,(4)_b,(2)_b\}\,,
\end{aligned} \tag{4.12}$$

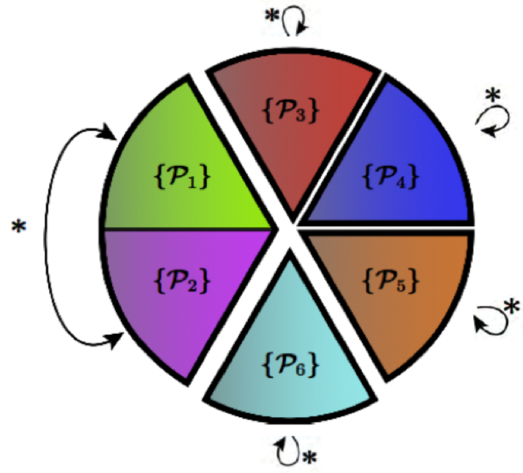

Figure 3. Orbit space of $\mathcal{P}$ permutation matrices under the $*$ map.

$$
\begin{aligned}
\boldsymbol{S}_{\boldsymbol{\mathcal{P}}4}[\alpha] :& \{(0)_b,(10)_b,(12)_b,(6)_b\},\{(0)_b,(12)_b,(6)_b,(10)_b\},\{(2)_b,(8)_b,(14)_b,(4)_b\},\\
&\{(2)_b,(14)_b,(4)_b,(8)_b\},\{(4)_b,(8)_b,(2)_b,(14)_b\},\{(4)_b,(14)_b,(8)_b,(2)_b\},\\
&\{(6)_b,(10)_b,(0)_b,(12)_b\},\{(6)_b,(12)_b,(10)_b,(0)_b\},\{(8)_b,(2)_b,(4)_b,(14)_b\},\\
&\{(8)_b,(4)_b,(14)_b,(2)_b\},\{(10)_b,(0)_b,(6)_b,(12)_b\},\{(10)_b,(6)_b,(12)_b,(0)_b\},\\
&\{(12)_b,(0)_b,(10)_b,(6)_b\},\{(12)_b,(6)_b,(0)_b,(10)_b\},\{(14)_b,(2)_b,(8)_b,(4)_b\},\\
&\{(14)_b,(4)_b,(2)_b,(8)_b\}\,,
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
\boldsymbol{S}_{\boldsymbol{\mathcal{P}}5}[\alpha] :& \{(0)_b,(6)_b,(10)_b,(12)_b\},\{(0)_b,(10)_b,(12)_b,(6)_b\},\{(2)_b,(4)_b,(8)_b,(14)_b\},\\
&\{(2)_b,(8)_b,(14)_b,(4)_b\},\{(4)_b,(2)_b,(14)_b,(8)_b\},\{(4)_b,(14)_b,(8)_b,(2)_b\},\\
&\{(6)_b,(0)_b,(12)_b,(10)_b\},\{(6)_b,(12)_b,(10)_b,(0)_b\},\{(8)_b,(2)_b,(4)_b,(14)_b\},\\
&\{(8)_b,(14)_b,(2)_b,(4)_b\},\{(10)_b,(0)_b,(6)_b,(12)_b\},\{(10)_b,(12)_b,(0)_b,(6)_b\},\\
&\{(12)_b,(6)_b,(0)_b,(10)_b\},\{(12)_b,(10)_b,(6)_b,(0)_b\},\{(14)_b,(4)_b,(2)_b,(8)_b\},\\
&\{(14)_b,(8)_b,(4)_b,(2)_b\}\,,
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
\boldsymbol{S}_{\boldsymbol{\mathcal{P}}6}[\alpha] :& \{(0)_b,(6)_b,(12)_b,(10)_b\},\{(0)_b,(10)_b,(6)_b,(12)_b\},\{(2)_b,(4)_b,(14)_b,(8)_b\},\\
&\{(2)_b,(8)_b,(4)_b,(14)_b\},\{(4)_b,(2)_b,(8)_b,(14)_b\},\{(4)_b,(14)_b,(2)_b,(8)_b\},\\
&\{(6)_b,(0)_b,(10)_b,(12)_b\},\{(6)_b,(12)_b,(0)_b,(10)_b\},\{(8)_b,(2)_b,(14)_b,(4)_b\},\\
&\{(8)_b,(14)_b,(4)_b,(2)_b\},\{(10)_b,(0)_b,(12)_b,(6)_b\},\{(10)_b,(12)_b,(6)_b,(0)_b\},\\
&\{(12)_b,(6)_b,(10)_b,(0)_b\},\{(12)_b,(10)_b,(0)_b,(6)_b\},\{(14)_b,(4)_b,(8)_b,(2)_b\},\\
&\{(14)_b,(8)_b,(2)_b,(4)_b\}\,.
\end{aligned}
\tag{4.15}
$$

The notation is designed to elicit the fact that for each choice of $\boldsymbol{\mathcal{P}}_\Lambda$, there are sixteen possible choices of $\boldsymbol{S}_{\boldsymbol{\mathcal{P}}_\Lambda}[\alpha]$ where the index α enumerates those choices taking on values $1, \ldots 16$.

At this stage, we have distributed all of the elements of BC_4 among the partitions. This, however, does not saturate the number of adinkras that were found by the code enabled search. There are more quartets whose existence is due to "color flips". In order to define "color flips", it is first convenient to define "antonym pairs" of Boolean factors. Given a Boolean factor $(\#)_b$, its antonym is given by $(15 - \#)_b$. In order to illustrate this, a few examples suffice.

Each Boolean factor is equivalent to a real diagonal matrix that squares to the identity. Using the conventions set up in [11]. The Boolean factors $(3)_b$, and $(6)_b$ correspond to respectively to the equations

$$(3)_b = \begin{bmatrix} (-1) & 0 & 0 & 0 \\ 0 & (-1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (6)_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (-1) & 0 & 0 \\ 0 & 0 & (-1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.16}$$

which possess the respective antonyms given by

$$(12)_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (-1) & 0 \\ 0 & 0 & 0 & (-1) \end{bmatrix}, \quad (9)_b = \begin{bmatrix} (-1) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (-1) \end{bmatrix}. \tag{4.17}$$

As we have already described, it takes a quartet of Boolean factors to construct a representation of the Garden Algebra. We now make an observation.

> *If a specified Boolean factor quartet (together with a permutation partition) satisfies the Garden Algebra, then replacing any of the Boolean factors by their antonyms leads to another Boolean factor quartet that satisfies the Garden Algebra.*

Let us illustrate the import of this by examining the Boolean factor quartet $\{(0)_b, (6)_b, (12)_b, (10)_b\}$ and all of its Boolean factor quartet antonyms shown below.

$$\begin{aligned} &\{(0)_b, (6)_b, (12)_b, (10)_b\} : \\ &\quad \{(0)_b, (6)_b, (12)_b, (5)_b\}, \{(0)_b, (6)_b, (3)_b, (5)_b\}, \{(0)_b, (6)_b, (3)_b, (10)_b\}, \\ &\quad \{(0)_b, (9)_b, (12)_b, (5)_b\}, \{(0)_b, (9)_b, (3)_b, (5)_b\}, \{(0)_b, (9)_b, (3)_b, (10)_b\}, \\ &\quad \{(0)_b, (9)_b, (12)_b, (10)_b\}, \{(15)_b, (9)_b, (12)_b, (5)_b\}, \{(15)_b, (9)_b, (3)_b, (5)_b\}, \\ &\quad \{(15)_b, (9)_b, (3)_b, (10)_b\}, \{(15)_b, (9)_b, (12)_b, (10)_b\}, \{(15)_b, (9)_b, (12)_b, (5)_b\}, \\ &\quad \{(15)_b, (6)_b, (3)_b, (5)_b\}, \{(15)_b, (6)_b, (3)_b, (10)_b\}, \{(15)_b, (6)_b, (12)_b, (10)_b\}. \end{aligned} \tag{4.18}$$

On the first line of this expression we have the specified Boolean factor quartet and under this we list all of its Boolean factor quartet antonyms.

For the first listed antonym, only the fourth Boolean factor entry or the "fourth color" was replaced by its antonym. This is what is meant by a single "color flip". For the second listed antonym, the third and fourth Boolean factor entries or the "third color" and "fourth color" were replaced by their antonyms. This is what is meant by "flipping" two colors. For the third listed antonym, only the third Boolean factor entry or the "third color" was replaced by its antonym. This is again a "flipping" of one color.

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Concentrating now once more only on the Boolean factor quartet $\{(0)_b, (6)_b, (12)_b, (10)_b\}$, we can see among the antonyms one is related to it in a special manner. The antonym Boolean factor quartet $\{(15)_b, (9)_b, (3)_b, (5)_b\}$ has all four of its colors flipped with respect to the initial Boolean factor quartet. Thus we call $\{(0)_b, (6)_b, (12)_b, (10)_b\}$ and $\{(15)_b, (9)_b, (3)_b, (5)_b\}$ antipodal Boolean factor quartet pairs. Among the sixteen Boolean factor quartets shown in (4.18) eight such pairs occur. Thus, for each value of α, one must specify which of the antonyms is used to construct and L-matrix. For this purpose, we use an index β_a which takes on eight values.

Given two quartets $(\mathrm{L}_{\mathrm{I}}^{(\mathcal{R})})_i{}^{\hat{\jmath}}$ and $(\mathrm{L}_{\mathrm{I}}^{(\mathcal{R}')})_i{}^{\hat{\jmath}}$ where the elements in the second set are related to the first by replacing at least one Boolean factor by its antonym, there exist a 4 $\times$ 4 Boolean factor matrix denoted by $[\mathcal{A}(\mathcal{R},\, \mathcal{R}')]_{\mathrm{I}}{}^{\mathrm{J}}$ which acts on the color space of the links (i.e. the indices of the I, J, $\ldots$ type) such that

$$(\mathrm{L}_{\mathrm{I}}^{(\mathcal{R})})_i{}^{\hat{\jmath}} \;=\; \big[\mathcal{A}(\mathcal{R},\, \mathcal{R}')\big]_{\mathrm{I}}{}^{\mathrm{J}} (\mathrm{L}_{\mathrm{J}}^{(\mathcal{R}')})_i{}^{\hat{\jmath}}\,, \tag{4.19}$$

and as a consequence of (4.19) we see

$$(\mathrm{R}_{\mathrm{I}}^{(\mathcal{R})})_{\hat{\jmath}}{}^{i} \;=\; \big[\mathcal{A}(\mathcal{R},\, \mathcal{R}')\big]_{\mathrm{I}}{}^{\mathrm{J}} (\mathrm{R}_{\mathrm{J}}^{(\mathcal{R}')})_{\hat{\jmath}}{}^{i}\,. \tag{4.20}$$

It should also be noted that the definition of the antonyms imply that the representations $\mathcal{R}'$ and $\mathcal{R}$ that appear in (4.19) and (4.20) must belong to the same partition sector shown in diagram shown in figure 2. The equations in (3.2), (4.19) and (4.20) imply

$$\begin{aligned} \ell_{\mathrm{IJ}}^{(\mathcal{R})\,\hat{a}} &= \big[\mathcal{A}(\mathcal{R},\, \mathcal{R}')\big]_{\mathrm{I}}{}^{\mathrm{K}}\,\big[\mathcal{A}(\mathcal{R},\, \mathcal{R}')\big]_{\mathrm{J}}{}^{\mathrm{L}}\,\ell_{\mathrm{KL}}^{(\mathcal{R}')\,\hat{a}} \\ \widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})\,\hat{a}} &= \big[\mathcal{A}(\mathcal{R},\, \mathcal{R}')\big]_{\mathrm{I}}{}^{\mathrm{K}}\,\big[\mathcal{A}(\mathcal{R},\, \mathcal{R}')\big]_{\mathrm{J}}{}^{\mathrm{L}}\,\widetilde{\ell}_{\mathrm{KL}}^{(\mathcal{R}')\,\hat{a}}\,. \end{aligned} \tag{4.21}$$

Let us observe the distinction between the Boolean quartet factors that appear in (4.10)–(4.15) and all of their antonyms is not intrinsic, but is an artifact of the choices made to discuss this aspect of the construction. It may be possible to provide a more symmetrical treatment of the (4.10)–(4.15) and all of their antonyms. However, we have not been able to create such a formulation.

Now the meaning of the "representation label", first written in (3.1), can be explicitly discussed. Each value of $\mathcal{R}$ corresponds to a specification of the pairs of indices $(\Lambda,\, \alpha|\beta_a)$. This implies there are 6 $\times$ 16 $\times$ 8 = 6 $\times$ 128 = 768 quartets which satisfy the Garden Algebra conditions. Notice that 1,536/762 = 2 which shows the algorithmic counting did not remove antipodal Boolean factor quartets.

Finally, let us note all discussions in this section are totally disconnected from considerations of four dimensional supersymmetry representations. We have simply enunciated the rich mathematical structure imposed on the Coxeter Group BC_4 when analyzed through the lens of the "Garden Algebra" $\mathcal{GR}$(4,4).

5 Explicit values for ℓ and $\widetilde{\ell}$ coefficients

In this section, for all of the elements of BC_4, the explicit values of the coefficients $\ell_{\mathrm{IJ}}^{(\mathcal{R})1}$, $\ell_{\mathrm{IJ}}^{(\mathcal{R})2}$, $\ell_{\mathrm{IJ}}^{(\mathcal{R})3}$, $\widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})1}$, $\widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})2}$, and $\widetilde{\ell}_{\mathrm{IJ}}^{(\mathcal{R})3}$ which are related to each of the representation $(\mathcal{R})$ in the order shown in

$\boldsymbol{\mathcal{P}}_1:$						χ_o
$\ell_{12}^{(1)2}=1$	$\ell_{13}^{(1)3}=1$	$\ell_{14}^{(1)1}=1$	$\ell_{23}^{(1)1}=1$	$\ell_{24}^{(1)3}=-1$	$\ell_{34}^{(1)2}=1$	1
$\tilde{\ell}_{12}^{(2)3}=1$	$\tilde{\ell}_{13}^{(2)2}=1$	$\tilde{\ell}_{14}^{(2)1}=1$	$\tilde{\ell}_{23}^{(2)1}=-1$	$\tilde{\ell}_{24}^{(2)2}=1$	$\tilde{\ell}_{34}^{(2)3}=-1$	-1
$\ell_{12}^{(3)2}=1$	$\ell_{13}^{(3)3}=1$	$\ell_{14}^{(3)1}=1$	$\ell_{23}^{(3)1}=1$	$\ell_{24}^{(3)3}=-1$	$\ell_{34}^{(3)2}=1$	1
$\tilde{\ell}_{12}^{(4)3}=1$	$\tilde{\ell}_{13}^{(4)2}=1$	$\tilde{\ell}_{14}^{(4)1}=1$	$\tilde{\ell}_{23}^{(4)1}=-1$	$\tilde{\ell}_{24}^{(4)2}=1$	$\tilde{\ell}_{34}^{(4)3}=-1$	-1
$\ell_{12}^{(5)2}=1$	$\ell_{13}^{(5)3}=1$	$\ell_{14}^{(5)1}=1$	$\ell_{23}^{(5)1}=1$	$\ell_{24}^{(5)3}=-1$	$\ell_{34}^{(5)2}=1$	1
$\tilde{\ell}_{12}^{(6)3}=1$	$\tilde{\ell}_{13}^{(6)2}=1$	$\tilde{\ell}_{14}^{(6)1}=1$	$\tilde{\ell}_{23}^{(6)1}=-1$	$\tilde{\ell}_{24}^{(6)2}=1$	$\tilde{\ell}_{34}^{(6)3}=-1$	-1
$\ell_{12}^{(7)2}=1$	$\ell_{13}^{(7)3}=1$	$\ell_{14}^{(7)1}=1$	$\ell_{23}^{(7)1}=1$	$\ell_{24}^{(7)3}=-1$	$\ell_{34}^{(7)2}=1$	1
$\tilde{\ell}_{12}^{(8)3}=1$	$\tilde{\ell}_{13}^{(8)2}=1$	$\tilde{\ell}_{14}^{(8)1}=1$	$\tilde{\ell}_{23}^{(8)1}=-1$	$\tilde{\ell}_{24}^{(8)2}=1$	$\tilde{\ell}_{34}^{(8)3}=-1$	-1
$\tilde{\ell}_{12}^{(9)3}=1$	$\tilde{\ell}_{13}^{(9)2}=1$	$\tilde{\ell}_{14}^{(9)1}=1$	$\tilde{\ell}_{23}^{(9)1}=-1$	$\tilde{\ell}_{24}^{(9)2}=1$	$\tilde{\ell}_{34}^{(9)3}=-1$	-1
$\ell_{12}^{(10)2}=1$	$\ell_{13}^{(10)3}=1$	$\ell_{14}^{(10)1}=1$	$\ell_{23}^{(10)1}=1$	$\ell_{24}^{(10)3}=-1$	$\ell_{34}^{(10)2}=1$	1
$\tilde{\ell}_{12}^{(11)3}=1$	$\tilde{\ell}_{13}^{(11)2}=1$	$\tilde{\ell}_{14}^{(11)1}=1$	$\tilde{\ell}_{23}^{(11)1}=-1$	$\tilde{\ell}_{24}^{(11)2}=1$	$\tilde{\ell}_{34}^{(11)3}=-1$	-1
$\ell_{12}^{(12)2}=1$	$\ell_{13}^{(12)3}=1$	$\ell_{14}^{(12)1}=1$	$\ell_{23}^{(12)1}=1$	$\ell_{24}^{(12)3}=-1$	$\ell_{34}^{(12)2}=1$	1
$\tilde{\ell}_{12}^{(13)3}=1$	$\tilde{\ell}_{13}^{(13)2}=1$	$\tilde{\ell}_{14}^{(13)1}=1$	$\tilde{\ell}_{23}^{(13)1}=-1$	$\tilde{\ell}_{24}^{(13)2}=1$	$\tilde{\ell}_{34}^{(13)3}=-1$	-1
$\ell_{12}^{(14)2}=1$	$\ell_{13}^{(14)3}=1$	$\ell_{14}^{(14)1}=1$	$\ell_{23}^{(14)1}=1$	$\ell_{24}^{(14)3}=-1$	$\ell_{34}^{(14)2}=1$	1
$\tilde{\ell}_{12}^{(15)3}=1$	$\tilde{\ell}_{13}^{(15)2}=1$	$\tilde{\ell}_{14}^{(15)1}=1$	$\tilde{\ell}_{23}^{(15)1}=-1$	$\tilde{\ell}_{24}^{(15)2}=1$	$\tilde{\ell}_{34}^{(15)3}=-1$	-1
$\ell_{12}^{(16)2}=1$	$\ell_{13}^{(16)3}=1$	$\ell_{14}^{(16)1}=1$	$\ell_{23}^{(16)1}=1$	$\ell_{24}^{(16)3}=-1$	$\ell_{34}^{(16)2}=1$	1

$\boldsymbol{\mathcal{P}}_2:$						$\chi_{\rm o}$
$\tilde{\ell}_{12}^{(1)3}=1$	$\tilde{\ell}_{13}^{(1)2}=1$	$\tilde{\ell}_{14}^{(1)1}=1$	$\tilde{\ell}_{23}^{(1)1}=-1$	$\tilde{\ell}_{24}^{(1)2}=1$	$\tilde{\ell}_{34}^{(1)3}=-1$	-1
$\ell_{12}^{(2)2}=1$	$\ell_{13}^{(2)3}=1$	$\ell_{14}^{(2)1}=1$	$\ell_{23}^{(2)1}=1$	$\ell_{24}^{(2)3}=-1$	$\ell_{34}^{(2)2}=1$	1
$\tilde{\ell}_{12}^{(3)3}=1$	$\tilde{\ell}_{13}^{(3)2}=1$	$\tilde{\ell}_{14}^{(3)1}=1$	$\tilde{\ell}_{23}^{(3)1}=-1$	$\tilde{\ell}_{24}^{(3)2}=1$	$\tilde{\ell}_{34}^{(3)3}=-1$	-1
$\ell_{12}^{(4)2}=1$	$\ell_{13}^{(4)3}=1$	$\ell_{14}^{(4)1}=1$	$\ell_{23}^{(4)1}=1$	$\ell_{24}^{(4)3}=-1$	$\ell_{34}^{(4)2}=1$	1
$\ell_{12}^{(5)2}=1$	$\ell_{13}^{(5)3}=1$	$\ell_{14}^{(5)1}=1$	$\ell_{23}^{(5)1}=1$	$\ell_{24}^{(5)3}=-1$	$\ell_{34}^{(5)2}=1$	1
$\tilde{\ell}_{12}^{(6)3}=1$	$\tilde{\ell}_{13}^{(6)2}=1$	$\tilde{\ell}_{14}^{(6)1}=1$	$\tilde{\ell}_{23}^{(6)1}=-1$	$\tilde{\ell}_{24}^{(6)2}=1$	$\tilde{\ell}_{34}^{(6)3}=-1$	-1
$\ell_{12}^{(7)2}=1$	$\ell_{13}^{(7)3}=1$	$\ell_{14}^{(7)1}=1$	$\ell_{23}^{(7)1}=1$	$\ell_{24}^{(7)3}=-1$	$\ell_{34}^{(7)2}=1$	1
$\tilde{\ell}_{12}^{(8)3}=1$	$\tilde{\ell}_{13}^{(8)2}=1$	$\tilde{\ell}_{14}^{(8)1}=1$	$\tilde{\ell}_{23}^{(8)1}=-1$	$\tilde{\ell}_{24}^{(8)2}=1$	$\tilde{\ell}_{34}^{(8)3}=-1$	-1
$\tilde{\ell}_{12}^{(9)3}=1$	$\tilde{\ell}_{13}^{(9)2}=1$	$\tilde{\ell}_{14}^{(9)1}=1$	$\tilde{\ell}_{23}^{(9)1}=-1$	$\tilde{\ell}_{24}^{(9)2}=1$	$\tilde{\ell}_{34}^{(9)3}=-1$	-1
$\ell_{12}^{(10)2}=1$	$\ell_{13}^{(10)3}=1$	$\ell_{14}^{(10)1}=1$	$\ell_{23}^{(10)1}=1$	$\ell_{24}^{(10)3}=-1$	$\ell_{34}^{(10)2}=1$	1
$\tilde{\ell}_{12}^{(11)3}=1$	$\tilde{\ell}_{13}^{(11)2}=1$	$\tilde{\ell}_{14}^{(11)1}=1$	$\tilde{\ell}_{23}^{(11)1}=-1$	$\tilde{\ell}_{24}^{(11)2}=1$	$\tilde{\ell}_{34}^{(11)3}=-1$	-1
$\ell_{12}^{(12)2}=1$	$\ell_{13}^{(12)3}=1$	$\ell_{14}^{(12)1}=1$	$\ell_{23}^{(12)1}=1$	$\ell_{24}^{(12)3}=-1$	$\ell_{34}^{(12)2}=1$	1
$\ell_{12}^{(13)2}=1$	$\ell_{13}^{(13)3}=1$	$\ell_{14}^{(13)1}=1$	$\ell_{23}^{(13)1}=1$	$\ell_{24}^{(13)3}=-1$	$\ell_{34}^{(13)2}=1$	1
$\tilde{\ell}_{12}^{(14)3}=1$	$\tilde{\ell}_{13}^{(14)2}=1$	$\tilde{\ell}_{14}^{(14)1}=1$	$\tilde{\ell}_{23}^{(14)1}=-1$	$\tilde{\ell}_{24}^{(14)2}=1$	$\tilde{\ell}_{34}^{(14)3}=-1$	-1
$\ell_{12}^{(15)2}=1$	$\ell_{13}^{(15)3}=1$	$\ell_{14}^{(15)1}=1$	$\ell_{23}^{(15)1}=1$	$\ell_{24}^{(15)3}=-1$	$\ell_{34}^{(15)2}=1$	1
$\tilde{\ell}_{12}^{(16)3}=1$	$\tilde{\ell}_{13}^{(16)2}=1$	$\tilde{\ell}_{14}^{(16)1}=1$	$\tilde{\ell}_{23}^{(16)1}=-1$	$\tilde{\ell}_{24}^{(16)2}=1$	$\tilde{\ell}_{34}^{(16)3}=-1$	-1

$\boldsymbol{\mathcal{P}}_3$:						χ_o
$\tilde{\ell}_{12}^{(1)3} = -1$	$\tilde{\ell}_{13}^{(1)2} = 1$	$\tilde{\ell}_{14}^{(1)1} = -1$	$\tilde{\ell}_{23}^{(1)1} = 1$	$\tilde{\ell}_{24}^{(1)2} = 1$	$\tilde{\ell}_{34}^{(1)3} = 1$	-1
$\ell_{12}^{(2)2} = -1$	$\ell_{13}^{(2)3} = -1$	$\ell_{14}^{(2)1} = 1$	$\ell_{23}^{(2)1} = 1$	$\ell_{24}^{(2)3} = 1$	$\ell_{34}^{(2)2} = -1$	1
$\tilde{\ell}_{12}^{(3)3} = -1$	$\tilde{\ell}_{13}^{(3)2} = 1$	$\tilde{\ell}_{14}^{(3)1} = -1$	$\tilde{\ell}_{23}^{(3)1} = 1$	$\tilde{\ell}_{24}^{(3)2} = 1$	$\tilde{\ell}_{34}^{(3)3} = 1$	-1
$\ell_{12}^{(4)2} = -1$	$\ell_{13}^{(4)3} = -1$	$\ell_{14}^{(4)1} = 1$	$\ell_{23}^{(4)1} = 1$	$\ell_{24}^{(4)3} = 1$	$\ell_{34}^{(4)2} = -1$	1
$\tilde{\ell}_{12}^{(5)3} = -1$	$\tilde{\ell}_{13}^{(5)2} = 1$	$\tilde{\ell}_{14}^{(5)1} = -1$	$\tilde{\ell}_{23}^{(5)1} = 1$	$\tilde{\ell}_{24}^{(5)2} = 1$	$\tilde{\ell}_{34}^{(5)3} = 1$	-1
$\ell_{12}^{(6)2} = -1$	$\ell_{13}^{(6)3} = -1$	$\ell_{14}^{(6)1} = 1$	$\ell_{23}^{(6)1} = 1$	$\ell_{24}^{(6)3} = 1$	$\ell_{34}^{(6)2} = -1$	1
$\tilde{\ell}_{12}^{(7)3} = -1$	$\tilde{\ell}_{13}^{(7)2} = 1$	$\tilde{\ell}_{14}^{(7)1} = -1$	$\tilde{\ell}_{23}^{(7)1} = 1$	$\tilde{\ell}_{24}^{(7)2} = 1$	$\tilde{\ell}_{34}^{(7)3} = 1$	-1
$\ell_{12}^{(8)2} = -1$	$\ell_{13}^{(8)3} = -1$	$\ell_{14}^{(8)1} = 1$	$\ell_{23}^{(8)1} = 1$	$\ell_{24}^{(8)3} = 1$	$\ell_{34}^{(8)2} = -1$	1
$\ell_{12}^{(9)2} = -1$	$\ell_{13}^{(9)3} = -1$	$\ell_{14}^{(9)1} = 1$	$\ell_{23}^{(9)1} = 1$	$\ell_{24}^{(9)3} = 1$	$\ell_{34}^{(9)2} = -1$	1
$\tilde{\ell}_{12}^{(10)3} = -1$	$\tilde{\ell}_{13}^{(10)2} = 1$	$\tilde{\ell}_{14}^{(10)1} = -1$	$\tilde{\ell}_{23}^{(10)1} = 1$	$\tilde{\ell}_{24}^{(10)2} = 1$	$\tilde{\ell}_{34}^{(10)3} = 1$	-1
$\ell_{12}^{(11)2} = -1$	$\ell_{13}^{(11)3} = -1$	$\ell_{14}^{(11)1} = 1$	$\ell_{23}^{(11)1} = 1$	$\ell_{24}^{(11)3} = 1$	$\ell_{34}^{(11)2} = -1$	1
$\tilde{\ell}_{12}^{(12)3} = -1$	$\tilde{\ell}_{13}^{(12)2} = 1$	$\tilde{\ell}_{14}^{(12)1} = -1$	$\tilde{\ell}_{23}^{(12)1} = 1$	$\tilde{\ell}_{24}^{(12)2} = 1$	$\tilde{\ell}_{34}^{(12)3} = 1$	-1
$\ell_{12}^{(13)2} = -1$	$\ell_{13}^{(13)3} = -1$	$\ell_{14}^{(13)1} = 1$	$\ell_{23}^{(13)1} = 1$	$\ell_{24}^{(13)3} = 1$	$\ell_{34}^{(13)2} = -1$	1
$\tilde{\ell}_{12}^{(14)3} = -1$	$\tilde{\ell}_{13}^{(14)2} = 1$	$\tilde{\ell}_{14}^{(14)1} = -1$	$\tilde{\ell}_{23}^{(14)1} = 1$	$\tilde{\ell}_{24}^{(14)2} = 1$	$\tilde{\ell}_{34}^{(14)3} = 1$	-1
$\ell_{12}^{(15)2} = -1$	$\ell_{13}^{(15)3} = -1$	$\ell_{14}^{(15)1} = 1$	$\ell_{23}^{(15)1} = 1$	$\ell_{24}^{(15)3} = 1$	$\ell_{34}^{(15)2} = -1$	1
$\tilde{\ell}_{12}^{(16)3} = -1$	$\tilde{\ell}_{13}^{(16)2} = 1$	$\tilde{\ell}_{14}^{(16)1} = -1$	$\tilde{\ell}_{23}^{(16)1} = 1$	$\tilde{\ell}_{24}^{(16)2} = 1$	$\tilde{\ell}_{34}^{(16)3} = 1$	-1

$\boldsymbol{\mathcal{P}}_4:$						$\chi_{\rm o}$
$\ell_{12}^{(1)2}=1$	$\ell_{13}^{(1)3}=1$	$\ell_{14}^{(1)1}=1$	$\ell_{23}^{(1)1}=1$	$\ell_{24}^{(1)3}=-1$	$\ell_{34}^{(1)2}=1$	1
$\tilde{\ell}_{12}^{(2)3}=1$	$\tilde{\ell}_{13}^{(2)2}=1$	$\tilde{\ell}_{14}^{(2)1}=1$	$\tilde{\ell}_{23}^{(2)1}=-1$	$\tilde{\ell}_{24}^{(2)2}=1$	$\tilde{\ell}_{34}^{(2)3}=-1$	-1
$\ell_{12}^{(3)2}=1$	$\ell_{13}^{(3)3}=1$	$\ell_{14}^{(3)1}=1$	$\ell_{23}^{(3)1}=1$	$\ell_{24}^{(3)3}=-1$	$\ell_{34}^{(3)2}=1$	1
$\tilde{\ell}_{12}^{(4)3}=1$	$\tilde{\ell}_{13}^{(4)2}=1$	$\tilde{\ell}_{14}^{(4)1}=1$	$\tilde{\ell}_{23}^{(4)1}=-1$	$\tilde{\ell}_{24}^{(4)2}=1$	$\tilde{\ell}_{34}^{(4)3}=-1$	-1
$\tilde{\ell}_{12}^{(5)3}=1$	$\tilde{\ell}_{13}^{(5)2}=1$	$\tilde{\ell}_{14}^{(5)1}=1$	$\tilde{\ell}_{23}^{(5)1}=-1$	$\tilde{\ell}_{24}^{(5)2}=1$	$\tilde{\ell}_{34}^{(5)3}=-1$	-1
$\ell_{12}^{(6)2}=1$	$\ell_{13}^{(6)3}=1$	$\ell_{14}^{(6)1}=1$	$\ell_{23}^{(6)1}=1$	$\ell_{24}^{(6)3}=-1$	$\ell_{34}^{(6)2}=1$	1
$\tilde{\ell}_{12}^{(7)3}=1$	$\tilde{\ell}_{13}^{(7)2}=1$	$\tilde{\ell}_{14}^{(7)1}=1$	$\tilde{\ell}_{23}^{(7)1}=-1$	$\tilde{\ell}_{24}^{(7)2}=1$	$\tilde{\ell}_{34}^{(7)3}=-1$	-1
$\ell_{12}^{(8)2}=1$	$\ell_{13}^{(8)3}=1$	$\ell_{14}^{(8)1}=1$	$\ell_{23}^{(8)1}=1$	$\ell_{24}^{(8)3}=-1$	$\ell_{34}^{(8)2}=1$	1
$\ell_{12}^{(9)2}=1$	$\ell_{13}^{(9)3}=1$	$\ell_{14}^{(9)1}=1$	$\ell_{23}^{(9)1}=1$	$\ell_{24}^{(9)3}=-1$	$\ell_{34}^{(9)2}=1$	1
$\tilde{\ell}_{12}^{(10)3}=1$	$\tilde{\ell}_{13}^{(10)2}=1$	$\tilde{\ell}_{14}^{(10)1}=1$	$\tilde{\ell}_{23}^{(10)1}=-1$	$\tilde{\ell}_{24}^{(10)2}=1$	$\tilde{\ell}_{34}^{(10)3}=-1$	-1
$\ell_{12}^{(11)2}=1$	$\ell_{13}^{(11)3}=1$	$\ell_{14}^{(11)1}=1$	$\ell_{23}^{(11)1}=1$	$\ell_{24}^{(11)3}=-1$	$\ell_{34}^{(11)2}=1$	1
$\tilde{\ell}_{12}^{(12)3}=1$	$\tilde{\ell}_{13}^{(12)2}=1$	$\tilde{\ell}_{14}^{(12)1}=1$	$\tilde{\ell}_{23}^{(12)1}=-1$	$\tilde{\ell}_{24}^{(12)2}=1$	$\tilde{\ell}_{34}^{(12)3}=-1$	-1
$\tilde{\ell}_{12}^{(13)3}=1$	$\tilde{\ell}_{13}^{(13)2}=1$	$\tilde{\ell}_{14}^{(13)1}=1$	$\tilde{\ell}_{23}^{(13)1}=-1$	$\tilde{\ell}_{24}^{(13)2}=1$	$\tilde{\ell}_{34}^{(13)3}=-1$	-1
$\ell_{12}^{(14)2}=1$	$\ell_{13}^{(14)3}=1$	$\ell_{14}^{(14)1}=1$	$\ell_{23}^{(14)1}=1$	$\ell_{24}^{(14)3}=-1$	$\ell_{34}^{(14)2}=1$	1
$\tilde{\ell}_{12}^{(15)3}=1$	$\tilde{\ell}_{13}^{(15)2}=1$	$\tilde{\ell}_{14}^{(15)1}=1$	$\tilde{\ell}_{23}^{(15)1}=-1$	$\tilde{\ell}_{24}^{(15)2}=1$	$\tilde{\ell}_{34}^{(15)3}=-1$	-1
$\ell_{12}^{(16)2}=1$	$\ell_{13}^{(16)3}=1$	$\ell_{14}^{(16)1}=1$	$\ell_{23}^{(16)1}=1$	$\ell_{24}^{(16)3}=-1$	$\ell_{34}^{(16)2}=1$	1

$\boldsymbol{\mathcal{P}}_5$:						$\chi_{\rm o}$
$\ell_{12}^{(1)2}=1$	$\ell_{13}^{(1)3}=1$	$\ell_{14}^{(1)1}=1$	$\ell_{23}^{(1)1}=1$	$\ell_{24}^{(1)3}=-1$	$\ell_{34}^{(1)2}=1$	1
$\tilde{\ell}_{12}^{(2)3}=1$	$\tilde{\ell}_{13}^{(2)2}=1$	$\tilde{\ell}_{14}^{(2)1}=1$	$\tilde{\ell}_{23}^{(2)1}=-1$	$\tilde{\ell}_{24}^{(2)2}=1$	$\tilde{\ell}_{34}^{(2)3}=-1$	-1
$\ell_{12}^{(3)2}=1$	$\ell_{13}^{(3)3}=1$	$\ell_{14}^{(3)1}=1$	$\ell_{23}^{(3)1}=1$	$\ell_{24}^{(3)3}=-1$	$\ell_{34}^{(3)2}=1$	1
$\tilde{\ell}_{12}^{(4)3}=1$	$\tilde{\ell}_{13}^{(4)2}=1$	$\tilde{\ell}_{14}^{(4)1}=1$	$\tilde{\ell}_{23}^{(4)1}=-1$	$\tilde{\ell}_{24}^{(4)2}=1$	$\tilde{\ell}_{34}^{(4)3}=-1$	-1
$\ell_{12}^{(5)2}=1$	$\ell_{13}^{(5)3}=1$	$\ell_{14}^{(5)1}=1$	$\ell_{23}^{(5)1}=1$	$\ell_{24}^{(5)3}=-1$	$\ell_{34}^{(5)2}=1$	1
$\tilde{\ell}_{12}^{(6)3}=1$	$\tilde{\ell}_{13}^{(6)2}=1$	$\tilde{\ell}_{14}^{(6)1}=1$	$\tilde{\ell}_{23}^{(6)1}=-1$	$\tilde{\ell}_{24}^{(6)2}=1$	$\tilde{\ell}_{34}^{(6)3}=-1$	-1
$\ell_{12}^{(7)2}=1$	$\ell_{13}^{(7)3}=1$	$\ell_{14}^{(7)1}=1$	$\ell_{23}^{(7)1}=1$	$\ell_{24}^{(7)3}=-1$	$\ell_{34}^{(7)2}=1$	1
$\tilde{\ell}_{12}^{(8)3}=1$	$\tilde{\ell}_{13}^{(8)2}=1$	$\tilde{\ell}_{14}^{(8)1}=1$	$\tilde{\ell}_{23}^{(8)1}=-1$	$\tilde{\ell}_{24}^{(8)2}=1$	$\tilde{\ell}_{34}^{(8)3}=-1$	-1
$\tilde{\ell}_{12}^{(9)3}=1$	$\tilde{\ell}_{13}^{(9)2}=1$	$\tilde{\ell}_{14}^{(9)1}=1$	$\tilde{\ell}_{23}^{(9)1}=-1$	$\tilde{\ell}_{24}^{(9)2}=1$	$\tilde{\ell}_{34}^{(9)3}=-1$	-1
$\ell_{12}^{(10)2}=1$	$\ell_{13}^{(10)3}=1$	$\ell_{14}^{(10)1}=1$	$\ell_{23}^{(10)1}=1$	$\ell_{24}^{(10)3}=-1$	$\ell_{34}^{(10)2}=1$	1
$\tilde{\ell}_{12}^{(11)3}=1$	$\tilde{\ell}_{13}^{(11)2}=1$	$\tilde{\ell}_{14}^{(11)1}=1$	$\tilde{\ell}_{23}^{(11)1}=-1$	$\tilde{\ell}_{24}^{(11)2}=1$	$\tilde{\ell}_{34}^{(11)3}=-1$	-1
$\ell_{12}^{(12)2}=1$	$\ell_{13}^{(12)3}=1$	$\ell_{14}^{(12)1}=1$	$\ell_{23}^{(12)1}=1$	$\ell_{24}^{(12)3}=-1$	$\ell_{34}^{(12)2}=1$	1
$\tilde{\ell}_{12}^{(13)3}=1$	$\tilde{\ell}_{13}^{(13)2}=1$	$\tilde{\ell}_{14}^{(13)1}=1$	$\tilde{\ell}_{23}^{(13)1}=-1$	$\tilde{\ell}_{24}^{(13)2}=1$	$\tilde{\ell}_{34}^{(13)3}=-1$	-1
$\ell_{12}^{(14)2}=1$	$\ell_{13}^{(14)3}=1$	$\ell_{14}^{(14)1}=1$	$\ell_{23}^{(14)1}=1$	$\ell_{24}^{(14)3}=-1$	$\ell_{34}^{(14)2}=1$	1
$\tilde{\ell}_{12}^{(15)3}=1$	$\tilde{\ell}_{13}^{(15)2}=1$	$\tilde{\ell}_{14}^{(15)1}=1$	$\tilde{\ell}_{23}^{(15)1}=-1$	$\tilde{\ell}_{24}^{(15)2}=1$	$\tilde{\ell}_{34}^{(15)3}=-1$	-1
$\ell_{12}^{(16)2}=1$	$\ell_{13}^{(16)3}=1$	$\ell_{14}^{(16)1}=1$	$\ell_{23}^{(16)1}=1$	$\ell_{24}^{(16)3}=-1$	$\ell_{34}^{(16)2}=1$	1

$\boldsymbol{\mathcal{P}}_6$:						$\chi_{\rm o}$
$\tilde{\ell}_{12}^{(1)3} = 1$	$\tilde{\ell}_{13}^{(1)2} = 1$	$\tilde{\ell}_{14}^{(1)1} = 1$	$\tilde{\ell}_{23}^{(1)1} = -1$	$\tilde{\ell}_{24}^{(1)2} = 1$	$\tilde{\ell}_{34}^{(1)3} = -1$	-1
$\ell_{12}^{(2)2} = 1$	$\ell_{13}^{(2)3} = 1$	$\ell_{14}^{(2)1} = 1$	$\ell_{23}^{(2)1} = 1$	$\ell_{24}^{(2)3} = -1$	$\ell_{34}^{(2)2} = 1$	1
$\tilde{\ell}_{12}^{(3)3} = 1$	$\tilde{\ell}_{13}^{(3)2} = 1$	$\tilde{\ell}_{14}^{(3)1} = 1$	$\tilde{\ell}_{23}^{(3)1} = -1$	$\tilde{\ell}_{24}^{(3)2} = 1$	$\tilde{\ell}_{34}^{(3)3} = -1$	-1
$\ell_{12}^{(4)2} = 1$	$\ell_{13}^{(4)3} = 1$	$\ell_{14}^{(4)1} = 1$	$\ell_{23}^{(4)1} = 1$	$\ell_{24}^{(4)3} = -1$	$\ell_{34}^{(4)2} = 1$	1
$\tilde{\ell}_{12}^{(5)3} = 1$	$\tilde{\ell}_{13}^{(5)2} = 1$	$\tilde{\ell}_{14}^{(5)1} = 1$	$\tilde{\ell}_{23}^{(5)1} = -1$	$\tilde{\ell}_{24}^{(5)2} = 1$	$\tilde{\ell}_{34}^{(5)3} = -1$	-1
$\ell_{12}^{(6)2} = 1$	$\ell_{13}^{(6)3} = 1$	$\ell_{14}^{(6)1} = 1$	$\ell_{23}^{(6)1} = 1$	$\ell_{24}^{(6)3} = -1$	$\ell_{34}^{(6)2} = 1$	1
$\tilde{\ell}_{12}^{(7)3} = 1$	$\tilde{\ell}_{13}^{(7)2} = 1$	$\tilde{\ell}_{14}^{(7)1} = 1$	$\tilde{\ell}_{23}^{(7)1} = -1$	$\tilde{\ell}_{24}^{(7)2} = 1$	$\tilde{\ell}_{34}^{(7)3} = -1$	-1
$\ell_{12}^{(8)2} = 1$	$\ell_{13}^{(8)3} = 1$	$\ell_{14}^{(8)1} = 1$	$\ell_{23}^{(8)1} = 1$	$\ell_{24}^{(8)3} = -1$	$\ell_{34}^{(8)2} = 1$	1
$\ell_{12}^{(9)2} = 1$	$\ell_{13}^{(9)3} = 1$	$\ell_{14}^{(9)1} = 1$	$\ell_{23}^{(9)1} = 1$	$\ell_{24}^{(9)3} = -1$	$\ell_{34}^{(9)2} = 1$	1
$\tilde{\ell}_{12}^{(10)3} = 1$	$\tilde{\ell}_{13}^{(10)2} = 1$	$\tilde{\ell}_{14}^{(10)1} = 1$	$\tilde{\ell}_{23}^{(10)1} = -1$	$\tilde{\ell}_{24}^{(10)2} = 1$	$\tilde{\ell}_{34}^{(10)3} = -1$	-1
$\ell_{12}^{(11)2} = 1$	$\ell_{13}^{(11)3} = 1$	$\ell_{14}^{(11)1} = 1$	$\ell_{23}^{(11)1} = 1$	$\ell_{24}^{(11)3} = -1$	$\ell_{34}^{(11)2} = 1$	1
$\tilde{\ell}_{12}^{(12)3} = 1$	$\tilde{\ell}_{13}^{(12)2} = 1$	$\tilde{\ell}_{14}^{(12)1} = 1$	$\tilde{\ell}_{23}^{(12)1} = -1$	$\tilde{\ell}_{24}^{(12)2} = 1$	$\tilde{\ell}_{34}^{(12)3} = -1$	-1
$\ell_{12}^{(13)2} = 1$	$\ell_{13}^{(13)3} = 1$	$\ell_{14}^{(13)1} = 1$	$\ell_{23}^{(13)1} = 1$	$\ell_{24}^{(13)3} = -1$	$\ell_{34}^{(13)2} = 1$	1
$\tilde{\ell}_{12}^{(14)3} = 1$	$\tilde{\ell}_{13}^{(14)2} = 1$	$\tilde{\ell}_{14}^{(14)1} = 1$	$\tilde{\ell}_{23}^{(14)1} = -1$	$\tilde{\ell}_{24}^{(14)2} = 1$	$\tilde{\ell}_{34}^{(14)3} = -1$	-1
$\ell_{12}^{(15)2} = 1$	$\ell_{13}^{(15)3} = 1$	$\ell_{14}^{(15)1} = 1$	$\ell_{23}^{(15)1} = 1$	$\ell_{24}^{(15)3} = -1$	$\ell_{34}^{(15)2} = 1$	1
$\tilde{\ell}_{12}^{(16)3} = 1$	$\tilde{\ell}_{13}^{(16)2} = 1$	$\tilde{\ell}_{14}^{(16)1} = 1$	$\tilde{\ell}_{23}^{(16)1} = -1$	$\tilde{\ell}_{24}^{(16)2} = 1$	$\tilde{\ell}_{34}^{(16)3} = -1$	-1

On the basis of this list, algorithms and codes were written in order to calculate the values of the two quadratic forms, to be discussed later, on the $\widetilde{\ell}$ and ℓ adinkra parameter spaces. The results of these calculation provide the calculational foundation for the statements made subsequently.

6 A first quadratic form on the Adinkra parameter space

In some of our previous work, we have defined the "chi-oh" function $\chi_o(\boldsymbol{S}_{\boldsymbol{\mathcal{P}}_\Lambda}[\alpha] \cdot \boldsymbol{\mathcal{P}}_\Lambda)$ that maps the three quartets $(\boldsymbol{S}_{\boldsymbol{\mathcal{P}}_1}[1] \cdot \boldsymbol{\mathcal{P}}_1)$, $(\boldsymbol{S}_{\boldsymbol{\mathcal{P}}_2}[1] \cdot \boldsymbol{\mathcal{P}}_2)$, and $(\boldsymbol{S}_{\boldsymbol{\mathcal{P}}_3}[1] \cdot \boldsymbol{\mathcal{P}}_3)$ into $\boldsymbol{Z}_2$ according to

$$\chi_o(\boldsymbol{S}_{\boldsymbol{\mathcal{P}}_1}[1] \cdot \boldsymbol{\mathcal{P}}_1) ~=~ +\,1\,, \qquad \chi_o(\boldsymbol{S}_{\boldsymbol{\mathcal{P}}_2}[1] \cdot \boldsymbol{\mathcal{P}}_2) ~=~ \chi_o(\boldsymbol{S}_{\boldsymbol{\mathcal{P}}_3}[1] \cdot \boldsymbol{\mathcal{P}}_3) ~=~ -\,1\,. \tag{6.1}$$

Using the same definition, we can now extend the range of the "chi-oh" function throughout the entirety of BC_4. Expressed in terms of the ℓ and $\widetilde{\ell}$ parameters in (3.3) we have

$$\chi_o(\boldsymbol{S}_{\boldsymbol{\mathcal{P}}_\Lambda}[\alpha] \cdot \boldsymbol{\mathcal{P}}_\Lambda) ~=~ \frac{1}{24}\,\epsilon^{I\,J\,K\,L} \sum_{\rm I,J,K,L,\hat{a}} \Big[\, \ell_{\rm IJ}^{(\alpha)\hat{a}}\,\ell_{\rm KL}^{(\alpha)\hat{a}} ~+~ \widetilde{\ell}_{\rm IJ}^{(\alpha)\hat{a}}\,\widetilde{\ell}_{\rm JL}^{(\alpha)\hat{a}} \,\Big]\,, \tag{6.2}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	+ 1	-1	+1	-1	+1	-1	+1	-1	-1	+1	-1	+1	1-	+1	-1	+1
2	- 1	+1	-1	+1	+1	-1	+1	-1	-1	+1	-1	+1	+1	-1	+1	-1
3	- 1	+1	-1	+1	-1	+1	-1	+1	+1	-1	+1	-1	+1	-1	+1	-1
4	+1	-1	+1	-1	-1	+1	-1	+1	+1	-1	+1	-1	-1	+1	-1	+1
5	+1	-1	+1	-1	+1	-1	+1	-1	-1	+1	-1	+1	-1	+1	-1	+1
6	-1	+1	-1	+1	-1	+1	-1	+1	+1	-1	+1	-1	+1	-1	+1	-1

Table 1. Values of $\chi_o(\boldsymbol{S}_{\boldsymbol{\mathcal{P}}_\Lambda}[\alpha] \cdot \boldsymbol{\mathcal{P}}_\Lambda)$ in BC_4.

and we show the values of this function in table 1 where the numbers in the first row correspond to the values of α and the numbers in the first column correspond to Λ. We note the identities

$$\sum_\Lambda \chi_o(\boldsymbol{S}_{\boldsymbol{\mathcal{P}}_\Lambda}[\alpha] \cdot \boldsymbol{\mathcal{P}}_\Lambda) \ = \ \sum_\alpha \chi_o(\boldsymbol{S}_{\boldsymbol{\mathcal{P}}_\Lambda}[\alpha] \cdot \boldsymbol{\mathcal{P}}_\Lambda) \ = \ 0\,, \tag{6.3}$$

so that half the elements of BC_4 are mapped into $+1$ and half the elements of BC_4 are mapped into -1.

One result that follows from the emergence of these two distinct classes defined by the values of $\chi_o(\mathcal{R})$ can be combined with the discussion given in section 2 to make an important observation. If one begins solely with the adinkra's holoraumy matrices, it is necessary to have at least two sets of four-color adinkras with opposite values of $\chi_o(\mathcal{R})$ in order to reconstruct a set of $\boldsymbol{\gamma}$-matrices from the L-matrices and R-matrices associated with the adinkras.

Finally, we note that if $(\,\mathrm{L}_{\rm I}^{(\mathcal{R})}\,)_i{}^{\hat{\jmath}}$ and $(\,\mathrm{R}_{\rm I}^{(\mathcal{R})}\,)_{\hat{\jmath}}{}^i$ for representation $\mathcal{R}$ are related to $(\,\mathrm{L}_{\rm I}^{(\mathcal{R}')}\,)_i{}^{\hat{\jmath}}$ and $(\,\mathrm{R}_{\rm I}^{(\mathcal{R}')}\,)_{\hat{\jmath}}{}^i$ for representation $\mathcal{R}'$ as shown in (4.19) and (4.20), it follows that

$$\chi_o(\mathcal{R}) \ = \ \det\big|\big[\mathcal{A}(\mathcal{R},\,\mathcal{R}')\big]\big|\,\chi_o(\mathcal{R}')\,, \tag{6.4}$$

where $\det|[\mathcal{A}(\mathcal{R},\,\mathcal{R}')]|$ denotes the determinant of the matrix $[\mathcal{A}(\mathcal{R},\,\mathcal{R}')]_{\rm I}{}^{\rm J}$ and can only take on the values of ± 1.

7 A second quadratic form on the Adinkra parameter space

Defining a dot product in the parameters space as

$$\mathcal{G}\left[(\mathcal{R})\,,\,(\mathcal{R}')\right]_\ell \ \equiv \ \frac{1}{12}\sum_{\rm I,J,\hat{a}} \Big[\ \ell_{\rm IJ}^{(\mathcal{R})\hat{a}}\,\ell_{\rm IJ}^{(\mathcal{R}')\hat{a}} \ + \ \widetilde{\ell}_{\rm IJ}^{(\mathcal{R})\hat{a}}\,\widetilde{\ell}_{\rm IJ}^{(\mathcal{R}')\hat{a}}\ \Big]\,, \tag{7.1}$$

the 'angles' (denoted by $\theta[(\mathcal{R})\,,\,(\mathcal{R}')]_\ell$) between two representations $(\mathcal{R})$, and $(\mathcal{R}')$ specified by their values of ℓ and $\widetilde{\ell}$ in the 72-dimensional parameter space of (3.3) and (3.4), can be defined from this inner product in the usual way:

$$\cos\left\{\theta[(\mathcal{R})\,,\,(\mathcal{R}')]_\ell\right\} \ = \ \frac{\mathcal{G}[\,(\mathcal{R}),(\mathcal{R}')\,]_\ell}{\sqrt{\mathcal{G}[\,(\mathcal{R}),(\mathcal{R})\,]_\ell}\,\sqrt{\mathcal{G}[\,(\mathcal{R}'),(\mathcal{R}')\,]_\ell}}\,. \tag{7.2}$$

Here we discuss this dot products between each of the 16 representations within $\{\mathcal{P}_1\}$. We use these explicitly derived results to motivate their extension throughout all the sections $\{\mathcal{P}_1\}, \ldots, \{\mathcal{P}_6\}$.

For $\{\mathcal{P}_1\}$, these form two sets of mutually "orthogonal" representations:

$$
\begin{aligned}
&\mathcal{G}\left[(\mathcal{R})\,,\,(\mathcal{R}')\right]_{\,\ell} = 1 \qquad \text{for} \qquad \chi_{\rm o}(\mathcal{R}) \;=\; \chi_{\rm o}(\mathcal{R}') \\
&\mathcal{G}\left[(\mathcal{R})\,,\,(\mathcal{R}')\right]_{\,\ell} = 0 \qquad \text{for} \qquad \chi_{\rm o}(\mathcal{R}) \;=\; -\,\chi_{\rm o}(\mathcal{R}')\,.
\end{aligned}
\tag{7.3}
$$

These equations are very interesting from one perspective.

The definition of $\chi_{\rm o}$ implies that it is measuring a property of each individual representation without regard to any other representation. On the other hand, when $(\mathcal{R}) \neq (\mathcal{R}')$, the quantity $\mathcal{G}\,[(\mathcal{R})\,,\,(\mathcal{R}')]_{\,\ell}$ is measuring a property of the $(\mathcal{R})$ representation relative to the $(\mathcal{R}')$ representation. It is useful to make analogy here.

If we think about the elements of the adinkra space described by the $\widetilde{\ell}$ and ℓ as vectors and the quantity $\mathcal{G}\,[(\mathcal{R})\,,\,(\mathcal{R}')]_{\,\ell}$ as defining an inner product, then when $\mathcal{G}\,[(\mathcal{R})\,,\,(\mathcal{R}')]_{\,\ell} = 1$ but $(\mathcal{R}) \neq (\mathcal{R}')$, we would conclude that the two vectors associated with $(\mathcal{R})$ $(\mathcal{R}')$ are collinear and in fact the same.

Once more the cases when $({\rm L}_{\rm I}^{(\mathcal{R})})_{i}{}^{\hat{\jmath}}$ and $({\rm R}_{\rm I}^{(\mathcal{R})})_{\hat{\jmath}}{}^{i}$ are associated with representation $\mathcal{R}$ and $({\rm L}_{\rm I}^{(\mathcal{R}')})_{i}{}^{\hat{\jmath}}$ and $({\rm R}_{\rm I}^{(\mathcal{R}')})_{\hat{\jmath}}{}^{i}$ associated with representation $\mathcal{R}'$, subject to the relationships shown in (4.19) and (4.20) has a number of implications.

We are now in position to present a surprising theorem.

Let $(\mathcal{R})$ denote an adinkra representation constructed from elements of BC_4, as listed in section 4. Let $(\mathcal{R}')$ denote an adinkra representation constructed from the listed ones involving BC_4, but with the difference that $(\mathcal{R}')$ is constructed from $(\mathcal{R})$ by use of antonyms but excluding anti-podal copies. Under these conditions the only possible values of $\mathcal{G}\,[(\mathcal{R})\,,\,(\mathcal{R}')]_{\,\ell}$ are given by $-1/3$, 0, and 1. We will now give a proof by construction.

Using the definition of the gadget we recall

$$
\mathcal{G}\left[(\mathcal{R})\,,\,(\mathcal{R}')\right]_{\,\ell} \;\equiv\; \frac{1}{12}\sum_{\rm I,J,\hat{a}} \left[\; \ell_{\rm IJ}^{(\mathcal{R})\hat{a}}\,\ell_{\rm IJ}^{(\mathcal{R}')\hat{a}} \;+\; \widetilde{\ell}_{\rm IJ}^{(\mathcal{R})\hat{a}}\,\widetilde{\ell}_{\rm IJ}^{(\mathcal{R}')\hat{a}} \;\right]\,,
\tag{7.4}
$$

where the parameters $\ell_{\rm IJ}^{(\mathcal{R})\hat{a}}$, $\widetilde{\ell}_{\rm IJ}^{(\mathcal{R})\hat{a}}$, $\ell_{\rm IJ}^{(\mathcal{R}')\hat{a}}$, $\widetilde{\ell}_{\rm IJ}^{(\mathcal{R}')\hat{a}}$ are defined via equations of the form given in (3.3). We emphasize for adinkras constructed from BC_4, when the ℓ's are non-zero, the $\widetilde{\ell}$ vanish and vice-versa.

Furthermore, if the L-matrices and R-matrices associated with the two representations $(\mathcal{R})$ and $(\mathcal{R}')$ satisfy the conditions in (4.19) and (4.20) then

$$
\mathcal{G}[(\mathcal{R}),(\mathcal{R}')]_{\ell} \;\equiv\; \frac{1}{12}\sum_{\rm I,J,K,L,\hat{a}} [\mathcal{A}(\mathcal{R},\mathcal{R}')]_{\rm I}{}^{\rm K}[\mathcal{A}(\mathcal{R},\mathcal{R}')]_{\rm J}{}^{\rm L}\Big[\; \ell_{\rm KL}^{(\mathcal{R}')\hat{a}}\ell_{\rm IJ}^{(\mathcal{R}')\hat{a}} \;+\; \widetilde{\ell}_{\rm IJ}^{(\mathcal{R}')\hat{a}}\widetilde{\ell}_{\rm KL}^{(\mathcal{R}')\hat{a}} \;\Big].
\tag{7.5}
$$

Let us rewrite (7.4) more explicitly to find

$$
\begin{aligned}
\mathcal{G}\left[(\mathcal{R})\,,\,(\mathcal{R}')\right]_{\,\ell} \;\equiv\; \frac{1}{6}\sum_{\hat{a}}\Big[\; &\ell_{12}^{(\mathcal{R})\hat{a}}\,\ell_{12}^{(\mathcal{R}')\hat{a}} \;+\; \ell_{13}^{(\mathcal{R})\hat{a}}\,\ell_{13}^{(\mathcal{R}')\hat{a}} \;+\; \ell_{14}^{(\mathcal{R})\hat{a}}\,\ell_{14}^{(\mathcal{R}')\hat{a}} \\
&+\; \ell_{23}^{(\mathcal{R}')\hat{a}}\ell_{23}^{(\mathcal{R}')\hat{a}} \;+\; \ell_{24}^{(\mathcal{R})\hat{a}}\,\ell_{24}^{(\mathcal{R}')\hat{a}} \;+\; \ell_{34}^{(\mathcal{R})\hat{a}}\,\ell_{34}^{(\mathcal{R}')\hat{a}} \\
&+\; \widetilde{\ell}_{12}^{(\mathcal{R})\hat{a}}\,\widetilde{\ell}_{12}^{(\mathcal{R}')\hat{a}} \;+\; \widetilde{\ell}_{13}^{(\mathcal{R})\hat{a}}\,\widetilde{\ell}_{13}^{(\mathcal{R}')\hat{a}} \;+\; \widetilde{\ell}_{14}^{(\mathcal{R})\hat{a}}\,\widetilde{\ell}_{14}^{(\mathcal{R}')\hat{a}} \\
&+\; \widetilde{\ell}_{23}^{(\mathcal{R}')\hat{a}}\widetilde{\ell}_{23}^{(\mathcal{R}')\hat{a}} \;+\; \widetilde{\ell}_{24}^{(\mathcal{R})\hat{a}}\,\widetilde{\ell}_{24}^{(\mathcal{R}')\hat{a}} \;+\; \widetilde{\ell}_{34}^{(\mathcal{R})\hat{a}}\,\widetilde{\ell}_{34}^{(\mathcal{R}')\hat{a}} \;\Big]\,.
\end{aligned}
\tag{7.6}
$$

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$\widehat{\mathcal{R}}$	Φ	G	W_a
$\widehat{\chi}_{\rm o}(\widehat{\mathcal{R}})$	$+1$	-1	-1

Table 2. 4D Superfield Values of $\widehat{\chi}_{\rm o}(\widehat{\mathcal{R}})$.

Color flipping does not change the orthogonality between the distinct set of representations listed in section 4. However, when the dot products between any two pair of those representations are non-zero, color flipping of one with respect to the other may change the value of the dot product calculated. With a small amount of effort, we observed the changes occur as follows: within each set, flipping an odd number of colors for one representation in the dot product relative to the other representation used in the dot product implies the representations are orthogonal. Flipping two colors, the dot product becomes $-1/3$. Flipping four colors, the dot product maintains the value of 1. This is because flipping all four colors within a representation leads to no change with respect to the ℓ's and $\widetilde{\ell}$'s. Additional deliberation shows this same behavior extends throughout all six of the sectors $\{\mathcal{P}_1\}, \dots, \{\mathcal{P}_6\}$ together with their antonym extensions.

Finally, flipping an odd number of colors, as seen via the result in (6.4), implies that even for the antonym extended quartets, the condition $\chi_{\rm o}(\mathcal{R}) = -\chi_{\rm o}(\mathcal{R}')$ implies orthogonality with respect to the gadget.

8 4D supermultiplet numbers

As shown in [17], a parameter $\widehat{\chi}_{\rm o}$, that appears in the following definition

$$[\, 1 \,+\, \widehat{\chi}_{\rm o} \,] \Box \;=\; -\, \frac{1}{4} [\mathrm{I} + \gamma^5]^{a\,b} [\mathrm{I} - \gamma^5]^{c\,d} \left[\, D_a \, D_c \, D_d D_b \,\right] , \tag{8.1}$$

can be used on the chiral, vector and tensor supermultiplets in 4D by evaluation on the spinor components given respectively by

$$\psi_a \;\equiv\; D_a \, \Phi \, \big| \, , \qquad \chi_a \;\equiv\; D_a \, G \, \big| \, , \qquad \lambda_a \;\equiv\; W_a \, \big| \, , \tag{8.2}$$

and where Φ, G, and W_a are the usual superfields that describe the 4D chiral, tensor, and vector supermultiplets. Direct calculations yield the results shown in table 2. We have asserted that the 4D quantity $\widehat{\chi}_{\rm o}(\widehat{\mathcal{R}})$ is the analog of the first quadratic form defined on adinkras.

In the work of [10], it was shown that within 4D supersymmetrical theories involving the supermultiplets described by Φ, G, and W_a it is possible to define an analog of the second quadratic form that exists for adinkras. In order to define this analog, one must first define 4D holoraumy tensors denoted by $\left[\boldsymbol{H}^{\mu(\widehat{\mathcal{R}})} \right]_{a\,b\,c}{}^{d}$ over the superfield representations $\widehat{\mathcal{R}} = (\Phi,\, G,\, W_a)$. Once more direct calculations yield respectively

$$\begin{aligned} \mathrm{D}_a, \mathrm{D}_b t \, \psi_c &= -i\, (\gamma^5 \gamma^\nu)_{ab} (\gamma^5 [\gamma_\nu, \gamma^\mu])_c{}^d \partial_\mu \psi_d \\ &\equiv \left[\boldsymbol{H}^{\mu(CS)} \right]_{a\,b\,c}{}^{d} \, (\partial_\mu \psi_d\,) \;, \end{aligned} \tag{8.3}$$

$$
\begin{aligned}
\mathrm{D}_a, \mathrm{D}_b t\, \chi_c = {} & i2\, C_{ab}(\gamma^\mu)_c{}^d \partial_\mu \chi_d \;-\; i2\, (\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d \partial_\mu \chi_d \\
& + i2\, (\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d \partial_\mu \chi_d \\
\equiv {} & \left[\boldsymbol{H}^{\mu(TS)} \right]_{a\,b\,c}{}^d \; (\partial_\mu \chi_d\,)\;.
\end{aligned} \tag{8.4}
$$

$$
\begin{aligned}
\mathrm{D}_a, \mathrm{D}_b t\, \lambda_c = {} & -i2\, C_{ab}(\gamma^\mu)_c{}^d \partial_\mu \lambda_d \;-\; i2\, (\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d \partial_\mu \lambda_d \\
& - i2\, (\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d \partial_\mu \lambda_d \\
\equiv {} & \left[\boldsymbol{H}^{\mu(VS)} \right]_{a\,b\,c}{}^d \; (\partial_\mu \lambda_d\,)\;,
\end{aligned} \tag{8.5}
$$

A metric on the representation space of these supermultiplets can be defined by

$$
\begin{aligned}
\widehat{\mathcal{G}}[(\widehat{\mathcal{R}}), (\widehat{\mathcal{R}}')] \;=\; -\frac{1}{768}\Bigg\{ & \; [\boldsymbol{H}^{\mu(\widehat{\mathcal{R}})}]_{a\,b\,c}{}^d\, [\boldsymbol{H}_\mu{}^{(\widehat{\mathcal{R}}')}]^{a\,b}{}_d{}^c \\
& - \frac{1}{2}\, (\gamma^\alpha)_c{}^e\, [\boldsymbol{H}^{\mu(\widehat{\mathcal{R}})}]_{a\,b\,e}{}^f\, (\gamma_\alpha)_f{}^d\, [\boldsymbol{H}_\mu{}^{(\widehat{\mathcal{R}}')}]^{a\,b}{}_d{}^c \\
& - \frac{1}{2}\, (\gamma^5\gamma^\alpha)_c{}^e\, [\boldsymbol{H}^{\mu(\widehat{\mathcal{R}})}]_{a\,b\,e}{}^f\, (\gamma^5\gamma_\alpha)_f{}^d\, [\boldsymbol{H}_\mu{}^{(\widehat{\mathcal{R}}')}]^{a\,b}{}_d{}^c \Bigg\}\,,
\end{aligned} \tag{8.6}
$$

and this readily leads to a definition of the angles between the 4D supermultiplet representations

$$
\cos\left\{ \theta[(\widehat{\mathcal{R}})\,,\, (\widehat{\mathcal{R}}')] \right\} \;=\; \frac{\widehat{\mathcal{G}}[\,(\widehat{\mathcal{R}}), (\widehat{\mathcal{R}}')\,]}{\sqrt{\widehat{\mathcal{G}}[\,(\widehat{\mathcal{R}}), (\widehat{\mathcal{R}})\,]}\,\sqrt{\widehat{\mathcal{G}}[\,(\widehat{\mathcal{R}}'), (\widehat{\mathcal{R}}')]}}\;. \tag{8.7}
$$

When we display these angles in the form of a matrix where the rows and columns each take on the respective values Φ, G, and W_a we find

$$
\theta[(\widehat{\mathcal{R}})\,,\, (\widehat{\mathcal{R}}')] \;=\; \begin{bmatrix} 0 & \frac{\pi}{2} & \frac{\pi}{2} \\ \frac{\pi}{2} & 0 & \cos^{-1}\left(\frac{-1}{3}\right) \\ \frac{\pi}{2} & \cos^{-1}\left(\frac{-1}{3}\right) & 0 \end{bmatrix}\;. \tag{8.8}
$$

9 Culling & filtering

In a previous work [18], there was presented an obstruction that indicated when an adinkra with two colors was compatible with being the projection of a two dimensional supermultiplet. It was shown there exist what we may call the "no two-color ambidextrous bow tie" theorem which asserted if an adinkra graph contained a certain structure, then it was not possible to consistently "lift" the adinkra graph in such a way that it could be associated with a supermultiplet defined on a Minkowski space with a Lorentzian metric with diagonal entries of the $(-1, 1)$ variety.

Up until now we have made no similar comments about when an arbitrary four-color adinkra can be regarded as being associated with the projection of a supermultiplet defined on a Minkowski space with a Lorentzian metric with diagonal entries of the $(-1, 1, 1, 1)$ variety.

Due to our analysis of BC_4, we now have enough hints so as be comfortable laying out a set of analogous requirements for all adinkras based on BC_4 elements that correspond to valise adinkras. We are now ready to describe how a subset of the elements of BC_4 can be used to construct the off-shell minimal representations of 4D, $\mathcal{N}=1$ SUSY.

This section will depend crucially on some conjectures for which we do not have closed-form explicit mathematical proofs.

Conjecture 1. *Given an arbitrary element in BC_4 it is always possible to find three additional distinct elements so that this quartet of distinct elements satisfies the "Garden Algebra" with four colors.*

Conjecture 2. *Given an a quartet of elements in BC_4 that satisfies the "Garden Algebra" with four colors, their holoramy tensors always takes the form given in (3.3) and (3.4) with either all of the ℓ-coefficients equal to zero or all of the $\tilde{\ell}$-coefficients equal to zero.*

We do not have a closed-form explicit analytical mathematical proof of either. For the first conjecture, the explicit construction in the work of [11] gives us confidence in its validity. The code described therein constitutes a proof by exhaustive examination. For the second conjecture in this current work we *now* once more created algorithms upon which to make the claim throughout the elements of BC_4.

Under the two conjectures, the process of culling and filtering of the elements of BC_4 to consistently describe 4D, $\mathcal{N}=1$ spacetime supermultiplets only requires the application of the tools of the $*$-map and the holoraumy tensor $\widetilde{\boldsymbol{V}}_{\rm IJ}^{(\mathcal{R})}$.

One can pick an arbitrary element of BC_4 and examine how it behaves with respect to the $*$-map acting on the permutation upon which the element is constructed. The permutation associated with the element will be in one of the "even" sets ($\{\mathcal{P}_3\}$ thru $\{\mathcal{P}_6\}$) or one of the "odd" sets ($\{\mathcal{P}_1\}$ or $\{\mathcal{P}_2\}$). If the permutation associated with the element is in one of the "even" sets, we next calculate the holoraumy associated with it. Let us call the element our base element. By conjecture 2 this must take to the form of (3.3) with half of the coefficients vanishing.

Now there comes a subtlety. In going from (3.3) to (3.4) there is an ambiguity. To go from the former to the latter required the identifications made in (2.11) and (2.12). However, as we discussed below the latter equations, there is always an inherent ambiguity as identified in the discussion above (2.7). So using this ambiguity we can simply declare that whatever explicit matrices emerge from the holoraumy associated with this base element are associated with the orbital SU(2).

To the skeptical reader on this point, we should also note this also emphasizes that the 4D Lorentz symmetry is an emergent symmetry. Before the choice of which adinkra based SU(2) symmetry corresponds to the orbital SU(2) of Minkowski space, both adinkra based SU(2) symmetry groups are equivalent.

This choice immediately culls and filters the rest of the BC_4 elements dependent on even permutations. Given a second element dependent upon an even permutation, if its holoraumy tensor commutes with that of the base element, this second element does not provide an example that can be reached by projection of any 4D, $\mathcal{N}=1$ spacetime supermultiplet. On the other hand, given a second element dependent upon an even permutation, if its holoraumy tensor does not commute with that of the base element, this second element does provide an example that can be reached by projection of a 4D, $\mathcal{N} = 1$ spacetime supermultiplet.

The process we have described above provides a set theoretic definition of a 4D, $\mathcal{N} = 1$ spacetime vector supermultiplet based solely on the properties of elements of BC_4. We must still consider the BC_4 elements that depend on odd permutations.

For the BC_4 elements dependent on odd permutations, we now imagine calculating the holoraumy tensors. Given conjecture 2, some of these will have holoraumy tensors that commute with the vector supermultiplet holoraumy tensor as defined above. Others will have holoraumy tensors that do not commute with the vector supermultiplet holoraumy tensor as defined above.

If the BC_4 elements dependent on odd permutations possess holoraumy tensors that commute with the vector supermultiplet holoraumy tensor as defined above, then such elements describe the projections of 4D, $\mathcal{N} = 1$ spacetime chiral supermultiplets.

If the BC_4 elements dependent on odd permutations possess holoraumy tensors that do not commute with the vector supermultiplet holoraumy tensor as defined above, then such elements describe the projections of 4D, $\mathcal{N} = 1$ spacetime tensor supermultiplets.

Notice that the definitions given above depend only on structures that are intrinsic to BC_4. So these are "BC_4-centric" definitions of the chiral, vector, and tensor multiplet adinkras that do not require any information from the higher dimensional supermultiplets. Only the behavior of the BC_4 elements under the $*$-map and the holoraumy tensors associated with each BC_4 element have been used to define the respective adinkras to be associated with each off-shell supermultiplet. Although there is nothing in these definitions that depend on structures outside BC_4, the requirement on the commutativity or non-commutativity of the various holoraumies is motivated by the study [10] where these conditions were found to hold in four dimensional description of these supermultiplets.

The arguments are a little bit more involved if one begins the analysis from a starting point of picking an element of BC_4 that depends on odd permutations. But with appropriate modifications, the same final result occurs.

In this section, we have proposed a set of criterion and described a process by which one-half of all possible four color adinkras described by BC_4 can simultaneously describe results obtainable from a 0-brane reduction procedure applied to minimal off-shell 4D, $\mathcal{N} = 1$ supermultiplets. The proposed method for starting from some of adinkras and using them as the basis for constructing 4D supermultiplets may seem convoluted. We wish to motivate the proposed method. For this purpose, the image in figure 4 is useful to illustrate some points. On the left portion of the image, the Venn diagram contains the three minimal supermultiplets (denoted by dots within the Venn diagram) reduced to one dimension. Under a consistent reduction procedure (denoted by the blue lines), these supermultiplets lead to three adinkras represented by dots within the leftmost Venn diagram.

We know based on the construction from the permutation group, its Boolean quartets, and their antonyms, there are 768 adinkras and thus 765 of these are not obtained from the reduction procedure. In figure 4 these are represented by dots in the leftmost Venn diagram

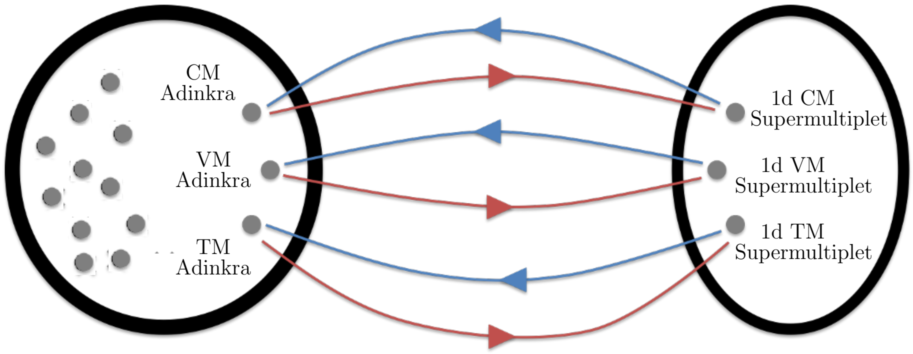

Figure 4. Adinkra Space, 1d Supermultiplet Space, Reduction & Enhancement.

that are unconnected to the supermultiplets represented by dots within the rightmost Venn diagram.

The important point to note is that precisely which three adinkras are obtained by the reduction procedure depends on many details of the reduction procedure itself as well as the gamma matrices defined in the four dimensional theory prior to the reduction. This is where the ambiguities discussed in section 2 play a role. If one changes the reduction procedure (e.g. re-order the fermions in going from 4D to 1d), uses it uniformly, then the reduction can lead to a different triplet of four color adinkras.

The proposal we have made for enhancing the adinkras from one dimensions to become the bases for four dimensional supermultiplets enforces a sort of "democracy" among the 768 adinkra quartets. Our proposal is essentially that any subset of them whose values calculated from $\chi_o(\mathcal{R})$ and $\theta[(\mathcal{R})\,,\,(\mathcal{R}')]_\ell$ that align precisely with the values of the 4D results shown in section 8 $\widehat{\chi}_o(\widehat{\mathcal{R}})$ and $\theta[(\widehat{\mathcal{R}})\,,\,(\widehat{\mathcal{R}}')]$ should be regarded as candidates for dimensional enhancement to 4D. Let us ensure the reader understands we have now checked our proposal on several, but not the entirety of, elements of the two spaces illustrated in figure 4.

10 Conclusion

In this paper, we have attempted to repeat the path pioneered by the work in [18] that showed how adinkras in one dimension can be extended to understand when such adinkras also allow the interpretation of being the reductions of 2D, $\mathcal{N} = 1$ supermultiplets. The work in [18] can be interpreted as the analog of the integration of a 1-cycle along a closed path. More recently [19, 20], however, there has been introduced another methodology only based on the codes. Older works, [21–23] had made note of the role of codes in defining irreducible representations of adinkras that descend from four dimensions. But the work in [19, 20] emphasizes that codes also play a role in understanding dimensional enhancement from 1D to 2D. In the light of the result in [17] on fermionic dimensional enhancement, it would be an interesting investigation to see how codes play a role in that result.

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p	$(4p!)/2^{(p+1)}$
1	6
2	5,040

Table 3. Sections by Quotienting.

It has been noted [27] the mathematical object shown in figure 2 is the case of $p = 1$ of the more general partitions given by $\boldsymbol{S}_{4p}/\{\boldsymbol{Z}_2^{(1)} \times \cdots \times \boldsymbol{Z}_2^{(p+1)}\}$ of $\boldsymbol{S}_{4p}$. In our discussion the next required step to obtaining L-matrices was to append to each section a set of Boolean factors. In appendix C of the work given in [28], this same sort of construction was used where the L-matrices were constructed by appending Boolean factors to the generators of $\boldsymbol{\mathcal{P}}_2$. If such quotients are taken as a starting point for higher order construction, this will present some challenges as noted in table 3.

Before leaving entirely the realm of conjectures, there is one more that we would like to present. This one is not confined to adinkras related to BC_4.

In a recent fascinating development [29, 30] in this general line of research on Garden Algebras, Adinkras, and codes (GAAC), there has appeared indications that adinkras can be interpreted as objects possessing algebraic geometrical descriptions as punctures of Riemann surfaces. This work also introduced the concept that monodromy matrices exist that are related to the bosonic fields of adinkras. With this occurrence, we now discuss how the holoraumy matrices defined by $\boldsymbol{V}_{\rm IJ}^{(\mathcal{R})}$ are related to the monodromy argument based in algebraic geometry.

As far back as the work in [11, 28], it was pointed out there are two sets of matrices which can be denoted by $\{\mathcal{A}\}$ and $\{\mathcal{B}\}$, each consisting of three elements ($\{\mathcal{A}\} = \{\boldsymbol{\alpha}^1$, $\boldsymbol{\alpha}^2$, $\boldsymbol{\alpha}^3\}$ and $\{\mathcal{B}\} = \{\boldsymbol{\beta}^1$, $\boldsymbol{\beta}^2$, $\boldsymbol{\beta}^3\}$), that play a significant role with regard to this class of adinkras. Using the nomenclature of this older work, these are described by

$$\begin{aligned} \boldsymbol{\alpha}^1 &= \; -i\,(12)_b(14)(23)\,, & \boldsymbol{\beta}^1 &= \; -i\,(10)_b(14)(23)\,, \\ \boldsymbol{\alpha}^2 &= \; -i\,(10)_b(12)(34)\,, & \boldsymbol{\beta}^2 &= \; -i\,(12)_b(13)(24)\,, \\ \boldsymbol{\alpha}^3 &= \; -i\,(6)_b(13)(24)\,, & \boldsymbol{\beta}^3 &= \; -i\,(6)_b(12)(34)\,, \end{aligned} \tag{10.1}$$

and when we multiply by i and then drop sign factors $(\#)_b$ (or equivalently take absolute values) we see that

$$\left|i\,\boldsymbol{\alpha}^1\right| = \left|i\,\boldsymbol{\beta}^1\right|, \qquad \left|i\,\boldsymbol{\alpha}^2\right| = \left|i\,\boldsymbol{\beta}^3\right|, \qquad \left|i\,\boldsymbol{\alpha}^3\right| = \left|i\,\boldsymbol{\beta}^2\right|, \tag{10.2}$$

which together with the identity matrix form the vierergruppe. This further implies

$$\begin{aligned} \boldsymbol{\mu}_4 &= \left|i\,\boldsymbol{\alpha}^1\right| = \left|i\,\boldsymbol{\beta}^1\right|, \\ \boldsymbol{\mu}_3 &= \left|i\,\boldsymbol{\alpha}^3\right| = \left|i\,\boldsymbol{\beta}^2\right|, \\ \boldsymbol{\mu}_2 &= \left|i\,\boldsymbol{\alpha}^2\right| = \left|i\,\boldsymbol{\beta}^3\right|. \end{aligned} \tag{10.3}$$

relating the three elements $\boldsymbol{\mu}_4$, $\boldsymbol{\mu}_3$, and $\boldsymbol{\mu}_2$ [29, 30] of the monodromy to elements in the $\{\mathcal{A}\}$ and $\{\mathcal{B}\}$ matrix sets.

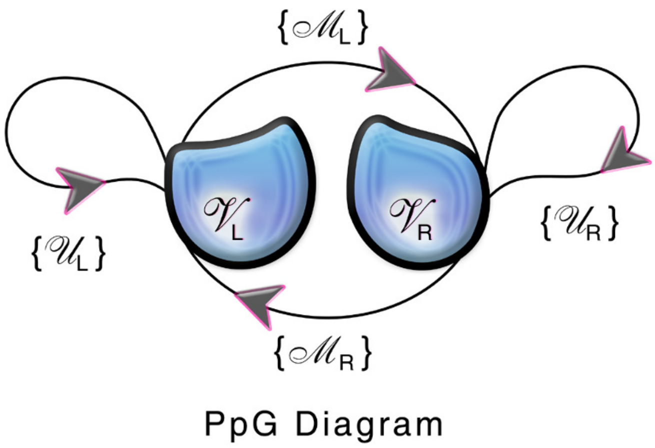

Figure 5. Placement-putting Graph.

To continue in our discussion, we now reintroduce an image (see figure 5) from the work of [14] which show the relationships between all the mathematical constructs of the Garden Algebra. The space of all the bosonic nodes is illustrated by the set $\mathcal{V}_L$, the space of all the fermionic nodes is illustrated by the set $\mathcal{V}_R$, the space of all linear maps that relate bosonic elements to fermionic ones is illustrated by the set $\mathcal{M}_R$, and finally the space of all linear maps that relate fermionic elements to bosonic ones is illustrated by the set $\mathcal{M}_L$. The L-matrices are contained in the space $\mathcal{M}_L$, the R-matrices are contained in the space $\mathcal{M}_R$, the (V_{IJ}) matrices are contained in the space $\mathcal{U}_L$, and the $(\widetilde{V}_{IJ})$ matrices are contained in the space $\mathcal{U}_R$. In particular, considering the bosonic holoraumy matrices, it has been noted [6, 7]

$$\boldsymbol{V}_{\rm IJ}^{(\mathcal{R})} = \left[\kappa_{\rm IJ}^{(\mathcal{R})1}\, \boldsymbol{\alpha^1} + \kappa_{\rm IJ}^{(\mathcal{R})2}\, \boldsymbol{\alpha^2} + \kappa_{\rm IJ}^{(\mathcal{R})3}\, \boldsymbol{\alpha^3} \right] + \left[\widetilde{\kappa}_{\rm IJ}^{(\mathcal{R})1}\, \boldsymbol{\beta^1} + \widetilde{\kappa}_{\rm IJ}^{(\mathcal{R})2}\, \boldsymbol{\beta^2} + \widetilde{\kappa}_{\rm IJ}^{(\mathcal{R})3}\, \boldsymbol{\beta^3} \right] , \tag{10.4}$$

where the quantities $\kappa_{\rm IJ}^{(\mathcal{R})1}$, $\kappa_{\rm IJ}^{(\mathcal{R})2}$, $\kappa_{\rm IJ}^{(\mathcal{R})3}$, $\widetilde{\kappa}_{\rm IJ}^{(\mathcal{R})1}$, $\widetilde{\kappa}_{\rm IJ}^{(\mathcal{R})2}$, and $\widetilde{\kappa}_{\rm IJ}^{(\mathcal{R})3}$ are parameters analogous to the ℓ and $\widetilde{\ell}$ discussed earlier. Multiplying this last equation by factors of i on both sides and taking absolute values, we find,

$$\begin{aligned} \left| i\, \boldsymbol{V}_{\rm IJ}^{(\mathcal{R})} \right| &= \left[\kappa_{\rm IJ}^{(\mathcal{R})1} \left| i\, \boldsymbol{\alpha^1} \right| + \kappa_{\rm IJ}^{(\mathcal{R})2} \left| i\, \boldsymbol{\alpha^2} \right| + \kappa_{\rm IJ}^{(\mathcal{R})3} \left| i\, \boldsymbol{\alpha^3} \right| \right] \\ &\quad + \left[\widetilde{\kappa}_{\rm IJ}^{(\mathcal{R})1} \left| i\, \boldsymbol{\beta^1} \right| + \widetilde{\kappa}_{\rm IJ}^{(\mathcal{R})2} \left| i\, \boldsymbol{\beta^2} \right| + \widetilde{\kappa}_{\rm IJ}^{(\mathcal{R})3} \left| i\, \boldsymbol{\beta^3} \right| \right] \\ &= \left[(\kappa_{\rm IJ}^{(\mathcal{R})1} + \widetilde{\kappa}_{\rm IJ}^{(\mathcal{R})1})\, \boldsymbol{\mu}_4 + (\kappa_{\rm IJ}^{(\mathcal{R})2} + \widetilde{\kappa}_{\rm IJ}^{(\mathcal{R})3})\, \boldsymbol{\mu}_2 + (\kappa_{\rm IJ}^{(\mathcal{R})3} + \widetilde{\kappa}_{\rm IJ}^{(\mathcal{R})2})\, \boldsymbol{\mu}_3 \right] \end{aligned} \tag{10.5}$$

and thus the bosonic holoraumy matrix on the l.h.s. above is related to the monodromy matrices on the r.h.s. of the equation. We thus assert that the monodromy calculations that arise from the view of Riemann surface and adinkra chromotopologies lead to information about the absolute values of the entries in the bosonic holoraumy matrices.

With this occurrence, we conjecture the *full* bosonic holoraumy matrices defined by $\boldsymbol{V}_{\rm IJ}^{(\mathcal{R})}$ will also likely be related to a set of extensions of monodromy matrices [29, 30].

Conjecture 3. *The holoraumy matrices* $\mathbf{V}_{\rm IJ}^{(\mathcal{R})}$*, that can be constructed from the L-matrices and R-matrices of the "Garden Algebra", may be obtained from an algebraic geometrical construction based on monodromy matrices including data about dashing.*

> "A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas".
>
> G.H. Hardy

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