

# High-temperature expansion of supersymmetric partition functions

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**ABSTRACT:** Di Pietro and Komargodski have recently demonstrated a four-dimensional counterpart of Cardy's formula, which gives the leading high-temperature ( $\beta \rightarrow 0$ ) behavior of supersymmetric partition functions  $Z^{\text{SUSY}}(\beta)$ . Focusing on superconformal theories, we elaborate on the subleading contributions to their formula when applied to free chiral and U(1) vector multiplets. In particular, we see that the high-temperature expansion of  $\ln Z^{\text{SUSY}}(\beta)$  terminates at order  $\beta^0$ . We also demonstrate how their formula must be modified when applied to SU( $N$ ) toric quiver gauge theories in the planar ( $N \rightarrow \infty$ ) limit. Our method for regularizing the one-loop determinants of chiral and vector multiplets helps to clarify the relation between the 4d  $\mathcal{N} = 1$  superconformal index and its corresponding supersymmetric partition function obtained by path-integration.

**KEYWORDS:** Supersymmetric gauge theory, Anomalies in Field and String Theories, AdS-CFT Correspondence, 1/N Expansion

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**1 Introduction**

Some time ago, Cardy famously employed modular invariance to obtain the high-temperature behavior of conformal field theory (CFT) partition functions in two dimensions [1]. This result has since been exploited in a variety of contexts, including the statistical physics of black holes [2–4]. Cardy’s formula gives the leading order divergence of the CFT partition function  $Z(\beta)$  as the inverse temperature  $\beta$  goes to zero:

$$\ln Z(\beta) \sim \frac{\pi^2 c_L}{6\beta}. \tag{1.1}$$

Here,  $c_L$  is the left-handed CFT central charge, and we have focused on the holomorphic sector for simplicity. Note that the term “Cardy formula” is often applied to the expression, derived from the above relation, for the micro-canonical entropy of a 2d CFT at high energies. However, in the present work, by “Cardy formula” we always refer to the above canonical version for the asymptotic high-temperature expansion of  $\ln Z$ .

Similar formulae had been long sought in higher dimensions without much success, partly because Cardy’s main tool, modular invariance, has no known higher-dimensional counterpart. Recently, Di Pietro and Komargodski have combined ideas from supersymmetry and hydrodynamics [5, 6] to obtain the high-temperature behavior of supersymmetric (SUSY) partition functions in four and six dimensions [7]. Here we expand on their result in the context of four-dimensional superconformal field theories (SCFTs).

By SUSY partition function we mean the one computed with periodic boundary conditions for fermions along the thermal circle; this amounts to an insertion of  $(-1)^F$  when the partition function is represented as a weighted sum over the states, and makes it independent of exactly marginal couplings [8]. Therefore, one might anticipate that the partition function displays universal high-temperature behavior depending only on the 4d central charges. This was realized by Di Pietro and Komargodski, who demonstrated the relation [7]

$$\ln Z^{\text{SUSY}}(\beta) \approx \frac{16\pi^2(c-a)}{3\beta}, \tag{1.2}$$

where  $c$  and  $a$  are the central charges of the 4d SCFT, and where the spatial manifold is taken to be the round  $S^3$ . (In the main text we focus on the round  $S^3$ , while relegating the case of the squashed sphere to appendix A.)

The formula (1.2) can be thought of as the leading order result in a high-temperature expansion. In this paper we explore the subleading corrections to eq. (1.2) and provide evidence that it receives only “non-perturbative” corrections in  $\beta$  (of the type  $e^{-1/\beta}$ ), and  $\mathcal{O}(\log \beta)$  and  $\mathcal{O}(\beta^0)$  corrections. While these corrections have already been pointed out by Di Pietro and Komargodski in [7], we conjecture that the series expansion of  $\ln Z^{\text{SUSY}}(\beta)$  around  $\beta = 0$  terminates at  $\mathcal{O}(1)$ , and that no corrections arise at order  $\beta$  or higher.

To explore the subleading behavior of SUSY partition functions, it proves helpful to understand the relation between their path-integral representation and their representation as a weighted sum. The latter is called the superconformal index [9, 10], and may be defined with two fugacities as

$$\mathcal{I}(p, q) = \text{Tr} \left[ (-1)^F e^{-\hat{\beta}(\Delta - 2j_2 - \frac{3}{2}r)} p^{j_1 + j_2 + \frac{1}{2}r} q^{-j_1 + j_2 + \frac{1}{2}r} \right]. \tag{1.3}$$

Here the trace is over the Hilbert space of the SCFT in radial quantization,  $\Delta$  is the conformal dimension of the state,  $r$  is its R-charge, and  $(j_1, j_2)$  are its  $\text{SO}(4) = \text{SU}(2)_1 \times \text{SU}(2)_2$  quantum numbers. Only states with  $\Delta - 2j_2 - \frac{3}{2}r = 0$  contribute to the index, so it is independent of  $\hat{\beta}$ . The index may be related to the partition function on the round  $S^3 \times S^1$  by taking  $p = q = e^{-\beta}$ , where  $\beta$  is identified with the radius of the  $S^1$ . We thus have

$$\mathcal{I}(\beta) = \mathcal{I}(e^{-\beta}, e^{-\beta}) = \text{Tr} \left[ (-1)^F e^{-\beta(\Delta - \frac{1}{2}r)} \right]. \tag{1.4}$$

The generalization to squashed 3-sphere (and therefore non-equal fugacities in the index) will be discussed in appendix A.

In the following, we will refer to  $Z^{\text{SUSY}}$  obtained by path-integration as the “SUSY partition function”, and to  $\mathcal{I}$  as the “index”. The relation between these two quantities

is [11–13]<sup>1</sup>

$$\mathcal{I}(\beta) = e^{\beta E_{\text{susy}}} Z^{\text{SUSY}}(\beta) = e^{\frac{4(3c+a)}{27}\beta} Z^{\text{SUSY}}(\beta), \quad (1.5)$$

where the supersymmetric Casimir energy [11, 13, 14] is given by  $E_{\text{susy}} = 4(3c + a)/27$ . It is worth noting that in [12, 13], an extra  $\mathcal{O}(1/\beta)$  factor was present in the exponent of the prefactor. However, as mentioned in [7, 14], and as highlighted below, an alternative regularization of the computations in [12–14] would eliminate that extra factor.

The relation (1.5), when combined with our claim that  $\ln Z^{\text{SUSY}}(\beta)$  has no  $\mathcal{O}(\beta)$  term in its asymptotic high-temperature expansion, implies an  $\mathcal{O}(\beta)$  term (namely  $\frac{4(3c+a)}{27}\beta$ ) in the high-temperature expansion of  $\ln \mathcal{I}(\beta)$ . This was conjectured in [15].

The fact that the path-integral and the trace representation of the partition function are related by anomaly-dependent factors has a well-known counterpart in 2d CFT which we will review below.

Also well-known for 2d CFTs is the breakdown (except for sparse CFTs [4]) of Cardy’s formula in the limit of large central charge. The analogous situation for the Di Pietro-Komargodski formula (1.2) was noted in [16] from a case-by-case study of some holographic SCFTs, and also the  $A_k$  SQCD fixed points. It was observed that in the planar limit, which is the 4d gauge theory counterpart of the 2d large- $c$  limit, the 4d index has a rather non-universal high-temperature behavior which is not dictated solely by the central charges. In this paper we systematically investigate large- $N$  toric quivers, and see the modification of (1.2) explicitly. This modification can be intuitively understood as a kind of non-commutativity between the high-temperature limit and the planar limit.

In appendix A, we generalize our computations to the case with squashed 3-sphere as the spatial manifold. Among other things, we derive a powerful identity (given in eq. (A.6), generalizing eq. (3.8)) which relates the elliptic Gamma function to the non-compact quantum dilogarithm, and makes the high-temperature behavior of the index of a chiral multiplet quite transparent and its connection with the 3d partition function manifest.

The results of appendix A will be employed in appendix B to demonstrate the relation between the high-temperature expansion of the index and the holographically derived prescriptions of [15] for extracting the central charges from the single-trace index. In particular, we show that the prescriptions of [15] probe only the  $\mathcal{O}(\beta)$  term in the high-temperature expansion of  $\ln \mathcal{I}$ , and are insensitive to the leading  $\mathcal{O}(1/\beta)$  behavior.

The organization of this paper is as follows. In the next section, for the purpose of orientation and also to highlight later some analogies with 4d SCFTs, we review Cardy’s formula and the subleading corrections it receives in a high-temperature expansion. In section 3 we consider free chiral and U(1) vector multiplets and examine the subleading corrections to the Di Pietro-Komargodski formula. Section 4 contains the discussion of large- $N$  toric quivers, and our concluding remarks are presented in the last section. SCFTs with squashed three-sphere as their spatial manifold are treated in appendix A, and in appendix B the connection between our findings in the present paper and the proposals of [15] are clarified.

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<sup>1</sup>See eq. (A.11) for the relation between the index with two fugacities and the SUSY partition function on the squashed 3-sphere as spatial manifold.

## 2 Cardy's formula for 2d CFTs

Before proceeding to four dimensions, we review some well-known facts about Cardy's formula for 2d CFTs. Cardy's formula [1] is obtained using the modular invariance of 2d CFT partition functions

$$Z_{PI}(\tau) = Z_{PI}(-1/\tau), \tag{2.1}$$

where  $-2\pi i\tau = \beta$ , and we have added a subscript *PI* since we assume the partition function is computed by a path integral. For simplicity we focus on the holomorphic sector of the CFT. If the theory has a gapped spectrum, with its lightest state having energy  $\Delta_0$  with respect to the vacuum, the low-temperature ( $\beta \rightarrow \infty$ ) partition function is dominated by the vacuum contribution  $e^{c_L\beta/24}$ , while the next contribution is down by a factor  $e^{-\beta\Delta_0}$ . Using (2.1) we arrive at Cardy's relation in (1.1), which we rewrite as

$$\ln Z_{PI}(\beta) \sim \frac{\pi^2 c_L}{6\beta}. \tag{2.2}$$

This formula leads to the micro-canonical entropy  $2\pi\sqrt{c_L(L_0 - c_L/24)}/6$ , which matches that of the Strominger-Vafa black hole [2].

The subleading correction to (2.2) is down by a factor  $e^{-4\pi^2\Delta_0/\beta}$ , which is non-analytic in  $\beta$ . In other words Cardy's formula is correct to all orders in a high-temperature expansion, and only receives "non-perturbative" corrections in  $\beta$ . Below, we will repeatedly use the symbol  $\sim$  to denote such all-orders equalities.

To highlight the analogy with 4d SCFTs we now assume the 2d CFT has a single left-handed conserved U(1) current  $J$  whose Laurent modes satisfy the following commutation relations

$$\begin{aligned} [L_n, J_m] &= -mJ_{m+n}, \\ [J_m, J_n] &= 2km\delta_{m+n,0}, \end{aligned} \tag{2.3}$$

where  $L_n$  are the Laurent modes of the energy-momentum tensor. For example, a (2,0) SCFT has such a conserved current with  $k = c_L/6$ , sometimes referred to as the R-current.

Adding a chemical potential  $\mu = 2\pi iz/\beta$  for the U(1) charge of the CFT states, one can define the following grand-canonical partition function

$$\mathcal{I}_{2d}(\tau, z) = \text{Tr}(q^{L_0} y^{J_0}), \tag{2.4}$$

where  $q = e^{2\pi i\tau}$ ,  $y = e^{2\pi iz}$ , and we have considered a holomorphic CFT (with  $c_R = 0$ ) for simplicity. This partition function is not modular invariant, but its path-integral representation  $Z_{PI}(\tau, z)$  is invariant under the modular transformation  $\{\tau, z\} \rightarrow \{-1/\tau, z/\tau\}$ . The two are related via [17]

$$\mathcal{I}_{2d}(\tau, z) = e^{-i\pi k \frac{z^2}{\tau} + 2\pi i\tau \frac{c_L}{24}} Z_{PI}(\tau, z). \tag{2.5}$$

Hence, modular invariance of  $Z_{PI}$  can be combined with the assumption that  $\mathcal{I}_{2d}$  is dominated at low temperatures by the vacuum, to give the high-temperature behavior

$$\mathcal{I}_{2d}(\tau, z) \rightarrow e^{-4\pi^2 k \frac{z^2}{\beta} - \frac{c_L}{24}\beta} e^{\frac{\pi^2 c_L}{6\beta}} = e^{\mu^2 k \beta - \frac{c_L}{24}\beta} e^{\frac{\pi^2 c_L}{6\beta}}. \tag{2.6}$$

For  $(2, 0)$  SCFTs with  $k = c_L/6$ , this variant of Cardy’s formula yields the micro-canonical entropy  $2\pi\sqrt{c_L(L_0 - c_L/24)/6 - J_0^2/4}$ , which reproduces<sup>2</sup> the Bekenstein-Hawking entropy of spinning generalizations of the Strominger-Vafa black hole [17, 18].

For future reference, note that the  $\mathcal{O}(\beta)$  term in the high-temperature expansion

$$\ln \mathcal{I}_{2d}(\tau, z) \sim \frac{\pi^2 c_L}{6\beta} + \mu^2 k \beta - \frac{c_L}{24} \beta. \tag{2.7}$$

differs from the one in

$$\ln Z_{\text{PI}}(\tau, z) \sim \frac{\pi^2 c_L}{6\beta} + \mu^2 \frac{k}{2} \beta. \tag{2.8}$$

In particular, when  $\mu = 0$ , the difference is entirely due to the familiar  $-c_L/24$  Casimir energy on the torus; this Casimir energy is computed by  $Z_{\text{PI}}$ , but not taken into account in our definition of  $\mathcal{I}_{2d}$  in (2.4).

Finally, we remind the reader that Cardy’s formula (2.2) fails for general 2d CFTs in which the  $c \rightarrow \infty$  limit is taken before (or at the same time as) the  $\beta \rightarrow 0$  limit; the asymptotic high-temperature expansion may clash with (2.2). This is because a given large- $c$  CFT may have too many light states so that its low-temperature partition function is no longer dominated by the vacuum contribution alone. CFTs with a “sparse” spectrum of low lying states avoid this breakdown. For related recent discussions see [4, 19, 20]. Similar sparseness conditions for the grand-canonical partition function (2.4) remain to be formulated.

### 3 Subleading corrections to the Di Pietro-Komargodski formula

We now return to four-dimensions and explore the subleading corrections to the Di Pietro-Komargodski result, (1.2). We claim that the high-temperature expansion of the SUSY partition function on the round  $S^3 \times S^1$  has the form

$$\ln Z^{\text{SUSY}}(\beta) \sim \frac{16\pi^2(c - a)}{3\beta} - 4(2a - c) \ln(\beta/2\pi) + \ln Z_{3d}, \tag{3.1}$$

which terminates at  $\mathcal{O}(\beta^0)$ , and is exact up to non-analytic terms of the type  $e^{-1/\beta}$ . Here  $Z_{3d}$  is the supersymmetric partition function of the dimensionally reduced theory on  $S^3$ , which in favorable cases can be computed by localization [21–23]. Using (1.5), the above relation leads to the expansion of the index

$$\ln \mathcal{I}(\beta) \sim \frac{16\pi^2(c - a)}{3\beta} - 4(2a - c) \ln(\beta/2\pi) + \ln Z_{3d} + \frac{4(3c + a)\beta}{27}. \tag{3.2}$$

The linear term on the r.h.s. of (3.2) was conjectured in [15], based on holographically derived relations between central charges and the index. That the high-temperature expansions of  $\ln \mathcal{I}(\beta)$  and  $\ln Z^{\text{SUSY}}(\beta)$  differ only by an  $\mathcal{O}(\beta)$  term is somewhat analogous to the 2d story sketched in eqs. (2.7) and (2.8).

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<sup>2</sup>Note that the actual derivation of this result is a bit more subtle [18], as it relies not on  $c_R = 0$ , but on taking the right-handed sector to be in a Ramond ground state.

Upon squashing the  $S^3$ , the above relations are generalized to (A.9) and (A.13). An important feature arises in the expansion of the index in (A.9), that we would like to highlight already: the linear term in  $\beta$  encodes *two linear combinations* of  $a$  and  $c$ , separated by their different dependence on the squashing parameter. Therefore both central charges can be distilled at high temperatures from the  $\mathcal{O}(\beta)$  term of  $\ln \mathcal{I}$ . In appendix B we will demonstrate that this observation is essential for making contact with the proposals of [15].

We now provide support for the claims (3.1) and (3.2) by investigating free chiral and  $U(1)$  vector multiplets. Of course, a general proof would require studying non-abelian gauge theories and going beyond the free cases.

### 3.1 Free chiral multiplet

Consider now the concrete case of a free chiral multiplet of off-shell R-charge  $R$  (we say off-shell because a free chiral multiplet is known to have R-charge  $2/3$  “on-shell”). Its index can be written as [24]

$$\mathcal{I}_\chi(R, p, q) = \Gamma(z, \tau, \sigma), \tag{3.3}$$

where  $p = e^{2\pi i \tau}$ ,  $q = e^{2\pi i \sigma}$ , and  $z = (\tau + \sigma)R/2$ . Here  $\Gamma(z, \tau, \sigma)$  is the elliptic Gamma function defined by

$$\Gamma(z, \tau, \sigma) = \prod_{j, k \geq 0} \frac{1 - e^{2\pi i((j+1)\tau + (k+1)\sigma - z)}}{1 - e^{2\pi i(j\tau + k\sigma + z)}}. \tag{3.4}$$

In order to investigate the high-temperature limit of (3.3), we make use of the  $SL(3, \mathbb{Z})$  modular property [25]

$$\Gamma(z, \tau, \sigma) = e^{-i\pi M(z; \tau, \sigma)} \frac{\Gamma\left(\frac{z}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right)}{\Gamma\left(\frac{z-\tau}{\sigma}, -\frac{1}{\sigma}, -\frac{\tau}{\sigma}\right)}, \tag{3.5}$$

where

$$M(z; \tau, \sigma) = \frac{z^3}{3\tau\sigma} - \frac{\tau + \sigma - 1}{2\tau\sigma} z^2 + \frac{\tau^2 + \sigma^2 + 3\tau\sigma - 3\tau - 3\sigma + 1}{6\tau\sigma} z + \frac{1}{12}(\tau + \sigma - 1)(\tau^{-1} + \sigma^{-1} - 1). \tag{3.6}$$

This is analogous to how the  $SL(2, \mathbb{Z})$  properties of 2d partition functions allow a Cardy-type analysis, as briefly sketched in the previous section.

Restricting to the case of equal fugacities (i.e. the round  $S^3$ ), the index can be written as

$$\mathcal{I}_\chi(R, \beta) = \Gamma(R\tau, \tau, \tau), \tag{3.7}$$

where  $\tau = i\beta/2\pi$ . In order to study its  $\tau \rightarrow 0$  limit, we resort to Theorem 5.2 of [25] which is derived from (3.5), along with some straightforward manipulation, to rewrite  $\Gamma(R\tau, \tau, \tau)$  as

$$\Gamma(R\tau, \tau, \tau) = \frac{e^{-i\pi M(\tau, R)}}{\psi(-(R-1))} \prod_{n=1}^{\infty} \frac{\psi\left(\frac{n+(R-1)\tau}{\tau}\right)}{\psi\left(\frac{n-(R-1)\tau}{\tau}\right)}, \tag{3.8}$$

where

$$M(\tau, R) = \left(\frac{R-1}{6}\right) \frac{1}{\tau} + \left(\frac{R^2}{2} - R + \frac{5}{12}\right) + \left(\frac{R^3}{3} - R^2 + \frac{5R}{6} - \frac{1}{6}\right) \tau. \quad (3.9)$$

The  $\psi$  functions present on the r.h.s. of eq. (3.8) can be expressed as

$$\ln \psi(R) = R \ln(1 - e^{-2\pi i R}) - \frac{1}{2\pi i} Li_2(e^{-2\pi i R}). \quad (3.10)$$

$\psi(R)$  has a zero of order  $j$  at  $R = j$ , and a pole of the same order at  $R = -j$ , for  $j \in \mathbb{Z}^{>0}$ .

The reader familiar with the 3d localization literature may notice that  $\psi(R)$  is related to the function  $\ell(R)$  that Jafferis uses in [22] via

$$\ell(R) = \ln \psi(-R) + \frac{i\pi R^2}{2} - \frac{i\pi}{12}. \quad (3.11)$$

$\ell(R)$  has the useful property that  $\ell(-R) = -\ell(R)$ . From the information on poles and zeros of  $\psi(R)$ , we see that  $\ell(R)$  is singular at  $R \in \mathbb{Z} - \{0\}$ . For future reference, we add that  $\ell(R)$  is related to the function  $s_{b=1}(R)$  in [23] (see also appendix A) via

$$\ell(R) = \ln s_{b=1}(iR). \quad (3.12)$$

To obtain the high-temperature behavior of (3.8) we utilize the fact that the  $\psi$  function is exponentially close to one when its argument has a large negative imaginary part [25]. This means that in the limit  $\beta = -2\pi i \tau \rightarrow 0$ , the infinite product in eq. (3.8) can be replaced with one, yielding

$$\ln \mathcal{I}_\chi(R, \beta) = \ln \Gamma(R\tau, \tau, \tau) \sim -i\pi M(\tau, R) - \ln \psi(-(R-1)). \quad (3.13)$$

Recall that  $\sim$  means to all orders in a high-temperature expansion; non-analytic corrections of the type  $e^{-1/\beta}$  coming from the infinite product on the r.h.s. of (3.8) are present but are not part of the perturbative expansion. Substituting in (3.9) for  $M(\tau, R)$  and making use of (3.11) then gives

$$\ln \mathcal{I}_\chi(R, \beta) \sim \frac{-\pi^2(R-1)}{3\beta} + \ell(-(R-1)) + \beta \left(\frac{R^3}{6} - \frac{R^2}{2} + \frac{5R}{12} - \frac{1}{12}\right), \quad (3.14)$$

in perfect agreement with the conjectured form of the index, (3.2). Importantly, there are no terms of order  $\beta^2$  or higher on the r.h.s. . Also, since for a chiral multiplet  $2a - c = 0$ , there is no  $\mathcal{O}(\log \beta)$  term here, unlike in the case of a free U(1) vector multiplet [7].

The temperature-independent term  $\ell(-(R-1))$  in (3.14) is precisely the log of the partition function of a 3d chiral multiplet [22]; this is the well-known result that the  $\mathcal{N} = 1$  4d index reduces, as  $\beta \rightarrow 0$ , to the 3d partition function, after (and only after) its  $\mathcal{O}(1/\beta)$  divergent exponent is removed. Related discussions can be found in [26–31]. The argument above is, however, in our opinion the most transparent derivation of the reduction result for a chiral multiplet (see also appendix A.1 for the case with squashing).

Having established the expansion of the index, we now turn to the high-temperature behavior of  $Z_\chi^{\text{SUSY}}(R, \beta)$ , the SUSY partition function of a chiral multiplet with R-charge



$R$ . The computation is done by KK compactification of the theory on the thermal circle, calculating the contribution to the free energy of the  $n$ -th KK modes  $\ln Z_\chi^{(n)}(R, \beta)$ , and then summing up over  $n$ . A similar calculation was performed in the appendix of [7] to obtain the leading high-temperature behavior of the SUSY partition function. For  $\ln Z_\chi^{(n)}(R, \beta)$ , we may use the results of [22, 23]

$$\ln Z_\chi^{(n)}(R, \beta) = \ln s_{b=1} \left( i - iR - \frac{2\pi n}{\beta} \right) = \ell \left( 1 - R + \frac{2\pi in}{\beta} \right). \quad (3.15)$$

Summing over the KK tower now gives

$$\begin{aligned} \ln Z_\chi^{\text{SUSY}}(R, \beta) &= \sum_{n \in \mathbb{Z}} \ell \left( 1 - R + \frac{2\pi in}{\beta} \right) \\ &= \ell(-(R-1)) + \sum_{n>0} [\ell(1 - R + 2\pi in/\beta) - \ell(-1 + R + 2\pi in/\beta)], \end{aligned} \quad (3.16)$$

where we have used the property  $\ell(-x) = -\ell(x)$ . With the aid of (3.11) we can write the above result in terms of the  $\psi$  functions as follows

$$\begin{aligned} \ln Z_\chi^{\text{SUSY}}(R, \beta) &= \sum_{n>0} \ln \frac{\psi(R-1-2\pi in/\beta)}{\psi(1-R-2\pi in/\beta)} + 2 \sum_{n>0} \frac{2\pi^2(R-1)n}{\beta} + \ell(-(R-1)) \\ &= \sum_{n>0} \ln \frac{\psi(R-1-2\pi in/\beta)}{\psi(1-R-2\pi in/\beta)} - \frac{\pi^2(R-1)}{3\beta} + \ell(-(R-1)), \end{aligned} \quad (3.17)$$

where, following [7], we have used zeta function regularization in order to arrive at the result in the last line.

Note that our computation differs from that given in the appendix of [7] in a few respects. This affects the subleading order, but not the leading order result, which was the focus of [7]. First of all, in contrast with [7], we have assembled the KK contributions before taking the high-temperature limit. Secondly, while the 3d bosonic partition functions in [7] and [23] are identical, agreement of the 3d fermionic partition functions is more subtle. The Dirac spectrum in (4.6) of [23] has two pieces; if the contribution from the second term is summed over *after shifting* the related quantum number, and a relative minus sign is introduced, then (A.4) of [7] is recovered.<sup>3</sup> To avoid subtleties with the fermion reduction (such as the mixing of the reduced fermions), our logic above is to reduce the bosons on  $S^1$  to obtain the 3d bosonic Lagrangian. Then, instead of reducing the fermions, we simply appeal to the SUSY completion of the 3d action of the KK bosons. The resulting 3d partition function for the  $n$ -th KK modes is now that reported in the 3d localization literature. This method of computing the 4d partition function is equivalent to that of [12, 13].

Comparing (3.17) with (3.8), (3.9), and (3.11), now yields the relation

$$\mathcal{I}_\chi(R, \beta) = e^{\left(\frac{R^3}{6} - \frac{R^2}{2} + \frac{5R}{12} - \frac{1}{12}\right)\beta} Z_\chi^{\text{SUSY}}(R, \beta). \quad (3.18)$$

Since a chiral multiplet with R-charge  $R$  has

$$\frac{R^3}{6} - \frac{R^2}{2} + \frac{5R}{12} - \frac{1}{12} = \frac{4}{27}(3c+a), \quad (3.19)$$

---

<sup>3</sup>Note that  $l, q, n$ , and  $\sigma$  in [23] correspond to  $r_3, R, l$ , and  $n/r_1$  in [7], respectively.

we confirm the relation (1.5) between the index and the SUSY partition function for this case. Combining (1.5) with the high-temperature expansion of the index in (3.14) then gives the expansion for  $\ln Z^{\text{SUSY}}$  presented in (3.1). Note in particular that the last term on the r.h.s. of eq. (3.17) is the contribution of the zero-modes. Our computation above therefore gives some understanding for why the finite part of the 4d SUSY partition function reduces to the 3d partition function upon taking the  $\beta \rightarrow 0$  limit; this is because the  $n \neq 0$  KK modes only contribute to the  $\mathcal{O}(1/\beta)$  term in  $\ln Z^{\text{SUSY}}$ , besides giving transcendently small corrections to it that are negligible in the high-temperature limit.

### 3.2 Free U(1) vector multiplet

Our next case study is the theory of a single free U(1) vector multiplet. The index of this theory is given by [24]

$$\mathcal{I}_v(p, q) = (p; p)(q; q), \tag{3.20}$$

where  $(a; q) = \prod_{k=0}^{\infty} (1 - aq^k)$  is the  $q$ -Pochhammer symbol. We are, of course, mainly interested in the case of equal fugacities, in which case

$$\mathcal{I}_v(\beta) = (q; q)^2. \tag{3.21}$$

Note that  $(q; q)$  is related to Dedekind's eta function via

$$\eta(\beta) = q^{1/24}(q; q). \tag{3.22}$$

The high-temperature expansion may be obtained by invoking the familiar  $SL(2, \mathbb{Z})$  modular property  $\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)$ . We find that at high temperatures  $\eta \rightarrow e^{-\pi^2/6\beta} \sqrt{\frac{2\pi}{\beta}}$ , which leads to

$$\mathcal{I}_v(\beta) \rightarrow e^{-\pi^2/3\beta} \left(\frac{2\pi}{\beta}\right) e^{\beta/12}. \tag{3.23}$$

Taking the logarithm of this equation gives

$$\ln \mathcal{I}_v \sim -\pi^2/3\beta - \ln\left(\frac{\beta}{2\pi}\right) + \beta/12, \tag{3.24}$$

where again  $\sim$  means correct to all orders in  $\beta$ , but excluding non-analytic corrections of the type  $e^{-1/\beta}$ . This is almost in full agreement with (3.2), since for a single vector multiplet  $c = 1/8$  and  $a = 3/16$ . In particular, (3.24) confirms the conjecture in [15] regarding the linear term. Also, terms of order  $\beta^2$  or higher are absent. To make the agreement with (3.2) complete, however, we need to have  $\ln Z_{3d} = 0$ . As explained in [7] the dimensionally reduced theory of a vector multiplet is quantum-mechanically ill-defined, and the logarithmic term in (3.24) is signalling a problem; the 3d vector multiplet has non-compact moduli, over which the vacuum state can not be properly normalized (see also the related discussion in [32]). There would not be anything puzzling with our finding  $\ln Z_{3d} = 0$ , however, if we think of (3.21) as the index of a U(1) vector multiplet with the zero-modes removed (see [33] for a similar terminology in a 2d context).

Now consider the SUSY partition function of the same theory. It is given in eq. (G.11) of [13] (with  $r_G = 1$ )

$$Z_v^{\text{SUSY}}(\beta) = e^{\frac{i\pi}{2}\Psi(0,\tau,\tau)}(q; q)^2 = e^{\frac{i\pi}{2}\Psi(0,\tau,\tau)}\mathcal{I}_v(\beta), \quad (3.25)$$

where  $\Psi(w, \tau, \tau)$  is defined through

$$F(w, \tau, \tau)F(-w, \tau, \tau) = \frac{e^{i\pi\Psi(w,\tau,\tau)}}{\Gamma(w, \tau, \tau)\Gamma(-w, \tau, \tau)}, \quad (3.26)$$

with

$$F(w, \tau, \sigma) = \prod_{n_0 \in \mathbb{Z}} \prod_{n_1, n_2 \geq 0} \frac{w + n_0 + \frac{\tau+\sigma}{2} - \frac{\tau+\sigma}{2} - n_1\tau - n_2\sigma}{w + n_0 + \frac{\tau+\sigma}{2} + \frac{\tau+\sigma}{2} + n_1\tau + n_2\sigma}. \quad (3.27)$$

Combining the previous two equations, we arrive at

$$F(w, \tau, \sigma)F(-w, \tau, \sigma) = \prod_{n_0 \in \mathbb{Z}} \left[ \prod_{n_1, n_2 \geq 0} \frac{(-w - n_0 - \frac{\tau+\sigma}{2}) + \frac{\tau+\sigma}{2} + n_1\tau + n_2\sigma}{(w + n_0 + \frac{\tau+\sigma}{2}) + \frac{\tau+\sigma}{2} + n_1\tau + n_2\sigma} \right. \\ \left. \times \frac{(w - n_0 - \frac{\tau+\sigma}{2}) + \frac{\tau+\sigma}{2} + n_1\tau + n_2\sigma}{(-w + n_0 + \frac{\tau+\sigma}{2}) + \frac{\tau+\sigma}{2} + n_1\tau + n_2\sigma} \right]. \quad (3.28)$$

The r.h.s. can be written in terms of the function  $s_b$  in [23]

$$F(w, \tau, \tau)F(-w, \tau, \tau) = \prod_{n_0 \in \mathbb{Z}} \left[ s_{b=1} \left( -i + \frac{2\pi}{\beta}(n_0 + w) \right) s_{b=1} \left( -i + \frac{2\pi}{\beta}(n_0 - w) \right) \right] \\ = \prod_{n_0 \in \mathbb{Z}} \left[ s_{b=1} \left( i - \frac{2\pi}{\beta}(n_0 + w) \right) s_{b=1} \left( i - \frac{2\pi}{\beta}(n_0 - w) \right) \right]^{-1}. \quad (3.29)$$

In the last step we have used  $s_b(-x) = 1/s_b(x)$ . Comparing with (3.15) makes it now clear that

$$F(w, \tau, \tau)F(-w, \tau, \tau) = \frac{1}{Z_\chi^{\text{SUSY}}(R = w/\tau, \beta)Z_\chi^{\text{SUSY}}(R = -w/\tau, \beta)}. \quad (3.30)$$

The conversion factor that we derived between  $Z_\chi^{\text{SUSY}}$  and  $\mathcal{I}_\chi$  in (3.18) leads therefore to the correct function  $\Psi$  mediating  $Z_v^{\text{SUSY}}$  and  $\mathcal{I}_v$ . Explicit calculation by combining (3.7), (3.18), (3.26) and (3.30) shows

$$\Psi(w, \tau, \tau) = 2\frac{w^2}{\tau} + \frac{\tau}{3}. \quad (3.31)$$

Plugging back into (3.25) gives

$$Z_v^{\text{SUSY}}(\beta) = \eta(q)^2 = e^{-\beta/12}\mathcal{I}_v(\beta), \quad (3.32)$$

which confirms (1.5) for the free vector case. Using eq. (3.24) we can then write down the following high-temperature expansion

$$\ln Z_v^{\text{SUSY}} \sim -\pi^2/3\beta - \ln \left( \frac{\beta}{2\pi} \right). \quad (3.33)$$

## 4 A Di Pietro-Komargodski formula for large- $N$ toric quivers

Toric quiver theories are a much-studied subset of supersymmetric gauge theories whose field content can be efficiently summarized in a quiver diagram. These are directed graphs with nodes representing  $\mathcal{N} = 1$  vector multiplets and edges representing  $\mathcal{N} = 1$  chiral multiplets. The nodes at the ends of an edge represent vector multiplets under which the chiral multiplet represented by the edge is charged. The direction of the edge encodes further information about the representation of the gauge group according to which the chiral multiplet transforms. The toric condition puts further constraints on the theory, thereby guaranteeing some nice properties such as existence of a non-trivial IR fixed point with a holographic dual describable by “toric geometry” (see for instance [34]). A canonical example is the  $\mathcal{N} = 4$  SYM with  $SU(N)$  gauge group, which can be represented by one node (standing for the  $SU(N)$  vector multiplet), and three directed edges (standing for the three  $\mathcal{N} = 1$  chiral multiplets in the adjoint) that both emanate from and end on that one node.

In this section we show that for the large- $N$  limit of toric quiver gauge theories the relation (1.2) is modified to

$$\begin{aligned} \ln \mathcal{I}_{\text{quiver}}^{N \rightarrow \infty}(\beta) \sim & \frac{\pi^2}{6\beta} \sum_{i=1}^{n_z} \frac{1}{r_i} + \frac{16\pi^2}{3\beta} \sum_{\text{adj}} (\delta c_{\text{adj}} - \delta a_{\text{adj}}) \\ & + \frac{n_z}{2} \ln(\beta/2\pi) + \ln Y + \frac{4(3\delta c + \delta a)\beta}{27}, \end{aligned} \quad (4.1)$$

where  $r_i$  are the R-charges of extremal BPS mesons in the quiver [35, 36],  $n_z$  is the number of such mesons (or the number of corresponding zigzag paths in the brane-tiling picture [36]),  $\delta c$  and  $\delta a$  denote the  $\mathcal{O}(1)$  contributions to the full central charges (while  $\delta c_{\text{adj}}$  and  $\delta a_{\text{adj}}$  denote only the contributions from any chiral adjoint matter) and  $\ln Y = \frac{1}{2} \sum_{i=1}^{n_z} \ln r_i + \sum_{\text{adj}} \ell(R_{\text{adj}} - 1)$ . See equations (3.11) and (3.10) for the definition of the function  $\ell$ . The way to determine  $r_i$  and  $n_z$  for a given quiver is explained in eq. (4.8) below.

With the aid of a conjecture in [16], we can write<sup>4</sup>

$$\sum_{i=1}^{n_z} \frac{1}{r_i} = \frac{3}{16\pi^3} \left( 19\text{vol}(SE) + \frac{1}{8}\text{Riem}^2(SE) \right), \quad (4.2)$$

where  $SE$  denotes the Sasaki-Einstein 5-manifold dual to the quiver gauge theory. The above conjecture was motivated by the finding in [37] that one can “hear the shape of the dual geometry” in the asymptotics of the Hilbert series of mesonic operators in the SCFT. We note that the leading high-temperature behavior of the index of toric quivers is contained in the first two terms of (4.1). The first term, according to (4.2), is dictated by the geometry of the dual internal manifold, while the second is given by the  $\mathcal{O}(1)$  part of the contribution of adjoint matter to  $c - a$ . The latter is hence the only part of the finite- $N$

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<sup>4</sup>In [16] the conjectured expression was given for  $a_0 - b_0$ , the difference of two coefficients appearing in the high-temperature expansion of the single-trace index. However, in all cases considered there  $a_0 - b_0 = \sum 1/r_i$ , so the conjecture can be alternatively stated as in (4.2).

Di Pietro-Komargodski formula that escapes metamorphosis into “geometry” in the planar limit. In addition, while at zero squashing both of these terms have the same dependence on  $\beta$ , as displayed in (A.24) they each have distinct dependence on the squashing parameter and can therefore be distinguished.

As an illustrative example, let us consider the  $\mathcal{N} = 4$  theory, and see how the conjecture (4.2) works for this case. In this theory  $n_z = 3$  and<sup>5</sup>  $r_{1,2,3} = 2/3$ . The conjecture (4.2) reads

$$\frac{9}{2} = \sum_{i=1}^3 \frac{1}{r_i} = \frac{3}{16\pi^3} \left( 19\text{vol}(S^5) + \frac{1}{8}\text{Riem}^2(S^5) \right) = \frac{3}{16\pi^3} \left( 19(\pi^3) + \frac{1}{8}(40\pi^3) \right), \quad (4.3)$$

where we have used the geometrical data in table 2 of [37] to evaluate the r.h.s. . Similar tests can be successfully performed for all the  $\text{SE}_5$  manifolds listed in table 2 of [37].

### 4.1 Derivation

Our starting point for computing the large- $N$  index is the following expression, valid when the nodes of the quiver have  $\text{SU}(N)$  gauge groups [38]

$$\ln \mathcal{I}_{\text{quiver}}^{N \rightarrow \infty}(q) = - \sum_{k=1}^{\infty} \frac{\text{tr } i(q^k)}{k} - \ln \prod_{k=1}^{\infty} \det(1 - i(q^k)). \quad (4.4)$$

The matrix  $i$  has the single-letter index of the fields transforming in the fundamental representation of the  $j$ -th node and the anti-fundamental representation of the  $k$ -th node as its  $jk$  entry. On its diagonal it has the single-letter index of the corresponding vector multiplets and the adjoint matter. The first term on the r.h.s. of (4.4) is the subtracted contribution of the  $\text{U}(1)$ 's from the  $\text{U}(N)$  answer given by the second term. We neglect the first term until eq. (4.11) where it is re-introduced.

To obtain an expression for the second term on the r.h.s. of (4.4) we use [35]

$$(1 - i(q)) = \frac{\chi(q)}{(1 - q)^2}, \quad (4.5)$$

with the  $n_v \times n_v$  matrix  $\chi$  (where  $n_v$  is the number of nodes in the quiver) being a purely graph-theoretic object given by

$$\chi(q) = 1 - q^2 - M_Q(q) + q^2 M_Q(q^{-1}). \quad (4.6)$$

Here  $M_Q(q)$  is the weighted adjacency matrix

$$M_Q(q) = \sum_e q^{R(e)} E_{h(e),t(e)}, \quad (4.7)$$

with  $R(e)$  the R-charge of the edge  $e$  in the quiver and  $E_{v,w}$  is a matrix such that the  $(v, w)$  entry is 1 and all other entries are zero.

---

<sup>5</sup>Note that  $r = 2/3$  is the R-charge of the trace of the adjoint matter. This exemplifies the fact that the “extremal BPS mesons” that play a role in (4.1) are in general mesons of the theory with  $\text{U}(N)$  gauge group. The language of zigzag paths [36] might therefore be preferable when studying  $\text{SU}(N)$  quivers.

Our following manipulations are made possible by the remarkable factorization [35, 36]

$$\det\chi(t) = \prod_{i=1}^{n_z} (1 - t^{r_i}). \quad (4.8)$$

The above identity is proven for a subset of all toric quivers in [35], but is conjectured to be valid more generally [36]. It allows an efficient rewriting of the index of the quiver theories in terms of  $(q^{r_i}; q^{r_i})$ .

The second term on the r.h.s. of (4.4) can be written as

$$-\ln \prod_{k=1}^{\infty} \det \left( \frac{\chi(q^k)}{(1 - q^k)^2} \right) = -\ln \prod_{k=1}^{\infty} \left( \frac{1}{(1 - q^k)^{2n_v}} \right) \det(\chi(q^k)), \quad (4.9)$$

which with the aid of the  $q$ -Pochhammer symbol and eq. (3.21) can be cast into

$$2n_v \ln(q; q) - \sum_{i=1}^{n_z} \ln(t^{r_i}; t^{r_i}) = n_v \ln \mathcal{I}_v(\beta) - \frac{1}{2} \sum_{i=1}^{n_z} \ln \mathcal{I}_v(\beta r_i). \quad (4.10)$$

Now we re-introduce the first term on the r.h.s. of (4.4), whose contribution from the vector multiplets happens to kill the first term on the r.h.s. of the above equation. However, the contribution from the adjoint matter remains, so that

$$\ln \mathcal{I}_{\text{quiver}}^{N \rightarrow \infty}(\beta) = -\frac{1}{2} \sum_{i=1}^{n_z} \ln \mathcal{I}_v(\beta r_i) - \sum_{\text{adj}} \ln \mathcal{I}_\chi(R_{\text{adj}}, \beta). \quad (4.11)$$

Using  $\sum_{i=1}^{n_z} r_i = 2n_v$ , and also employing (3.14) and (3.24), the high-temperature expansion given in (4.1) is obtained.

## 4.2 The $\mathcal{N} = 4$ theory as an example

The  $\mathcal{N} = 4$  theory has one vector multiplet and three adjoint chiral multiplets of R-charge  $R = 2/3$ . Application of eq. (4.8) gives for this case  $n_z = 3$  and  $r_{1,2,3} = 2/3$ . For the theory with  $SU(N)$  gauge group we find from (4.11) that

$$\begin{aligned} \ln \mathcal{I}_{\mathcal{N}=4}^{N \rightarrow \infty}(\beta) &= -\frac{3}{2} \ln \mathcal{I}_v(2\beta/3) - 3 \ln \mathcal{I}_\chi(R = 2/3, \beta) \\ &= -3 \ln(q^{2/3}; q^{2/3}) - 3 \ln \Gamma(2\tau/3, \tau, \tau). \end{aligned} \quad (4.12)$$

The asymptotic high-temperature expansion can be derived from the expressions in the previous section to be

$$\ln \mathcal{I}_{\mathcal{N}=4}^{N \rightarrow \infty}(\beta) \sim \frac{5}{2} \frac{\pi^2}{6\beta} + \frac{3}{2} \ln \left( \frac{\beta}{2\pi} \right) + \left( \frac{3}{2} \ln \frac{2}{3} - 3\ell(1/3) \right) - \frac{4}{27}\beta, \quad (4.13)$$

which is exact up to non-analytic corrections of the type  $e^{-1/\beta}$ .

Note that for the  $\mathcal{N} = 4$  theory with  $U(N)$  gauge group we would only need the second term in eq. (4.4), whose representation in (4.10) can be employed to obtain

$$\ln \mathcal{I}_{U(N) \mathcal{N}=4}^{N \rightarrow \infty}(\beta) \sim \frac{1}{2} \frac{\pi^2}{6\beta} + \frac{1}{2} \ln \left( \frac{\beta}{2\pi} \right) + \frac{3}{2} \ln \frac{2}{3}, \quad (4.14)$$

with no  $\mathcal{O}(\beta)$  term on the r.h.s., and also no appearance of the  $\ell$  function. Both of these features are shared by all  $U(N)$  quivers described by (4.10).

## 5 Discussion

In this note we have considered free chiral and  $U(1)$  vector multiplets, and shown the robustness of the Di Pietro-Komargodski formula. When written as the high-temperature expansion of  $\ln Z^{\text{SUSY}}(\beta)$ , it receives — aside from transcendentally small contributions — only  $\mathcal{O}(\ln \beta)$  and  $\mathcal{O}(\beta^0)$  corrections. It is tempting to speculate that similar statements apply to general Lagrangian SCFTs. It may be possible to investigate this robustness by defining a “holomorphic temperature” and using holomorphy on  $S^3 \times S^1$  [39]. Another suspicion is that a yet more robust version of the Di Pietro-Komargodski formula may exist for SUSY partition functions obtained by path-integration over *holomorphically normalized*, as opposed to canonically normalized, gauge fields. To explore these possibilities, extending the “effective gauge coupling” technique of [40], and “holomorphic gauge coupling” technique of [41] to the curved-space supersymmetric case may prove helpful.

The reader may ask why in our discussion of finite- $N$  theories we have emphasized that the theories under study are free, while the index of any Lagrangian theory is independent of the couplings and can be evaluated easily even if the theory flows to an interacting SCFT in the IR. The reason is that such interacting SCFTs can not be constructed with only chiral and  $U(1)$  vector multiplets; one needs asymptotic freedom in the Lagrangian, and therefore non-abelian gauge fields. We have not studied non-abelian gauge theories in this work because their index is significantly more difficult to analyze, involving contour integrals that are hard to evaluate analytically [24]. We hope that the understanding gained in this work eventually help analyzing the high-temperature behavior of non-abelian gauge fields.

The subleading  $\mathcal{O}(\ln \beta)$  term in (3.1) and (3.2) has some resemblance to the results of [32, 42]; it may be possible to make immediate progress generalizing (3.1) and (3.2) for  $\mathcal{N} = 2$  SCFTs by their methods. It would be very interesting if an analysis along those lines shows that the coefficient of  $\ln \beta$  in (3.1) and (3.2) depends — in contrast to what we claimed — not only on the central charges, but also on some “non-universal” information, such as the dimensions of certain operators.

Another direction to study is examining the high-temperature behavior of the index of non-Lagrangian SCFTs. The particular case of  $E_6$  SCFT is readily in analytical reach [43]. In fact, since the proof of Di Pietro and Komargodski applies only to Lagrangian theories [7], it would be very interesting to see if even the leading high-temperature behavior pans out for the  $E_6$  SCFT.

A somewhat different path to explore is that of large- $N$  gauge theories. For toric quivers we presented in eq. (4.1) the explicit form of the modified Di Pietro-Komargodski formula, including its subleading corrections to all orders in  $\beta$ . As eqs. (4.1) and (4.2) show, understanding the high-temperature behavior of the index of holographic quivers involves elements of graph theory, geometry, and the theory of modular forms (or perhaps a matrix generalization thereof, as eq. (4.4) suggests). It would be nice to have a more general understanding of the connection between these elements beyond the toric case.

A topic we did not touch upon in our discussion of large- $N$  quivers is that of the SUSY partition functions  $Z_{\text{quiver}}^{N \rightarrow \infty}(\beta)$  obtained by path integration, in the planar limit. In analogy



with the finite- $N$  case, we expect such partition functions to be proportional to  $\mathcal{I}_{\text{quiver}}^{N \rightarrow \infty}(\beta)$  with an anomaly-dependent coefficient mediating the relation. We leave a careful study of this problem to the future, and simply note that a modification of the finite- $N$  version in (1.5) may have implications for the Casimir energy mismatch puzzle raised in [44].

Non-holographic theories in the planar limit present another playground in which to observe potential modifications of the Di Pietro-Komargodski formula, and explore its subleading corrections. Let us discuss one example of this class, namely the SQCD fixed point with  $x = N_c/N_f = 1/2$  in the Veneziano limit. The index of this theory can be easily obtained from the expressions given in [24]. We find

$$\begin{aligned} \ln \mathcal{I}_{\text{SQCD}, x=1/2}^{N_c \rightarrow \infty}(\beta) &= -N_f^2 \ln(q; q) + (N_f^2 - 1) \ln(q^2; q^2) \\ &\sim \frac{N_f^2 + 1}{2} \left( \frac{\pi^2}{6\beta} \right) + \frac{1}{2} \ln \left( \frac{\beta}{2\pi} \right) - \frac{N_f^2 - 1}{2} \ln 2 + \frac{N_f^2 - 2}{24} \beta. \end{aligned} \tag{5.1}$$

The coefficient of  $\pi^2/6\beta$  in the second line is known (as the  $a_0$  coefficient in table 2 of [16]) to be  $2N_c^2 + 1/2$  for general  $x$ . An application of the finite- $N$  Di Pietro-Komargodski formula would give instead  $32(c - a) = 2N_c^2 + 2$ , which although correct at order  $N_c^2$ , differs at order one from the actual value. On the other hand the coefficient of  $\beta$  above precisely matches with  $4(3c + a)/27$  predicted by the finite- $N$  formula (3.2). Importantly, unlike for the holographic quivers, this term includes the full central charges, and not just their  $\mathcal{O}(1)$  piece. This is related to the observations made in [15] regarding the possibility of extracting the full central charges from the large- $N$  index of  $A_k$  SQCD fixed points (see appendix B).

Finally, modular properties of the SUSY partition functions discussed above hint toward a general modular structure in four dimensions. In section 2 we presented some 2d relations that bear striking resemblance to those in four dimensions. The resemblance is, however, far from perfect. For example,  $Z_{\text{PI}}$  in section 2 was modular invariant, but the four-dimensional  $Z^{\text{SUSY}}$  is apparently not. A deeper understanding of the differences with the 2d modular structure may shed light on the 4d/2d relations [45–50].

**Note added:** shortly after the first version of the present paper appeared on arXiv, two important related developments happened. The authors of [51] formulated a sparseness condition for the elliptic genera of 2d CFTs with (2,2) supersymmetry, partially addressing the problem mentioned in the last sentence of section 2. In [52] the supersymmetric Casimir energy is studied in great detail, and also the regularization of SUSY partition functions on Hopf surfaces is clarified.

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## A SCFTs with squashed three-sphere as spatial manifold

In the main body, we have focused on the round  $S^3 \times S^1$ . Here we demonstrate that the results can be extended to the more general case where the fugacities  $p$  and  $q$  are not necessarily equal. The index with two fugacities is given in (1.3), and the corresponding SUSY partition function is (see for instance [31]) the one computed on  $S_b^3 \times S^1$ , with the first factor representing a squashed 3-sphere with squashing parameter  $b$ , which is related to the fugacities by  $p = e^{2\pi i \tau} = e^{-\beta b^{-1}}$  and  $q = e^{2\pi i \sigma} = e^{-\beta b}$ .

In the following we will need some mathematical notation that we now introduce. An important role will be played below by the “double sine” function  $s_b$  in [53], which can be represented as

$$s_b(-ix) = \prod_{m,n \geq 0} \frac{mb + nb^{-1} + \frac{Q}{2} - x}{mb + nb^{-1} + \frac{Q}{2} + x}, \tag{A.1}$$

with  $Q \equiv b + b^{-1}$ . In parallel with (3.12) we can define  $\ell_b(x) = \ln s_b(ix)$ . To generalize (3.11) we then define the function  $\psi_b$  through

$$\ell_b(x) = \ln \psi_b(-x) + \frac{i\pi x^2}{2} - \frac{i\pi}{24}(b^2 + b^{-2}). \tag{A.2}$$

$\psi_b$  is related to the “non-compact quantum dilogarithm” function  $e_b$  in [54] via  $\psi_b(x) = e_b(-ix)$ . Using eq. (15) in [54] we can express  $\psi_b(x)$  for  $|\operatorname{Re} x| < Q/2$  by

$$\ln \psi_b(x) = \int_{-\infty}^{+\infty} \frac{dt}{4t} \frac{e^{-2xt}}{\sinh(bt) \sinh(t/b)}, \tag{A.3}$$

where the singularity at  $t = 0$  is put below the contour of integration. Explicit evaluation of the above contour integral for  $b = 1$ , gives  $\psi$  as written in (3.10). In other words,  $\psi_{b=1}$  equals the  $\psi$  function in the main text. Importantly for our computations below, eq. (21) in [54] can be used to write

$$\psi_b(x) = \frac{(e^{-2\pi i x b + i\pi Q b}; e^{2\pi i b^2})}{(e^{-2\pi i x b^{-1} - i\pi Q b^{-1}}; e^{-2\pi i b^{-2}})} = \prod_{k \geq 0} \frac{1 - e^{-2\pi i x b + i\pi Q b + 2\pi i k b^2}}{1 - e^{-2\pi i x b^{-1} - i\pi Q b^{-1} - 2\pi i k b^{-2}}}. \tag{A.4}$$

Note that when  $x$  has a large negative imaginary part of order  $1/\beta$ , the expression (A.4) shows that  $\ln \psi_b(x) \sim 0$ , where  $\sim$  means equality to all orders in  $\beta$  but excluding non-analytic corrections of the type  $e^{-1/\beta}$ .

### A.1 Free chiral multiplet

Consider a chiral multiplet of R-charge  $R$ . The index is given by [24]

$$\mathcal{I}_\chi(R, \beta, b) = \Gamma(z, \tau, \sigma), \tag{A.5}$$

with  $z = R \frac{Q}{2} \frac{i\beta}{2\pi}$ ,  $\tau = \frac{i\beta}{2\pi} b^{-1}$ , and  $\sigma = \frac{i\beta}{2\pi} b$ . We will show below that (3.8) can be generalized to

$$\Gamma(z, \tau, \sigma) = \frac{e^{-i\pi M(z; \tau, \sigma)}}{\psi_b(-(R-1)\frac{Q}{2})} \prod_{n=1}^{\infty} \frac{\psi_b(-\frac{2\pi i n}{\beta} + (R-1)\frac{Q}{2})}{\psi_b(-\frac{2\pi i n}{\beta} - (R-1)\frac{Q}{2})}, \quad (\text{A.6})$$

where

$$\begin{aligned} -i\pi M(z; \tau, \sigma) = & -\frac{\pi^2(R-1)Q}{3\beta} \frac{Q}{2} - \left[ \frac{i\pi}{2} \left( (R-1)\frac{Q}{2} \right)^2 - \frac{i\pi}{24} (b^2 + b^{-2}) \right] \\ & + \beta \left( -\frac{(R-1)}{48} (b + b^{-1} + b^3 + b^{-3}) + \frac{(R-1)^3}{48} (b + b^{-1})^3 \right). \end{aligned} \quad (\text{A.7})$$

Then an argument similar to that in the main text gives the high-temperature expansion

$$\begin{aligned} \ln \mathcal{I}_\chi(R, \beta, b) \sim & -\frac{\pi^2(R-1)}{3\beta} \left( \frac{b + b^{-1}}{2} \right) + \ell_b \left( -(R-1)\frac{b + b^{-1}}{2} \right) \\ & + \beta \left( -\frac{(R-1)}{48} \left( b + \frac{1}{b} + b^3 + \frac{1}{b^3} \right) + \frac{(R-1)^3}{48} \left( b + \frac{1}{b} \right)^3 \right). \end{aligned} \quad (\text{A.8})$$

In particular, the index reduces to the 3d partition function,  $\exp(\ell_b(-(R-1)\frac{b+b^{-1}}{2})) = s_b(-i(R-1)\frac{b+b^{-1}}{2})$ , in the limit  $\beta \rightarrow 0$ , after its  $\mathcal{O}(1/\beta)$  divergent exponent is removed.

The expansion (A.8) can be written in terms of the central charges as (note that for chiral multiplets  $2a - c = 0$ )

$$\begin{aligned} \ln \mathcal{I}(\beta, b) \sim & \frac{16\pi^2(c-a)}{3\beta} \left( \frac{b + b^{-1}}{2} \right) - 4(2a - c) \ln(\beta/2\pi) + \ln Z_{3d} \\ & + \beta \left( \frac{2}{27} (b + b^{-1})^3 (3c - 2a) + \frac{2}{3} (b + b^{-1})(a - c) \right). \end{aligned} \quad (\text{A.9})$$

In section A.2 we will see that the above form applies (as in the main text, up to the  $\ln Z_{3d}$  term) to a U(1) vector multiplet as well. Importantly, both for a free chiral and a free U(1) vector multiplet, the linear term in the high-temperature expansion of  $\ln \mathcal{I}(\beta, b)$  encodes enough information to allow extracting  $a$  and  $c$  separately from the index. As demonstrated in appendix B, this observation is crucial for making contact with the prescriptions in [15].

The calculation of the SUSY partition function (highlighted below) goes through very similarly to that in the main text, and this time yields

$$\ln Z_\chi^{\text{SUSY}}(R, \beta, b) = \beta \left[ \frac{(R-1)}{48} \left( b + \frac{1}{b} + b^3 + \frac{1}{b^3} \right) - \frac{(R-1)^3}{48} \left( b + \frac{1}{b} \right)^3 \right] + \ln \mathcal{I}_\chi(R, \beta, b), \quad (\text{A.10})$$

which can be written alternatively in terms of the central charges of the chiral multiplet as

$$\mathcal{I}(\beta, b) = \exp \left[ \beta \left( \frac{2}{27} (b + b^{-1})^3 (3c - 2a) + \frac{2}{3} (b + b^{-1})(a - c) \right) \right] Z^{\text{SUSY}}(\beta, b). \quad (\text{A.11})$$

This agrees with the prefactors proposed in [12, 13], *except for the absence of the  $\mathcal{O}(1/\beta)$  term* in the exponent.

Combining (A.10) and (A.8) we obtain the high-temperature expansion

$$\ln Z_\chi^{\text{SUSY}}(R, \beta, b) \sim -\frac{\pi^2(R-1)}{3\beta} \left( \frac{b+b^{-1}}{2} \right) + \ell_b \left( -(R-1) \frac{b+b^{-1}}{2} \right), \quad (\text{A.12})$$

or in terms of the central charges

$$\ln Z^{\text{SUSY}}(\beta, b) \sim \frac{16\pi^2(c-a)}{3\beta} \left( \frac{b+b^{-1}}{2} \right) - 4(2a-c) \ln(\beta/2\pi) + \ln Z_{3d}, \quad (\text{A.13})$$

with no  $\mathcal{O}(\beta)$  term on the r.h.s. . Again, we will see in section A.2 that (aside from the  $\ln Z_{3d}$  term) the above expansion applies to a U(1) vector multiplet as well.

We now turn to the proof of (A.6) by starting with the modular property of the Gamma function (3.5). We rewrite this expression in terms of  $R, b, \beta, Q$ , expand using (3.4), and manipulate as follows

$$\begin{aligned} \Gamma(z, \tau, \sigma) &= e^{-i\pi M} \frac{\Gamma\left(\frac{RQb}{2}, \frac{2\pi ib}{\beta}, b^2\right)}{\Gamma\left(\frac{RQb^{-1}}{2} - b^{-2}, \frac{2\pi ib^{-1}}{\beta}, -b^{-2}\right)} \\ &= e^{-i\pi M} \prod_{n,k=0}^{\infty} \frac{1 - e^{2\pi i((n+1)(\frac{2\pi ib}{\beta}) + (k+1)b^2 - \frac{RQb}{2})}}{1 - e^{2\pi i(n(\frac{2\pi ib}{\beta}) + kb^2 + \frac{RQb}{2})}} \times \\ &\quad \times \frac{1 - e^{2\pi i(n(\frac{2\pi ib^{-1}}{\beta}) - kb^{-2} + \frac{RQb^{-1}}{2} - b^{-2})}}{1 - e^{2\pi i((n+1)(\frac{2\pi ib^{-1}}{\beta}) - (k+1)b^{-2} - \frac{RQb^{-1}}{2} + b^{-2})}} \\ &= e^{-i\pi M} \prod_{n,k=0}^{\infty} \frac{1 - e^{-\frac{4\pi^2}{\beta}(n+1)b - i\pi(R-1)Qb + i\pi Qb + 2\pi ikb^2}}{1 - e^{-\frac{4\pi^2}{\beta} \frac{(n+1)}{b} - i\pi(R-1)\frac{Q}{b} - i\pi\frac{Q}{b} - \frac{2\pi ik}{b^2}}} \times \\ &\quad \times \frac{1 - e^{-\frac{4\pi^2}{\beta} \frac{n}{b} + i\pi(R-1)\frac{Q}{b} - i\pi\frac{Q}{b} - \frac{2\pi ik}{b^2}}}{1 - e^{-\frac{4\pi^2}{\beta} nb + i\pi(R-1)Qb + i\pi Qb + 2\pi ikb^2}} \\ &= e^{-i\pi M} \prod_{k \geq 0} \frac{1 - e^{i\pi(R-1)Qb^{-1} - i\pi Qb^{-1} - 2\pi ikb^{-2}}}{1 - e^{i\pi(R-1)Qb + i\pi Qb + 2\pi ikb^2}} \\ &\quad \times \prod_{n > 0, k \geq 0} \frac{1 - e^{-\frac{4\pi^2}{\beta} nb - i\pi(R-1)Qb + i\pi Qb + 2\pi ikb^2}}{1 - e^{-\frac{4\pi^2}{\beta} nb^{-1} - i\pi(R-1)Qb^{-1} - i\pi Qb^{-1} - 2\pi ikb^{-2}}} \\ &\quad \times \prod_{n > 0, k \geq 0} \frac{1 - e^{-\frac{4\pi^2}{\beta} nb^{-1} + i\pi(R-1)Qb^{-1} - i\pi Qb^{-1} - 2\pi ikb^{-2}}}{1 - e^{-\frac{4\pi^2}{\beta} nb + i\pi(R-1)Qb + i\pi Qb + 2\pi ikb^2}}. \end{aligned} \quad (\text{A.14})$$

Our claim in eq. (A.6) then follows from using the expression (A.4) for the function  $\psi_b$ .

The computation of the SUSY partition function starts, as in (3.15), with the contribution of the  $n$ -th KK mode [53]

$$\ln Z_\chi^{(n)}(R, \beta, b) = \ln s_b \left( -i(R-1) \frac{Q}{2} - \frac{2\pi n}{\beta} \right) = \ell_b \left( -(R-1) \frac{Q}{2} + \frac{2\pi in}{\beta} \right). \quad (\text{A.15})$$

Then following a similar line of argument as in subsection 3.1, and using the relation (A.2) between  $\ell_b$  and  $\psi_b$ , we have

$$\begin{aligned} \ln Z_\chi^{\text{SUSY}}(R, \beta, b) &= \sum_{n>0} \ln \frac{\psi_b((R-1)\frac{Q}{2} - 2\pi in/\beta)}{\psi_b((1-R)\frac{Q}{2} - 2\pi in/\beta)} \\ &\quad + 2 \sum_{n>0} \frac{2\pi^2(R-1)n}{\beta} \frac{Q}{2} + \ell_b \left( -(R-1)\frac{Q}{2} \right) \\ &= \sum_{n>0} \ln \frac{\psi_b((R-1)\frac{Q}{2} - 2\pi in/\beta)}{\psi_b((1-R)\frac{Q}{2} - 2\pi in/\beta)} \\ &\quad - \frac{\pi^2(R-1)Q}{3\beta} \frac{Q}{2} + \ell_b \left( -(R-1)\frac{Q}{2} \right). \end{aligned}$$

This combined with (A.6) proves (A.10).

## A.2 Free U(1) vector multiplet

The superconformal index of a single free U(1) vector multiplet is [24]

$$\mathcal{I}_v(p, q) = (p; p)(q; q). \quad (\text{A.16})$$

Now, following the same line of argument that led to (3.24) we arrive at

$$\mathcal{I}_v(p, q) \rightarrow e^{-\pi^2(b+b^{-1})/6\beta} \left( \frac{2\pi}{\beta} \right) e^{(b+b^{-1})\beta/24}, \quad (\text{A.17})$$

or

$$\ln \mathcal{I}_v(\beta, b) \sim -\frac{\pi^2(b+b^{-1})}{6\beta} - \ln \left( \frac{\beta}{2\pi} \right) + \frac{\beta(b+b^{-1})}{24}. \quad (\text{A.18})$$

To study  $Z_v^{\text{SUSY}}(p, q)$ , we need  $\Psi(w, \tau, \sigma)$ . A computation similar to that which led to (3.31), but now employing (A.10), gives

$$\Psi(w, \tau, \sigma) = w^2 \left( \frac{1}{\tau} + \frac{1}{\sigma} \right) + \frac{1}{6}(\tau + \sigma). \quad (\text{A.19})$$

For  $w = 0$ , when combined with (G.11) of [13], this implies

$$Z_v^{\text{SUSY}}(p, q) = p^{1/24} q^{1/24} \mathcal{I}_v(p, q) = \eta(p)\eta(q), \quad (\text{A.20})$$

in accord with (A.11). The high-temperature expansion is simply

$$\ln Z_v(\beta, b) \sim -\frac{\pi^2(b+b^{-1})}{6\beta} - \ln \left( \frac{\beta}{2\pi} \right). \quad (\text{A.21})$$

Note that eqs. (A.10) and (A.19), when combined with the localization results of [13], yield the relation (A.11) between  $\mathcal{I}$  and  $Z^{\text{SUSY}}$  even in presence of interactions and non-abelian gauge fields.

### A.3 Toric quivers in the planar limit

To generalize our results in section 4 to the index with two fugacities, we need the following more general form of eq. (4.5):

$$(1 - i(p, q)) = \frac{\chi(t)}{(1-p)(1-q)}, \quad (\text{A.22})$$

where  $t = \sqrt{pq}$ . Eq. (4.8) remains unchanged. The rest of the computation in section 4 goes through with little change, and one arrives at

$$\ln \mathcal{I}_{\text{quiver}}^{N \rightarrow \infty}(t, y) = -\frac{1}{2} \sum_{i=1}^{n_z} \ln \mathcal{I}_v(t_v = t^{r_i}, y_v = 1) - \sum_{\text{adj}} \ln \mathcal{I}_\chi(R = R_{\text{adj}}, t_\chi = t, y_\chi = y), \quad (\text{A.23})$$

where  $y = \sqrt{p/q}$ .

We can now use the results in appendices A.1 and A.2 to write down the asymptotic high-temperature expansion of the large- $N$  index in (A.23). A simple calculation using (A.8) and (A.18) shows

$$\begin{aligned} \ln \mathcal{I}_{\text{quiver}}^{N \rightarrow \infty}(\beta, b) &\sim \frac{\pi^2}{6\beta \left(\frac{b+b^{-1}}{2}\right)} \sum_{i=1}^{n_z} \frac{1}{r_i} \\ &+ \frac{16\pi^2 \left(\frac{b+b^{-1}}{2}\right)}{3\beta} \sum_{\text{adj}} (\delta c_{\text{adj}} - \delta a_{\text{adj}}) + \frac{n_z}{2} \ln(\beta/2\pi) + \ln Y_b \\ &+ \beta \left( \frac{2}{27} (b+b^{-1})^3 (3\delta c - 2\delta a) + \frac{2}{3} (b+b^{-1})(\delta a - \delta c) \right), \end{aligned} \quad (\text{A.24})$$

where the notation is similar to that in (4.1), except for  $\ln Y_b = \frac{1}{2} \sum_{i=1}^{n_z} \ln(r_i \left(\frac{b+b^{-1}}{2}\right)) + \sum_{\text{adj}} \ell_b((R_{\text{adj}} - 1) \left(\frac{b+b^{-1}}{2}\right))$ . See eqs. (A.2) and (A.3) for the definition of the function  $\ell_b$ .

## B Single-trace index and the central charges

The single-trace index is defined as the plethystic log [55] of the index (1.3)

$$I_{\text{s.t.}}(\beta, b) \equiv \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \ln \mathcal{I}(n\beta, b), \quad (\text{B.1})$$

where  $\mu(n)$  is the Möbius function. The adjective “single-trace” is particularly appropriate for theories that admit a planar limit in which single-trace operators are weakly interacting. For such cases if in the definition of the index in (1.3) one restricts the trace to the “single-trace states” in the Hilbert space, one obtains the single-trace index as defined above. In AdS/CFT, the weakly interacting mesons of the SCFT at large 't Hooft coupling map to the KK supergravity modes in the bulk. Therefore the single-trace index is quite natural from the bulk point of view.

In [15], building on the holographic results of [56], prescriptions were proposed for extracting the central charges  $a$  and  $c$  from the single-trace index of an SCFT. It was

observed that for holographic theories the prescriptions reproduce the  $\mathcal{O}(1)$  piece of the central charges, denoted by  $\delta a$  and  $\delta c$ , while for  $A_k$  SQCD fixed points at the Veneziano limit and for finite- $N$  theories they give the full central charges and not just their  $\mathcal{O}(1)$  piece. It was also suspected that there may be a relation between those prescriptions and the Di Pietro-Komargodski formula (1.2).

In this appendix we show that the proposals in [15] probe in fact only the  $\mathcal{O}(\beta)$  term in the high-temperature expansion of the indices (A.9) and (A.24), and have nothing to do with their leading  $\mathcal{O}(1/\beta)$  behavior. This also explains the applicability of the formulas in [15] to large- $N$  theories: while the leading  $\mathcal{O}(1/\beta)$  behavior of the large- $N$  indices can be drastically different from the finite- $N$  proposal of Di Pietro and Komargodski, the  $\mathcal{O}(\beta)$  term is either completely inherited from the finite- $N$  theory (as in the case of  $A_k$  SQCD fixed points, an example of which appears in eq. (5.1)), or at least its  $\mathcal{O}(1)$  piece survives (as in the case of holographic quivers described in eq. (A.24)).

To simplify comparison with [15] we start by the expansion of the single-trace index  $I_{\text{s.t.}}$ , first around  $y = 1$  and then around  $t_t = 1$  (see section IV of [16])

$$\begin{aligned}
 I_{\text{s.t.}} = & \left( \frac{a_0}{t_t - 1} + a_1 + a_2(t_t - 1) + \dots \right) \\
 & + (y - 1)^2 \left( \frac{b_0}{(t_t - 1)^3} + \frac{b_1}{(t_t - 1)^2} + \frac{b_2}{t_t - 1} + \dots \right) \\
 & + \dots,
 \end{aligned}
 \tag{B.2}$$

where  $t_t = 1/t = e^{\beta Q/2}$  is the  $t$ -variable defined in [15, 16] and  $y = e^{-\beta(\frac{b-1}{2})}$ . Then the prescriptions in [15] read

$$\begin{aligned}
 \hat{a} &= \frac{9(a_0 - b_0)}{32(t_t - 1)^2} - \frac{3(a_0 + 12a_2) - 9(b_0 - b_1 + b_2)}{32} + \dots, \\
 \hat{c} &= -\frac{3(a_0 - b_0)}{32(t_t - 1)^2} - \frac{2(a_0 + 12a_2) + 3(b_0 - b_1 + b_2)}{32} + \dots,
 \end{aligned}
 \tag{B.3}$$

where the functions  $\hat{a}$  and  $\hat{c}$  have a second-order pole in the high-temperature ( $t_t \rightarrow 1$ ) limit and their finite piece in this limit gives the central charges (or in the case of holographic quivers, their  $\mathcal{O}(1)$  pieces).

### B.1 Finite- $N$ theories

We start with the high-temperature expansion in (A.9), and take its plethystic log as defined in eq. (B.1). The following sums are needed in the process [57]

$$\begin{aligned}
 \sum \frac{\mu(n)}{n^2} &= \frac{1}{\zeta(2)} = \frac{6}{\pi^2}, \\
 \sum \frac{\mu(n)}{n} &= 0, \\
 \sum \frac{\mu(n) \ln n}{n} &= -1, \\
 \sum \mu(n) &\rightarrow \frac{1}{\zeta(0)} = -2.
 \end{aligned}
 \tag{B.4}$$

The result for the high-temperature expansion of the single-trace index is

$$I_{\text{s.t.}} \sim \frac{32(c-a)}{2\beta} (b+b^{-1}) + 4(2a-c) - \beta \left( \frac{4}{27}(b+b^{-1})^3(3c-2a) + \frac{4}{3}(b+b^{-1})(a-c) \right). \quad (\text{B.5})$$

We now wish to demonstrate that the leading  $\mathcal{O}(1/\beta)$  behavior of the single-trace index, or its subleading  $\mathcal{O}(\beta^0)$  term, do not play a role in evaluating the central charges via (B.3). To illustrate this, we will assume the following form for the single-trace index

$$I_{\text{s.t.}} \sim \frac{G}{2\beta} (b+b^{-1}) + C - \beta \left( \frac{4}{27}(b+b^{-1})^3(3c-2a) + \frac{4}{3}(b+b^{-1})(a-c) \right). \quad (\text{B.6})$$

Using this, we will demonstrate that the prescriptions (B.3) are independent of the coefficients  $G, C$ .

To compare (B.6) with the expansion (B.2), we need the dictionary

$$b+b^{-1} = \frac{2 \ln t_t}{\sqrt{\ln(t_t y) \ln(t_t/y)}}, \quad \beta = \sqrt{\ln(t_t y) \ln(t_t/y)}. \quad (\text{B.7})$$

Substituting the above relations in (B.6), expanding first around  $y = 1$  and then around  $t_t = 1$ , and finally comparing with (B.2), gives

$$\begin{aligned} a_0 &= G, & a_1 &= \frac{1}{2}G + C, & a_2 &= -\frac{8}{27}a - \frac{8}{9}c - \frac{1}{12}G, \\ b_0 &= G, & b_1 &= \frac{3}{2}G, & b_2 &= \frac{64}{27}a - \frac{32}{9}c + \frac{1}{2}G. \end{aligned} \quad (\text{B.8})$$

Plugging the above values in (B.3) yields

$$\hat{a} = a + \dots, \quad \hat{c} = c + \dots, \quad (\text{B.9})$$

where the ellipses denote terms that vanish at  $t_t = 1$ . Note that the  $G$ -dependence of the coefficients (B.8) drops out when evaluating (B.3). Moreover, the only  $C$ -dependent coefficient in (B.8) is  $a_1$ , which does not show up in (B.3).

## B.2 $A_k$ SQCD fixed points in the Veneziano limit

A natural generalization of standard SQCD is achieved by adding a single adjoint chiral multiplet  $X$  to SQCD, and turning on a simple superpotential  $\text{Tr} X^{k+1}$ . This leads to the so-called  $A_k$  SQCD theories [58].

We have not been able to compute the high-temperature expansion of the index of these theories when  $b \neq 1$ . In fact even for  $b = 1$ , SQCD (corresponding to  $k = 1$ ) with  $x = 1/2$  has been the only example whose high-temperature expansion we have completely evaluated; it is shown in (5.1). Nevertheless, since the single-trace index of these theories were computed explicitly in [16], we can go in the reverse direction to that in the previous subsection, and use the high-temperature expansion of  $I_{\text{s.t.}}(\beta, b)$  to gain information on the expansion of  $\ln \mathcal{I}(\beta, b)$ . This is, in fact, how the linear term in (3.2) was conjectured in [15].

Note that, as stated in [15], going from the high-temperature expansion of the single-trace index to that of  $\ln \mathcal{I}(\beta, b)$ , one can not reproduce the  $\mathcal{O}(\beta^0)$  term in the latter. However, the other claim in [15], that even the  $\mathcal{O}(\ln \beta)$  term of  $\ln \mathcal{I}(\beta, b)$  can not be obtained from the high-temperature expansion of the single-trace index, is not quite correct. Here we see, from the third line of (B.4), that the coefficient of  $\ln \beta$  in the expansion of  $\ln \mathcal{I}(\beta, b)$  is simply given by minus the  $\mathcal{O}(\beta^0)$  term in the expansion of  $I_{\text{s.t.}}(\beta, b)$ . For instance, by negating the second term on the r.h.s. of relation (B.5) one obtains the coefficient of the log term on the r.h.s. of (A.9).

We leave out the details, and only report an ansatz which is confirmed by the type of analysis mentioned in the previous paragraph

$$\begin{aligned} \ln \mathcal{I}_{A_k}^{N_c \rightarrow \infty \text{ SQCD}}(\beta, b) &\sim \frac{2k^3 + 3k^2 - 1}{4k(1+k)} \left( \frac{\pi^2}{6\beta \left(\frac{b+b^{-1}}{2}\right)} \right) \\ &+ \frac{16kN_c^2 - 8k^2 + 8k}{4k(1+k)} \left( \frac{\pi^2 \left(\frac{b+b^{-1}}{2}\right)}{6\beta} \right) + \frac{1}{2} \ln \left( \frac{\beta}{2\pi} \right) + \mathcal{O}(\beta^0) \quad (\text{B.10}) \\ &+ \beta \left( \frac{2}{27} (b+b^{-1})^3 (3c-2a) + \frac{2}{3} (b+b^{-1})(a-c) \right). \end{aligned}$$

In particular, the leading  $\mathcal{O}(1/\beta)$  term is different from the finite- $N$  Di Pietro-Komargodski formula in two important respects: *i*) it is not entirely determined by the central charges, and *ii*) the dependence on  $b + b^{-1}$  is slightly more complicated than in the finite- $N$  version, (A.9). While at finite  $N$  the leading term depends on  $b + b^{-1}$  only through  $\frac{b+b^{-1}}{\beta}$ , the large- $N$  version has another  $\mathcal{O}(1/\beta)$  term which is proportional to  $\frac{1}{(b+b^{-1})\beta}$ . Similar observations could be made equally as well for the high-temperature expansion of the index of large- $N$  toric quivers, presented in (A.24). In both cases, the term proportional to  $\frac{1}{(b+b^{-1})\beta}$ , which would be absent at finite  $N$ , is responsible for the poles showing up in the prescriptions (B.3). This provides some insight into the divergences that were encountered in [15, 16] when extracting the central charges of large- $N$  theories.

In the next subsection we focus on large- $N$  quivers and demonstrate the aforementioned connection between the poles in (B.3) and the term proportional to  $\frac{1}{(b+b^{-1})\beta}$  in the expansion of  $\ln \mathcal{I}(\beta, b)$ . A completely similar analysis can be applied to the  $A_k$  SQCD theories.

### B.3 Toric quivers in the planar limit

As in appendix B.1, we start with the high-temperature expansion of the index, take its plethystic log to arrive at the expansion of the single-trace index, rewrite it in terms of  $t_i, y$ , and extract its coefficients to plug in the prescriptions (B.3). Taking the plethystic log of (A.24), we arrive at

$$\begin{aligned} I_{\text{s.t. quiver}}^{N \rightarrow \infty} &\sim \frac{2}{\beta(b+b^{-1})} \sum_{i=1}^{n_z} \frac{1}{r_i} + \frac{32(b+b^{-1})}{2\beta} \sum_{\text{adj}} (\delta c_{\text{adj}} - \delta a_{\text{adj}}) - \frac{n_z}{2} \\ &- \beta \left( \frac{4}{27} (b+b^{-1})^3 (3\delta c - 2\delta a) + \frac{4}{3} (b+b^{-1})(\delta a - \delta c) \right). \quad (\text{B.11}) \end{aligned}$$



As in subsection B.1, we now assume the following expansion instead of (B.11)

$$I_{\text{s.t. quiver}}^{N \rightarrow \infty} \sim \frac{2H}{\beta(b+b^{-1})} + \frac{G(b+b^{-1})}{2\beta} + C - \beta \left( \frac{4}{27}(b+b^{-1})^3(3\delta c - 2\delta a) + \frac{4}{3}(b+b^{-1})(\delta a - \delta c) \right), \quad (\text{B.12})$$

and argue that the prescriptions (B.3) are independent of  $G, H, C$ , except that  $H$  determines the pole terms that according to the prescription of [15] one should drop.

Following similar steps as in subsection B.1 leads this time to the coefficients

$$\begin{aligned} a_0 &= G + H, & a_1 &= \frac{1}{2}G + \frac{1}{2}H + C, & a_2 &= -\frac{8}{27}\delta a - \frac{8}{9}\delta c - \frac{1}{12}G - \frac{1}{12}H, \\ b_0 &= G, & b_1 &= \frac{3}{2}G, & b_2 &= \frac{64}{27}\delta a - \frac{32}{9}\delta c + \frac{1}{2}G. \end{aligned} \quad (\text{B.13})$$

Importantly, this time  $a_0 - b_0 = H \neq 0$ ; this proves our claim that the poles in (B.3) (or alternatively, the divergences encountered in [15, 16] for large- $N$  theories) are due to the term proportional to  $\frac{1}{\beta(b+b^{-1})}$  in the high-temperature expansion of  $\ln \mathcal{I}(\beta, b)$ .

Plugging the above set of coefficients in (B.3) leads to

$$\hat{a} = \frac{9H}{32(t_t - 1)^2} + a + \dots, \quad \hat{c} = -\frac{3H}{32(t_t - 1)^2} + c + \dots, \quad (\text{B.14})$$

as expected, where again the dependence on  $G$  has dropped out. There is no dependence on  $C$  either, as one could anticipate by noting that  $C$  enters only  $a_1$  in (B.13), while  $a_1$  does not appear in (B.3).

Finally, we would like to point out that the above discussion generalizes (up to the matching of the indices of bulk and boundary, and assuming (4.8)) the AdS/CFT matching of the  $\mathcal{O}(1)$  piece of the central charges to any toric quiver. In [15] such a matching was demonstrated only for toric quivers dual to smooth  $SE_5$  manifolds and without adjoint matter. The expression (A.24) on the other hand applies also to toric quivers with adjoint matter and with singular dual geometry, as it hinges on the factorization (4.8) whose validity is demonstrated in several singular cases and in presence of adjoints as well [36].

The matching mentioned above could be alternatively demonstrated by applying the prescriptions of [15] to the single-trace index of a general toric quiver, which can be deduced from eq. (A.23) to be

$$I_{\text{s.t. quiver}}^{N \rightarrow \infty}(t, y) = -\frac{1}{2} \sum_{i=1}^{n_z} i_v(t_v = t^{r_i}, y_v = 1) - \sum_{\text{adj}} i_\chi(R = R_{\text{adj}}, t_\chi = t, y_\chi = y), \quad (\text{B.15})$$

with  $i_v(t_v, y_v)$  and  $i_\chi(R, t_\chi, y_\chi)$  the single-letter indices of the vector and chiral multiplets

$$i_v(t_v, y_v) = \frac{2t_v^2 - t_v(y_v + y_v^{-1})}{(1 - t_v y_v)(1 - t_v y_v^{-1})}, \quad i_\chi(R, t_\chi, y_\chi) = \frac{t_\chi^R - t_\chi^{2-R}}{(1 - t_\chi y_\chi)(1 - t_\chi y_\chi^{-1})}. \quad (\text{B.16})$$

Plugging the above explicit expressions in (B.15) we obtain

$$I_{\text{s.t. quiver}}^{N \rightarrow \infty}(t, y) = \sum_{i=1}^{n_z} \frac{t^{r_i}}{1 - t^{r_i}} - \sum_{\text{adj}} \frac{t^{R_{\text{adj}}} - t^{2-R_{\text{adj}}}}{(1 - ty)(1 - t/y)}, \quad (\text{B.17})$$

to which the prescriptions of [15] can be successfully applied.

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## References

- [1] J.L. Cardy, *Operator content of two-dimensional conformally invariant theories*, *Nucl. Phys. B* **270** (1986) 186 [[INSPIRE](#)].
- [2] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett. B* **379** (1996) 99 [[hep-th/9601029](#)] [[INSPIRE](#)].
- [3] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett. B* **379** (1996) 99 [[hep-th/9601029](#)] [[INSPIRE](#)].
- [4] T. Hartman, C.A. Keller and B. Stoica, *Universal spectrum of 2D conformal field theory in the large  $c$  limit*, *JHEP* **09** (2014) 118 [[arXiv:1405.5137](#)] [[INSPIRE](#)].
- [5] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla and T. Sharma, *Constraints on fluid dynamics from equilibrium partition functions*, *JHEP* **09** (2012) 046 [[arXiv:1203.3544](#)] [[INSPIRE](#)].
- [6] K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz and A. Yarom, *Towards hydrodynamics without an entropy current*, *Phys. Rev. Lett.* **109** (2012) 101601 [[arXiv:1203.3556](#)] [[INSPIRE](#)].
- [7] L. Di Pietro and Z. Komargodski, *Cardy formulae for SUSY theories in  $d = 4$  and  $d = 6$* , *JHEP* **12** (2014) 031 [[arXiv:1407.6061](#)] [[INSPIRE](#)].
- [8] E. Witten, *Constraints on supersymmetry breaking*, *Nucl. Phys. B* **202** (1982) 253 [[INSPIRE](#)].
- [9] C. Romelsberger, *Counting chiral primaries in  $N = 1$ ,  $D = 4$  superconformal field theories*, *Nucl. Phys. B* **747** (2006) 329 [[hep-th/0510060](#)] [[INSPIRE](#)].
- [10] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209 [[hep-th/0510251](#)] [[INSPIRE](#)].
- [11] H.-C. Kim and S. Kim, *M5-branes from gauge theories on the 5-sphere*, *JHEP* **05** (2013) 144 [[arXiv:1206.6339](#)] [[INSPIRE](#)].
- [12] C. Closset and I. Shamir, *The  $\mathcal{N} = 1$  chiral multiplet on  $T^2 \times S^2$  and supersymmetric localization*, *JHEP* **03** (2014) 040 [[arXiv:1311.2430](#)] [[INSPIRE](#)].
- [13] B. Assel, D. Cassani and D. Martelli, *Localization on Hopf surfaces*, *JHEP* **08** (2014) 123 [[arXiv:1405.5144](#)] [[INSPIRE](#)].
- [14] J. Lorenzen and D. Martelli, *Comments on the Casimir energy in supersymmetric field theories*, *JHEP* **07** (2015) 001 [[arXiv:1412.7463](#)] [[INSPIRE](#)].
- [15] A.A. Ardehali, J.T. Liu and P. Szepietowski, *Central charges from the  $\mathcal{N} = 1$  superconformal index*, *Phys. Rev. Lett.* **114** (2015) 091603 [[arXiv:1411.5028](#)] [[INSPIRE](#)].
- [16] A.A. Ardehali, J.T. Liu and P. Szepietowski,  *$c$ -a from the  $\mathcal{N} = 1$  superconformal index*, *JHEP* **12** (2014) 145 [[arXiv:1407.6024](#)] [[INSPIRE](#)].
- [17] P. Kraus and F. Larsen, *Partition functions and elliptic genera from supergravity*, *JHEP* **01** (2007) 002 [[hep-th/0607138](#)] [[INSPIRE](#)].
- [18] J.C. Breckenridge, R.C. Myers, A.W. Peet and C. Vafa, *D-branes and spinning black holes*, *Phys. Lett. B* **391** (1997) 93 [[hep-th/9602065](#)] [[INSPIRE](#)].

- [19] A. Belin, C.A. Keller and A. Maloney, *String universality for permutation orbifolds*, *Phys. Rev. D* **91** (2015) 106005 [[arXiv:1412.7159](#)] [[INSPIRE](#)].
- [20] F.M. Haehl and M. Rangamani, *Permutation orbifolds and holography*, *JHEP* **03** (2015) 163 [[arXiv:1412.2759](#)] [[INSPIRE](#)].
- [21] A. Kapustin, B. Willett and I. Yaakov, *Exact results for Wilson loops in superconformal Chern-Simons theories with matter*, *JHEP* **03** (2010) 089 [[arXiv:0909.4559](#)] [[INSPIRE](#)].
- [22] D.L. Jafferis, *The exact superconformal R-symmetry extremizes Z*, *JHEP* **05** (2012) 159 [[arXiv:1012.3210](#)] [[INSPIRE](#)].
- [23] N. Hama, K. Hosomichi and S. Lee, *Notes on SUSY gauge theories on three-sphere*, *JHEP* **03** (2011) 127 [[arXiv:1012.3512](#)] [[INSPIRE](#)].
- [24] F.A. Dolan and H. Osborn, *Applications of the superconformal index for protected operators and q-hypergeometric identities to N = 1 dual theories*, *Nucl. Phys. B* **818** (2009) 137 [[arXiv:0801.4947](#)] [[INSPIRE](#)].
- [25] G. Felder and A. Varchenko, *The elliptic gamma function and  $SL(3, Z) \times Z^3$* , *Adv. Math.* **156** (2000) 44 [[math/9907061](#)].
- [26] F.A.H. Dolan, V.P. Spiridonov and G.S. Vartanov, *From 4D superconformal indices to 3D partition functions*, *Phys. Lett. B* **704** (2011) 234 [[arXiv:1104.1787](#)] [[INSPIRE](#)].
- [27] A. Gadde and W. Yan, *Reducing the 4D index to the  $S^3$  partition function*, *JHEP* **12** (2012) 003 [[arXiv:1104.2592](#)] [[INSPIRE](#)].
- [28] Y. Imamura, *Relation between the 4D superconformal index and the  $S^3$  partition function*, *JHEP* **09** (2011) 133 [[arXiv:1104.4482](#)] [[INSPIRE](#)].
- [29] V. Niarchos, *Seiberg dualities and the 3d/4d connection*, *JHEP* **07** (2012) 075 [[arXiv:1205.2086](#)] [[INSPIRE](#)].
- [30] P. Agarwal, A. Amariti, A. Mariotti and M. Siani, *BPS states and their reductions*, *JHEP* **08** (2013) 011 [[arXiv:1211.2808](#)] [[INSPIRE](#)].
- [31] O. Aharony, S.S. Razamat, N. Seiberg and B. Willett, *3d dualities from 4d dualities*, *JHEP* **07** (2013) 149 [[arXiv:1305.3924](#)] [[INSPIRE](#)].
- [32] M. Buican, T. Nishinaka and C. Papageorgakis, *Constraints on chiral operators in  $\mathcal{N} = 2$  SCFTs*, *JHEP* **12** (2014) 095 [[arXiv:1407.2835](#)] [[INSPIRE](#)].
- [33] F. Benini, R. Eager, K. Hori and Y. Tachikawa, *Elliptic genera of two-dimensional  $N = 2$  gauge theories with rank-one gauge groups*, *Lett. Math. Phys.* **104** (2014) 465 [[arXiv:1305.0533](#)] [[INSPIRE](#)].
- [34] S. Franco, A. Hanany, D. Martelli, J. Sparks, D. Vegh and B. Wecht, *Gauge theories from toric geometry and brane tilings*, *JHEP* **01** (2006) 128 [[hep-th/0505211](#)] [[INSPIRE](#)].
- [35] R. Eager, J. Schmude and Y. Tachikawa, *Superconformal indices, Sasaki-Einstein manifolds and cyclic homologies*, *Adv. Theor. Math. Phys.* **18** (2014) 129 [[arXiv:1207.0573](#)] [[INSPIRE](#)].
- [36] P. Agarwal, A. Amariti and A. Mariotti, *A zig-zag index*, [arXiv:1304.6733](#) [[INSPIRE](#)].
- [37] R. Eager, M. Gary and M.M. Roberts, *Can you hear the shape of dual geometries?*, *JHEP* **10** (2013) 209 [[arXiv:1011.5231](#)] [[INSPIRE](#)].
- [38] A. Gadde, L. Rastelli, S.S. Razamat and W. Yan, *On the superconformal index of  $N = 1$  IR fixed points: a holographic check*, *JHEP* **03** (2011) 041 [[arXiv:1011.5278](#)] [[INSPIRE](#)].
- [39] G. Festuccia and N. Seiberg, *Rigid supersymmetric theories in curved superspace*, *JHEP* **06** (2011) 114 [[arXiv:1105.0689](#)] [[INSPIRE](#)].

- [40] L.G. Yaffe and B. Svetitsky, *First order phase transition in the SU(3) gauge theory at finite temperature*, *Phys. Rev. D* **26** (1982) 963 [INSPIRE].
- [41] N. Arkani-Hamed and H. Murayama, *Holomorphy, rescaling anomalies and exact  $\beta$ -functions in supersymmetric gauge theories*, *JHEP* **06** (2000) 030 [hep-th/9707133] [INSPIRE].
- [42] A.D. Shapere and Y. Tachikawa, *Central charges of  $N = 2$  superconformal field theories in four dimensions*, *JHEP* **09** (2008) 109 [arXiv:0804.1957] [INSPIRE].
- [43] A. Gadde, L. Rastelli, S.S. Razamat and W. Yan, *The superconformal index of the  $E_6$  SCFT*, *JHEP* **08** (2010) 107 [arXiv:1003.4244] [INSPIRE].
- [44] D. Cassani and D. Martelli, *The gravity dual of supersymmetric gauge theories on a squashed  $S^1 \times S^3$* , *JHEP* **08** (2014) 044 [arXiv:1402.2278] [INSPIRE].
- [45] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, *Lett. Math. Phys.* **91** (2010) 167 [arXiv:0906.3219] [INSPIRE].
- [46] N. Wyllard,  *$A_{N-1}$  conformal Toda field theory correlation functions from conformal  $N = 2$  SU( $N$ ) quiver gauge theories*, *JHEP* **11** (2009) 002 [arXiv:0907.2189] [INSPIRE].
- [47] N. Nekrasov and E. Witten, *The  $\Omega$  deformation, branes, integrability and Liouville theory*, *JHEP* **09** (2010) 092 [arXiv:1002.0888] [INSPIRE].
- [48] A. Gadde, L. Rastelli, S.S. Razamat and W. Yan, *The 4D superconformal index from  $q$ -deformed 2D Yang-Mills*, *Phys. Rev. Lett.* **106** (2011) 241602 [arXiv:1104.3850] [INSPIRE].
- [49] M. Yamazaki, *Quivers, YBE and 3-manifolds*, *JHEP* **05** (2012) 147 [arXiv:1203.5784] [INSPIRE].
- [50] Y. Terashima and M. Yamazaki, *Emergent 3-manifolds from 4D superconformal indices*, *Phys. Rev. Lett.* **109** (2012) 091602 [arXiv:1203.5792] [INSPIRE].
- [51] N. Benjamin, M.C.N. Cheng, S. Kachru, G.W. Moore and N.M. Paquette, *Elliptic genera and 3D gravity*, arXiv:1503.04800 [INSPIRE].
- [52] B. Assel, D. Cassani, L. Di Pietro, Z. Komargodski, J. Lorenzen and D. Martelli, *The Casimir energy in curved space and its supersymmetric counterpart*, arXiv:1503.05537 [INSPIRE].
- [53] N. Hama, K. Hosomichi and S. Lee, *SUSY gauge theories on squashed three-spheres*, *JHEP* **05** (2011) 014 [arXiv:1102.4716] [INSPIRE].
- [54] L.D. Faddeev, R.M. Kashaev and A. Yu. Volkov, *Strongly coupled quantum discrete Liouville theory. 1. Algebraic approach and duality*, *Commun. Math. Phys.* **219** (2001) 199 [hep-th/0006156] [INSPIRE].
- [55] S. Benvenuti, B. Feng, A. Hanany and Y.-H. He, *Counting BPS operators in gauge theories: quivers, syzygies and plethystics*, *JHEP* **11** (2007) 050 [hep-th/0608050] [INSPIRE].
- [56] M. Beccaria and A.A. Tseytlin, *Higher spins in  $AdS_5$  at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT*, *JHEP* **11** (2014) 114 [arXiv:1410.3273] [INSPIRE].
- [57] C. Nanxian, *Möbius inversion in physics*, World Scientific, Singapore (2010).
- [58] D. Kutasov and A. Schwimmer, *On duality in supersymmetric Yang-Mills theory*, *Phys. Lett. B* **354** (1995) 315 [hep-th/9505004] [INSPIRE].