

On the global symmetries of 6D superconformal field theories

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ABSTRACT: We study global symmetry groups of six-dimensional superconformal field theories (SCFTs). In the Coulomb branch we use field theoretical arguments to predict an upper bound for the global symmetry of the SCFT. We then analyze global symmetry groups of F-theory constructions of SCFTs with a one-dimensional Coulomb branch. While in the vast majority of cases, all of the global symmetries allowed by our Coulomb branch analysis can be realized in F-theory, in a handful of cases we find that F-theory models fail to realize the full symmetry of the theory on the Coulomb branch. In one particularly mysterious case, F-theory models realize several distinct maximal subgroups of the predicted group, but not the predicted group itself.

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1 Introduction

The discovery of nontrivial six-dimensional superconformal field theories (SCFTs) nearly twenty years ago [1–3] was quite surprising. On the one hand, it was known that such theories could exist in principle [4], but on the other hand it was argued that they could not be Lagrangian theories, i.e., that they could not be studied as perturbations of free field theory.

Instead, a variety of indirect constructions were given, all derived from string theory and its cousins: theories of point-like heterotic instantons [2, 5, 6] (sometimes located at singularities [7, 8]), theories on five-branes (sometimes located at singularities [9–11]), theories built from NS-branes and D-branes in type IIA [12–14], theories built from local F-theory models [7, 8, 15–18], and so on. More recently, AdS_7 duals have been found for some infinite families of these theories [19, 20]. Examples have been studied with either $(2,0)$ or $(1,0)$ supersymmetry; in this paper, we focus on theories with minimal $(1,0)$ supersymmetry in six dimensions.

It is generally difficult to directly identify the degrees of freedom in these theories. One important clue is provided by the global symmetry group of the theory, together with its action on the fields of the theory, so it is desirable to determine this group whenever possible.¹ In many of the string-related constructions of these theories, the global symmetry group is manifest, but in one key case — constructions from F-theory — the global symmetry group is difficult to determine. In this paper, we investigate global symmetry groups of these F-theory constructions, determining them in some cases and constraining them in others.

The motivation for focusing on this particular type of six-dimensional SCFTs comes from recent dramatic progress in classifying such theories [21–23].² The classification result of [21] shows that for any 6D SCFT constructed from F-theory and having a Coulomb branch, the base of the F-theory model is an orbifold of the form \mathbb{C}^2/Γ , so we begin in section 2 with a discussion of F-theory on orbifold bases. We then turn in section 3 to a more general consideration of the local F-theory constructions of 6D SCFTs, explaining that the constructed theory is not directly conformal but rather flows to an interacting conformal theory under renormalization. In fact, the F-theory construction itself contains parameters which are irrelevant in the conformal limit, and for special values of those parameters a global symmetry group may be manifest. However, the global symmetry group is not manifest for generic values, and so in most cases the global symmetries of the theory must be viewed as emergent symmetries in the IR limit (from the F-theory perspective).

In section 4, we study these theories in their Coulomb branch, where they take the form of a conventional 6D supersymmetric field theory with hypermultiplets, vector multiplets, and tensor multiplets. All global symmetries of the SCFT act as global symmetries of the field theory on the Coulomb branch, so the global symmetry group of the latter constrains the global symmetry group of the SCFT. For theories whose Coulomb branch contains one tensor multiplet, we present these field theoretic global symmetry groups in section 5 (using results of [18, 27, 28]), in the case that the theory on the Coulomb branch is a gauge theory.

Finally, in the remainder of the paper, we determine the possible manifest global symmetries of F-theory models whose Coulomb branch contains precisely one tensor multiplet. In many cases, we find that all of the global symmetries of the field theory can be realized in some particular F-theory model. In some cases, however, we show that this is not possible,³ and we determine instead the (relatively) maximal global symmetry groups which can be realized in F-theory. For one particularly mysterious SCFT, there is more than one such relatively maximal subgroup of the field theoretic global symmetry group. We summarize these results in subsection 6.7.

In section 7 we state our conclusions and outline open problems.

¹Knowing the global symmetry group may also aid in identifying a possible AdS_7 dual for a family of theories.

²As has been pointed out to us privately by several different people, the classification of [21–23] implicitly assumes that there are no “frozen” seven-branes [24–26] in the F-theory models. Inclusion of such seven-branes might possibly expand the list of known theories.

³As this paper was nearing completion, a beautiful paper appeared [29] which showed on field theory grounds that in one of the cases for which we find a mismatch between the Coulomb branch prediction and the F-theory possibilities, the SCFT itself does not have the same global symmetry as on the Coulomb branch, but instead has the restricted global symmetry found in F-theory.

2 F-theory on orbifolds

Recently, there has been effort in classifying six-dimensional superconformal field theories (SCFTs) [21, 22] from F-theory. The data necessary to define an F-theory model are given by specifying a compact base space B , a line bundle over B and sections f and g of the fourth and sixth powers of this line bundle. These data determine the Weierstrass model of a fibration $\pi : X \rightarrow B$, and the discriminant $4f^3 + 27g^2$ of the Weierstrass model whose zeros (the “discriminant locus”) describe where the fibration is singular.⁴ To obtain a conventional F-theory model (i.e., one not involving tensionless strings), the base B must be nonsingular and the multiplicities of f and g cannot simultaneously exceed 4 and 6 at any point of B . But it is also possible to use a Weierstrass model to directly study theories having tensionless strings.

It is convenient to generalize this construction by considering a local model for B , in other words, to relax the condition that B be compact. In this case, if we focus on a particular point P of B or if we consider a collection of curves $\Sigma_i \subset B$ which are contractible to a point, there will be a scaling limit in which gravity decouples and a conformal fixed point emerges.

In the case of a point P in B , if the point is a nonsingular one at which the multiplicity of f is at most 3 or the multiplicity of g is at most 5, then the corresponding conformal fixed point will simply be a free theory. However, if the theory had tensionless strings before rescaling, then one expects an interacting SCFT. Something similar happens if the point P is itself a singular point of B .

The classification results of [21–23] begin with an F-theory model in which the singularities of B and any points for which (f, g) have high multiplicity have been blown up. That is, the starting point is the so-called⁵ *Coulomb branch* of the SCFT, in which nonzero expectation values have been given to the tensor multiplets in the theory. Concretely, this amounts to choosing a nonsingular base B and a collection of compact curves $\{\Sigma_i\}$ to serve as (some of) the components of the discriminant locus, such that the collection of curves can be contracted to a point, which might be a singular point. The basic idea is that when blowing down the compact curves to obtain the CFT, there are, in general, non-compact curves carrying non-abelian gauge algebras, which in the singular limit give rise to the global symmetry of the SCFT. This identification of global symmetries goes back to the early discussions of an F-theory realization of the small $E_8 \times E_8$ instanton [15, 16, 30], and has been used in a number of recent papers [21, 22, 31].

The classification result in [21] shows that for any local F-theory model of a 6D SCFT having a Coulomb branch, the base (before blowing up) must have at worst an orbifold singularity \mathbb{C}^2/Γ for some $\Gamma \subset U(2)$. Since we are interested in studying the Weierstrass models of such theories directly, we will explain in this section how to formulate F-theory on such bases, reviewing and extending the discussion in [32].

For any base B , the data of a supersymmetric F-theory model are specified by Weierstrass coefficients f and g which are sections of $\mathcal{O}(-4K_B)$ and $\mathcal{O}(-6K_B)$, respectively.

⁴We are assuming that there are no “frozen” seven-branes [24–26].

⁵This is sometimes referred to as the *tensor branch*.

Moreover, the total space of the elliptic fibration is the compactification of a threefold inside

$$\mathcal{O}(-2K_B) \oplus \mathcal{O}(-3K_B),$$

defined by an equation in Weierstrass form

$$-y^2 + x^3 + fx + g = 0, \tag{2.1}$$

where (x, y) are local coordinates taking values in $\mathcal{O}(-2K_B) \oplus \mathcal{O}(-3K_B)$, where K_B is the canonical class of B . The challenge is interpreting the sheaves $\mathcal{O}(-mK_B)$ when the base B is allowed to have orbifold singularities.

When B is nonsingular, each $\mathcal{O}(-mK_B)$ is a line bundle, i.e., a locally free sheaf of rank one. In the presence of orbifold singularities, however, these sheaves may only be “reflexive” sheaves of rank one (see, for example, [33]). In practice, this means that while for any point $Q \neq P$ there is a section of the sheaf which does not vanish at Q , it may be necessary to use several different sections to ensure that at least one of them is nonzero at each point in a small neighborhood of P . Once appropriate bases have been determined, the Weierstrass equation can be interpreted as an equation with coefficients in the coordinate ring of B .

We illustrate this point with the example of $\mathbb{C}^2/\mathbb{Z}_3$, with the generator of the group acting via

$$(z, w) \mapsto (e^{2\pi i/3}z, e^{2\pi i/3}w).$$

On the complement of $z = 0$, local coordinates on the quotient are provided by $s = z^3$ and $t = w/z$ (with $s \neq 0$). On the other hand, on the complement of $w = 0$, local coordinates are provided by $u = z/w$ and $v = w^3$ (with $v \neq 0$). The coordinate change map is given by

$$s = u^3v, \quad t = \frac{1}{u}, \tag{2.2}$$

which implies that $ds \wedge dt = udu \wedge dv$.

To find generating sections of $\mathcal{O}(-mK)$, we consider an expression of the form

$$\frac{s^i t^j}{(ds \wedge dt)^{\otimes m}} = \frac{u^{3i-j-m} v^i}{(du \wedge dv)^{\otimes m}}. \tag{2.3}$$

It is apparent that this gives a regular section whenever $j \geq 0$ and $3i - j - m \geq 0$.

If we consider the case of $m = 0$, we are simply computing the invariant functions. These are generated by

$$1, \quad \alpha := s = u^3v, \quad \beta := st = u^2v, \quad \gamma := st^2 = uv, \quad \delta := st^3 = v. \tag{2.4}$$

There are several relations among these generators:

$$\beta^2 = \alpha\gamma, \quad \gamma^2 = \beta\delta, \quad \text{and} \quad \alpha\delta = \beta\gamma, \tag{2.5}$$

and it is known that the coordinate ring is

$$\mathcal{R} := \mathbb{C}[\alpha, \beta, \gamma, \delta]/(\beta^2 - \alpha\gamma, \gamma^2 - \beta\delta, \alpha\delta - \beta\gamma). \tag{2.6}$$

Now, a general element g of $\mathcal{O}(-6K_B)$ is simply given by

$$\frac{g_0 s^2}{(ds \wedge dt)^{\otimes 6}} = \frac{g_0 v^2}{(du \wedge dv)^{\otimes 6}}, \quad (2.7)$$

where $g_0 \in \mathcal{R}$. (This is because $\mathcal{O}(-6K_B)$ is locally free.) The form of a general element f of $\mathcal{O}(-4K_B)$ is more complicated; it is given by

$$\frac{f_0 s^2 + f_1 s^2 t + f_2 s^2 t^2}{(ds \wedge dt)^{\otimes 4}} = \frac{f_0 u^2 v^2 + f_1 u v^2 + f_2 v^2}{(du \wedge dv)^{\otimes 4}}, \quad (2.8)$$

where f_0, f_1 , and f_2 are all elements of \mathcal{R} . Note, however, that the f_j are not uniquely determined by f ; an alteration which replaces (f_0, f_1, f_2) by

$$(f_0 + \beta\varphi_1 + \gamma\varphi_2 + \delta\varphi_3, f_1 - \alpha\varphi_1 - \beta(\varphi_2 + \psi_1) - \gamma(\varphi_3 + \psi_2) - \delta\psi_3, f_2 + \alpha\psi_1 + \beta\psi_2 + \gamma\psi_3), \quad (2.9)$$

where $\phi_i, \psi_i \in \mathcal{R}$, does not change f .

Writing the Weierstrass equation involves variable elements $x_0, x_1, y_0 \in \mathcal{R}$ with

$$x = \frac{x_0 s + x_1 s t}{(ds \wedge dt)^{\otimes 2}} = \frac{x_0 u v + x_1 v}{(du \wedge dv)^{\otimes 2}} \quad \text{and} \quad y = \frac{y_0 s}{(ds \wedge dt)^{\otimes 3}} = \frac{y_0 v}{(du \wedge dv)^{\otimes 3}}, \quad (2.10)$$

representing arbitrary sections of $\mathcal{O}(-2K_B)$ and $\mathcal{O}(-3K_B)$, respectively. Note that the pair (x_0, x_1) is not uniquely determined by x , but that alterations of the form

$$(x_0, x_1) \mapsto (x_0 + \beta\xi_1 + \gamma\xi_2 + \delta\xi_3, x_1 - \alpha\xi_1 - \beta\xi_2 - \gamma\xi_3), \quad (2.11)$$

where ξ_i are elements of \mathcal{R} , leave x unchanged.

Substituting into the usual Weierstrass equation (2.1) gives a section of $\mathcal{O}(-6K_B)$ which must vanish. Writing this as a multiple of the generator

$$\frac{s^2}{(ds \wedge dt)^{\otimes 6}} = \frac{v^2}{(du \wedge dv)^{\otimes 6}} \quad (2.12)$$

gives an equation in \mathcal{R} , namely,

$$y_0^2 = \alpha(x_0^3 + f_0 x_0) + \beta(3x_0^2 x_1 + f_1 x_0 + f_0 x_1) + \gamma(3x_0 x_1^2 + f_2 x_0 + f_1 x_1) + \delta(x_1^3 + f_2 x_1) + g_0, \quad (2.13)$$

which must be satisfied in addition to the relations (2.5) in order to determine an elliptic fibration over B . (Note that the equivalence relation (2.11) must also be imposed.)

3 6D SCFTs from local F-theory models

Most studies of 6D SCFTs constructed from F-theory have studied the models on the Coulomb branch, where there are no tensionless strings and where one thus expects a conventional field theory once gravity has been decoupled.

We propose instead that 6D SCFTs are most naturally studied on a singular local model which has tensionless strings associated with a particular point in the geometry, because this brings us directly into contact with the conformal field theory. The simplest example of this

is the F-theory realization of the small E_8 instanton, which consists of a point on the base B at which the multiplicity of f is 4 and the multiplicity of g is 6. Traditionally, this theory is studied by blowing up the point to obtain the Coulomb branch of the theory, and then rescaling the metric to zoom in on a neighborhood of the curve (which is contracted back to a point in the process, if the scaling is appropriately chosen). Here, we suggest that this scaling should be done directly on the singular F-theory model, zooming in on the point P supporting the singularity and only considering an appropriately rescaled metric in a neighborhood of that point. (This extends the analysis of [34] to the case of a noncompact base.)

The key features of F-theory at the point P after zooming are captured by the leading terms in the Weierstrass equation, truncating f and g appropriately:

$$y^2 = x^3 + \tilde{f}(z, w)x + \tilde{g}(z, w),$$

where \tilde{f} and \tilde{g} contain only the low-order terms of f and g . (Precisely which low-order terms are needed depends on the type of superconformal fixed point.) What is perhaps surprising is that varying the coefficients of the polynomials \tilde{f} and \tilde{g} , or including higher-order terms, does not affect the SCFT. The coefficients must be sufficiently generic to avoid converging to a different SCFT, and in particular cannot be zero, but any generic choice will do.

One may wonder if this phenomenon has some five-dimensional interpretation upon circle compactification. It does not: the new degrees of freedom in the five-dimensional theory live in vector multiplets (and geometrically correspond to the possibility of blowing up singularities in the total space of the elliptic fibration). However, this phenomenon of extra polynomial coefficients not needed to describe the CFT actually corresponds to hypermultiplets in a compact model of F-theory which would still be hypermultiplets upon circle compactification. These hypermultiplets decouple in the CFT limit.

Our interpretation of this observation is that the local F-theory data are specifying more than just the SCFT, they are specifying an irrelevant deformation of the SCFT, with the coefficients in f and g choosing which irrelevant deformation is being made.⁶ In other words, the local F-theory data specify a point in the configuration space for the field theories which lies to the ultraviolet of the SCFT and flows to it under renormalization.

Some of the local F-theory realizations of a given SCFT may make the global symmetries of the SCFT manifest, but this is not a requirement. When present, the global symmetries are visible in an F-theory realization in the form of a component of the discriminant locus $\{4f^3 + 27g^2 = 0\}$ which passes through the point P . After rescaling, this component is necessarily non-compact, and the coupling of the associated gauge group has gone to zero, leaving a global symmetry.

For example, the particular realization of the small E_8 instanton with equation

$$y^2 = x^3 + z^4x + z^5w, \tag{3.1}$$

has discriminant locus $\{z^{10}(4z^2 + 27w^2) = 0\}$ and has a manifest E_8 global symmetry along $z = 0$. However, the generic realization of this same SCFT has *no* manifest global symmetry.

⁶It is known that 6D SCFTs do not admit marginal or relevant deformations, but can admit irrelevant ones [35].

In this paper, we shall determine the maximal manifest global symmetry which can be obtained in F-theory realizations of SCFTs with a one-dimensional Coulomb branch. We do our calculations on the Coulomb branch, but our conclusions are properly drawn after the distinguished curve (created by blowing up) is contracted.

4 Global symmetries in the Coulomb branch

All global symmetries of the SCFT act as global symmetries of the field theory on the Coulomb branch, so the global symmetry group of the latter constrains the global symmetry group of the SCFT.⁷ We determine these field theoretic global symmetry groups (using results of [18, 27, 28]) for gauge theories with a one-dimensional Coulomb branch.

As described in section 2, we specify an F-theory model in six dimensions via sections f and g of appropriate rank-one reflexive sheaves on a base B which is a complex surface. These sections in turn determine an elliptically fibered Calabi-Yau threefold $\pi : X \rightarrow B$ with a section, typically singular, by means of a Weierstrass equation

$$y^2 = x^3 + fx + g . \tag{4.1}$$

This equation has an associated discriminant

$$\Delta := 4f^3 + 27g^2 , \tag{4.2}$$

and discriminant locus

$$\{\Delta = 0\} , \tag{4.3}$$

which indicates where the fibration is singular.⁸ These quantities are sections of certain line bundles

$$f \leftrightarrow \mathcal{O}(-4K_B) , \quad g \leftrightarrow \mathcal{O}(-6K_B) , \quad \Delta \leftrightarrow \mathcal{O}(-12K_B) . \tag{4.4}$$

The above fibration corresponds to a non-singular F-theory model when it is possible to blow-up the singularities in (4.1) to obtain a smooth elliptically fibered Calabi-Yau \tilde{X} . F-theory compactified on \tilde{X} will yield a six-dimensional theory, whose gauge group G is determined by the singularity structure of the elliptic fibration. We will focus our analysis on the non-abelian part of the Lie algebra \mathfrak{g} , which is characterized by the codimension one singularities, and on the matter content of the theory, which is encoded in the codimension two singularities. The degrees of vanishing of the quantities in (4.4) along a curve Σ suffice in most cases to determine the gauge algebra along Σ according to the Kodaira classification [36–38]. In some cases, however, information regarding monodromy is needed to determine the precise form of the gauge algebra. This is summarized in table 1. The last entry in the table is designated “non-minimal” since after the singularities in the fibers

⁷It appears possible for a direct factor of the global symmetry group of the SCFT to act trivially on the Coulomb branch, but the only known cases where this might possibly happen are not gauge theories, so we do not consider this possibility here. We will return to this point at the end of subsection 6.7.

⁸In the language of type IIB string theory, the discriminant locus indicates the location of seven-branes wrapping divisors on B .

ord(f)	ord(g)	ord(Δ)	type	singularity	non-abelian algebra
≥ 0	≥ 0	0	I_0	none	none
0	0	1	I_1	none	none
0	0	$n \geq 2$	I_n	A_{n-1}	$\mathfrak{su}(n)$ or $\mathfrak{sp}([n/2])$
≥ 1	1	2	II	none	none
1	≥ 2	3	III	A_1	$\mathfrak{su}(2)$
≥ 2	2	4	IV	A_2	$\mathfrak{su}(3)$ or $\mathfrak{su}(2)$
≥ 2	≥ 3	6	I_0^*	D_4	$\mathfrak{so}(8)$ or $\mathfrak{so}(7)$ or \mathfrak{g}_2
2	3	$n \geq 7$	I_{n-6}^*	D_{n-2}	$\mathfrak{so}(2n-4)$ or $\mathfrak{so}(2n-5)$
≥ 3	4	8	IV*	\mathfrak{e}_6	\mathfrak{e}_6 or \mathfrak{f}_4
3	≥ 5	9	III*	\mathfrak{e}_7	\mathfrak{e}_7
≥ 4	5	10	II*	\mathfrak{e}_8	\mathfrak{e}_8
≥ 4	≥ 6	≥ 12	non-minimal	-	-

Table 1. Singularity types with associated non-abelian algebras.

are resolved, the resolution contains a curve which can be blown down, i.e., it is not a “minimal model” in the sense of birational geometry. In fact, blowing that curve down is associated to a new Weierstrass model in which the degrees of vanishing of (f, g, Δ) have been reduced by $(4, 6, 12)$. For this reason we need not consider such cases.

The “monodromy” which determines which gauge algebra occurs for a given singularity type is part of Tate’s algorithm [39], first discussed in the physics literature in [40]. A general specification for determining the gauge algebra was given in table 4 of [28], which we review here. In each case, there is a covering of the curve Σ which can be described by means of an algebraic equation in an auxiliary variable ψ (which takes its values in a line bundle over Σ). If Σ is defined by $z = 0$, then these monodromy covers are given as follows [28]:

type	equation of monodromy cover
$I_m, m \geq 3$	$\psi^2 + (9g/2f) _{z=0}$
IV	$\psi^2 - (g/z^2) _{z=0}$
I_0^*	$\psi^3 + (f/z^2) _{z=0} \cdot \psi + (g/z^3) _{z=0}$
$I_{2n-5}^*, n \geq 3$	$\psi^2 + \frac{1}{4}(\Delta/z^{2n+1})(2zf/9g)^3 _{z=0}$
$I_{2n-4}^*, n \geq 3$	$\psi^2 + (\Delta/z^{2n+2})(2zf/9g)^2 _{z=0}$
IV*	$\psi^2 - (g/z^4) _{z=0}$

For all cases except I_0^* , the equation of the monodromy cover takes the form “ $\psi^2 - \text{something}$ ”, and this cover splits (leading to no monodromy) if and only if the expression “something” is a perfect square. In the remaining case I_0^* , the monodromy cover equation defines a degree 3 cover of Σ , and one must analyze this further to determine if that cover is irreducible ($\mathfrak{g}(\Sigma) = \mathfrak{g}_2$), splits into two components ($\mathfrak{g}(\Sigma) = \mathfrak{so}(7)$), or splits into three

components ($\mathfrak{g}(\Sigma) = \mathfrak{so}(8)$). Each of these possibilities is reflected in the behavior of the residual discriminant, as we will now describe.

First, suppose that the monodromy cover splits completely (the case of gauge algebra $\mathfrak{so}(8)$). That is, suppose that

$$\psi^3 + (f/z^2)|_{z=0} \cdot \psi + (g/z^3)|_{z=0} = (\psi - \alpha)(\psi - \beta)(\psi - \gamma), \quad (4.5)$$

with $\alpha + \beta + \gamma = 0$. It is easy to calculate the residual discriminant $(\Delta/z^6)|_{z=0}$ in this case; it is

$$\begin{aligned} 4(\alpha\beta + \beta\gamma + \gamma\delta)^3 + 27\alpha^2\beta^2\gamma^2 &= -(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 \\ &= -(\alpha - \beta)^2(\alpha + 2\beta)^2(2\alpha + \beta)^2. \end{aligned} \quad (4.6)$$

This form of the residual discriminant is a necessary consequence of having gauge algebra $\mathfrak{so}(8)$. Notice that the analysis of [28] associates to the vanishing of the factors $\alpha - \beta$, $\beta - \gamma$, and $\gamma - \alpha$ the three different 8-dimensional representations of $\mathfrak{so}(8)$.

Second, suppose that the monodromy cover splits into a linear factor and a quadratic factor (the case of gauge algebra $\mathfrak{so}(7)$). That is, suppose that

$$\psi^3 + (f/z^2)|_{z=0} \cdot \psi + (g/z^3)|_{z=0} = (\psi - \lambda)(\psi^2 + \lambda\psi + \mu). \quad (4.7)$$

We can calculate the residual discriminant $(\Delta/z^6)|_{z=0}$ as

$$4(\mu - \lambda^2)^3 + 27\lambda^2\mu^2 = (\mu + 2\lambda^2)^2(4\mu - \lambda^2). \quad (4.8)$$

The factor $\varphi := 4\mu - \lambda^2$ is the discriminant of $\psi^2 + \lambda\psi + \mu = 0$, so it captures the ramification points of the double cover. (Moreover, if φ is a square, then the double cover splits and we are back to the case of $\mathfrak{so}(8)$.) The analysis of [28] then shows that these intersection points are associated to the 7-dimensional representation of $\mathfrak{so}(7)$, while the other points (the zeros of $\mu + 2\lambda^2$) are associated to the spinor representation.

Notice that if the monodromy cover splits completely in (4.7), we may choose one of the roots (say α) to use as λ , and then we have $\mu = \beta\gamma$. It follows that

$$\begin{aligned} \mu + 2\lambda^2 &= \beta\gamma + 2\alpha^2 = (2\alpha + \beta)(\alpha - \beta), \\ 4\mu - \lambda^2 &= 4\beta\gamma - \alpha^2 = -(\alpha + 2\beta)^2 \end{aligned} \quad (4.9)$$

(using $\gamma = -\alpha - \beta$), which allows for easy comparison between the two forms of the discriminant.

5 Prediction from field theory

In this section we want to describe the global symmetries that we would expect from field theory on the Coulomb branch for gauge theories with a one-dimensional Coulomb branch. Let us consider a curve Σ with gauge algebra \mathfrak{g} . The possible gauge algebras and matter representations were determined from anomaly cancellation in [18, 27]; a convenient place

to find these results in F-theory language is table 10 of [28].⁹ Field theory predicts (see, for example, [41]) that N hypermultiplets in a complex representation of the gauge group support an $SU(N)$ symmetry group, which is enhanced to $SO(2N)$ for quaternionic (i.e., pseudoreal) representations, and to $Sp(N)$ for real representations.¹⁰ In the quaternionic case, the underlying complex representation is a representation by half-hypermultiplets, so that N is allowed to be a half-integer. (We will emphasize this by declaring that the representation consists of M copies of $\frac{1}{2}V$, where M is allowed to be an arbitrary integer, leading to global symmetry $SO(M)$.)

Our conventions here differ slightly with the ones used in [28]: to compare the two, one needs to substitute $-K_B$ for L in [28]. Moreover, we interpret $(\dots)|_\Sigma$ as an intersection number $(\dots) \cdot \Sigma$, and do not worry about the algebraic cycle class discussed in [28]. Also, we note that the adjoint representation never occurs in our models since Σ is rational, and thus we will simply substitute

$$L|_\Sigma = -K_B \cdot \Sigma = 2 + \Sigma^2, \tag{5.1}$$

into the formulas of [28]. Note that [28] would have included $(1 + \Sigma^2)\Lambda_{\text{irr}}^2$ as part of the representation content of $\mathfrak{sp}(n)$, $n \geq 2$, but since $\Sigma^2 < 0$ we conclude that $\Sigma^2 = -1$ and this representation does not occur.

One of the observations of [28] is that in some cases there can be more than one matter representation with the same effect on anomaly cancellation. In the case of $\mathfrak{su}(6)$ and $\Sigma^2 = -1$, Λ^3 plus two fundamentals (denoted by $\mathfrak{su}(6)^*$) has the same anomaly content as two copies of Λ^2 (denoted by $\mathfrak{su}(6)$). As discussed in [28, 43], these different matter representations arise in enhancements from A_5 to E_6 . However, it is easy to see that for any other gauge group and a single contractible rational curve, none of these “alternate” possibilities can occur. In fact, table 3 of [28] shows that in order to get one of these alternate possibilities, there must be at least two copies of Λ^2 or Λ_{irr}^2 (for $\mathfrak{su}(n)$ or $\mathfrak{sp}(n)$, respectively). As we can read off from table 2, this representation does not appear for $\mathfrak{sp}(n)$, while for $\mathfrak{su}(n)$ there are $2 + \Sigma^2$ copies of Λ^2 , but since $\Sigma^2 < 0$ this means that there is at most one copy. This argument does not apply to $\mathfrak{su}(6)$ because in this case Λ^3 is a pseudo-real representation.

A key point we want to emphasize is that the global symmetry only depends on the gauge algebra supported on Σ , and not on the Kodaira type that realizes it. For example, we can realize $\mathfrak{su}(2)$ with Kodaira types I_2 , I_3 , III and IV (with appropriate monodromies), and in all cases the global symmetry is $\mathfrak{so}(32 + 12\Sigma^2)$. For this reason table 2 lists only the algebras and not the Kodaira types.

⁹Our notation for representations follows [28] and is fairly standard: for $\mathfrak{su}(n)$ and $\mathfrak{sp}(n)$, F denotes the fundamental representation and Λ^k denotes its exterior powers. Note that for $\mathfrak{g} = \mathfrak{sp}(n)$, the representation Λ^2 contains a one-dimensional summand; the remainder is denoted by Λ_{irr}^2 . For $\mathfrak{so}(n)$, V denotes the vector representation, S_+ and S_- denote half-spin representations, and S_* (according to [28]) denotes a spin or half-spin representation, depending on the parity of n . For the exceptional algebras, representations are denoted by their dimension (in bold face type).

¹⁰Our convention is that $Sp(N)$ is a subgroup of $SU(2N)$.

\mathfrak{g}	representation	global symmetry
$\mathfrak{su}(2)$	$(32 + 12\Sigma^2)\frac{1}{2}F$	$\mathfrak{so}(32 + 12\Sigma^2)$
$\mathfrak{su}(3)$	$(18 + 6\Sigma^2)F$	$\mathfrak{su}(18 + 6\Sigma^2)$
$\mathfrak{su}(4)$	$(16 + 4\Sigma^2)F + (2 + \Sigma^2)\Lambda^2$	$\mathfrak{su}(16 + 4\Sigma^2) \oplus \mathfrak{sp}(2 + \Sigma^2)$
$\mathfrak{su}(5)$	$(16 + 3\Sigma^2)F + (2 + \Sigma^2)\Lambda^2$	$\mathfrak{su}(16 + 3\Sigma^2) \oplus \mathfrak{su}(2 + \Sigma^2)$
$\mathfrak{su}(6)$	$(16 + 2\Sigma^2)F + (2 + \Sigma^2)\Lambda^2$	$\mathfrak{su}(16 + 2\Sigma^2) \oplus \mathfrak{su}(2 + \Sigma^2)$
$\mathfrak{su}(6)^*$	$(16 + \Sigma^2)F + \frac{1}{2}(2 + \Sigma^2)\Lambda^3$	$\mathfrak{su}(16 + \Sigma^2) \oplus \mathfrak{so}(2 + \Sigma^2)$
$\mathfrak{su}(n), n \geq 7$	$(16 + (8 - n)\Sigma^2)F + (2 + \Sigma^2)\Lambda^2$	$\mathfrak{su}(16 + (8 - n)\Sigma^2) \oplus \mathfrak{su}(2 + \Sigma^2)$
$\mathfrak{sp}(n), n \geq 2$	$(16 + 4n)\frac{1}{2}F$	$\mathfrak{so}(16 + 4n)$
$\mathfrak{so}(7)$	$(3 + \Sigma^2)V + 2(4 + \Sigma^2)S_*$	$\mathfrak{sp}(3 + \Sigma^2) \oplus \mathfrak{sp}(8 + 2\Sigma^2)$
$\mathfrak{so}(8)$	$(4 + \Sigma^2)V + (4 + \Sigma^2)(S_+ + S_-)$	$\mathfrak{sp}(4 + \Sigma^2) \oplus \mathfrak{sp}(4 + \Sigma^2) \oplus \mathfrak{sp}(4 + \Sigma^2)$
$\mathfrak{so}(9)$	$(5 + \Sigma^2)V + (4 + \Sigma^2)S_*$	$\mathfrak{sp}(5 + \Sigma^2) \oplus \mathfrak{sp}(4 + \Sigma^2)$
$\mathfrak{so}(10)$	$(6 + \Sigma^2)V + (4 + \Sigma^2)S_*$	$\mathfrak{sp}(6 + \Sigma^2) \oplus \mathfrak{su}(4 + \Sigma^2)$
$\mathfrak{so}(11)$	$(7 + \Sigma^2)V + (4 + \Sigma^2)\frac{1}{2}S_*$	$\mathfrak{sp}(7 + \Sigma^2) \oplus \mathfrak{so}(4 + \Sigma^2)$
$\mathfrak{so}(12)$	$(8 + \Sigma^2)V + (4 + \Sigma^2)\frac{1}{2}S_*$	$\mathfrak{sp}(8 + \Sigma^2) \oplus \mathfrak{so}(4 + \Sigma^2)$
$\mathfrak{so}(13)$	$(9 + \Sigma^2)V + (2 + \frac{1}{2}\Sigma^2)\frac{1}{2}S_*$	$\mathfrak{sp}(9 + \Sigma^2) \oplus \mathfrak{so}(2 + \frac{1}{2}\Sigma^2)$
$\mathfrak{so}(n), n \geq 14$	$(n - 8)V$	$\mathfrak{sp}(n - 8)$
\mathfrak{e}_6	$(6 + \Sigma^2)\mathbf{27}$	$\mathfrak{su}(6 + \Sigma^2)$
\mathfrak{e}_7	$(8 + \Sigma^2)\frac{1}{2}\mathbf{56}$	$\mathfrak{so}(8 + \Sigma^2)$
\mathfrak{e}_8	none	none
\mathfrak{f}_4	$(5 + \Sigma^2)\mathbf{26}$	$\mathfrak{sp}(5 + \Sigma^2)$
\mathfrak{g}_2	$(10 + 3\Sigma^2)\mathbf{7}$	$\mathfrak{sp}(10 + 3\Sigma^2)$

Note: $\Sigma^2 = -1$ for $\mathfrak{su}(6)^*$; $\Sigma^2 = -1$ for $\mathfrak{sp}(n), n \geq 2$; $\Sigma^2 = -4$ for $\mathfrak{so}(n), n \geq 14$; and $\Sigma^2 = -12$ for \mathfrak{e}_8 .

Table 2. Global symmetries as predicted from field theory.

We have a minor correction, or reinterpretation, to offer concerning table 10 of [28]: for $\mathfrak{so}(8)$, S_* must be interpreted as $\frac{1}{2}(S_+ + S_-)$.¹¹ Also, the analysis in example 13 of [28] shows that $4 + \Sigma^2$ must be divisible by 2 for $\mathfrak{so}(13)$ and must be divisible by 4 for $\mathfrak{so}(14)$. Thus, $\Sigma^2 = -4$ for $\mathfrak{so}(14)$ and $\Sigma^2 = -2, -4$ for $\mathfrak{so}(13)$. In addition, as explained in [28], for Kodaira type I_m^* , $m \geq 4$ (which corresponds to $\mathfrak{so}(n), n \geq 15$), we must have $(-2K_B - \Sigma)|_{\Sigma} = 0$, i.e., $\Sigma^2 = -4$. Similarly, for Kodaira type II^* (which corresponds to \mathfrak{e}_8), $(-6K_B - 5\Sigma)|_{\Sigma} = 0$, i.e., $\Sigma^2 = -12$.

The reality conditions for $\mathfrak{su}(n)$ are slightly subtle: the fundamental representation is quaternionic for $n = 2$ and is complex for $n > 2$; the anti-symmetric representation

¹¹In fact, this change could be made uniformly for all $\mathfrak{so}(2n)$ cases if desired, since S_+ is Casimir equivalent to S_- in all cases except $2n = 8$.

is real for $n = 4$ and is complex for $n > 4$. In the $\mathfrak{sp}(n)$ case the situation is simpler, as the fundamental representation is always quaternionic and the irreducible part of the anti-symmetric representation (which does not occur) is always real.

We also used the following reality conditions for the spinor representation(s) of $\mathfrak{so}(n)$:

- If $n = 0 \pmod 8$, the two spinor representations are real.
- If $n = 1, 7 \pmod 8$, the unique spinor representation is real.
- If $n = 2, 6 \pmod 8$, the two spinor representations are complex.
- If $n = 3, 5 \pmod 8$, the unique spinor representation is quaternionic.
- If $n = 4 \pmod 8$, the two spinor representations are quaternionic.

It is amusing to verify that table 2 respects the exceptional isomorphisms of Lie algebras. If we extend the $\mathfrak{so}(n)$ formula to the representation of $\mathfrak{so}(6)$, we find $(2 + \Sigma^2)V + 4(4 + \Sigma^2)S_*$, which agrees with the entry for $\mathfrak{su}(4)$ once we identify (V, S_*) with (Λ, F) . Similarly, in the case of $\mathfrak{so}(5)$, we find $(1 + \Sigma^2)V + 8(4 + \Sigma^2)\frac{1}{2}S_*$, which implies that $\Sigma^2 = -1$ and V does not occur, and which then agrees with the entry for $\mathfrak{sp}(2)$ once we identify S_* of $\mathfrak{so}(5)$ with F of $\mathfrak{sp}(2)$.

6 Constraints from F-theory

We now consider how global symmetries may be realized in F-theory. As previously discussed, if we have an F-theory model defined on a neighborhood of a contractible collection of curves, there may be additional non-compact curves meeting the contractible collection over which the elliptic curve degenerates further. If there were a larger open set containing a compactification of one of those curves, then the Kodaira-Tate classification would have determined a gauge group for that curve. In the limit where the curve becomes non-compact, the gauge coupling goes to zero and we see a global symmetry rather than a gauge symmetry in our local model. These are the F-theory global symmetries we wish to compute. In all cases, we will verify our *basic proposition*: each maximal group from an F-theory construction is a subgroup of the field-theoretic group determined in section 5. In most cases, we will find that F-theory can realize the field-theoretic group.

Note that in many cases we cannot determine from the local analysis whether the Kodaira fibers over the non-compact curve have monodromy or not, so we assume the maximal global symmetry group compatible with the data (i.e., the one without monodromy). In a few cases, though, the nature of the intersection constrains the non-compact curve to have monodromy, thereby reducing the size of the global symmetry group.

This is a good moment to point out a small assumption we have been forced to make in our analysis. The discussion of Tate's algorithm in [42] and [43] is incomplete for Kodaira types I_n , $7 \leq n \leq 9$, and it is therefore conceivable that our analysis misses some cases associated with those Kodaira types.

Let us consider an irreducible effective divisor $\Sigma = \{z = 0\} \subseteq \{\Delta = 0\}$ with self-intersection number $\Sigma \cdot \Sigma = -m$. We introduce the condensed notation $(a, b, d)_\Sigma$ to indicate

the orders of vanishing of f , g and Δ along Σ , respectively. Σ must be topologically a \mathbb{P}^1 and its genus is $g = 0$, thus we have (as in (5.1))

$$K_B \cdot \Sigma = -2 + m . \tag{6.1}$$

The quantities

$$\tilde{f} \equiv \frac{f}{z^a}, \quad \tilde{g} \equiv \frac{g}{z^b}, \quad \tilde{\Delta} \equiv \frac{\Delta}{z^d}, \tag{6.2}$$

are sections of the following line bundles

$$\tilde{f} \leftrightarrow \mathcal{O}(-4K_B - a\Sigma), \quad \tilde{g} \leftrightarrow \mathcal{O}(-6K_B - b\Sigma), \quad \tilde{\Delta} \leftrightarrow \mathcal{O}(-12K_B - d\Sigma) . \tag{6.3}$$

We define the *residual vanishings* on Σ as

$$\begin{aligned} \tilde{a}_\Sigma &\equiv (-4K_B - a\Sigma) \cdot \Sigma = -4(m - 2) + ma, \\ \tilde{b}_\Sigma &\equiv (-6K_B - b\Sigma) \cdot \Sigma = -6(m - 2) + mb, \\ \tilde{d}_\Sigma &\equiv (-12K_B - d\Sigma) \cdot \Sigma = -12(m - 2) + md . \end{aligned} \tag{6.4}$$

These values, which are required to be non-negative since \tilde{f} , \tilde{g} , $\tilde{\Delta}$ do not contain Σ as a component, count the number of zeros (with multiplicity) of \tilde{f} , \tilde{g} and $\tilde{\Delta}$, respectively, when restricted to Σ . We will refer to $\tilde{\Delta}$ as the *residual discriminant*. Extending the notation above, we will indicate the triple (6.4) as $(\tilde{a}, \tilde{b}, \tilde{d})_\Sigma$. Our analysis aims to determine which collections of local configurations are globally compatible with (6.4).

The first step towards our goal is to tackle the intersection between two curves. Let $\Sigma = \{z = 0\}$ and $\Sigma' = \{\sigma = 0\}$ be two such curves intersecting at a point $P \equiv \Sigma \cap \Sigma'$. We describe this situation locally by a Weierstrass model, i.e., by specifying the quantities in (4.1). Following the discussion above we demand that the multiplicities of f and g at P do not exceed 4 and 6, respectively.¹² This basic requirement yields a first set of constraints. In fact, we can exclude all pairwise intersections that satisfy

$$a_\Sigma + a_{\Sigma'} \geq 4, \quad b_\Sigma + b_{\Sigma'} \geq 6 . \tag{6.5}$$

As an example of this, we see from table 1 that a curve carrying singularity type II^* is in principle only allowed to have intersections with curves carrying singularity type I_n . These remaining cases are fully treated in appendix A. In general, the situation is a bit more complicated, as

$$\text{ord}_P f \geq a_\Sigma + a_{\Sigma'}, \quad \text{ord}_P g \geq b_\Sigma + b_{\Sigma'}, \quad \text{ord}_P \Delta \geq d_\Sigma + d_{\Sigma'} . \tag{6.6}$$

It is worth noting that there are cases where the strict inequalities hold; that is, we find additional contributions to the multiplicity at P beyond those coming from the degrees of vanishing along the two curves. Geometrically, this means that our local model describes

¹²If that is not the case, i.e., if the multiplicity of f at P is at least 4 and the multiplicity of g at P is at least 6, it follows that the multiplicity of Δ at P is at least 12.

a situation in which other components of the quantities f , g and Δ intersect at P . Hence, our criterion to discard pairwise intersections now reads

$$\text{ord}_P f \geq 4, \quad \text{ord}_P g \geq 6. \quad (6.7)$$

For each intersection, it is then natural to define

$$\tilde{a}_P \equiv \text{ord}_P \tilde{f}|_{z=0}, \quad \tilde{b}_P \equiv \text{ord}_P \tilde{g}|_{z=0}, \quad \tilde{d}_P \equiv \text{ord}_P \tilde{\Delta}|_{z=0}, \quad (6.8)$$

and we will use the notation $(\tilde{a}_P, \tilde{b}_P, \tilde{d}_P)_\Sigma$.

The second step in our analysis is to determine how to glue these local models for pairwise intersections into globally well-defined configurations. Let $\Sigma = \{z = 0\}$ be a curve as above, and let Σ_k , $k = 1, \dots, N$, be a collection of curves, each transversely intersecting pairwise with Σ at the points P_k . We implement the global constraints on the assembly of the local configurations; these read

$$\tilde{a}_\Sigma \geq \sum_k \tilde{a}_{P_k}, \quad \tilde{b}_\Sigma \geq \sum_k \tilde{b}_{P_k}, \quad \tilde{d}_\Sigma \geq \sum_k \tilde{d}_{P_k}. \quad (6.9)$$

Before we proceed to a more detailed case by case investigation, we wish to make a couple of general remarks. We say that $\Sigma = \{z = 0\}$ carries *odd* type whenever the discriminant has the form

$$\frac{\Delta}{z^d} = \left(4\tilde{f}^3 + 27z^p\tilde{g}^2\right), \quad (6.10)$$

for some $p > 0$ and $z \nmid \tilde{f}$. We indicate this as $(a, b + B, d)_\Sigma$, where $B = 0, 1, \dots$. Setting $z = 0$, the second term in the r.h.s. vanishes identically and we find that $\tilde{d}_P = 3\tilde{a}_P$. Similarly, we say Σ carries *even* type when $(a + A, b, d)_\Sigma$, $A = 0, 1, \dots$, and the residual discriminant has the form

$$\frac{\Delta}{z^d} = \left(4z^p\tilde{f}^3 + 27\tilde{g}^2\right), \quad (6.11)$$

for some $p > 0$ and $z \nmid \tilde{g}$. In this case we have instead $\tilde{d}_P = 2\tilde{b}_P$. We describe the remaining cases, i.e., I_n and I_n^* , as *hybrid* types, since both $\tilde{f}|_{z=0}$ and $\tilde{g}|_{z=0}$ in principle contribute to the residual discriminant.

In the even and odd cases, we are able to treat the general cases at once, i.e., for any value of A and B . This is done by relaxing the first condition in (6.9) for the even case and the second in the odd case. For hybrid curves, this is however not possible, though we will argue in the relevant case (type I_0^*) that the maximal group is obtained for $A = B = 0$.

6.1 Type I_n

Let $\Sigma = \{z = 0\}$ carry type I_n for $n \geq 2$ with self-intersection number $\Sigma^2 = -m$. The residual vanishings on Σ are given by

$$(\tilde{a}, \tilde{b}, \tilde{d})_\Sigma = (8 - 4m, 12 - 6m, 24 + (n - 12)m), \quad (6.12)$$

where $m = 1, 2$. From the analysis in [42] and [43] (reviewed in appendix A) we have

$$f|_{z=0} = -\frac{1}{48}\phi_n^2, \quad g|_{z=0} = \frac{1}{864}\phi_n^3, \quad (6.13)$$

for some locally-defined function ϕ_n , which depends on n . Except when $n = 2$, where there is no monodromy issue, we can additionally write¹³ $\phi_n = \mu\phi_0^2$ with μ square-free and the monodromy is determined by whether μ vanishes¹⁴ somewhere along Σ .

The case $m = 2$ is very simple and we consider it first. Notice that $\tilde{a}_\Sigma = \tilde{b}_\Sigma = 0$, implying that neither quantity in (6.13) can vanish. This means that μ cannot vanish anywhere along Σ , so the case with monodromy cannot be realized. This matches with the fact that $\Sigma^2 = -1$ for $\mathfrak{sp}(n)$ as derived in section 5. Hence, there is no monodromy along Σ and the only allowed configurations are chains of type $\mathcal{C}_{n_1, \dots, n_N} \equiv [I_{n_1}, \dots, I_{n_N}]$ with associated algebras $\mathfrak{su}(n_1) \oplus \dots \oplus \mathfrak{su}(n_N)$. In particular, the maximal configuration will be $\mathcal{C}_{\tilde{d}_\Sigma}$, yielding the algebra $\mathfrak{su}(\tilde{d}_\Sigma)$ where $\tilde{d}_\Sigma = 2n$. We easily see that the basic proposition is verified here, as the algebra from field theory is $\mathfrak{so}(8)$ for $n = 2$ and $\mathfrak{su}(2n)$ for $n \geq 3$.

In the rest of this section, we will deal with the case $m = 1$, for which $(\tilde{a}, \tilde{b})_\Sigma = (4, 6)$. We start by proving some simple but useful results for $n \geq 3$, where from (6.13) we have $\deg \phi_n = 2$ and

$$\deg \mu + 2 \deg \phi_0 = 2. \tag{6.14}$$

If Σ has monodromy, then $\deg \mu > 0$ and by (6.14) we get $\deg \mu = 2$ and $\deg \phi_0 = 0$. Otherwise, Σ has no monodromy and $\deg \mu = 0$, $\deg \phi_0 = 1$.

We wish to study the restrictions that this global constraint imposes on different Kodaira singularity types when present in a configuration over Σ . Let $n = 2i$ or $n = 2i + 1$ and let $\Sigma' = \{\sigma = 0\}$ carry type $I_{n'}$, where $n' = 2j$ or $n' = 2j + 1$. To determine the maximal allowed value for n' we will consider for simplicity the general forms given in (A.25). The divisibility conditions read

$$z, \sigma \nmid u, \quad z^i \sigma^j \mid v, \quad z^{2i} \sigma^{2j} \mid w. \tag{6.15}$$

The residual discriminant for Σ is

$$\frac{\Delta}{z^n} \Big|_{z=0} = u^2 \Big|_{z=0} \underbrace{(4uw - v^2)}_{\hat{\Delta}} \Big|_{z=0}, \tag{6.16}$$

and by (6.15) we have that $\sigma^{n'} \mid \hat{\Delta}$; that is the degree of vanishing of $\hat{\Delta}$ at $\sigma = 0$ is at least n' . In other words, though $u|_{z=0}$ could vanish at $\sigma = 0$ as well (therefore contributing to \tilde{d}_P), it does not increase the degree of vanishing of Δ along the transverse type $I_{n'}$ curve. In the remainder of this section, we assume without loss of generality that $u|_{z=0}$ does not vanish at the intersection points between Σ and curves of type $I_{n'}$.

Now we wish to discuss the implications of a transverse intersection with a curve carrying any of the remaining Kodaira singularity types. If Σ has no monodromy, we observe from table 6 that type I_0^* and type I_n^* are not allowed in any configurations, while type IV is forbidden for $n \geq 4$ and type III for $n \geq 5$. Moreover, here $\deg \phi_0 = 1$, thus there can be at most one curve carrying type other than $I_{n'}$, which will intersect Σ at $\phi_0|_{z=0} = 0$.

¹³Of course, μ and ϕ_0 depend on n , but we will not keep track of this index.

¹⁴Note that our previously stated criterion for monodromy in this case asked whether $-9g/2f$, when restricted to $z = 0$, is a square or not. Since $-9g/2f|_{z=0} = \frac{1}{4}\phi_n = \frac{1}{4}\mu\phi_0^2$, this is clearly equivalent.

In the monodromy case, $\deg \mu = 2$ and configurations can admit up to two curves carrying type other than $I_{n'}$ intersecting Σ precisely at the roots of $\mu|_{z=0}$. To summarize, the relevant configurations are chains of the type $\mathcal{C}_{n_1, \dots, n_N}$ with one or two other singularity types attached. However, since $\mathcal{C}_{n_1, \dots, n_N}$ yields the algebra $\oplus_i \mathfrak{su}(n_i)$, when looking for the maximal global symmetry group we will just consider $\mathcal{C}_{\sum_i n_i}$ with algebra $\mathfrak{su}(\sum_i n_i)$.

Perhaps a comment is useful here. We stated above that Kodaira types other than $I_{n'}$ can only intersect Σ at the roots of $\phi_n|_{\Sigma}$. There is an example that seems to contradict this. Let us consider an intersection with type IV along the locus $\{\sigma = 0\}$, then both f and g vanish to order 2 in σ and it seems that $\phi_n = -9g/2f|_{z=0}$ need not vanish. However, the point is that the quantity g/f is evaluated along Σ , where $g|_{z=0}$ and $f|_{z=0}$ can have higher orders of vanishing. These are determined carefully in appendix A, and it turns out that for all intersections, except with type $I_{n'}$, ϕ_n does indeed vanish.

Having summarized some of the general features of the pertinent local models, let us proceed to a detailed analysis.

$n = 2$

In this case, there is no monodromy ambiguity and we see from table 6 that types III, IV, I_0^* and $I_{n'}^*$ can be part of a configuration on Σ . Here,

$$\frac{\Delta}{z^2} \Big|_{z=0} = \phi^2 (\phi \tilde{g}_2 - f_1^2) , \tag{6.17}$$

where the quantities on the r.h.s. are restricted to the locus $\{z = 0\}$. In the language of (6.16), we define

$$\hat{\Delta} \equiv \phi \tilde{g}_2 - f_1^2 . \tag{6.18}$$

From (6.13), it follows $\deg \phi = 2$ and there can be up to two curves carrying type other than $I_{n'}$ in a configuration. Let $\Sigma' = \{\sigma = 0\}$ carry a given Kodaira type and let $P \equiv \Sigma \cap \Sigma'$. Our strategy is to determine the lowest order of vanishing of $\hat{\Delta}$ at P, and introducing a non-standard notation, we will denote this value by $\text{ord}_P \hat{\Delta}$. This is easily done by recalling the general form for type I_2

$$f = -\frac{1}{48} \phi^2 + f_1 z + O(z^2), \quad g = \frac{1}{864} \phi^3 - \frac{1}{12} \phi f_1 z + \left(\tilde{g}_2 - \frac{1}{12} \phi f_2 \right) z^2 + O(z^3), \tag{6.19}$$

as well as the data from table 1. In fact, let Σ' carry type III, then $\sigma|f$ and $\sigma^2|g$, making $\sigma|\{\phi, f_1\}$ and $\sigma^2|\tilde{g}_2$. Plugging these into (6.18), we obtain $\sigma^2|\hat{\Delta}$. If Σ' carries type IV, then $\sigma^2|f$ and $\sigma^2|g$, and hence $\sigma^2|f_1$ while everything else is unchanged from the case above, in turn yielding $\text{ord}_P \hat{\Delta} = 3$. Next, we consider the case of Σ' carrying type I_p^* for $p \geq 0$. The argument here is somewhat different, as it suffices to realize that

$$6 + p = \text{ord}_P \Delta = 2 \text{ord}_P \phi + \text{ord}_P \hat{\Delta} . \tag{6.20}$$

Now, the solution (6.19) is equivalent to (A.25) with the following identifications

$$u = \frac{1}{4} \phi, \quad v_i = f_i, \quad i \geq 1, \quad w_j = \tilde{g}_j, \quad j \geq 2 . \tag{6.21}$$

The Tate form for I_p^* (regardless of monodromy) prescribes $\sigma|u$ but $\sigma^2 \nmid u$, hence $\text{ord}_P \phi = 1$.¹⁵ This determines $\text{ord}_P \widehat{\Delta} = 4 + p$. We summarize all of this in the following table

	I_0^*	I_p^*	III	IV	$I_{n'}$
$\text{ord}_P \widehat{\Delta}$	4	$4 + p$	2	3	n'

This also allows us to reduce the types of configurations we need to consider. From the data above, as far as algebras are concerned

$$[\dots, \text{III}, \dots, I_{n'}, \dots] \subset [\dots, I_{n'+2}, \dots], \quad [\dots, \text{IV}, \dots, I_{n'}, \dots] \subset [\dots, I_{n'+3}, \dots]. \quad (6.22)$$

Hence, we avoid considering type III and type IV. Finally, recalling that $\deg \widehat{\Delta} = 10$, we obtain the following relevant configurations

configuration(s)	algebra(s)
I_{10}	$\mathfrak{su}(10)$
$I_p^*, I_{6-p}, 0 \leq p \leq 6$	$\mathfrak{so}(8 + 2p) \oplus \mathfrak{su}(6 - p)$
$I_p^*, I_q^*, I_{2-p-q}, 0 \leq \{p, q, p + q\} \leq 2$	$\mathfrak{so}(8 + 2p) \oplus \mathfrak{so}(8 + 2q) \oplus \mathfrak{su}(2 - p - q)$

We conclude that the global symmetry of the maximal configuration $[I_6^*]$ agrees with the field-theoretic prediction of $\mathfrak{so}(20)$.

$n \geq 3$ odd

Here $\widetilde{d}_\Sigma = 12 + n$, and from appendix A we have

$$\frac{\Delta}{z^n} \Big|_{z=0} = u_0^2 (4u_0 w_{2i+1} + 4u_1 w_{2i} - 2v_i v_{i+1}), \quad (6.23)$$

where $u_0 = \mu \phi_0^2$ and $n = 2i + 1$. As before, we assume throughout the rest of this section the generic form (A.25), and we can rewrite the last expression as

$$\frac{\Delta}{z^n} \Big|_{z=0} = \frac{1}{16} \mu^3 \phi_0^4 (\phi_0^2 w_{2i+1} + t_i^2 u_1 - \phi_0 t_i v_{i+1}), \quad (6.24)$$

denoting by $\widehat{\Delta}$ the term inside the brackets. Even for the cases $n = 3$ and $n = 5$ the expressions above are completely general, as there is an isomorphisms between the explicit forms (A.11) and (A.20) and the inductive form (A.25), given by (for $n = 5$)

$$u_0 = \frac{1}{4} \mu \phi_0^2, \quad u_1 = \phi_1, \quad v_2 = \frac{1}{2} \mu \phi_0 \psi_2, \quad v_k = f_k, \quad k \geq 3, \quad w_4 = \frac{1}{4} \mu \psi_2^2, \quad w_l = \widehat{g}_l, \quad l \geq 5. \quad (6.25)$$

First, let us suppose that there is monodromy on Σ . From above we have that $\deg \mu = 2$ and $\deg \phi_0 = 0$; we set $\phi_0 \equiv 1$ for concreteness. This implies $\deg \widehat{\Delta} = \widetilde{d}_\Sigma - 6 = 6 + n$, yielding I_{6+n} as the largest type $I_{n'}$ allowed in any configuration.

¹⁵In other words, if $\text{ord}_P \phi > 1$ then we are in a different brach of Kodaira's classification.

To take into account the presence of other singularity types we need to determine, as before, the lowest vanishing order of $\widehat{\Delta}$ along $\Sigma' = \{\sigma = 0\}$. We recall from (A.28) the expressions

$$\begin{aligned} u &= \frac{1}{4}\mu + u_1z + O(z^2), \\ v &= \frac{1}{2}\mu t_i z^i + v_{i+1}z^{i+1} + O(z^{i+2}), \\ w &= \frac{1}{4}\mu t_i^2 z^{2i} + w_{2i+1}z^{2i+1} + O(z^{2i+2}). \end{aligned} \tag{6.26}$$

When Σ' carries type III, we have that $\sigma|\{u, v\}$ and $\sigma^2|w$. Since σ^2 must divide μt_i^2 and μ is square-free, it follows that $\sigma|t_i$. All of this implies that $\sigma^2|\widehat{\Delta}$ for type III. When Σ' carries type IV without monodromy (if there is monodromy, the gauge algebra is $\mathfrak{su}(2)$ and the discussion follows the one above for type III), we have $\sigma|u$ and $\sigma^2|\{v, w\}$. Hence, the terms in

$$g = \frac{2}{27}u^3 - \frac{1}{3}uv + w \tag{6.27}$$

have degrees of vanishing 3, 3 and 2 along $\{\sigma = 0\}$, respectively. Monodromy forces $g/\sigma^2|_{\sigma=0} = w$ to be a square, and from (6.26), we obtain

$$\frac{w}{\sigma^2}\Big|_{\sigma=0} = \widetilde{w}_{2i+1}\Big|_{\sigma=0}z^{2i+1} + \dots, \tag{6.28}$$

where $w_{2i+1} = \sigma^2\widetilde{w}_{2i+1}$. Whether (6.28) is a square or not is contingent upon whether \widetilde{w}_{2i+1} vanishes identically or it is further divisible by σ . In either case, it is straightforward to verify that $\text{ord}_P\widehat{\Delta} = 3$.

In the case Σ' supports type I_0^* , there are two cases we need to distinguish. In fact, while $\sigma|u$ and $\sigma^2|v$ always hold, we have that $\sigma^3|w$ for I_0^{ns} (algebra \mathfrak{g}_2) and $\sigma^4|w$ for both I_0^{ss} and I_0^{rs} (algebras $\mathfrak{so}(7)$ and $\mathfrak{so}(8)$, respectively). For the latter case, we will consider only I_0^{rs} as it leads to the largest global symmetry. It follows that $\sigma|t_i$ for I_0^{ns} and $\sigma^2|t_i$ for I_0^{rs} , and (6.24) determines $\sigma^3|\widehat{\Delta}$ for I_0^{ns} and $\sigma^4|\widehat{\Delta}$ for I_0^{rs} .

The remaining case we need to analyze is when Σ' carries type I_p^* for $p \geq 1$. First suppose that $p = 2n' + 1$; then $\sigma|u$, $\sigma^{n'+3}|v$ and $\sigma^{2n'+4}|w$, which in turn implies $\sigma^{n'+2}|t_i$. This leads to $\text{ord}_P\widehat{\Delta} = 2n' + 4 = p + 3$, and this holds for I_p^{ns} . The condition for monodromy is whether

$$\frac{w}{\sigma^{2n'+4}}\Big|_{\sigma=0} = \widetilde{w}_{2i+1}\Big|_{\sigma=0}z^{2i+1} + \dots, \tag{6.29}$$

where $w_{2i+1} = \sigma^{2n'+4}\widetilde{w}$, has a square root. As before, in order to have no monodromy $\widetilde{w}_{2i+1}|_{\sigma=0} = 0$, and this makes $\text{ord}_P\widehat{\Delta} = 2n' + 5 = p + 4$. The even case, $p = 2n'$, is similar, but we repeat the argument for completeness. Here we have $\sigma|u$, $\sigma^{n'+2}|v$ and $\sigma^{2n'+3}|w$, which leads to $\sigma^{n'+1}|t_i$. It is convenient to define $\mu = \sigma\tilde{\mu}$, $u_1 = \sigma\tilde{u}_1$, $t_i = \sigma^{n'+1}\tilde{t}_i$, $v_{i+1} = \sigma^{n'+2}\tilde{v}_{i+1}$ and $w_{2i+1} = \sigma^{2n'+3}\tilde{w}_{2i+1}$. This gives $\text{ord}_P\widehat{\Delta} = 2n' + 3 = p + 3$, valid for I_p^{ns} . In order to have no monodromy, the quantity

$$4\left(\frac{u}{\sigma}\right)\Big|_{\sigma=0} \left(\frac{w}{\sigma^{2n'+3}}\right)\Big|_{\sigma=0} - \frac{v}{\sigma^{2n'+2}}\Big|_{\sigma=0} = \tilde{\mu}(\tilde{t}_i^2\tilde{u}_1 + \tilde{w}_{2i+1} - \tilde{t}_i\tilde{v}_{i+1})\Big|_{\sigma=0}z^{2i+1} + O(z^{2i+2}), \tag{6.30}$$

must be a square. Therefore, the coefficient of the first term on the r.h.s. of (6.30) must vanish and in particular it coincides, up to an irrelevant factor of $\tilde{\mu}$, with (6.24), giving $\text{ord}_P \widehat{\Delta} = p + 4$. We summarize these results in the following table

	I_0^{*ns}	I_0^{*s}	I_p^{*ns}	I_p^{*s}	IV	III	$I_{n'}$
$\text{ord}_P \widehat{\Delta}$	3	4	$3 + p$	$4 + p$	3	2	n'

The argument we provided in the $n = 2$ case for type III and type IV goes through here as well, since for these, the values of $\text{ord}_P \widehat{\Delta}$ are identical. Thus, we will exclude configurations containing them. The remaining relevant configurations are

configuration(s)	algebra(s)
I_{n+6}	$\mathfrak{su}(n + 6)$
$I_p^{*s}, I_{n+2-p}, 0 \leq p \leq n + 2$	$\mathfrak{so}(8 + 2p) \oplus \mathfrak{su}(n + 2 - p)$
$I_p^{*ns}, I_{n+3-p}, 0 \leq p \leq n + 3$	$\mathfrak{so}(7 + 2p) \oplus \mathfrak{su}(n + 3 - p)$
$I_p^{*s}, I_q^{*s}, I_{n-2-p-q}, 0 \leq \{p, q, p+q\} \leq n-2$	$\mathfrak{so}(8+2p) \oplus \mathfrak{so}(8+2q) \oplus \mathfrak{su}(n-2-p-q)$
$I_p^{*s}, I_q^{*ns}, I_{n-1-p-q}, 0 \leq \{p, q, p+q\} \leq n-1$	$\mathfrak{so}(8+2p) \oplus \mathfrak{so}(7+2q) \oplus \mathfrak{su}(n-1-p-q)$
$I_p^{*ns}, I_q^{*ns}, I_{n-p-q}, 0 \leq \{p, q, p+q\} \leq n$	$\mathfrak{so}(7+2p) \oplus \mathfrak{so}(7+2q) \oplus \mathfrak{su}(n-p-q)$

By inspection, we see that within each row of the above table, except the first, the algebra is maximal when p or $p + q$ assume the highest allowed value, i.e., when the last curve is I_0 . This also tells us that $[I_p^{*s}, I_{n-2-p}^{*s}] \subset [I_{n+2}^{*s}]$ and $[I_p^{*s}, I_{n-1-p}^{*ns}] \subset [I_{n+3}^{*ns}]$ and of course $[I_{n+6}] \subset [I_{n+2}^{*s}] \subset [I_{n+3}^{*ns}]$. The locally maximal configurations are then given by

configuration(s)	algebra(s)
I_{n+3}^{*ns}	$\mathfrak{so}(13 + 2n)$
$I_p^{*ns}, I_{n-p}^{*ns}, 0 \leq p \leq \frac{n+1}{2}$	$\mathfrak{so}(7 + 2p) \oplus \mathfrak{so}(7 + 2n - 2p)$

The predicted algebra is $\mathfrak{so}(16+2(n-1))$. It is easy to check that the basic proposition holds for each value of n . This example shows two important characteristics of our findings, as we mentioned in the discussion above. First, we verified explicitly that the algebra predicted through field theory is not always realizable in F-theory. Second, this model exhibits multiple relatively maximal algebras, each subalgebras of the maximal algebra from field theory.

If there is no monodromy on Σ , we set $\mu \equiv 1$ and $\text{deg } \phi_0 = 1$. Then $\text{deg } \widehat{\Delta} = \widetilde{d}_\Sigma - 4 = 8 + n$. We recall that type I_0^* and type I_n^* are forbidden in all configurations. However, as mentioned above, for $n = 3$ type III and type IV are still allowed, so we obtain

configuration	algebra	(6.31)
I_{11}	$\mathfrak{su}(11)$	
IV, I_8	$\mathfrak{su}(3) \oplus \mathfrak{su}(8)$	
III, I_9	$\mathfrak{su}(2) \oplus \mathfrak{su}(9)$	

These are all subalgebras of $\mathfrak{su}(11)$ (which is in turn a subalgebra of field-theoretic prediction $\mathfrak{su}(12)$). For $n \geq 5$, the only relevant configurations are of the form $\mathcal{C}_{n_1, \dots, n_N}$, and

the maximal algebra among these results from $\mathcal{C}_{\tilde{d}-4}$. This yields $\mathfrak{su}(8+n)$, which coincides with the prediction from field theory.

$n \geq 4$ even

In this case, we set $n = 2i$ and the residual discriminant assumes the form

$$\frac{\Delta}{z^n} \Big|_{z=0} = u_0^2 (4u_0 w_{2i} - v_i^2) , \tag{6.32}$$

where again $u_0 = \frac{1}{4}\mu\phi_0^2$, and

$$u = \frac{1}{4}\mu\phi_0^2 + u_1 z + \dots , \quad v = v_i z^i + v_{i+1} z^{i+1} + \dots , \quad w = w_{2i} z^{2i} + w_{2i+1} z^{2i+1} + \dots . \tag{6.33}$$

We define $\widehat{\Delta} \equiv 4u_0 w_{2i} - v_i^2$.

Suppose first that Σ has monodromy. In this case the form (6.32) is generic within the assumptions of this paper. In fact, we have already showed that the inductive form (A.25) reproduces the generic form for type I_5 regardless of monodromy, and in particular for type I_4 . For type I_6 this is not true in general, but it holds when we impose that Σ has monodromy. In this case, recall that $\deg \mu = 2$ and $\deg \phi_0 = \deg \alpha + \deg \beta = 0$ and we set wlog $\alpha = \beta \equiv 1$. The identification between the two sets of local functions is

$$\begin{aligned} u_0 &= \frac{1}{4}\mu , & u_1 &= \nu , & u_2 &= \phi_2 , \\ v_3 &= \frac{1}{3}\nu\phi_2 - 3\lambda , & v_4 &= f_4 + \frac{1}{3}\phi_2^2 , & v_k &= f_k , \quad k \geq 5 , \\ w_6 &= \widehat{g}_6 + \frac{1}{3}f_4\phi_2 + \frac{1}{27}\phi_2^3 , & w_l &= \widehat{g}_l , \quad l \geq 7 . \end{aligned} \tag{6.34}$$

Hence, the only case which is not captured in full generality¹⁶ by (6.32) is $n = 6$ with no monodromy, which we will discuss separately.

Let us proceed and determine the lowest order of vanishing of $\widehat{\Delta}$ for intersections with different Kodaira types. Given the simpler form of the solution (6.33) with respect to (6.26), the analysis will be less involved. Let $\Sigma' = \{\sigma = 0\}$ carry type III, then $\sigma| \{u, v\}$, $\sigma^2|w$ and $\sigma^2|\widehat{\Delta}$. When Σ' carries type IV, the only modification is that now $\sigma^2|v$, yielding $\sigma^3|\widehat{\Delta}$. Now let Σ' carry type I_p^* , where $p \geq 0$. In what follows, we do not need to distinguish between the different monodromy cases, so we will just assume the largest induced algebra. If $p = 2n' + 1$ we have $\sigma|u$, $\sigma^{n'+3}|v$, $\sigma^{2n'+4}|w$ and $\sigma^{2n'+5}|\widehat{\Delta}$; if $p = 2n'$ we have $\sigma|u$, $\sigma^{n'+2}|v$, $\sigma^{2n'+3}|w$ and $\sigma^{2n'+4}|\widehat{\Delta}$. We summarize these results in the following table

	I_0^*	I_p^*	IV	III	$I_{n'}$	(6.35)
$\text{ord}_P \widehat{\Delta}$	4	$4 + p$	3	2	n'	

¹⁶As stated above, we are making an assumption for $n = 7, 8, 9$.

As before, we do not need to consider configurations involving type III and type IV. Since $\deg \widehat{\Delta} = \widetilde{d}_\Sigma - 4 = 8 + n$, we obtain

configuration(s)	algebra(s)
I_{8+n}	$\mathfrak{su}(8+n)$
$I_p^*, I_{4+n-p}, \quad 0 \leq p \leq n+4$	$\mathfrak{so}(8+2p) \oplus \mathfrak{su}(4+n-p)$
$I_p^*, I_q^*, I_{n-p-q}, \quad 0 \leq \{p, q, p+q\} \leq n$	$\mathfrak{so}(8+2p) \oplus \mathfrak{so}(8+2q) \oplus \mathfrak{su}(n-p-q)$

We see that there is a maximal algebra, namely $\mathfrak{so}(16+2n)$, given by $[I_{n+4}^*]$. This coincides with the prediction for each n .

If there is no monodromy on Σ (and $n \neq 6$), the only other singularity type other than $I_{n'}$ is type III for $n = 4$, for which $\widetilde{d}_P \geq 4$, as can be read from (6.35) and (6.32). The relevant configurations in this case are therefore

configuration	algebra
I_{12}	$\mathfrak{su}(12)$
III, I_{10}	$\mathfrak{su}(2) \oplus \mathfrak{su}(10)$

The prediction from field theory is $\mathfrak{su}(12)$, hence we have verified our basic proposition. If $n > 4$, the analysis is similar to the odd case: the only relevant configurations are $\mathcal{C}_{n_1, \dots, n_N}$ and the maximal is $\mathcal{C}_{\widetilde{d}-4}$. This again yields $\mathfrak{su}(8+n)$, matching the field theory result.

The remaining case to be analyzed is $n = 6$ without monodromy. From (A.17) we have

$$\widetilde{f}|_{z=0} = -\frac{1}{48}\alpha^4\beta^4, \quad \widetilde{g}|_{z=0} = \frac{1}{864}\alpha^6\beta^6, \quad (6.36)$$

as well as

$$\widetilde{\Delta}|_{z=0} = \frac{1}{432}\alpha^4\beta^3 \underbrace{(27\alpha^2\beta^3\widehat{g}_6 + 9\alpha^2\beta^2\phi_2f_4 + \alpha^2\phi_2^3 - 243\lambda^2\beta^3 + 54\phi_2\nu\lambda\beta^2 - 3\beta\nu^2\phi_2^2)}_{\widehat{\Delta}}. \quad (6.37)$$

Now, since $\Sigma^2 = -1$ we have that $(\widetilde{a}, \widetilde{b})_\Sigma = (4, 6)$, which together with (6.36) implies

$$\deg \alpha + \deg \beta = 1. \quad (6.38)$$

Also, we have that $\widetilde{d}_\Sigma = 18$ and

$$\widetilde{d}_\Sigma = 4 \deg \alpha + 3 \deg \beta + \deg \widehat{\Delta}. \quad (6.39)$$

We recall that the relevant configuration $\mathcal{C}_{n_1, \dots, n_N}$ only has support on $\widehat{\Delta}$, that is $\sigma_i \nmid \{\alpha, \beta\}$, where $\{\sigma_i = 0\}$ supports the i -th curve in the configuration. We can then rewrite (6.39) as

$$18 - \sum_i n_i \geq 4 \deg \alpha + 3 \deg \beta. \quad (6.40)$$

There are two possibilities:

1. $\deg \alpha = 1, \deg \beta = 0$. Then $\sum_i n_i \leq 14$ and the maximal configuration is $\mathcal{C}_{\widetilde{d}_\Sigma-4}$ with algebra $\mathfrak{su}(14)$. This is the algebra associated to the $\mathfrak{su}(6)$ row of table 2.
2. $\deg \alpha = 0, \deg \beta = 1$. Here, we instead have $\sum_i n_i \leq 15$ and the maximal configuration is $\mathcal{C}_{\widetilde{d}_\Sigma-3}$ with algebra $\mathfrak{su}(15)$. This corresponds to the alternative possibility $\mathfrak{su}(6)^*$.

6.2 Type III

Let $\Sigma = \{z = 0\}$ carry singularity type III and $\Sigma \cdot \Sigma = -m$. Plugging $(a, b, d)_\Sigma = (1, 2 + B, 3)_\Sigma$ in (6.4) we compute $(\tilde{a}, \tilde{b}, \tilde{d})_\Sigma = (8 - 3m, 12 - 4m + Bm, 24 - 9m)$. We now proceed to separately analyze the two cases $m = 1, 2$.

Let us start with $m = 2$. First of all, we need to determine which curves locally admit allowed intersections. From (6.5) and appendix B we see that type IV*, III* and II* are forbidden, as well as type I_n* for $n \geq 1$. From table 6, we see that an intersection with I_n is possible for each n , which together with type III, IV and I₀* constitute the blocks for building configurations. As we remarked above, we do not need to worry about \tilde{b}_P for odd types. Thus, we list the above singularity types with their contributions to \tilde{a}_P , borrowing the appropriate entries from table 1 and table 6

	III	IV	I ₀ *	I _n
\tilde{a}_P	1	2	≥ 2	$\lceil n/2 \rceil$

Now, since $\tilde{a}_\Sigma = 2$ we must exclude I_{n \geq 5}, and the various possibilities compatible with (6.9) are

configuration	algebra
III, III	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
I ₂ , III	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
I ₂ , I ₂	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
I ₃	$\mathfrak{su}(3)$
I ₄	$\mathfrak{su}(4)$
IV	$\mathfrak{su}(3)$
I ₀ *	$\mathfrak{so}(7)$ or \mathfrak{g}_2

Note that we cannot assume that an I₀* meeting a type III supports an $\mathfrak{so}(8)$ algebra because that would require all intersection points with the residual discriminant (even the ones not near the current intersection) to have even multiplicity. And indeed, the analysis of [29] shows that it can be at most $\mathfrak{so}(7)$. Explicitely, let $\Sigma' = \{\sigma = 0\}$ carry type I₀*, then

$$f = z\sigma^2 f_0, \quad g = z^2\sigma^3 g_0(\sigma), \quad (6.41)$$

where f_0 is a non-zero constant and $\deg g_0(\sigma) = 2B + 1$ such that $g_0(0) \neq 0$. The residual discriminant for Σ' reads

$$\frac{\Delta}{\sigma^6} \Big|_{\sigma=0} = z^3 (4f_0^3 + 27zg_0^2(0)), \quad (6.42)$$

which is not a square in z and the algebra supported on Σ' cannot be $\mathfrak{so}(8)$. We conclude that there is an algebra, namely $\mathfrak{so}(7)$, that admits as a subalgebra every other entry in the list and is the desired maximal global symmetry algebra in this case.

Let us turn to the case $m = 1$. The local models for the intersections are the same as above, but this time we have $\tilde{a}_\Sigma = 5$. This leads to a larger set of possibilities for the

intersecting curves, which we do not reproduce here. The result is that there are three different models that exhibit maximal global symmetry:

configuration	algebra
I_0^*, I_0^*, I_2	$\mathfrak{so}(7) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$
I_0^*, I_6	$\mathfrak{so}(7) \oplus \mathfrak{su}(6)$
I_{10}	$\mathfrak{su}(10)$

As before, we have indicated the monodromy choices in each case making the global symmetry algebra relatively maximal and we used the fact that type I_0^* cannot support an $\mathfrak{so}(8)$ algebra. We notice that these algebras are all contained in the algebra $\mathfrak{so}(20)$ predicted from field theory.

6.3 Type IV

We now turn to the analysis of a Kodaira type IV curve along $\Sigma = \{z = 0\}$ with $(a, b, d)_\Sigma = (2 + A, 2, 4)$, for which the residual vanishings are given by $(\tilde{a}, \tilde{b}, \tilde{d})_\Sigma = (8 - 2m + Am, 12 - 4m, 24 - 8m)$. The possibilities for the self-intersection are $m = 1, 2, 3$. Moreover, there are two possibilities for the monodromy on a type IV, the condition being whether $\tilde{g}|_{z=0}$ is a square. If $m = 3$, the gauge algebra must be $\mathfrak{su}(3)$, and there is no matter (and hence no global symmetry).

For $m = 2$, we have $\tilde{b}_\Sigma = 4$. Let us start by listing configurations that satisfy (6.9). These are

configuration	algebra
I_0^*	$\mathfrak{so}(7)$ or \mathfrak{g}_2
III, III	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
I_3, I_3	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$
IV, IV	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$
IV, I_3	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$
I_4	$\mathfrak{sp}(2)$

Now we will determine how the monodromy will impose different restrictions in each case.

- $\mathfrak{su}(2)$. In this case, $\tilde{g}|_{z=0}$ is not allowed to have all double roots. Therefore, we discard the configurations for which each intersection yields an even value for \tilde{b}_P . All configurations in the list above except the first fall into this category. As in the case of type III, the type I_0^* gives $\mathfrak{so}(7)$ (or possibly \mathfrak{g}_2) as global algebra and it is maximal in this case.
- $\mathfrak{su}(3)$. In this case, $\tilde{g}|_{z=0}$ is a square and hence all its roots are double roots. Moreover, Σ is not allowed to intersect $\Sigma' = \{\sigma = 0\}$ carrying type I_0^* since $\tilde{b}_P \geq 3$, where $P \equiv \Sigma \cap \Sigma'$, is increased at least to $\tilde{b}_P = 4$. Thus, at P we are in the case (6.5) and the intersection is non-minimal. The remaining configurations are in principle allowed and yield $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ and $\mathfrak{sp}(2)$ as maximal global symmetries, which indeed are both subalgebras of the algebra $\mathfrak{su}(6)$ predicted from field theory.

If $m = 1$, then $\tilde{b}_\Sigma = 8$ and the list of possible configurations is

configuration	algebra
I_0^*, I_0^*, IV	$\mathfrak{so}(7)^{\oplus 2} \oplus \mathfrak{su}(3)$
I_0^*, I_0^*, I_3	$\mathfrak{so}(7)^{\oplus 2} \oplus \mathfrak{su}(3)$
I_0^*, IV, IV	$\mathfrak{so}(7) \oplus \mathfrak{su}(3)^{\oplus 2}$
I_0^*, I_5	$\mathfrak{so}(7) \oplus \mathfrak{sp}(2)$
IV, IV, IV, IV	$\mathfrak{su}(3)^{\oplus 4}$
IV, I_6	$\mathfrak{su}(3) \oplus \mathfrak{sp}(3)$
I_8	$\mathfrak{sp}(4)$
I_4, IV, IV	$\mathfrak{sp}(2) \oplus \mathfrak{su}(3)^{\oplus 2}$

Again, we discuss the two monodromy cases separately.

- $\mathfrak{su}(2)$. Let us start by showing that the configuration $[I_0^*, I_0^*, IV]$ leads to a global symmetry that is not larger than $\mathfrak{so}(7) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(3)$. In fact, let $\{\sigma_1 = 0\}$, $\{\sigma_2 = 0\}$ and $\{\sigma_3 = 0\}$ be the curves carrying Kodaira type IV and the two type I_0^* , respectively. Then we have the following divisibility conditions

$$f = z^2 \sigma_1^2 \sigma_2^2 \sigma_3^2 f_0(\sigma), \quad g = z^2 \sigma_1^2 \sigma_2^3 \sigma_3^3 g_0, \quad (6.43)$$

where g_0 is a non-zero constant and $\deg f_0(\sigma) = \alpha$ and is such that $f_0(\sigma_i = 0) \neq 0$ for $i = 1, 2, 3$. Looking at $g/\sigma_1^2|_{\sigma_1=0}$, we see it is a perfect square in z , thus the gauge algebra along the intersecting IV is indeed $\mathfrak{su}(3)$ (this holds since the set-up already fills out \tilde{b}_Σ). Now, to determine the monodromy on the I_0^* curves, we look at the residual discriminant restricted to one of these curves, say $\{\sigma_2 = 0\}$, which reads

$$\frac{\Delta}{\sigma_2^6} \Big|_{\sigma_2=0} = z^4 (4z^2 \sigma_1^6 \sigma_3^6 f_0^3 + 27 \sigma_1^4 \sigma_3^6 g_0^2) \Big|_{\sigma_2=0}, \quad (6.44)$$

which is not a perfect square in z . Thus, the algebra cannot be $\mathfrak{so}(8)$ for either of the two curves carrying type I_0^* . We are therefore restricted to at most $\mathfrak{so}(7) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(3)$. This argument actually applies for any configuration that contains type I_0^* , in particular for the first four configurations of the above list. Finally, as before, we discard configurations in which each intersection yields an even value for \tilde{b}_P , for example a configuration of four type IV. This leaves $\mathfrak{so}(7) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(3)$ as the maximal algebra.

- $\mathfrak{su}(3)$. Here we recall that a configuration containing type I_0^* is not allowed. Therefore, we are left with the last four configurations of the list above, and all of these lead to subalgebras of the field-theoretic prediction $\mathfrak{su}(12)$.

6.4 Type I_0^*

Let us turn our attention to $\Sigma = \{z = 0\}$ carrying type I_0^* with $\Sigma^2 = -m$ and $(2 + A, 3 + B, 6)_\Sigma$, where A and B cannot be both nonzero. The residual vanishings on Σ

are $(\tilde{a}, \tilde{b}, \tilde{d})_\Sigma = (8 - 2m + Am, 12 - 3m + Bm, 24 - 6m)$. The local analysis derived in appendix A still applies, but it must be supplemented by a more careful treatment of the global constraints to be imposed on the Weierstrass model. We are going to prove some simple results about the *global* models involving intersections between type I_0^* and various curves carrying type I_n , for some n . In particular, we look for upper bounds on n such that type I_n is allowed to intersect type I_0^* , although *locally* the intersection is consistent for any n as can be seen from table 6.

There are three possibilities for the gauge algebra on Kodaira type I_0^* , determined by the behavior of the monodromy cover

$$\psi^3 + \frac{f}{z^2} \Big|_{z=0} \psi + \frac{g}{z^3} \Big|_{z=0} . \tag{6.45}$$

If the monodromy cover is irreducible, the gauge algebra is \mathfrak{g}_2 ; if it has two irreducible components, we have $\mathfrak{so}(7)$ and if it has three components, the gauge algebra is $\mathfrak{so}(8)$. Since the prediction from field theory as well as the results we find here differ for these cases, we will treat them separately.

We are going to argue that one needs only to consider configurations of the form $\mathcal{C}_{2n_1, \dots, 2n_N} \equiv [I_{2n_1}, \dots, I_{2n_N}]$. Concerning type III and type IV, we recall from the previous section that type IV intersecting type I_0^* always has monodromy, i.e., it carries algebra $\mathfrak{su}(2)$, so at least as far as global symmetry is concerned, we can replace $[\dots, IV, \dots, III, \dots] \rightarrow [\dots, I_2, \dots, I_2, \dots]$. Moreover, both type III and type IV yield higher values for \tilde{d}_P with respect to type I_2 , resulting in stronger constraints on the remainder of the configuration. Finally, type I_n intersecting type I_0^* always has monodromy as summarized in table 6. Thus, type I_{2n} and type I_{2n+1} contribute the same algebra summand, although the former yields a smaller value for \tilde{d}_P . Hence, in the remaining of this section we will restrict our attention to configurations of the form $\mathcal{C}_{2n_1, \dots, 2n_N}$ as defined above.

This has a nice additional consequence; as can be seen from the type I_0^* entry in table 5, the local models are highly constrained if either A or B are non-zero. Thus, we conclude that the maximal global symmetry will be realized for configurations $\mathcal{C}_{2n_1, \dots, 2n_N}$ for $A = B = 0$, as anticipated above.

Gauge algebra $\mathfrak{so}(8)$. Let us start by supposing that there is no monodromy on Σ , that is, type I_0^* on Σ carries $\mathfrak{so}(8)$ algebra. In this case, the monodromy cover takes the form

$$(\psi - \alpha)(\psi - \beta)(\psi - \gamma) , \tag{6.46}$$

where α, β and γ are sections of $-2K_B - \Sigma$ and $\alpha + \beta + \gamma = 0$. Let σ be a transverse coordinate to Σ in some open set, then we can represent the quantities α, β and γ as polynomials of an appropriate degree in σ . The degree is determined by how many times the sections vanish (with multiplicity) on Σ . This means that they vanish exactly

$$(-2K_B - \Sigma) \cdot \Sigma = 4 - m \tag{6.47}$$

times on Σ . In particular, $\deg \alpha = \deg \beta = \deg \gamma = 4 - m$. Now, the residual discriminant takes the form

$$\frac{\Delta}{z^6} \Big|_{z=0} = -(\alpha - \beta)^2 (2\alpha + \beta)^2 (\alpha + 2\beta)^2 . \tag{6.48}$$

By construction, $\alpha - \beta, 2\alpha + \beta, \alpha + 2\beta$ are also sections of $-2K_B - \Sigma$ and in particular they cannot coincide pairwise (otherwise the third vanishes identically or $\beta = 0$). The maximal algebra will be achieved if we can tune the above sections such that each of the factors of (6.48) has a (different) unique zero of multiplicity $4 - m$. If that were the case, the configuration would therefore be $\mathcal{C}_{8-2m,8-2m,8-2m}$, yielding the global symmetry algebra $\mathfrak{sp}(4 - m) \oplus \mathfrak{sp}(4 - m) \oplus \mathfrak{sp}(4 - m)$, which would match the predictions from field theory.

Unfortunately, this tuning is only possible when $4 - m = 1$, and cannot be carried out in the other cases. To see this, consider two homogeneous polynomials of degree $4 - m$ on \mathbb{CP}^1 , each with a zero of multiplicity $4 - m$ (representing two of the three factors appearing in (6.48)). Introducing an appropriate local coordinate t , we may assume that the zeros of one polynomial are at $t = 0$ and those of the other polynomial are at $t = 1$. For the third factor, then, we seek constants c_0 and c_1 such that

$$F(t) := c_0 t^{4-m} + c_1 (t - 1)^{4-m} \tag{6.49}$$

has a zero of multiplicity $4 - m$ at some value of t other than 0 or 1. In particular, if $4 - m > 1$, then (6.49) must have a multiple root so that $F(t)$ and $F'(t)$ must share a zero (other than $t = 0$ or $t = 1$). But since $F(t) - \frac{t}{4-m} F'(t) = -c_1 (t - 1)^{3-m}$, the only possible location of a common zero is $t = 1$, contradicting our assumption that it could be located at a third point. This shows that $F(t)$ cannot have roots with multiplicity higher than one; the $4 - m$ roots of $F(t)$ of multiplicity one could in principle support $4 - m$ curves of type I_2 .

The conclusion is that we can achieve $\mathfrak{sp}(4 - m) \oplus \mathfrak{sp}(4 - m) \oplus \mathfrak{sp}(1)^{\oplus(4-m)}$ symmetry.

Gauge algebra $\mathfrak{so}(7)$. In this case, the monodromy cover (6.45) takes the form

$$(\psi - \lambda)(\psi^2 + \lambda\psi + \mu), \tag{6.50}$$

where λ is a section of $-2K_B - \Sigma$ and μ is a section of $-4K_B - 2\Sigma$. As before, we think of them as polynomials of the appropriate degree in some transverse coordinate σ in some open set of Σ , and their degrees are given by

$$\deg \lambda = (-2K_B - \Sigma) \cdot \Sigma = 4 - m, \quad \deg \mu = (-4K_B - 2\Sigma) \cdot \Sigma = 8 - 2m. \tag{6.51}$$

The residual discriminant takes the form

$$\tilde{\Delta} \equiv \frac{\Delta}{z^6} \Big|_{z=0} = -(\lambda^2 - 4\mu)(2\lambda^2 - \mu)^2, \tag{6.52}$$

where $\varphi \equiv \lambda^2 - 4\mu$ is not a square (otherwise the algebra is $\mathfrak{so}(8)$). We notice that by construction φ and $2\lambda^2 - \mu$ cannot have common zeros. It is possible that φ has a factor that is a square, and we write $\varphi = \delta\gamma^2$, where δ is not a square. In particular, this means that $\deg \delta$ cannot vanish. Note that for $m = 3$, necessarily $\gamma \equiv 1$. The degrees of these quantities are as follows

$$\deg(2\lambda^2 - \mu) = 8 - 2m, \quad 8 - 2m = 2 \deg \gamma + \deg \delta \geq 2 \deg \gamma + 2. \tag{6.53}$$

This means that $(2\lambda^2 - \mu)^2$ can support at most type I_{16-4m} , while γ^2 can support at most type I_{6-2m} . The maximal algebra will be achieved if we can tune the above sections as described, yielding the configurations $\mathcal{C}_{16-4m,6-2m}$, with algebras $\mathfrak{sp}(8 - 2m) \oplus \mathfrak{sp}(3 - m)$. These coincide with the predictions from field theory.

Gauge algebra \mathfrak{g}_2 . When Σ carries gauge algebra \mathfrak{g}_2 , the Tate form prescribes the divisibility conditions

$$\tilde{u} \equiv u/z, \quad \tilde{v} \equiv v/z^2, \quad \tilde{w} \equiv w/z^3, \quad (6.54)$$

where (see appendix A)

$$f = -\frac{1}{3}u^2 + v, \quad g = \frac{2}{27}u^3 - \frac{1}{3}uv + w. \quad (6.55)$$

In terms of (6.54) the residual discriminant reads

$$\tilde{\Delta}\Big|_{z=0} = \frac{\Delta}{z^6}\Big|_{z=0} = 4\tilde{u}^3\tilde{w} - \tilde{u}^2\tilde{v}^2 - 18\tilde{u}\tilde{v}\tilde{w} + 4\tilde{v}^3 + 27\tilde{w}^2. \quad (6.56)$$

Let $\mathcal{C}_{2n_1, \dots, 2n_N}$ be a configuration of the type defined above, where $\Sigma_i = \{\sigma_i = 0\}$ carries type I_{2n_i} . If $2\sum_i n_i = \tilde{d}_\Sigma$, then the residual discriminant must assume the form $\tilde{\Delta} = c\sigma_1^{2n_1} \cdots \sigma_N^{2n_N}$, where c is a constant. That is, $\tilde{\Delta}$ is a square and the algebra along Σ cannot be \mathfrak{g}_2 . This yields a first constraint

$$\sum_i n_i \leq \frac{1}{2}\tilde{d}_\Sigma - 1 = 11 - 3m. \quad (6.57)$$

For each curve in $\mathcal{C}_{2n_1, \dots, 2n_N}$ (following our discussion in appendix A), we define the quantities

$$U_{(i)} \equiv \tilde{u}|_{z=0}, \quad V_{(i)} \equiv \frac{\tilde{v}}{\sigma^{n_i}}\Big|_{z=0}, \quad W_{(i)} \equiv \frac{\tilde{w}}{\sigma^{2n_i}}\Big|_{z=0}. \quad (6.58)$$

We are going to prove the following:

Lemma 1 *Given $\Sigma = \{z = 0\}$ carrying type I_0^* with gauge algebra \mathfrak{g}_2 , any configuration $\mathcal{C}_{2n_1, \dots, 2n_N}$ is such that $n_i \leq 10 + 3\Sigma^2$.*

This follows from the global constraints on \tilde{a}_Σ and \tilde{b}_Σ and on the degrees of U, V, W . Let type I_{2n} be supported along $\Sigma' = \{\sigma = 0\}$ and be part of a configuration on type I_0^* . First, we show that for $n > 10 - 3m$ the following hold

$$\deg V + n = 2 \deg U, \quad \deg W + 2n = 3 \deg U. \quad (6.59)$$

While this is trivial in the $m = 1$ and $m = 2$ cases, we need to be careful when $m = 3$. In fact, let $m = 3$ and $n = 2$. Imposing

$$8 - 2m = \deg \tilde{f}\Big|_{z=0} = \deg \left(-\frac{1}{3}U^2 + V\sigma^n \right), \quad (6.60)$$

it follows that either $\deg U \leq 1$ and $\deg V = 0$ or $\deg V = 2 \deg U - 2$. The second constraint

$$12 - 3m = \deg \tilde{g}\Big|_{z=0} = \deg \left(\frac{2}{27}U^3 - \frac{1}{3}UV\sigma^n + W\sigma^{2n} \right), \quad (6.61)$$

instead forces

$$\deg W + 4 = 3 \deg U \quad \text{or} \quad \deg W + 4 = 2 + \deg U + \deg V . \quad (6.62)$$

The first possibility trivially leads to $\deg U > 1$. The second implies $\deg U + \deg V \geq 2$, which again forces $\deg U > 1$ or $\deg V > 0$. When $n = 3$ (the only other possibility for $m = 3$), the claim is again trivial. Therefore, we verified that (6.59) hold in all cases of interest for this lemma.

Let $m = 3$. We need to show that type I_4 is forbidden. This follows easily from the above since for this curve, we have $\deg U \leq 1$, but (6.59) forces $\deg U \geq 2$.

Let $m = 2$. To show that type I_{10} is not allowed, we explicitly check that (6.60) and (6.61) cannot be both satisfied. Here, we have $\deg U \leq 4$ but (6.59) (for $n = 5$) fixes $\deg U = 4$, $\deg V = 3$ and $\deg W = 2$. We expand these quantities accordingly as

$$U = u_0 + u_1\sigma + u_2\sigma^2 + u_3\sigma^3 + u_4\sigma^4, \quad V = v_0 + v_1\sigma + v_2\sigma^2 + v_3\sigma^3, \quad W = w_0 + w_1\sigma + w_2\sigma^2, \quad (6.63)$$

where $u_4, v_3, w_2 \neq 0$. By imposing (6.60), we obtain

$$v_2 = \frac{2}{3}u_3u_4, \quad v_3 = \frac{1}{3}u_4^2, \quad v_1 = \frac{1}{3}u_3^2 + \frac{2}{3}u_2u_4, \quad v_0 = \frac{2}{3}u_2u_3 - \frac{2}{3}u_1u_4, \quad (6.64)$$

and in order for (6.61) to hold, we must satisfy

$$u_3(u_1u_3 - 2u_4) = 0, \quad u_2u_3^2 + u_4^2 = 0, \quad u_3^3 + 6u_2u_3u_4 + 3u_1u_4^2 = 0. \quad (6.65)$$

The only solution to the above system of equations is $u_3 = u_4 = 0$, which is unacceptable. Thus, $\deg g|_{z=0} \geq 7$ contradicting (6.61).

If $m = 1$, we need to show that I_{16} is forbidden. It is possible to treat this case as well as in the above, that is, by expanding U , V and W to the appropriate degrees and showing that $\deg f|_{z=0} = 6$ and $\deg f|_{z=0} = 9$ cannot both be satisfied. Since the procedure is identical but the algebra is more involved, we do not report these calculations here. This concludes the proof of the lemma.

We can generalize the lemma above in the following way. We want to show that a configuration such that $\sum_i n_i = 11 - 3m$ cannot be supported on type I_0^* with \mathfrak{g}_2 gauge algebra. Our argument goes as follows. Suppose $\sum_i n_i = 11 - 3m$, then the residual discriminant takes the following form

$$\tilde{\Delta} = (\sigma_1^{n_1} \cdots \sigma_N^{n_N})^2 F_{[2]}, \quad (6.66)$$

where $F_{[2]}$ can be represented as a polynomial of degree 2 in the local variable σ . If (6.66) can be rewritten as (6.52), this means that the underlying gauge algebra must be enhanced to at least $\mathfrak{so}(7)$. This amounts to investigating whether we can find appropriate sections α and β as in (6.51). As it happens, we can simply count parameters. In fact, for a configuration $\mathcal{C}_{2n_1, \dots, 2n_N}$ the number of parameters that we need to completely specify the form of (6.66) is $N + 3$, where the first summand corresponds to the locations of the N transverse curves and the additional 3 parameters characterize the polynomial $F_{[2]}$. We

already see that the most constraining configurations will be the “longest”, that is $n_i = 1$ for each $i = 1, \dots, N = 11 - 3m$. In particular, this means that in every case

$$\#(\tilde{\Delta}) \leq 11 - 3m + 3 = 14 - 3m . \tag{6.67}$$

On the other hand, (6.51) implies that the numbers of parameters that specify the local sections α and β are

$$\#(\alpha) = 4 - m + 1 , \qquad \#(\beta) = 8 - 2m + 1 . \tag{6.68}$$

Therefore, $\#(\alpha) + \#(\beta) \geq \#(\tilde{\Delta})$ always holds. This gives us our final bound,

$$\sum_i n_i \leq 10 - 3m , \tag{6.69}$$

on allowed configurations. We emphasize the fact that we did not make any statements regarding the realizability of these configurations; we only argued that if the configuration can be consistently constructed, the algebra must be at least $\mathfrak{so}(7)$ (this happens for example in the case $m = 3$ with $\mathcal{C}_{2,2}$).

With these results at our disposal, we conclude that the maximal configurations will be \mathcal{C}_{20-6m} , yielding the algebras $\mathfrak{sp}(10 - 3m)$. These agree with the predictions from field theory.

6.5 Type I_n^*

In this section, we analyze configurations on a curve $\Sigma = \{z = 0\}$ carrying type I_n^* , $n \geq 1$, and $(\tilde{a}, \tilde{b}, \tilde{d})_\Sigma = (8 - 2m, 12 - 3m, 24 + (n - 6)m)$, where $\Sigma^2 = -m$. (Recall that when $n \geq 4$, we have $m = 4$ and for I_3^* with monodromy, we have $m = 2$ or $m = 4$.) From table 7, we see that configurations involving curves carrying other than type II and type $I_{n'}$ are forbidden, and we discard the former since it does not contribute to the global symmetry. The degrees of vanishing of f and g along Σ are 2 and 3 respectively, thus the results in appendix A for type I_0^* (for $A = B = 0$) naturally extend here, that is, each curve $I_{n'}$ in the configuration has monodromy and the relevant configurations are again $\mathcal{C}_{2n_1, \dots, 2n_N}$ with the associated algebras $\mathfrak{sp}(n_1) \oplus \dots \oplus \mathfrak{sp}(n_N)$.

The following result will uniquely determine the maximal length of the configuration:

Result 1 *Let $\Sigma' = \{\sigma = 0\}$ be a curve carrying any Kodaira type and $P \equiv \Sigma \cap \Sigma'$. Then $(\tilde{a}_P, \tilde{b}_P)_\Sigma = (2k, 3k)$ for $k = 0, 1, 2, \dots$*

In fact, from appendix B we recall

$$\left. \frac{f}{z^2} \right|_{z=0} = -\frac{1}{3}u_1^2, \qquad \left. \frac{g}{z^3} \right|_{z=0} = \frac{2}{27}u_1^3, \tag{6.70}$$

where u_1 is a locally defined function in a neighborhood of Σ . Let k be the degree of u_1 in the variable σ , than the result immediately follows from (6.8).

We will now discuss the cases for different values of n .

$n < 4$ **odd**. Here the residual determinant reads

$$\frac{\Delta}{z^{2i+5}} \Big|_{z=0} = 4u_1^3 w_{2i+2}, \tag{6.71}$$

where $n = 2i - 1$. From appendix A and result 1 we have $\sigma_1^{2n_1} \dots \sigma_N^{2n_N} | w_{2i+2}$, and it is then convenient to define the local function ϕ such that

$$w_{2i+2} \equiv \phi \sigma_1^{2n_1} \dots \sigma_N^{2n_N}. \tag{6.72}$$

In terms of degrees, (6.71) reads

$$\tilde{d}_\Sigma - 2 \sum_i n_i = 3 \deg u_1 + \deg \phi. \tag{6.73}$$

As noted above, $\deg u_1 = \tilde{a}_\Sigma/2$, and the monodromy on type I_n^* for n odd is determined by whether w_{2n+2} has a square root. This means that $\deg \phi$ must be even for the case without monodromy (indicated by the upperscript “s”), while necessarily $\deg \phi > 0$ for the monodromy case (indicated by the upperscript “ns”). In order to impose the least constraint on $\sum_i n_i$, we will then assume $\deg \phi = 0$ for I_n^{*s} and $\deg \phi = 2$ for I_n^{*ns} . For $n = 1$, we have the following table:

m	\tilde{d}_Σ	$\deg u_1$	$(\sum_i n_i)^s$	max. algebra	$(\sum_i n_i)^{ns}$	max. algebra
1	19	3	5	$\mathfrak{sp}(5)$	4	$\mathfrak{sp}(4)$
2	14	2	4	$\mathfrak{sp}(4)$	3	$\mathfrak{sp}(3)$
3	9	1	3	$\mathfrak{sp}(3)$	2	$\mathfrak{sp}(2)$
4	4	0	2	$\mathfrak{sp}(2)$	1	$\mathfrak{sp}(1)$

From table 2, we read off the algebras predicted from field theory: $\mathfrak{sp}(6 - m) \oplus \mathfrak{sp}(4 - m)$ and $\mathfrak{sp}(5 - m) \oplus \mathfrak{sp}(4 - m)$ for the two monodromy cases. We have just determined that our configurations yield algebras that are contained in the first summands.

For $n = 3$, the analysis is even simpler, as $\tilde{d}_\Sigma - 3 \deg u_1 = 12$ for any value of m . Therefore, maximal configurations yield $\mathfrak{sp}(6)$ in the case without monodromy and $\mathfrak{sp}(5)$ in the monodromy case. The prediction from field theory is $\mathfrak{sp}(6)$ for the former case and $\mathfrak{sp}(9 - m) \oplus \mathfrak{so}(2 - \frac{1}{2}m)$ for the latter (with $m = 2$ or $m = 4$).

$n = 2$. The analysis for this case differs from the previous section as the residual discriminant now reads

$$\frac{\Delta}{z^8} \Big|_{z=0} = u_1^2 (4u_1 w_5 - v_3^2). \tag{6.75}$$

Considering again a configuration of the form $\mathcal{C}_{2n_1, \dots, 2n_N}$, we define

$$(4u_1 w_5 - v_3^2) \equiv \phi \sigma_1^{2n_1} \dots \sigma_N^{2n_N}, \tag{6.76}$$

and we have the following relation

$$\tilde{d}_\Sigma - 2 \sum_i n_i = 2 \deg u_1 + \deg \phi. \tag{6.77}$$

The monodromy condition is determined by whether $(4u_1w_7 - v_3^2)$ admits a square root, and we rephrase this in terms of a (minimal) constraint on the degree of ϕ : $\deg \phi = 0$ for I_2^{*s} and $\deg \phi = 2$ for I_2^{*ns} . The results are

m	\tilde{d}_Σ	$\deg u_1$	$(\sum_i n_i)^s$	max. algebra	$(\sum_i n_i)^{ns}$	max. algebra (6.78)
1	20	3	7	$\mathfrak{sp}(7)$	6	$\mathfrak{sp}(6)$
2	16	2	6	$\mathfrak{sp}(6)$	5	$\mathfrak{sp}(5)$
3	12	1	5	$\mathfrak{sp}(5)$	4	$\mathfrak{sp}(4)$
4	8	0	4	$\mathfrak{sp}(4)$	3	$\mathfrak{sp}(3)$

Again, these are subalgebras of the first summands of the field theoretical algebras from table 2.

$n \geq 4$. As before, there are two monodromy cases here, yielding gauge algebras $\mathfrak{so}(2n+7)$ for I_n^{*ns} and $\mathfrak{so}(2n+8)$ for I_n^{*s} . The corresponding predictions from table 2 are $\mathfrak{sp}(2n-1)$ and $\mathfrak{sp}(2n)$, respectively.

Since $m = 4$, we have $\deg u_1 = 0$ and

$$2 \sum_i n_i = \tilde{d}_\Sigma - \deg \phi = 4n - \deg \phi, \tag{6.79}$$

where ϕ is defined as in (6.72) for the odd case or as in (6.76) in the even case. With monodromy, we have also $\deg \phi > 0$ and the maximal configuration is given by $\mathcal{C}_{d_\Sigma-2}$, yielding global symmetry algebra $\mathfrak{sp}(2n-1)$; in the no monodromy case, we assume $\deg \phi = 0$ as above and \mathcal{C}_{d_Σ} is the maximal configuration with $\mathfrak{sp}(2n)$ as global symmetry. These both agree with the field theory predictions.

6.6 Type IV*, III* and II*

Here the situation is quite simple. These Kodaira types are not allowed to intersect any of the singularity types yielding a non-vanishing gauge algebra. Therefore, there is no global symmetry coming from F-theory. In particular, this result matches with the field theory prediction for type II*, for which no matter content is available.

6.7 Summary

Finally, we can summarize the results we have derived in this section. tables 3 and 4 list the (relatively) maximal symmetry algebras of F-theory constructions of the class of 6D SCFTs we consider in this work. We emphasize once more the nature of the results we find. Comparing with table 2, we conclude that in some cases the global symmetry from field theory can be realized; in other cases, this cannot happen and we determine the (relatively) maximal global symmetry algebras.

In fact, the agreement between the F-theory constructions and the Coulomb branch predictions is quite good, and we can easily list the cases where there is a mismatch. Note that in some cases, there is more than one way to produce the gauge algebra in F-theory; it is enough for our purposes if we can find agreement between the field theory prediction

and at least one of the F-theory realizations. With that being said, the cases where there is a mismatch are:

1. In the case of $\mathfrak{su}(2)$ with $m = 2$, we have obtained only $\mathfrak{so}(7)$ rather than the Coulomb branch expectation of $\mathfrak{so}(8)$. This is the case which has already been explained in [29], which found a field-theoretic reason that the SCFT should have a smaller global symmetry than the one observed on the Coulomb branch.
2. In the case of $\mathfrak{su}(3)$ with $m = 1$, we see the most severe mismatch: the predicted global symmetry is $\mathfrak{su}(12)$, but we find only a variety of different subalgebras of this (different ones for different realizations of the gauge algebra).
3. For $\mathfrak{so}(8)$ with $m = 1$ or $m = 2$, we only realized $\mathfrak{sp}(4-m) \oplus \mathfrak{sp}(4-m) \oplus \mathfrak{sp}(1)^{\oplus(4-m)}$ rather than the predicted $\mathfrak{sp}(4-m) \oplus \mathfrak{sp}(4-m) \oplus \mathfrak{sp}(4-m)$.
4. For $\mathfrak{so}(n)$, $9 \leq n \leq 13$ and $m < 4$, the predicted global symmetry associated to the spinor representation is never realized.
5. For $\mathfrak{so}(13)$ with $m = 2$, only a $\mathfrak{sp}(5)$ subalgebra of the predicted $\mathfrak{sp}(7)$ algebra is realized.
6. Finally, for \mathfrak{f}_4 , \mathfrak{e}_6 , and \mathfrak{e}_7 , none of the predicted global symmetries are realized.

One possible resolution to this mismatch is that, as in the case of $\mathfrak{su}(2)$ with $m = 2$, there is some as-yet-to-be-discovered field theoretic reason for the global symmetry algebra of the SCFT to be smaller than the one visible on the Coulomb branch. However, it is also conceivable that the “missing” global symmetries are present in the field theory, but cannot be realized in F-theory.

Finally, note that in this paper, we only addressed the case of gauge theories, completely ignoring the possibilities of Kodaira fibers of types I_0 , I_1 and II , but those possibilities do in fact occur. The known 6D SCFTs with no gauge symmetry include the “rank n E -string theories” and the (2,0) theories of type A, D, or E. Both of these families of examples have a construction in M-theory which shows the presence of an $SU(2)$ global symmetry¹⁷ (see, for example, [44]). However, this $SU(2)$ global symmetry has no known realization in F-theory, except for the case of the rank 2 E -string theory [45] and the (2,0) A_1 theory (which can be obtained from the proposal in [45] by moving to the point in the Coulomb moduli space of the rank 2 E -string theory which has an A_1 point). In general, the $SU(2)$ global symmetry has no visible action on the matter representation on the Coulomb branch, and so will be difficult to analyze.

7 Conclusions and outlook

In this work we have carried out a study of the global symmetries of 6D SCFTs; in particular, we compared the global symmetries arising from field theory considerations on the

¹⁷This global symmetry combines with the $SU(2)$ R -symmetry of a (1,0) SCFT to form an $SO(4)$ symmetry which is visible from the M-theory perspective.

type along Σ	algebra on Σ	$-\Sigma^2$	max. global symmetry algebra(s)
I_2	$\mathfrak{su}(2)$	2	$\mathfrak{su}(4)$
		1	$\mathfrak{so}(20)$
$I_{n \geq 3}, n \text{ odd}$	$\mathfrak{sp}([n/2])$	1	$\mathfrak{so}(13 + 2n)$ $\mathfrak{so}(7 + 2p) \oplus \mathfrak{so}(7 + 2n - 2p), 0 \leq p \leq \frac{n+1}{2}$
		$\mathfrak{su}(n)$	2
		1	$\mathfrak{su}(8 + n)$
$I_{n \geq 4}, n \text{ even}$	$\mathfrak{sp}(n/2)$	1	$\mathfrak{so}(16 + 2n)$
	$\mathfrak{su}(n)$	2	$\mathfrak{su}(2n)$
		1	$\mathfrak{su}(8 + n)$
I_6	$\mathfrak{su}(6)^*$	1	$\mathfrak{su}(15)$
III	$\mathfrak{su}(2)$	2	$\mathfrak{so}(7)$
		1	$\mathfrak{so}(7) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$ $\mathfrak{so}(7) \oplus \mathfrak{su}(6)$ $\mathfrak{su}(10)$
IV	$\mathfrak{su}(2)$	2	$\mathfrak{so}(7)$
		1	$\mathfrak{so}(7) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(3)$
	$\mathfrak{su}(3)$	3	-
		2	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ $\mathfrak{sp}(2)$
		1	$\mathfrak{su}(3)^{\oplus 4}$ $\mathfrak{su}(3)^{\oplus 2} \oplus \mathfrak{sp}(2)$ $\mathfrak{su}(3) \oplus \mathfrak{sp}(3)$ $\mathfrak{sp}(4)$
I_0^*	\mathfrak{g}_2	3	$\mathfrak{sp}(1)$
		2	$\mathfrak{sp}(4)$
		1	$\mathfrak{sp}(7)$
	$\mathfrak{so}(7)$	3	$\mathfrak{sp}(2)$
		2	$\mathfrak{sp}(4) \oplus \mathfrak{sp}(1)$
		1	$\mathfrak{sp}(6) \oplus \mathfrak{sp}(2)$
	$\mathfrak{so}(8)$	4	-
		3	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$
		2	$\mathfrak{sp}(2) \oplus \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)^{\oplus 2}$
1		$\mathfrak{sp}(3) \oplus \mathfrak{sp}(3) \oplus \mathfrak{sp}(1)^{\oplus 3}$	

Table 3. Global symmetries of F-theory models.

type along Σ	algebra on Σ	$-\Sigma^2$	max. global symmetry algebra
I_1^*	$\mathfrak{so}(9)$	4	$\mathfrak{sp}(1)$
		3	$\mathfrak{sp}(2)$
		2	$\mathfrak{sp}(3)$
		1	$\mathfrak{sp}(4)$
	$\mathfrak{so}(10)$	4	$\mathfrak{sp}(2)$
		3	$\mathfrak{sp}(3)$
		2	$\mathfrak{sp}(4)$
		1	$\mathfrak{sp}(5)$
I_2^*	$\mathfrak{so}(11)$	4	$\mathfrak{sp}(3)$
		3	$\mathfrak{sp}(4)$
		2	$\mathfrak{sp}(5)$
		1	$\mathfrak{sp}(6)$
	$\mathfrak{so}(12)$	4	$\mathfrak{sp}(4)$
		3	$\mathfrak{sp}(5)$
		2	$\mathfrak{sp}(6)$
		1	$\mathfrak{sp}(7)$
I_3^*	$\mathfrak{so}(13)$	2,4	$\mathfrak{sp}(5)$
	$\mathfrak{so}(14)$	4	$\mathfrak{sp}(6)$
$I_{n \geq 4}^*$	$\mathfrak{so}(2n+7)$	4	$\mathfrak{sp}(2n-1)$
	$\mathfrak{so}(2n+8)$	4	$\mathfrak{sp}(2n)$
IV^*	\mathfrak{f}_4	1-5	-
	\mathfrak{e}_6	1-6	-
III^*	\mathfrak{e}_7	1-8	-
II^*	\mathfrak{e}_8	12	-

Table 4. Global symmetries of F-theory models.

Coulomb branch of the SCFT to the constraints on global symmetries arising from F-theory constructions. We focused our analysis on gauge theories in which the Coulomb branch has dimension one. We found agreement in the vast majority of cases, and observed that one of the cases where there is disagreement has a recent field theory explanation in the literature [29]. These results should provide the first step in an analysis of global symmetries in the general case. When paired with the classification scheme of [22], the above results might allow one to compute the relatively maximal global symmetry algebra(s) of a given 6D SCFT from the conjecturally complete 6D SCFT dictionary found in [22]. Associating to a SCFT its relatively maximal global symmetry algebras could shed light on topics such as the structure of those discrete $U(2)$ subgroups associated to 6D SCFTs, as described in [21].

To be more precise, given a 6D SCFT meeting the hypotheses of [22], it has a description in F-theory in terms of an elliptic fibration degenerating over the discriminant locus, which in this more general context may be a linear or ‘D-type’ chain of \mathbb{P}^1 ’s with specified negative self-intersections obeying known restrictions. The action of a discrete $U(2)$ subgroup Γ on the base is also allowed, leaving an orbifold singularity in the base, $B \cong \mathbb{C}^2/\Gamma$. In terms of the complexity of the computation of the global symmetry algebra of a SCFT using an appropriate generalization of the above methods, one significant modification from the hypotheses of the present work is that one needs to consider more than a single compact curve as the discriminant locus; for a complete description of the rules these chains must obey, see [22]. Since we can associate a given 6D SCFT of type-A¹⁸ to a linear chain of pairwise transversely intersecting compact curves $\Sigma_1, \dots, \Sigma_m$ with specified self-intersection numbers, to compute the relatively maximal global symmetry algebra(s) of that SCFT, one might proceed in a fashion similar to that above, namely, by inspecting the set of allowed configurations of non-compact curves intersecting the configuration $\Sigma_1, \dots, \Sigma_m$ transversely. As above, this generalization should provide a family of relatively maximal global symmetry algebras realizable in F-theory for a 6D SCFT, each arising as the direct sum of the gauge algebras of some allowed configuration of transverse non-compact curves.

We hope to return to this question in future work.

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A The Tate algorithm for the I_n case

In this appendix, we determine the data necessary to describe a transverse intersection between a Kodaira type I_n supported on a divisor $\{\sigma = 0\}$ on the base and any other Kodaira type, which we assume to be supported on a divisor $\{z = 0\}$. For the methods of this appendix, type $I_{m \geq 1}^*$ can be treated on the same footing as type I_0^* , thus we will just refer to the latter in the following. For these cases, however, the analysis we present here do not suffice to completely solve these models, as they involve additional subtleties which we will deal with separately in the main body of the paper.

The authors of [43] went through a careful analysis of Weierstrass models for $\mathfrak{su}(n)$. Here we slightly generalize their approach in order to describe models for $\mathfrak{sp}(n/2)$ as well.

¹⁸A similar procedure can be applied to the curve clusters of type-D; here we restrict our discussion to type-A configurations for simplicity.

A similar approach was also employed in [42]. The strategy is to expand the quantities in (4.1) in powers of σ

$$f = \sum_i f_i \sigma^i, \quad g = \sum_i g_i \sigma^i, \quad \Delta = \sum_i \Delta_i \sigma^i, \quad (\text{A.1})$$

where the quantities f_i , g_i and Δ_i can be thought of as polynomials in z , at least locally in a neighborhood of $\{\sigma = 0\}$. The requirement for the divisor $\{\sigma = 0\}$ to carry type I_n implies that $\Delta_i = 0 \forall i < n$. As in [43], we assume that the divisor $\{\sigma = 0\}$ is non singular. This guarantees that any ring of local functions on a sufficiently small open subset of $\{\sigma = 0\}$ is a unique factorization domain.

In order to obtain a type I_1 singularity we need to impose

$$\Delta_0 = 4f_0^3 + 27g_0^2 = 0, \quad (\text{A.2})$$

that is, there exists a local function ϕ such that¹⁹

$$f_0 \sim -\frac{1}{48}\phi^2, \quad g_0 \sim \frac{1}{864}\phi^3. \quad (\text{A.3})$$

We set $f_0 = -\frac{1}{48}\phi^2$ and $g_0 = \frac{1}{864}\phi^3$ and we obtain:²⁰

Summary for I_1 .

$$\begin{aligned} f &= -\frac{1}{48}\phi^2 + f_1\sigma + O(\sigma^2), \\ g &= \frac{1}{864}\phi^3 + g_1\sigma + O(\sigma^2), \\ \Delta &= \frac{1}{192}\phi^3(12g_1 + \phi f_1)\sigma + O(\sigma^2). \end{aligned} \quad (\text{A.4})$$

The coefficients f_i , g_i above have been redefined to absorb the terms proportional to σ we might have introduced when substituting for f_0 and g_0 . Now, setting $\Delta_1 = 0$ implies

$$g_1 \sim -\frac{1}{12}\phi f_1. \quad (\text{A.5})$$

Moreover, we have that for each $i \geq 1$ there are the following standard terms in the expansion of the discriminant

$$\Delta_i = 4f_0^2 f_i + 27g_0 g_i + \dots. \quad (\text{A.6})$$

Hence, it turns out to be convenient to redefine

$$\tilde{g}_i = g_i + \frac{1}{12}\phi f_i, \quad i \geq 1. \quad (\text{A.7})$$

Summary for I_2 .

$$f = -\frac{1}{48}\phi^2 + f_1\sigma + f_2\sigma^2 + O(\sigma^3),$$

¹⁹Given two local functions ϕ_1 and ϕ_2 , the notation $\phi_1 \sim \phi_2$ indicates that the two functions are identical up to powers of σ .

²⁰The choice of the numerical coefficients aims to match the conventions in the existing literature.

$$\begin{aligned}
g &= \frac{1}{864}\phi^3 - \frac{1}{12}\phi f_1\sigma + \left(\tilde{g}_2 - \frac{1}{12}\phi f_2\right)\sigma^2 + O(\sigma^3), \\
\Delta &= \frac{1}{16}(\phi^3\tilde{g}_2 - \phi^2 f_1^2)\sigma^2 + O(\sigma^3).
\end{aligned}
\tag{A.8}$$

To ensure that the coefficient of σ^2 in Δ vanishes, we can set

$$\phi \sim \mu\phi_0^2, \quad f_1 \sim \frac{1}{2}\mu\phi_0\psi_1,
\tag{A.9}$$

with μ square-free (for appropriate locally-defined functions). Now, $\Delta_2 = 0$ is solved by

$$\tilde{g}_2 = \frac{1}{4}\mu\psi_1^2.
\tag{A.10}$$

Summary for I_3 .

$$\begin{aligned}
f &= -\frac{1}{48}\mu^2\phi_0^4 + \frac{1}{2}\mu\phi_0\psi_1\sigma + f_2\sigma^2 + f_3\sigma^3 + O(\sigma^4), \\
g &= \frac{1}{864}\mu^3\phi_0^6 - \frac{1}{24}\mu^2\phi_0^3\psi_1\sigma + \frac{1}{4}\left(\mu\psi_1^2 - \frac{1}{3}\mu\phi_0^2 f_2\right)\sigma^2 + \left(\tilde{g}_3 - \frac{1}{12}\mu\phi_0^2 f_3\right)\sigma^3 + O(\sigma^4), \\
\Delta &= \frac{1}{16}\mu^3\phi_0^3(\phi_0^3\tilde{g}_3 - \psi_1^3 - \phi_0^2\psi_1 f_2)\sigma^3 + O(\sigma^4).
\end{aligned}
\tag{A.11}$$

Now, in order to obtain type I_4 along $\{\sigma = 0\}$, we need to set

$$\Delta_3 = \frac{1}{16}\mu^3\phi_0^3(\phi_0^3\tilde{g}_3 - \psi_1^3 - \phi_0^2\psi_1 f_2) = 0.
\tag{A.12}$$

By writing the terms in the parenthesis as $\psi_1^3 + \phi_0(\dots)$, we have that $\phi_0|_{\sigma=0}$ must divide $\psi_1|_{\sigma=0}$, i.e.,

$$\psi_1 \sim -\frac{1}{3}\phi_0\phi_1.
\tag{A.13}$$

Now, $\Delta_3 = 0$ is solved by setting

$$\tilde{g}_3 = -\frac{1}{3}\phi_1 f_2 - \frac{1}{27}\phi_1^3.
\tag{A.14}$$

Again, for each $i \geq 3$ we have terms in the expansion of Δ_i of the form

$$\Delta_i = 24f_0 f_1 f_{i-1} + 27g_0 \tilde{g}_i + \dots,
\tag{A.15}$$

so that it is convenient to redefine

$$\hat{g}_i = \tilde{g}_i + \frac{1}{3}\phi_1 f_{i-1}, \quad \hat{f}_2 = f_2 + \frac{1}{3}\phi_1^2.
\tag{A.16}$$

Summary for I_4 .

$$\begin{aligned}
f &= -\frac{1}{48}\mu^2\phi_0^4 - \frac{1}{6}\mu\phi_0^2\phi_1\sigma + \left(\hat{f}_2 - \frac{1}{3}\phi_1^2\right)\sigma^2 + f_3\sigma^3 + f_4\sigma^4 + O(\sigma^5), \\
g &= \frac{1}{864}\mu^3\phi_0^6 + \frac{1}{72}\mu^2\phi_0^4\phi_1\sigma + \frac{1}{6}\left(\frac{1}{3}\mu\phi_0^2\phi_1^2 - \frac{1}{2}\mu\phi_0^2\hat{f}_2\right)\sigma^2
\end{aligned}$$

$$\begin{aligned}
 & + \left(-\frac{1}{3}\phi_1\widehat{f}_2 + \frac{2}{27}\phi_1^3 - \frac{1}{12}\mu\phi_0^2f_3 \right) \sigma^3 + \left(\widehat{g}_4 - \frac{1}{3}\phi_1f_3 - \frac{1}{12}\mu\phi_0^2f_4 \right) \sigma^4 + O(\sigma^5), \\
 \Delta = & \frac{1}{16}\mu^2\phi_0^4 \left(-\widehat{f}_2^2 + \mu\phi_0^2\widehat{g}_4 \right) \sigma^4 + O(\sigma^5). \tag{A.17}
 \end{aligned}$$

Next, in order for Δ_4 to vanish along $\{\sigma = 0\}$, we require that $\mu\phi_0|_{\sigma=0}$ divides $\widehat{f}_2|_{\sigma=0}$, i.e.,

$$\widehat{f}_2 \sim \frac{1}{2}\mu\phi_0\psi_2, \tag{A.18}$$

for some locally defined function ψ_2 . We obtain type I_5 along $\{\sigma = 0\}$ by setting

$$\widehat{g}_4 = \frac{1}{4}\mu\psi_2^2. \tag{A.19}$$

Summary for I_5 .

$$\begin{aligned}
 f = & -\frac{1}{48}\mu^2\phi_0^4 - \frac{1}{6}\mu\phi_0^2\phi_1\sigma + \left(\frac{1}{2}\mu\phi_0\psi_2 - \frac{1}{3}\phi_1^2 \right) \sigma^2 + f_3\sigma^3 + f_4\sigma^4 + f_5\sigma^5 + O(\sigma^6), \\
 g = & \frac{1}{864}\mu^3\phi_0^6 + \frac{1}{72}\mu^2\phi_0^4\phi_1\sigma + \frac{1}{6} \left(\frac{1}{3}\mu\phi_0^2\phi_1^2 - \frac{1}{4}\mu^2\phi_0^3\psi_2 \right) \sigma^2 \\
 & + \left(-\frac{1}{6}\mu\phi_0\phi_1\psi_2 + \frac{2}{27}\phi_1^3 - \frac{1}{12}\mu\phi_0^2f_3 \right) \sigma^3 + \left(\frac{1}{4}\mu\psi_2^2 - \frac{1}{3}\phi_1f_3 - \frac{1}{12}\mu\phi_0^2f_4 \right) \sigma^4 \\
 & + \left(\widehat{g}_5 - \frac{1}{3}\phi_1f_4 - \frac{1}{12}\mu\phi_0^2f_5 \right) \sigma^5 + O(\sigma^6), \\
 \Delta = & \frac{1}{16}\mu^3\phi_0^4 \left(\phi_0^2\widehat{g}_5 + \phi_1\psi_2^2 - \phi_0\psi_2f_3 \right) \sigma^5 + O(\sigma^6). \tag{A.20}
 \end{aligned}$$

We can go one step further and find a complete general form for type I_6 by setting $\Delta_5 = 0$, which reads

$$\phi_1\psi_2^2 - \phi_0\psi_2f_3 + \phi_0^2\widehat{g}_5 = 0. \tag{A.21}$$

Now, we factorize the roots of $\phi_0|_{\sigma=0}$ according to which ones divide $\phi_1|_{\sigma=0}$ or $\psi_2|_{\sigma=0}$, as

$$\phi_0 \sim \alpha\beta, \quad \psi_2 \sim -\frac{1}{3}\alpha\phi_2, \quad \phi_1 \sim \beta\nu, \tag{A.22}$$

where α, β, ϕ_2 and ν are locally-defined functions, and $\alpha|_{\sigma=0}$ is the greatest common divisor of $\phi_1|_{\sigma=0}$ and $\psi_2|_{\sigma=0}$. This implies that $\beta|_{\sigma=0}$ and $\phi_2|_{\sigma=0}$ do not share any common factor, and that $\Delta_5 = 0$ is satisfied by

$$f_3 \sim -\frac{1}{3}\nu\phi_2 - 3\beta\lambda, \quad \widehat{g}_5 \sim \phi_2\lambda, \tag{A.23}$$

for some locally-defined function λ . Finally

Summary for I_6 .

$$\begin{aligned}
 f = & -\frac{1}{48}\mu^2\alpha^4\beta^4 - \frac{1}{6}\mu\alpha^2\beta^3\nu\sigma + \frac{1}{3} \left(-\frac{1}{2}\mu\alpha^2\beta\phi_2 - \beta^2\nu^2 \right) \sigma^2 \\
 & + \left(-\frac{1}{3}\nu\phi_2 - 3\beta\lambda \right) \sigma^3 + f_4\sigma^4 + f_5\sigma^5 + f_6\sigma^6 + O(\sigma^7),
 \end{aligned}$$

$$\begin{aligned}
g &= \frac{1}{864}\mu^3\alpha^6\beta^6 + \frac{1}{72}\mu^2\alpha^4\beta^5\nu\sigma + \frac{1}{18}\left(\mu\alpha^2\beta^4\nu^2 + \frac{1}{4}\mu^2\alpha^4\beta^3\phi_2\right)\sigma^2 \\
&+ \left(\frac{1}{12}\mu\alpha^2\beta^2\nu\phi_2 + \frac{2}{27}\beta^3\nu^3 + \frac{1}{4}\mu\alpha^2\beta^3\lambda\right)\sigma^3 \\
&+ \left(\frac{1}{36}\mu\alpha^2\phi_2^2 + \frac{1}{9}\beta\nu^2\phi_2 + \beta^2\nu\lambda - \frac{1}{12}\mu\alpha^2\beta^2f_4\right)\sigma^4 \\
&+ \left(\phi_2\lambda - \frac{1}{3}\beta\nu f_4 - \frac{1}{12}\mu\alpha^2\beta^2f_5\right)\sigma^5 + \left(\widehat{g}_6 - \frac{1}{3}\beta\nu f_5 - \frac{1}{12}\mu\alpha^2\beta^2f_6\right)\sigma^6 + O(\sigma^7), \\
\Delta &= \frac{1}{432}\mu^2\alpha^4\beta^3\left(27\mu\alpha^2\beta^3\widehat{g}_6 + 9\mu\alpha^2\beta^2\phi_2f_4 + \mu\alpha^2\phi_2^3\right. \\
&\quad \left.- 243\lambda^2\beta^3 + 54\phi_2\nu\lambda\beta^2 - 3\beta\nu^2\phi_2^2\right)\sigma^6 + O(\sigma^7). \tag{A.24}
\end{aligned}$$

Solving algebraically $\Delta_6 = 0$ is a daunting challenge and we are not able to do it in complete generality. In order to deal with type I_n for $n \geq 7$, we implement the inductive argument of [42], of which we state here the results.

Summary for I_i , $i \geq 7$.

$$\begin{aligned}
f &= -\frac{1}{3}u^2 + v, \\
g &= \frac{2}{27}u^3 - \frac{1}{3}uv + w, \\
\Delta &= 4u^3w - u^2v^2 - 18uvw + 4v^3 + 27w^2, \tag{A.25}
\end{aligned}$$

where u , v and w are local functions such that

$$\sigma^{\lfloor i/2 \rfloor} \mid v, \quad \sigma^{2\lfloor i/2 \rfloor} \mid w, \quad \sigma \nmid u. \tag{A.26}$$

In the even case, i.e., for $i = 2n$, the expansion of u in powers of σ takes the form

$$u = \frac{1}{4}\mu\phi_0^2 + u_1\sigma + \dots + u_{n-1}\sigma^{n-1}, \tag{A.27}$$

where, as before, $\mu|_{\{\sigma=0\}}$ is square-free, while v and w are generic. In the odd case, $i = 2n + 1$, these have instead the expansions

$$\begin{aligned}
v &= \frac{1}{2}\mu\phi_0 t_n \sigma^n + v_{n+1}\sigma^{n+1} + \dots, \\
w &= \frac{1}{4}\mu t_n^2 \sigma^{2n} + w_{2n+1}\sigma^{2n+1} + \dots, \tag{A.28}
\end{aligned}$$

where t_n is a locally defined function. We note that this is indeed the most general solution for I_m , $m \geq 10$, while it is a solution, but not the most general one, for $m < 10$. We explicitly constructed above the most general solution for $m \leq 6$, so we have three cases, i.e., $m = 7, 8, 9$ for which we may not capture the full story. In this paper we treat these cases as described by (A.25).

Local models for intersections. We want to use the above expressions for type I_n in order to construct local models describing intersections of a divisor $\{\sigma = 0\}$ carrying type I_n and a divisor $\{z = 0\}$ carrying any other Kodaira type. In order to do this, let (a, b, d) be the degrees of vanishing of (f, g, Δ) along $\{z = 0\}$. Then we have the divisibility conditions

$$z^a \mid f, \quad z^b \mid g, \quad z^d \mid \Delta. \quad (\text{A.29})$$

Imposing these conditions on each term of the expansions (A.1) leads in turn to certain divisibility conditions on the various local functions describing the local model for the intersection. In particular, we must check whether the point of intersection $P = \{z = \sigma = 0\}$ has degrees of vanishing equal or higher than $(4, 6, 12)$, in other words, if it describes a non-minimal intersection.²¹ If that is that case, we discard this solution.

Monodromy. For type I_n , $n \geq 3$, there is a monodromy condition we need to test in order to specify the full algebra: in fact, the gauge algebra summand is either $\mathfrak{su}(n)$ or $\mathfrak{sp}([n/2])$. The condition for monodromy is determined by the function μ defined in (A.9): we are in the case without monodromy, i.e., the algebra is $\mathfrak{su}(n)$, when $\mu|_{\{\sigma=0\}}$ has no zeros, and conversely we are in the monodromy case when μ vanishes somewhere along $\{\sigma = 0\}$. As we will see in the example below, there are cases where the general form of the local model for the intersection forces μ to vanish somewhere. That is, we can prove that in order to have an allowed intersection the monodromy on the type I_n is fixed.

An example. Let us consider the intersection between a type I_4 along the divisor $\{\sigma = 0\}$ and a type I_0^* along the locus $\{z = 0\}$. We take the degrees of vanishing of f , g and Δ along $\{z = 0\}$ to be 2, 3 and 6 respectively. This means that

$$z^2 \mid f, \quad \text{but } z^3 \nmid f, \quad z^3 \mid g, \quad \text{but } z^4 \nmid g. \quad (\text{A.30})$$

Imposing these constraints on each term of the solution (A.17) implies that

$$z^2 \mid f_i, \quad i \geq 3, \quad z^3 \mid \widehat{g}_i, \quad i \geq 4, \quad z^2 \mid \left(\widehat{f}_2 - \frac{1}{3} \phi_1^2 \right) \implies z^2 \mid \widehat{f}_2, \quad z \mid \phi_1 \quad (\text{A.31})$$

and either $z \mid \phi_0$ or $z \mid \mu$. Let us distinguish between these two cases:

1. $z \mid \phi_0$ and $z \nmid \mu$. Let us rewrite our quantities by factoring out the explicit partial z dependence determined above; for simplicity of notation, we indicate the functions with the same symbol, as

$$\begin{aligned} f &= -\frac{1}{48} z^4 \mu^2 \phi_0^4 - \frac{1}{6} z^3 \mu \phi_0^2 \phi_1 \sigma + \left(z^2 \widehat{f}_2 - \frac{1}{3} z^2 \phi_1^2 \right) \sigma^2 + z^2 f_3 \sigma^3 + z^2 f_4 \sigma^4 + z^2 O(\sigma^5), \\ g &= \frac{1}{864} z^6 \mu^3 \phi_0^6 + \frac{1}{72} z^5 \mu^2 \phi_0^4 \phi_1 \sigma + \frac{1}{6} z^2 \mu \phi_0^2 \left(\frac{1}{3} z^2 \phi_1^2 - \frac{1}{2} z^2 \widehat{f}_2 \right) \sigma^2 \\ &+ \left(-\frac{1}{3} z^3 \phi_1 \widehat{f}_2 + \frac{2}{27} z^3 \phi_1^3 - \frac{1}{12} z^4 \mu \phi_0^2 f_3 \right) \sigma^3 + \left(z^3 \widehat{g}_4 - \frac{1}{3} z^3 \phi_1 f_3 - \frac{1}{12} z^4 \mu \phi_0^2 f_4 \right) \sigma^4 + z^3 O(\sigma^5). \end{aligned} \quad (\text{A.32})$$

The degrees of vanishing of f and g at P are 4 and 6, respectively. This solution is therefore non-minimal and it is not allowed.

²¹We refer to this condition as the “(4,6,12) condition”.

	$A = B = 0$	$A = 1$	$A \geq 2$	$B = 1$	$B \geq 2$
II*	0	0	0	-	-
III*	0	-	-	0	0
IV*	1	1	1	-	-
I ₀ *	∞	3	3	2	2
IV	∞	3	3	-	-
III	∞	-	-	∞	3
II	∞	∞	4	-	-

Table 5. Allowed intersections for type I_n .

2. $z|\mu$ and $z \nmid \phi_0$. In this case, we have

$$\begin{aligned}
 f &= -\frac{1}{48}z^2\mu^2\phi_0^4 - \frac{1}{6}z^2\mu\phi_0^2\phi_1\sigma + \left(z^2\widehat{f}_2 - \frac{1}{3}z^2\phi_1^2\right)\sigma^2 + z^2f_3\sigma^3 + z^2f_4\sigma^4 + z^2O(\sigma^5), \\
 g &= \frac{1}{864}z^3\mu^3\phi_0^6 + \frac{1}{72}z^3\mu^2\phi_0^4\phi_1\sigma + \frac{1}{6}z\mu\phi_0^2\left(\frac{1}{3}z^2\phi_1^2 - \frac{1}{2}z^2\widehat{f}_2\right)\sigma^2 \\
 &\quad + \left(-\frac{1}{3}z^3\phi_1\widehat{f}_2 + \frac{2}{27}z^3\phi_1^3 - \frac{1}{12}z^3\mu\phi_0^2f_3\right)\sigma^3 + \left(z^3\widehat{g}_4 - \frac{1}{3}z^3\phi_1f_3 - \frac{1}{12}z^3\mu\phi_0^2f_4\right)\sigma^4 + z^3O(\sigma^5), \\
 \Delta &= \frac{1}{16}z^2\mu^2\phi_0^4\left(-z^4\widehat{f}_2^2 + z^4\mu\phi_0^2\widehat{g}_4\right)\sigma^4 + z^6O(\sigma^5). \tag{A.33}
 \end{aligned}$$

Now, the degree of vanishing of f and g at P are 2 and 3, respectively, and therefore, this is an allowed intersection. Moreover, the fact that $\mu|_{\{\sigma=0\}}$ vanishes along the locus $\{z=0\}$ implies that we have monodromy and the gauge algebra for the type I_4 is $\mathfrak{sp}(2)$.

Summary. Following the procedure described above, we can summarize our results for local models describing intersections of type I_n with curves of even and odd type as well as I_0^* (this case can be extended to $I_{m \geq 1}^*$). Table 5 describes the maximum n such that type I_n is allowed by the (4,6,12) condition to intersect the singularity types listed below. The symbol - indicates that this possibility is not realizable, for example in all odd types we cannot have $\alpha \neq 0$.

The results summarized in this table have some immediate important consequences. First, configurations $\mathcal{C}_{n_1, \dots, n_N}$ of type I_{n_i} curves do not yield any non-abelian symmetry for type II*, III* and IV*. Second, in the most interesting case of type I_0^* , table 5 provides strong constraints if either $A \neq 0$ or $B \neq 0$. In both cases, by using the monodromy result in table 6, the only admissible algebra summand is $\mathfrak{sp}(1)$. This is trivially a sub-case of the more general $A = B = 0$ setting, to which we restricted our attention in the main body of the paper.

For each allowed model, we can compute the minimal local contributions and in some cases, as we have seen in the example above, we can gather information about the monodromy. We will restrict to the case $A = B = 0$, since from the above this is the relevant

	n	gauge algebra	\tilde{a}	\tilde{b}	\tilde{d}
I_0^*	≥ 2	$\mathfrak{sp}([n/2])$	0	0	6
IV	2	$\mathfrak{su}(2)$	0	2	4
IV	3	b.p.	1	2	4
IV	3	$\mathfrak{su}(2)$	0	3	6
IV	4	$\mathfrak{sp}(2)$	0	4	8
IV	5	$\mathfrak{sp}(2)$	0	5	10
IV	6	$\mathfrak{sp}(3)$	0	6	12
IV	≥ 7	$\mathfrak{sp}([n/2])$	0	n	$2n$
III	2	$\mathfrak{su}(2)$	1	1	3
III	3	b.p.	2	2	6
III	4	b.p.	2	1	6
III	5	$\mathfrak{sp}(2)$	3	3	9
III	6	$\mathfrak{sp}(3)$	3	3	9
III	≥ 7	$\mathfrak{sp}([n/2])$	$[n/2]$	$[n/2]$	$3[n/2]$

Table 6. Local contributions for type I_n .

case. We restrict our attention to intersections for type I_n , $n \geq 2$, since these are the ones carrying a non-abelian gauge algebra. In table 6, we used b.p. to indicate that both monodromies on type I_n are possible.

B The Tate algorithm for the I_n^* case

In this appendix, we collect the general forms for I_n^* for $n \geq 1$, taken from [42]. Let $\{\sigma = 0\}$ carry type I_n^* . We write the quantities f , g and Δ as expansions in the variable σ , where the coefficients are local functions in a sufficiently small open subset of $\{\sigma = 0\}$.

Summary for I_1^* .

$$\begin{aligned}
 f &= -\frac{1}{3}u_1^2\sigma^2 + f_3\sigma^3 + O(\sigma^4), \\
 g &= \frac{2}{27}u_1^3\sigma^3 + \left(\tilde{g}_4 - \frac{1}{3}u_1f_3\right)\sigma^4 + O(\sigma^5), \\
 \Delta &= 4u_1^3\tilde{g}_4\sigma^7 + O(\sigma^8),
 \end{aligned}
 \tag{B.1}$$

where we set $\tilde{g}_j = g_j + \frac{1}{3}u_1f_{j-1}$ for $j \geq 4$.

Summary for I_2^* .

$$\begin{aligned}
 f &= -\frac{1}{3}u_1^2\sigma^2 + f_3\sigma^3 + f_4\sigma^4 + O(\sigma^5), \\
 g &= \frac{2}{27}u_1^3\sigma^3 - \frac{1}{3}u_1f_3\sigma^4 + \left(\tilde{g}_5 - \frac{1}{3}u_1f_4\right)\sigma^5 + O(\sigma^6),
 \end{aligned}$$

	$A = B = 0$	$A = 1$	$A \geq 2$	$B = 1$	$B \geq 2$
II*	X	X	X	-	-
III*	X	-	-	X	X
IV*	X	X	X	-	-
I ₀ *	X	X	X	X	X
I _{p≥1} *	X	-	-	-	-
IV	X	X	X	-	-
III	X	-	-	X	X
II	1	1	1	-	-

Table 7. Allowed intersection for type I_n*

$$\Delta = u_1^2 (4u_1\tilde{g}_5 - f_3^2) \sigma^8 + O(\sigma^9) . \tag{B.2}$$

Summary for I₃*

$$\begin{aligned} f &= -\frac{1}{48}\mu_1^2 s_0^4 \sigma^2 + \frac{1}{2}\mu_1 s_0 t_2 \sigma^3 + f_4 \sigma^4 + f_5 \sigma^5 + O(\sigma^6) , \\ g &= \frac{1}{864}\mu_1^3 s_0^6 \sigma^3 - \frac{1}{24}\mu_1^2 s_0^3 t_2 \sigma^4 + \frac{1}{4}\mu_1 \left(t_2^2 - \frac{1}{3}s_0^2 f_4 \right) \sigma^5 + \left(\tilde{g}_6 - \frac{1}{12}\mu_1 s_0^2 f_5 \right) \sigma^6 + O(\sigma^7) , \\ \Delta &= \frac{1}{16}\mu_1^3 s_0^3 (\tilde{g}_6 s_0^3 - f_4 s_0^2 t_2 - t_2^3) \sigma^9 + O(\sigma^{10}) , \end{aligned} \tag{B.3}$$

where μ_1 is square-free.

The case $n \geq 4$ is fully captured by an induction argument analogous to the one presented in the previous appendix for type I_n.

Summary for I_{n≥4}*

For the even case, $n = 2m$, we have

$$\begin{aligned} u &= u_1 \sigma + u_2 \sigma^2 + \dots + u_m \sigma^m , \\ v &= v_{m+2} \sigma^{m+2} + v_{m+3} \sigma^{m+3} + \dots , \\ w &= w_{2m+3} \sigma^{2m+3} + w_{2m+4} \sigma^{2m+4} + \dots , \end{aligned} \tag{B.4}$$

while for the odd case, $n = 2m + 1$, we have

$$\begin{aligned} u &= u_1 \sigma + u_2 \sigma^2 + \dots + u_{m+1} \sigma^{m+1} , \\ v &= v_{m+3} \sigma^{m+3} + v_{m+4} \sigma^{m+4} + \dots , \\ w &= w_{2m+4} \sigma^{2m+4} + w_{2m+5} \sigma^{2m+5} + \dots , \end{aligned} \tag{B.5}$$

where f, g and Δ are as in (A.25).

Summary. We can summarize our results for local models on type I_n*. Table 7 lists the upper bounds on n such that the intersection between type I_n* with the various Kodaira singularities is not non-minimal. Again, by the symbol - we indicate that this possibility is not realizable, while the symbol X represents a non-minimal intersection $\forall n \geq 1$.

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