Towards unity of families: anti-SU(7) from Z_{12-I} orbifold compactification

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ABSTRACT: The problem of families, "Why are there three families of fermions?", is a long awaited question to be answered within a reasonable framework. We propose anti-$SU(N)$ groups for the unification of families in grand unification (GUT) groups, where the separation of color and weak gauge groups in the GUT is achieved by antisymmetric tensor Brout-Englert-Higgs boson instead of an adjoint representation. Theories of anti-$SU(N)$'s are proposed for the unification of families. The minimal model is found as $SU(7)_{anti2}$ GUT with the fermion representation $[3] + 2[2] + 8[\bar{1}]$. We present an example in a $\mathbf{Z}_{12-I}$ orbifold compactification, where the missing partner mechanism is also realized.

Contents

1   Introduction

The family problem, repeating fifteen chiral fields three times in the standard model (SM), has been known for almost four decades without an accepted theory so far. One family consists of fifteen chiral fields. In the Georgi-Glashow(GG) grand unification(GUT) model, the fifteen chiral fields in the first family are grouped into $\mathbf{10}\,(u^c, u, d, e^+)$ and $\mathbf{\bar{5}}\,(d^c, \nu_e, e)$ of SU(5)$_{\text{GG}}$ [1]. The family problem posited in GUTs is how $\mathbf{10} + \mathbf{\bar{5}}$ repeats exactly three times in an extended GUT. The gauge coupling unification [2] is the underlying principle for a GUT. A gauge model based on a simple group is a GUT. A gauge model based on a semi-simple group rendering one coupling constant (by some discrete symmetry) is a GUT also [3].

A chiral model with fifteen fields in a GUT is achieved by assigning fifteen in some complex representation of a GUT gauge group. Toward a solution of the family problem, Georgi formulated a principle on the unification of GUT-families with gauge groups containing SU(5) as a subgroup [4]. Absence of gauge anomaly is required. Then, any gauge group except SU(N) can be used for a GUT if they allow complex representations. In this try of anomaly-free groups, SO($2N$) with $N = 2n + 1$ allows complex representaions which are spinors of SO($2N$). The simplest case is SO(10) where the fifteen chiral fields plus a singlet neutrino are assigned in the spinor **16** [5, 6]. It is one family model. One may try SO(12) which however is not considered to be complex because the spinor **32** branches to **16** and $\overline{\mathbf{16}}$ of SO(10). So, the next step is a two family model where the spinor **64** of SO(14) is used [7, 8] where however non-standard charges must appear.[1] If we exclude the possibility beyond the SM charges, **64** of SO(14) is vectorlike and no chiral family is obtained. A scheme toward unification of GUT-families (UGUTF) came to a dead end within SO($2N$) groups.

To open a gate from the dead end alley toward the meadow, Georgi proposed UGUTFs in SU(N) groups [4], where the condition on the anomaly-freedom plays a central role. There have been some attempts along this line [11, 12]. After the string revolution with heterotic strings [13], the low energy gauge groups and spectra are computed through the compactification process [14–16], and UGUTFs did not attain much interest because three SM families could have been obtained through the compactification process [17, 18]. In this string scenario, the compactification schemes (such as orbifolds) basically choose what is the number of families. The drawback is that there are too many parameters to predict the fermion mass spectra.

In this paper, we attempt to realize Georgi's UGUTFs in string compactification. If possible, we will try to introduce some structure among three families so that the difference of the third family from the the first two is understood. Among heterotic strings, usually the $E_8 \times E_8'$ hetrotic string has been used because it contains spinors in the adjoint representation **248** of E_8. Then, the $E_8 \times E_8'$ heterotic string has been favored subconciously because of the embedding chain of the SO(10) spinor **16** in SO(10)→E_6 →E_7 →E_8. But, the SO(32) hetrotic string is also useful for phenomenology as we will briefly argue in this paper. The most severe obstacle in obtaining a realistic SU(5) GUT from heterotic string has been the difficulty, at the level 1 construction, of obtaining the adjoint representation **24** of SU(5) which is needed to break SU(5) down to the SM gauge group.

The first example without an adjoint representation for the Brout-Englert-Higgs(BEH) boson can be traced back to anti-SU(7) where antisymmetric tensor fields for BEH bosons, denoted as two and three index anti-symmetric tensor fields, $\Phi^{[\alpha\beta]}$, $\Phi_{[\alpha\beta]}$, $\Phi^{[\alpha\beta\gamma]}$, and $\Phi_{[\alpha\beta\gamma]}$, are used to reduce the rank of the SU(7)×U(1) GUT gauge group and separate the color and the weak parts [7, 8].[2] The sixteen chiral fields of the first family are grouped into $\mathbf{10}_1$ (d^c, u, d, N_1^0) and $\overline{\mathbf{5}}_{-3}$ (d^c, ν_e, e) and e_5^+ in SU(5)$_{\text{flip}}$. The structure is included in SU(7)$_{\text{anti2}}$. If it is applied to SO($2N$) gauge groups, we can call it anti-SU(N) where anti means that separation of color SU(3)$_c$ from weak SU(2)$_W$ is by the anti-symmetric tensor

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[1]There can exist a missing partner mechanism in this model, however, with certain assumptions. This was emphasized in [9]. Later, it was worked out in SO(10) [10], which is basically anti-SU(5)=SU(5)$_{\text{flip}}$.

[2]The acronym anti- was used in ref. [19] as anti-SU(5).

fields instead of the adjoint representation. For $N = 5$, it is now known as the flipped SU(5) [20]. It is obvious that the essential feature is included in the *word* 'anti-SU(N)'. Anti-SU(5) GUTs were obtained in string compactification [21, 22]. But, SU(5) GUTs are not UGUTFs.

In addition to the restriction on the number of families for $n_f = 3$, the R-parity is used for proton longevity and weakly interacting massive particle possibility for cold dark matter candidate. The R-parity can be a discrete subgroup of a U(1) gauge group which has been discussed in ref. [23, 24]. Another issue in supersymmetric GUTs is the problem of doublet-triplet splitting in the BEH multiplets containing H_u and H_d. We will realize the doublet-triplet splitting mechanism in SU(7)$_{\text{anti2}}$ anticipated in ref. [9].

In section 2, we recapitulate the old UGUTF scenario, and continue in section 3 to present a rationale that anti-SU(N)s are theories for UGUTFs. In section 4, we summarize an SU(7) realization of UGUTF in the $\mathbf{Z}_{12-I}$ orbifold compactification. Here, we present some details on how the massless spectra are obtained in $\mathbf{Z}_{12-I}$ so that line by line can be followed up in other orbifold constructions. We also point out how the missing-partner mechanism is realized in SU(7)$_{\text{anti2}}$. Section 5 is a conclusion. In appendix, we present the spectra not included in section 4.

2 Families unified in grand unification

Let the fundamental representations (or anti-symmetric representations) of SU(N) are bounded by square brackets. Representations [1] and [2] have the following matrix forms,

$$[1] \equiv \Phi^{[A]} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ f_6 \\ . \\ . \\ . \\ f_N \end{pmatrix}, \quad [2] \equiv \Phi^{[AB]} = \left(\begin{array}{cccc|ccc} 0, & \alpha_{12}, & \cdots, & \alpha_{15} & \epsilon_{16}, & \cdots, & \epsilon_{1N} \\ -\alpha_{12}, & 0, & \cdots, & \alpha_{25} & \epsilon_{26}, & \cdots, & \epsilon_{2N} \\ . & . & . & . & . & . & . \\ . & . & . & \alpha_{45} & . & . & . \\ -\alpha_{15}, & -\alpha_{25}, & \cdots, & 0 & \epsilon_{56}, & \cdots, & \epsilon_{5N} \\ \hline -\epsilon_{16}, & -\epsilon_{26} & \cdots, & -\epsilon_{56} & 0, & \cdots, & \beta_{6N} \\ . & . & . & . & . & . & . \\ -\epsilon_{1N}, & -\epsilon_{2N} & \cdots, & -\epsilon_{5N} & -\beta_{6N}, & \cdots, & 0 \end{array} \right)$$

(2.1)

where [1] means the dimension $\mathbf{N}$, [2] means the dimension $\left(\frac{\mathbf{N(N-1)}}{2!} \right)$ with two antisymmetric indices, [3] means the dimension $\left(\frac{\mathbf{N(N-1)(N-2)}}{3!} \right)$ with three antisymmetric indices, etc. We do not consider the symmetric indices such as $\{2\}$ $\left(= \frac{\mathbf{N(N+1)}}{2!} \right)$ because they will contain color sextets. [1] contains one $\mathbf{5}$, and [2] contains one $\mathbf{10}$ of SU(5). The number of the SU(5)$_{\text{GG}}$ families, i.e. that of $\mathbf{10}$ plus $\mathbf{\bar{5}}$, is counted by the number of $\mathbf{10}$ minus the number of $\mathbf{\overline{10}}$. The anomaly-freedom condition chooses the matching number of $\mathbf{\bar{5}}$'s. The numbers n_1 and n_2 for the vectorlike pairs $n_1(\mathbf{5} \oplus \mathbf{\bar{5}}) + n_2(\mathbf{10} \oplus \mathbf{\overline{10}})$ are not constrained

by the anomaly freedom. Thus, we count the number of families just by the net number of two index fermion representations in the $SU(5)_{GG}$ subgroup. For several $SU(N)$'s, we have the following family number n_f by counting the number of **10**'s,

$$
\begin{aligned}
SU(5):\quad & [2] \to n_f = 1 \\
SU(6):\quad & [3] \to n_f = 0, \quad [2] \to n_f = 1 \\
SU(7):\quad & [3] \to n_f = 1, \quad [2] \to n_f = 1 \\
SU(8):\quad & [4] \to n_f = 0, \quad [3] \to n_f = 2, \quad [2] \to n_f = 1 \\
SU(9):\quad & [4] \to n_f = 5, \quad [3] \to n_f = 3, \quad [2] \to n_f = 1 \\
SU(11):\quad & [5] \to n_f = -5, \quad [4] \to n_f = 9, \quad [3] \to n_f = 5, \quad [2] \to n_f = 1
\end{aligned} \tag{2.2}
$$

from which we define $[\overline{m}] = [N - m]$. Negative n_f gives the chiral family number $|n_f|$, which is also allowed. The anomaly units in $SU(N)$ are

$$
\mathcal{A}([m]) = \frac{(N-3)!(N-2m)}{(N-m-1)!(m-1)!}, \tag{2.3}
$$
$$
\mathcal{A}([1]) = 1, \ \mathcal{A}([2]) = N - 4, \ \mathcal{A}([3]) = \frac{(N-3)(N-6)}{2}, \ \text{etc.}
$$

where $m \geq 1$. For $[\bar{m}]$, we use $\mathcal{A}([\bar{m}]) = -\mathcal{A}([m])$. From now on, we frequently use the indices to represent the antisymmetric $[m]$ in terms of $\Psi^{[\alpha_1, \cdots, \alpha_m]}$ where $\alpha_1, \alpha_2, \cdots,$ and α_m are completely antisymmetrized.

Theory of families in GUTs does not allow repetition of the representation and gauge anomalies [4]. I. I. Rabi's terse question, "Who ordered that?" quipped after the 1936 discovery of muon, eloquently states the essence of the family problem. Let the matter representation be

$$
\mathcal{M} = \sum_{i=1}^{[N/2]} c_i [i] + \sum_{j=1}^{[N/2]} \bar{c}_j [\bar{j}] - \bar{c}_{N/2} [N/2] \delta_{N/2,\text{integer}}. \tag{2.4}
$$

where the elements of the integer set $\{c_i, \bar{c}_j\}$ do not have a common divisor. For three standard families of Georgi-Glashow GUT $SU(5)_{GG}$, Georgi required that all integers are 1 and found an $SU(11)$ unification of three $SU(5)_{GG}$ families [4], which was the first exampe of UGUTF. Allowing possiblities of $c_i > 1$ and $\bar{c}_j > 1$ but requiring no common divisor for all of them is a non-repetition of a set, which is a reasonable requirement for a solution of the family problem. Then, there are many possibilities [25]. If one requires renormalizable Yukawa couplings, in addition, with one irreducible type of representation containing the BEH doublet, an $SU(9)$ model seems minimal [12]. However, at the GUT scale it is possible to have some Planck mass suppressed nonrenormalizable Yukawa couplings. Thus, constraining models by renormalizable couplings is not warranted at this stage. The minimal chiral choice plus vectorlike representations for UGUTF by the rules of (2.2) is

$$
SU(8) \ : \ [3] \oplus [2] \oplus 9\,[\bar{1}] \oplus n_1([1] \oplus [\bar{1}]) \oplus n_2([2] \oplus [\bar{2}]) + \cdots. \tag{2.5}
$$
$$
SU(7) \ : \ [3] \oplus 2\,[2] \oplus 8\,[\bar{1}] \oplus n_1([1] \oplus [\bar{1}]) \oplus n_2([2] \oplus [\bar{2}]) + \cdots. \tag{2.6}
$$

where $n_i \neq 0$ for $i = 1, 2$ are needed for the BEH mechanism. For eq. (2.5) the number of non-singlet chiral fields is 156, while for eq. (2.6) the number of non-singlet chiral fields is 133. In this sense, eq. (2.6) is the minimal model.

2.1 Anti-SU(N)'s and SO(32) heterotic string

By the rules of (2.2), an SU(7) model with $[\bar{3}] \oplus [2] \oplus [\bar{1}]$ of ref. [7] cannot be a theory of UGUTF because family numbers in eqs. (2.2) do not allow funnily charged quarks and leptons. But it contains the key feature we explore in this paper. Spinor representations of SO($4n + 2$) can be complex and the next possibility beyond spinor **16** of SO(10) is spinor **64** of SO(14) which contains the SU(7) model with $[\bar{3}] \oplus [2] \oplus [\bar{1}]$. Under the SU(7) antisymmetric-tensor notation, it is $\Psi_{[\alpha\beta\gamma]} \oplus \Psi^{[\alpha\beta]} \oplus \Psi_{[\alpha]}$ [7]. It was the first example of anti-SU(N) where antisymmetric tensor BEH bosons, $\Phi^{[\alpha\beta]}$, $\Phi_{[\alpha\beta]}$, $\Phi^{[\alpha\beta\gamma]}$, and $\Phi_{[\alpha\beta\gamma]}$ are used to reduce the rank of the GUT gauge group. It allows two quark families and three lepton families, but it had a drawback due to the appearance of non-standard quarks and a doubly charged lepton. Note here that the rank $2n + 1$ of SO($4n + 2$) is reduced down to the rank $2n$ via the vacuum expectation values (VEVs) of $\Phi^{[\alpha\beta]}$ and $\Phi_{[\alpha\beta]}$ representations of the SU($2n + 1$)×U(1) subgroup of SO($4n + 2$).

Following the philosophy of ref. [7], we define SU(N)×U(1) subgroups of SO($2N$) by SU(N)$_{\text{anti2}}$, SU(N)$_{\text{anti3}}$, SU(N)$_{\text{anti4}}$, etc., which means that the spontaneous symmetry breaking of SO($2N$) takes the SU(N)×U(1)$_{2,3,4}$ subgroup route if $\Phi^{[AB]}$, $\Phi^{[ABC]}$, $\Phi^{[ABCD]}$ are used for the BEH bosons, respectively. The symmetry breaking of ref. [7] was the intersection of SU(7)$_{\text{anti2}}$ and SU(7)$_{\text{anti3}}$. In this language, SU(5)$_{\text{flip}}$ is identical to SU(5)$_{\text{anti2}}$, since there is no higher order anti-symmetric tensor field beyond $\Phi^{[AB]}$ for $N = 5$.[3] In this vain, we can define SU(N)$_{\text{GG}}$ as the symmetry breaking route of the BEH field Φ_B^A of SU(N)×U(1) subgroup of SO($2N$).

It looks like that the SM families are only possible from the spinor of SO(10), which has been a reason for a GUT extension chain, containing the spinor **16** of SO(10),

$$\text{SO(10)} \to \text{E}_6 \to \text{E}_7 \to \text{E}_8, \tag{2.7}$$

or

$$\text{SO(10)} \to \text{SO(12)} \to \text{SO(14)} \to \text{SO(16)} \cdots \tag{2.8}$$

Of course, SO($2N$) contains spinors, and spinors for $N > 5$ can produce **16**'s of SO(10). In string theory, gauge groups $\text{E}_8 \times \text{E}_8'$ and SO(32) are allowed only with their adjoint representations. This observation did not favor the SO(32) hetrotic string theory for the realistic purpose of obtaining **16** of SO(10), because the adjoint representation of SO(32) cannot produce any spinor representation in its subgroups.

Because no non-SM charges have been found up to the TeV scale, the SU(5)$_{\text{anti2}}$ rather than $[\bar{3}] \oplus [2] \oplus [\bar{1}]$ in SU(7) seems to contain some truth in it. But, for a UGUTF one should consider groups larger than SO(10). Here, we go beyond the SU(5)$_{\text{anti2}}$. Then, we can use anti-SU(N)'s for a theory of UGUTF. In particular, GUT symmetry breaking is easier in supersymmetric GUTs if we assign VEVs to $\Phi^{[45]}$ ($= \alpha_{45}$ of eq. (2.1)) and $\Phi_{[45]}$,

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[3]In ref. [19], SU(5)$_{\text{flip}}$ was called anti-SU(5).

i.e. in $SU(N)_{\text{anti2}}$. In addition, the flipped SU(5) GUTs have been obtained from string compactification [21, 22]. These string-derived anti- or flipped- SU(5)'s are assuming the symmetry breaking chain through the SO(10) route assuming the appearance of the spinor **16**, but the SO(10)-spinor chain is not possible in eq. (2.2). Therefore, for UGUTFs we follow the extension chain

$$
SU(5) \rightarrow SU(6) \rightarrow SU(7) \rightarrow SU(8) \rightarrow SU(9) \begin{cases} \rightarrow E_8 \\ \rightarrow SO(18) \rightarrow \cdots \rightarrow SO(32). \end{cases} \quad (2.9)
$$

In these chains, we do not follow the groups allowing only spinor representations. So, anti-SU(N) by anti-symmetric tensor fields [7] is the key in string compactification. The UGUTF of eq. (2.6) may be obtained in this way.

The SO(32) heterotic string is useful for phenomenology and UGUTF. Symmetry breaking of SO(32) through $SU(N)_{\text{anti2}}$ is possible because **496** of SO(32) contains the following SU(16) representations

$$
\Phi_B^A \oplus \Phi^{[AB]} \oplus \Phi_{[AB]}, \quad (A, B = 1, 2, \cdots, 16), \quad (2.10)
$$

whose dimensions are $(N^2 - 1)$, $\frac{N(N-1)}{2}$, and $\frac{N(N-1)}{2}$, respectively, of its SU(16) subgroup. In the orbifold compactification of SO(32), it will be easy to realize the representation $\Phi^{[AB]}$ and $\Phi_{[AB]}$ even at level 1, and the key UGUTF breaking, i.e. the separation of color $SU(3)_c$ and weak $SU(2)_W$, to the SM is possible by $\langle \Phi^{[45]} \rangle$ and $\langle \Phi_{[45]} \rangle$ of eq. (2.1). Because we allow only the SM fields, the fundamental representations $\Phi^{[A]}$ and $\Phi_{[A]}$ are also used to reduce the rank further by the VEVs at the locations $f_6, \cdots, f_N$ of eq. (2.1).

3 Toward a theory of family unification in orbifold compactification

As noted in the previous section, a minimal UGUTF needs a GUT allowing SU(7) as a subgroup. Compactification of heterotic string frequently needs an anti-SU(N) or $SU(N)_{\text{anti2}}$. The $E_8 \times E_8'$ heterotic string allows the rank 8 SU(9) which is considered to be a subgroup of E_8. We will try to realize eq. (2.6) from the string compactification. In addition, no degeneracy between families is left below the compactification scale. This means that we will introduce all the needed Wilson lines. As we will see, it is more difficult to obtain multi-index tensor fields in the twisted sectors. In particular, the three-index tensor field $\Psi^{[ABC]}$ cannot be obtained in the twisted sectors. So, $\Psi^{[ABC]}$, if they appear, is required to come from U.

In table 1, we list the number of fixed points for the non-prime orbifolds. For example, the cental number in $\mathbf{Z}_{6-II}$ and $\mathbf{Z}_{12-I}$ are 2 and 4, respectively, which mean that they have $\mathbf{Z}_{6/2}$ and $\mathbf{Z}_{12/4}$ symmetries, i.e. both have the $\mathbf{Z}_3$ symmetry in the second torus. They correspond to the gauge group phase in the untwisted sector matter $P \cdot V = \frac{2}{6}$ and $\frac{4}{12}$, respectively, and the untwisted sector multiplicity is 3. This is the easiest way to obtain three families from the U sector, i.e. $3(\mathbf{10} \oplus \overline{\mathbf{5}})$. In this case, the three families are not distinguished, and there must be an S_3 discrete symmetry. It may be broken by Higgsing at a GUT scale. Here, we do not follow this line of argument because there are too many possibilities.

If $\Psi^{[ABC]}$ appears from U, its multiplicities can be 1, 2, 3, or 4. Always, $p \cdot V = \frac{1}{N}$ gives the multiplicity 1 except in $\mathbf{Z}_4$. In $\mathbf{Z}_4$ (entries 1) and $\mathbf{Z}_{8-I}$ (entry 2), they give $\mathbf{Z}_4$

Order N	ϕ_s	ϕ_s^2	No. of fixed points
4	$\frac{1}{4}(2\ 1\ 1)$	$\frac{3}{8}$	16
6-I	$\frac{1}{6}(2\ 1\ 1)$	$\frac{1}{6}$	3
6-II	$\frac{1}{6}(3\ 2\ 1)$	$\frac{14}{36}$	12
8-I	$\frac{1}{8}(3\ 2\ 1)$	$\frac{14}{64}$	4
8-II	$\frac{1}{8}(4\ 3\ 1)$	$\frac{26}{64}$	8
12-I	$\frac{1}{12}(5\ 4\ 1)$	$\frac{42}{144}$	3
12-II	$\frac{1}{12}(6\ 5\ 1)$	$\frac{62}{144}$	4

Table 1. Number of fixed points of non-prime orbifolds.

U_i	Number of $\mathbf{10}$s	Tensor form	Chirality	$[p_{\text{spin}}]$ $(p_{\text{spin}} \cdot \phi_s)$
$U_1\,(p \cdot V = \frac{5}{12})$	1	$\Psi^{[ABC]}$	R	$[\oplus; +++]$ $\left(\frac{+5}{12}\right)$
$U_2\,(p \cdot V = \frac{4}{12})$	3	$\Psi^{[ABC]}$	L	$[\ominus; ++-]$ $\left(\frac{+4}{12}\right)$
$U_3\,(p \cdot V = \frac{1}{12})$	1	$\Psi^{[ABC]}$	L	$[\ominus; +-+]$ $\left(\frac{+1}{12}\right)$

Table 2. Number of $\mathbf{10}$'s from $\text{SU}(7)_{\text{anti2}}$ in $\mathbf{Z}_{12-I}$, and chirality for $P \cdot V = \phi_s \cdot s$ with $\phi_s = (\frac{5}{12}, \frac{4}{12}, \frac{1}{12})$ and even number of minuses from $s = (\ominus \text{ or } \oplus; \pm, \pm, \pm)$. For example, $s = (\oplus; +++)$ gives chirality R for $P \cdot V = p_{\text{spin}} \cdot \phi_s = \frac{5}{12}$.

which leads to multiplicity 2. In $\mathbf{Z}_{6-I}$ (entry 2), $\mathbf{Z}_{6-II}$ (entry 2) and $\mathbf{Z}_{12-I}$ (entry 4), they give $\mathbf{Z}_3$ which leads to multiplicity 3. In $\mathbf{Z}_4$ (entry 2), $\mathbf{Z}_{6-II}$ (entry 3), $\mathbf{Z}_{8-II}$ (entry 4) and $\mathbf{Z}_{12-II}$ (entry 6), they give $\mathbf{Z}_2$ which leads to multiplicity 4.

As an example, we show the U sector multiplicity of $\Psi^{[ABC]}$ in $\mathbf{Z}_{12-I}$ in table 2. If it is the representation of an $\text{SU}(8)$ GUT, then there are two $\mathbf{10}$'s (due to eq. (2.2)) from U. If it is the representation of an $\text{SU}(7)$ GUT, then there is a possibility of one $\mathbf{10}$ from U. Thus, we choose $\text{SU}(7)_{\text{anti2}}$. The remaining $\mathbf{10}$'s of $\text{SU}(5)$, i.e. $\mathbf{21}$'s of $\text{SU}(7)_{\text{anti2}}$, come from the twised sectors. However, it is not so easy to obtain two $\Psi^{[AB]}$'s of $\text{SU}(7)_{\text{anti2}}$ from T.

Summarizing the method to obtain three $\mathbf{10}$'s from $\text{SU}(7)_{\text{anti2}}$,

1. Matter representation $\Psi^{[ABC]}$ ($A = 1, 2, \cdots, 7$) must be present in the untwisted sector.

2. Matter $\Psi^{[AB]}$ must not appear in the untwisted sector.

3. Matter $\Psi^{[AB]}$ must be present in a twisted sector with the chirality that of $\Psi^{[ABC]}$.

$$(3.1)$$

Matter in the untwisted sector U_i occurs with $P \cdot V = \frac{N_i}{N}$. For example, $N_i = 5, 4, 1$ for $\mathbf{Z}_{12-I}$ is shown in table 1. In addition, one matter $\Psi^{[ABC]}$ is allowed only in U_1 and U_3 in $\mathbf{Z}_{12-I}$. We require that it has the spinor form, by choosing an appropriate V,

$$p \cdot V = \frac{1}{12}, \ U_3 : (\underline{- - - - + + +}; +)(0^8)',$$
$$p \cdot V = \frac{5}{12}, \ U_1 : (\underline{- - - - + + +}; +)(0^8)',$$

$$(3.2)$$

where the underline means permutations. Here, the torus or untwisted sector are called U_3 and U_1, respectively. Their CTP conjugates also appear in U as $(\underline{---+++}; -)(0^8)'$. Representations (3.2) satisfies **Condition 1**. To satisfy **Condition 2**, Wilson lines can be used if needed.

Condition 3 requires a full construction method. $\Psi^{[AB]}$ located in a twisted sector is the key part in this paper. In the $\mathbf{Z}_N$ orbifold, multiplicities in the k-th twisted sector $\mathcal{P}_k$ need to be calculated, which is given by[4]

$$\mathcal{P}_k = \frac{1}{N} \sum_{l=0}^{[N/2]} \tilde{\chi}(\theta^k, \theta^l) e^{i \, 2\pi l \Theta_0}, \tag{3.3}$$

where Θ_0 will be defined later. The chirality is given by the first entry of s, denoted as L- or R- movers, with the even number of total '$-$'s,

$$s = (s_0; \tilde{s}) = (\ominus \text{ or } \oplus; \pm, \pm, \pm). \tag{3.4}$$

In table 3, multiplicities are presented for $\mathbf{Z}_6, \mathbf{Z}_8$ and $\mathbf{Z}_{12}$ orbifolds. The multiplicities in $T_{N/4}$ are colored red. Matter $\Psi^{[AB]}$ can appear, if they do, only at $T_{N/4}$. From the red numbers in table 3, we note that $\mathbf{Z}_{8-II}$ and $\mathbf{Z}_{12-I}$ are the only allowed possibilities of obtaining multiplicity 2 at $T_{N/4}$, due to the large vacuum energy $2\tilde{c}_{N/4} = \frac{13}{8}$, as presented below in eqs. (3.14) and (3.15). In other models, it is impossible to house $\Psi^{[AB]}$. Out of $\mathbf{Z}_{8-II}$ and $\mathbf{Z}_{12-I}$, we choose the simpler case $\mathbf{Z}_{12-I}$ because we need to specify only one Wilson line. Note that in $\mathbf{Z}_{8-II}$ one has to specify two Wilson lines to specify the model completely without degeneracy. Then, at $T_{N/4}$, we must obtain two $\Psi^{[AB]}$'s. This condition is very restrictive and may rule out the $\mathbf{Z}_{12-I}$ possibility. Luckily, we find a model, satisfying this condition.

In the twisted sector, the masslessness conditions are satisfied for the phases contributed by the left- and right-movers [22],

$$2N_L^j \hat{\phi}_j + (P + kV) \cdot V - \frac{k}{2}V^2 = 2\tilde{c}_k, \text{ L movers}, \tag{3.5}$$

$$2N_R^j \hat{\phi}_j - \tilde{s} \cdot \phi_s + \frac{k}{2}\phi_s^2 = 2c_k, \text{ R movers}, \tag{3.6}$$

where j denotes the coordinate of the 6-dimensional compactified space running over $\{1, \bar{1}\}, \{2, \bar{2}\}, \{3, \bar{3}\}$, and $\hat{\phi}^j = \phi_s^j \cdot \text{sign}(\tilde{\phi}^j)$ with $\text{sign}(\phi^{\bar{j}}) = -\text{sign}(\tilde{\phi}^j)$. In calculating the multiplicities in Eq, (3.3), we use the phase Θ_0 with Δ_k,

$$\Theta_0 = \sum_j (N_L^j - N_R^j)\hat{\phi}^j - \frac{k}{2}(V_a^2 - \phi_s^2) + (P + kV_a) \cdot V_a - (\tilde{s} + k\phi_s) \cdot \phi_s + \text{integer},$$

$$= -\tilde{s} \cdot \phi_s + \Delta_k, \tag{3.7}$$

$$\Delta_k = (P + kV_a) \cdot V_a - \frac{k}{2}(V_a^2 - \phi_s^2) + \sum_j (N_L^j - N_R^j)\hat{\phi}^j$$

$$\equiv \Delta_k^0 + \Delta_k^N, \tag{3.8}$$

---

[4] $\tilde{\chi}(\theta^k, \theta^l)$ are presented in ref. [26].

	i	Mult. $\mathcal{P}_k(0)$	$\mathcal{P}_k(\frac{\pi}{3})$	$\mathcal{P}_k(\frac{2\pi}{3})$
		Mult.		
$\mathbf{Z}_6$–I	1	3	0	0
	2	15	0	0
	4	8	0	4
$\mathbf{Z}_6$–II	1	12	0	0
	2	6	0	0
	4	8	0	0

	i	Mult. $\mathcal{P}_k(0)$	$\mathcal{P}_k(\frac{\pi}{2})$	$\mathcal{P}_k(\pi)$
		Mult.		
$\mathbf{Z}_8$–I	1	4	0	0
	2	10	0	0
	3	4	0	0
	4	6	3	4
$\mathbf{Z}_8$–II	1	8	0	0
	2	3	1	1
	3	8	0	0
	4	6	3	4

	i	Mult. $\mathcal{P}_k(0)$	$\mathcal{P}_k(\frac{\pi}{3})$	$\mathcal{P}_k(\frac{2\pi}{3})$	$\mathcal{P}_k(\pi)$
		Mult.			
$\mathbf{Z}_{12}$–I	1	3	0	0	0
	2	3	0	0	0
	3	2	0	1	0
	4	9	0	0	6
	5	3	0	0	0
	6	4	2	3	2
$\mathbf{Z}_{12}$–II	1	4	0	0	0
	2	1	0	0	0
	3	8	0	0	0
	4	5	0	3	0
	5	4	0	0	0
	6	4	2	3	2

Table 3. Multiplicities in the $\mathbf{Z}_6, Z_8$, and $\mathbf{Z}_{12}$ twisted sectors.

where V_a is V distinguished by Wilson lines, and

$$\Delta_k^0 = P \cdot V_a + \frac{k}{2}(-V_a^2 + \phi_s^2),$$
$$\Delta_k^N = \sum_j (N_L^j - N_R^j)\hat{\phi}^j. \tag{3.9}$$

We choose $0 < \hat{\phi}^j \leq 1$ mod integer and oscillator contributions due to $(N_L - N_R)$ to the phase can be positive or negative, with $N_{L,R} \geq 0$. But each contribution to the vacuum energy $N_{L,R}^j \hat{\phi}^j$ is nonnegative. One oscillation contributes one number in ϕ_s. With the oscillator, the vacuum energy is shifted to

$$(P + kV_a)^2 = -2 \sum_j N_L^j \hat{\phi}^j + 2\tilde{c}_k$$
$$(p_{\text{vec}} + k\phi_s)^2 = -2 \sum_j N_R^j \hat{\phi}^j + 2c_k. \tag{3.10}$$

In eq. (3.10), instead of $p_{\rm vec}$ a four entry quantity $p_{\rm spin}$ of $\pm\frac{1}{2}$'s with even number of $-$'s is possible, but we do not find any example with $p_{\rm spin}$. The vacuum energy contributions in the twisted sectors in $\mathbf{Z}_{6,8,12}$ are given by[5]

$$\mathbf{Z}_{6-I}: \begin{cases} 2\tilde{c}_k: & \frac{3}{2}(k=1),\ \frac{4}{3}(k=2),\ \frac{3}{2}(k=3), \\ 2c_k: & \frac{1}{2}(k=1),\ \frac{1}{3}(k=2),\ \frac{1}{2}(k=3), \end{cases} \tag{3.11}$$

$$\mathbf{Z}_{6-II}: \begin{cases} 2\tilde{c}_k: & \frac{25}{18}(k=1),\ \frac{14}{9}(k=2),\ \frac{3}{2}(k=3), \\ 2c_k: & \frac{7}{18}(k=1),\ \frac{5}{9}(k=2),\ \frac{1}{2}(k=3), \end{cases} \tag{3.12}$$

$$\mathbf{Z}_{8-I}: \begin{cases} 2\tilde{c}_k: & \frac{47}{32}(k=1),\ \frac{11}{8}(k=2),\ \frac{47}{32}(k=3),\ \frac{3}{2}(k=4), \\ 2c_k: & \frac{15}{32}(k=1),\ \frac{3}{8}(k=2),\ \frac{15}{32}(k=1),\ \frac{1}{2}(k=3), \end{cases} \tag{3.13}$$

$$\mathbf{Z}_{8-II}: \begin{cases} 2\tilde{c}_k: & \frac{45}{32}(k=1),\ \frac{13}{8}(k=2),\ \frac{45}{32}(k=1),\ \frac{3}{2}(k=4), \\ 2c_k: & \frac{13}{32}(k=1),\ \frac{5}{8}(k=2),\ \frac{13}{32}(k=1),\ \frac{1}{2}(k=4). \end{cases} \tag{3.14}$$

$$\mathbf{Z}_{12-I}: \begin{cases} 2\tilde{c}_k: & \frac{210}{144}(k=1),\ \frac{216}{144}(k=2),\ \frac{234}{144}(k=3),\ \frac{192}{144}(k=4),\ \frac{210}{144}(k=5),\ \frac{216}{144}(k=6), \\ 2c_k: & \frac{11}{24}(k=1),\ \frac{1}{2}(k=2),\ \frac{5}{8}(k=3),\ \frac{1}{3}(k=4),\ \frac{11}{24}(k=5),\ \frac{1}{2}(k=6). \end{cases} \tag{3.15}$$

$$\mathbf{Z}_{12-II}: \begin{cases} 2\tilde{c}_k: & \frac{103}{72}(k=1),\ \frac{31}{18}(k=2),\ \frac{11}{8}(k=3),\ \frac{14}{9}(k=4),\ \frac{103}{72}(k=5),\ \frac{3}{2}(k=6), \\ 2c_k: & \frac{31}{72}(k=1),\ \frac{13}{18}(k=2),\ \frac{3}{8}(k=3),\ \frac{5}{9}(k=4),\ \frac{31}{72}(k=5),\ \frac{1}{2}(k=6). \end{cases} \tag{3.16}$$

Note that $2\tilde{c}_k - 2c_k = 1$ which is the required condition for $\mathcal{N} = 1$ supersymmetry in the 4 dimensional spectra.

After $\Psi^{[ABC]}$ and $\Psi^{[AB]}$ are obtained, the number of families is fixed. Namely, the number of $\Psi_{[A]}$ of SU(7)$_{\rm anti2}$ is fixed after fixing $\Psi^{[ABC]}$ and $\Psi^{[AB]}$. For $\Psi_R^{[ABC]} \oplus 2\Psi_R^{[AB]}$, there must be eight $\Psi_{[A]R}$'s. Locating $\mathbf{5}_{\rm BEH}$ and $\overline{\mathbf{5}}_{\rm BEH}$ of SU(5)$_{\rm GG}$ in SU(7)$_{\rm anti2}$ can be achieved in many ways.

Not to allow any left-over degeneracy, one must assign all possible Wilson lines. For $\mathbf{Z}_3$ and $\mathbf{Z}_4$, one must specify three Wilson lines, $a_1 = a_2, a_3 = a_4, a_5 = a_6$. So, they have the most complicated Wilson line structures. For $\mathbf{Z}_{6-II}, \mathbf{Z}_{8-I}$ and $\mathbf{Z}_{8-II}$, one must specify two Wilson lines. Specifying one Wilson line is enough in $\mathbf{Z}_{12-I}$, i.e. only $a_3 = a_4$ in the 2nd torus.

The Wilson loop integral is basically the Bohm-Aharanov effect in the internal space of two-torus,

$$\oint V^i\, dx_i = \frac{1}{2} \oint \left(\vec{\nabla} \times \vec{V}\right)^i_{0,+,-} \epsilon^{ijk}\, dx_{[jk]}. \tag{3.17}$$

If the $\mathbf{B}$-field (i.e. $\vec{\nabla} \times \vec{V}$) at the orbifold singularity is present, the phase through Δ_0 contributes in the multiplicity. For $\mathbf{Z}_{12-I}$, this is the case in $T_{1,2,4,5}$. The complication arises at the points with $3a_3 = 0$ mod. integer, i.e. at $T_{3,6}$ [22],[6] where the Bohm-Aharanov phase has to be taken into account explicitly. If there is no $\mathbf{B}$-field at the orbifold singularity, there is no Bohm-Aharanov phase, but then there for the (internal space) gauge symmetry we must require explicitly

$$(P + kV_0) \cdot a_3 = 0. \tag{3.18}$$

---

[5]Typos of appendix D of ref. [26] are corrected here.

[6]T_9 contains the CTP conjugate states of T_3.

This case applies to $T_3^{0,+,-}, T_6$, and of course at U also. We distinguish T_3 by $0, +$ and $-$ because the phase Δ_k^0 of eq. (3.9) contains an extra $\frac{k}{2}$ factor. Namely, eq. (3.18) is applied only at U, T_3 and T_6. We will comment more on this in subsections T_3 and T_6.

4   SU(7)$_{\text{anti2}}$ spectra

We calculate the SU(7) non-singlet spectra of SU(7)$_{\text{anti2}}$ in the $\mathbf{Z}_{12-I}$ orbifold. We choose the following model,

$$
V^a = \begin{cases} V_0 = \left(\frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}; \frac{+5}{12} \right) \left(\frac{4}{12}, \frac{4}{12}, \frac{4}{12}, \frac{4}{12}, 0, \frac{4}{12}, \frac{7}{12}, \frac{3}{12} \right)', \ V_0^2 = \frac{338}{144}, \\ a_3 = a_4 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3})(\frac{-1}{3}, \frac{-1}{3}, \frac{-1}{3}, \frac{-1}{3}, 0, 0, \frac{5}{3}, -1)'. \end{cases}
$$
$$(4.1)$$

Here, $a_3 \, (= a_4)$ is chosen to allow and/or forbid some spectra, and is composed of fractional numbers with the integer multiples of $\frac{1}{3}$ because the second torus has the $\mathbf{Z}_3$ symmetry. Shifted lattices by Wilson lines are given by V_+ and V_-,

$$
\begin{cases} V_+ = \left(\frac{-1}{12}, \frac{-1}{12}, \frac{-1}{12}, \frac{-1}{12}, \frac{-1}{12}, \frac{-1}{12}, \frac{-1}{12}; \frac{9}{12} \right) \left(0, 0, 0, 0, 0, \frac{4}{12}, \frac{27}{12}, \frac{-9}{12} \right)', \ V_+^2 = \frac{914}{144}, \\ V_- = \left(\frac{-9}{12}, \frac{-9}{12}, \frac{-9}{12}, \frac{-9}{12}, \frac{-9}{12}, \frac{-9}{12}, \frac{-9}{12}; \frac{+1}{12} \right) \left(\frac{8}{12}, \frac{8}{12}, \frac{8}{12}, \frac{8}{12}, 0, \frac{4}{12}, \frac{-13}{12}, \frac{15}{12} \right)', \ V_-^2 = \frac{1234}{144}. \end{cases}
$$
$$(4.2)$$

We anticipated to achieve the key spectra needed for SU(7)$_{\text{anti2}}$

$$
\Psi_{U_3}^{[ABC]} + 2\Psi_{T_3}^{[AB]} + (?) \left(\Phi_{T_3}^{[AB]} + \Phi_{[AB],T_3} \right) + \cdots,
$$
$$(4.3)$$

where the sectors they appear are marked as subscripts. At this point, we do not fix how many vectorlike pairs appear in T_3. The chiral representations, the candidates of fermion families, are represented by Ψ, and vectorlike reresentations, candidates for the BEH bosons, are represented by Φ.

The orbifold conditions, toward a low energy 4 dimensional (4D) effective theory, remove some weights of the original ten dimensional $E_8 \times E_8'$ weights. The remaining ones constitute the gauge multiplets and matter fields in the untwisted sector in the low energy 4D theory. Therefore, the weights in the U sector must satisfy $P^2 = 2$ as in the original $E_8 \times E_8'$ weights. Orbifold conditions produce singularities. They are typically represented in three two-dimensional tori. A loop of string can be twisted around these singularities and define twisted sectors $T_k \, (k = 1, 2, \cdots, 12)$. Twisting can introduce additional phases. Since T_{12-k} provides the anti-particles of T_k, we consider only T_k for $k = 1, 2, \cdots, 6$. T_6 contains both particles and anti-particles. T_6, not affected by Wilson lines, is like an untwisted sector. It contains the antiparticles also as in U. Since the Wilson lines can affect only in non-contractible loops as the hidden sector Aharanov-Bohm effect, the Wilson lines can affect only around the singularities in the twisted sectors, but have no effects in the untwisted sector.

In this section we consider two twisted sectors, T_3 and T_6, explicitly. The other twisted sectors, T_1, T_2, T_4, and T_5 will be listed in appendix.

4.1 Untwisted sector U

In U, we find the following nonvanishing roots of $SU(7) \times SU(4)'$

$$E_8 \text{ gauge multiplet}: \ P \cdot V = 0 \text{ mod. integer}$$
$$SU(7): \left\{ P = (\underline{+1 - 1\,0\,0\,0\,0\,0}; 0)(0^8)' \right.$$
$$E_8' \text{ gauge multiplet}: \ P \cdot V = 0 \text{ mod. integer}$$
$$SU(4)': \left\{ P = (0^8)(\underline{1 - 1\,0\,0}\ 0\ 0\ 0\ 0)'. \right. \tag{4.4}$$

In addition, there exists $U(1)^2 \times U(1)'^5$ symmetry. The non-singlet $SU(7)$ matter fields are

$$E_8 \text{ matter multiplet}: \ P \cdot V = \frac{5}{12}, \text{ mod. integer}$$
$$SU(7): \left\{ \ P = (\underline{+ + + - - - -}; +)(0^8)' \right\}: \ \Psi_R^{[ABC]}. \tag{4.5}$$

$\Psi^{[ABC]}$ contains one **10** of $SU(5)_{GG}$ family, which belongs to the first family, and is R-handed as shown in table 2. It is simple to find the E_8' hidden sector matter in U,

$$E_8' \text{ matter multiplet}:$$
$$SU(4)': \left\{ P = (0^8)(\underline{+ - - -}; + - - -)', \ P \cdot V = \tfrac{1}{12} \right\}: \ \mathbf{4}_L', \tag{4.6}$$

where $P \cdot a_3 = 0$ excludes the cases of $P = (0^8)(\underline{+ - - -}; + + + -)'$ ($P \cdot V = 0$) and $P = (0^8)(\underline{+ + - -}; - + + -)'$ ($P \cdot V = \frac{4}{12}$).

4.2 Twisted sectors T

In the twisted sectors, we list only $SU(7)$ or $SU(4)'$ non-singlets. Wilson lines distinguish three fixed points in the second torus, and the shift vectors we consider at T_k are split into three cases

$$T_k^{0,+,-}: \ kV_a = \begin{cases} kV \equiv kV_0 \\ k(V + a_3) \equiv kV_+ \\ k(V - a_3) \equiv kV_-, \end{cases} \tag{4.7}$$

where V_+ and V_- are given in eq. (4.2). Because $3a_3 = 0$ mod. integer and due to eq. (3.8), T_6 sector is not distingushed by the Wilson lines but $T_{3,9}$ are distinguished.

We select only the even lattices shifted from the untwisted lattices. They form even numbers for the sum of entries of each elements of P.

To obtain non-trivial number of families, we need two index tensor fields, $\Psi^{[AB]}$ and/or $\Psi_{[AB]}$. This possibility arises only in T_3 because at T_3 there appear fractional number with integer times $\frac{1}{4}$. In other twisted sectors, the entries are not multiples of $\frac{1}{4}$ in which case we cannot fulfil the masslessness condition with $2\tilde{c}_k$ given below.

In the k-th twisted sector, the masslessness condition to raise the tachyonic vacuum energy to zero is

$$(P + kV_a)^2 = 2\tilde{c}_k - \left(2 \sum_j N_L^j \hat{\phi}^j \right), \tag{4.8}$$

$$(p + k\phi_s)^2 = 2c_k - \left(2 \sum_j N_R^j \hat{\phi}^j \right), \tag{4.9}$$

where $2\tilde{c}_k$ and $2c_k$ are given in eq. (3.15), and the brackets must be taken into account when oscillators contribute. When the conditions (4.8), (4.9) are satisfied, we obtain the SUSY spectra for which the chirality and multiplicity are calculated from Θ_0 in the k-th twisted sector,

$$\Theta_0 = -\tilde{s} \cdot \phi_s + P \cdot V_a + \Delta_k^0 + \Delta_k^N + \left(p \cdot \phi_s + \delta_k^N \right), \tag{4.10}$$

where

$$\begin{aligned}
\Delta_k^0 &= \frac{k}{2}(\phi_s^2 - V_a^2), \\
\Delta_k^N &= 2 \sum_j N_L^j \, \hat{\phi}^j, \\
\delta_k^N &= -2 \sum_j N_R^j \, \hat{\phi}^j.
\end{aligned} \tag{4.11}$$

We choose $0 < \hat{\phi}^j \leq 1$ mod integer and oscillator contributions due to $(N_L - N_R)$ can be in principle positive or negative. As an example, consider the T_3 sector for the N_R^j contribution. Here, $3\phi_s = (\frac{5}{4}, \frac{4}{4}, \frac{1}{4})$, needing $p = (-1, -1, 0)$. So, $(p + 3\phi_s)^2 = \frac{1}{8}$, needing the N_R contribution $\frac{4}{8} = 2 \cdot \frac{3}{12}$ to make up $2c_3 = \frac{5}{8}$ [22]. Thus, the R-handed oscillator contribution is $\frac{6}{12} = 2 \cdot \frac{3}{12}$. Namely, δ^3 is $\frac{+3}{12}$ which is included in the tables.

We will select only the even lattices shifted from the untwisted lattices. They form even numbers if the entries of each elements of P are added. Because we consider $E_8 \times E_8'$, the product of E_8 and E_8' parts must be even. They need not be even separately. But, there is a distinction in even $\times$ even lattice and odd $\times$ odd lattice. In the former case, E_8 and E_8' gauge quatum numbers do not change, but in the latter case we change the signs of the quantum numbers. In the table captions, we take into account this fact.

4.2.1   Twisted sector T_3 ($\delta^3 = \frac{+3}{12}$)

In the multiplicity calculation in Θ_0, there is a factor $\frac{1}{2}$ between the lattice shifts by Wilson lines. Even though the Wilson lines cannot distinguish the fixed points, we consider V_+ and V_- also as if Wilson lines distinguish fixed points.

- Two index spinor form for V_0^3: the spinor form gives $(P + 3V_0)^2 = \frac{13}{8}$ but $(P + 3V_0) \cdot a_3 \neq 0$, and there is no allowed states.

- Two index vector form for V_0^3: for a vector form of P,

$$3V_0 = \left(\frac{-5}{4}, \frac{-5}{4}, \frac{-5}{4}, \frac{-5}{4}, \frac{-5}{4}, \frac{-5}{4}, \frac{-5}{4}; \frac{+5}{4} \right) \left(\frac{4}{4}, \frac{4}{4}, \frac{4}{4}, \frac{4}{4}, 0, \frac{4}{4}, \frac{7}{4}, \frac{3}{4} \right)', \ V_0^2 = \frac{338}{144}$$

$$P = (\underline{2, 2, 1, 1, 1, 1, 1}; -1)(-1, -1, -1, -1, 0, -1, -2, -1)', \ P \cdot V_0 = \frac{-87}{12}$$

$$P + 3V_0 = \left(\frac{3}{4} \frac{3}{4} \frac{-1}{4} \frac{-1}{4} \frac{-1}{4} \frac{-1}{4} \frac{-1}{4}; \frac{1}{4} \right) \left(0^6 \frac{-1}{4} \frac{-1}{4} \right)', \ (P + 3V_0) \cdot a_3 = 0,$$

$$3\phi_s = \left(\frac{5}{4}, \frac{4}{4}, \frac{1}{4} \right), \ p_{\text{vec}} = (-1, -1, 0), \ p_{\text{vec}} \cdot \phi_s = \frac{-9}{12}, \ p_{\text{vec}} + 3\phi_s = \left(\frac{1}{4}, 0, \frac{1}{4} \right).$$

$$\tag{4.12}$$

Chirality	$\tilde{s}$	$-\tilde{s}\cdot\phi_s$	$-p_{\text{vec}}\cdot\phi_s,\ P\cdot V_0$	$(3/2)\phi_s^2,\ -(3/2)V_0^2,$	$\Delta_3^N[\delta^3]$	$\Theta_0\,(\mathcal{P}_3^N)$
$\ominus = L$ $(---)$	$\frac{+5}{12}$	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+3}{12}],0[\frac{-3}{12}]$	$\frac{+1}{12}\,(0),\ \frac{+7}{12}\,(0)$
$\ominus = L$ $(-++)$	0	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+3}{12}],0[\frac{-3}{12}]$	$\frac{+8}{12}\,(3),\ \frac{+2}{12}\,(2)$
$\ominus = L$ $(+-+)$	$\frac{-1}{12}$	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+3}{12}],0[\frac{-3}{12}]$	$\frac{+7}{12}\,(0),\ \frac{+1}{12}\,(0)$
$\ominus = L$ $(++-)$	$\frac{-4}{12}$	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+3}{12}],0[\frac{-3}{12}]$	$\frac{+4}{12}\,(3),\ \frac{-2}{12}\,(2)$
$\oplus = R$ $(+++)$	$\frac{-5}{12}$	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+3}{12}],0[\frac{-3}{12}]$	$\frac{+3}{12}\,(0),\ \frac{-3}{12}\,(0)$
$\oplus = R$ $(+--)$	0	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+3}{12}],0[\frac{-3}{12}]$	$\frac{+8}{12}\,(3),\ \frac{+2}{12}\,(2)$
$\oplus = R$ $(-+-)$	$\frac{+1}{12}$	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+3}{12}],0[\frac{-3}{12}]$	$\frac{+9}{12}\,(0),\ \frac{+3}{12}\,(0)$
$\oplus = R$ $(--+)$	$\frac{+4}{12}$	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+8}{12}],0[\frac{-3}{12}]$	$\frac{+12}{12}\,(4),\ \frac{+6}{12}\,(2)$

Table 4. Two index vector for V_0^3: chiralities (in the first column) and multiplicities (in the last column) of $\Phi^{[AB]}$ in the T_3 sector of $\mathbf{Z}_{12-I}$ for $N = 12$. $\phi_s = (\frac{5}{12}, \frac{4}{12}, \frac{1}{12})$, $(3/2)\phi_s = (\frac{5}{8}, \frac{4}{8}, \frac{1}{8})$ and $12(P+3V)\cdot a_3 = 0$. In the last column, $\delta_R^k = \frac{3}{12}$ is added. Multiplicities of the masslessness states are given by the phase Θ_0. The allowed chiralities are colored red, $\Psi_R^{[AB]} + 10(\Phi_L^{[AB]} + \Phi_R^{[AB]})$.

Chirality	$\tilde{s}$	$-\tilde{s}\cdot\phi_s$	$-p_{\text{vec}}\cdot\phi_s,\ P\cdot V_+$	$(3/2)\phi_s^2,\ -(3/2)V_+^2,$	$\Delta_3^N[\delta^3]$	$\Theta_0\,(\mathcal{P}_3^N)$
$\ominus = L$ $(---)$	$\frac{+5}{12}$	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+3}{12}],0[\frac{-3}{12}]$	$\frac{+1}{12}\,(0),\ \frac{+7}{12}\,(0)$
$\ominus = L$ $(-++)$	0	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad (\frac{-1}{12})\frac{-457}{48}$	$0[\frac{+3}{12}],0[\frac{-3}{12}]$	$\frac{+8}{12}\,(3),\ \frac{+2}{12}\,(2)$
$\ominus = L$ $(+-+)$	$\frac{-1}{12}$	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+3}{12}],0[\frac{-3}{12}]$	$\frac{+7}{12}\,(0),\ \frac{+1}{12}\,(0)$
$\ominus = L$ $(++-)$	$\frac{-4}{12}$	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+3}{12}],0[\frac{-3}{12}]$	$\frac{+4}{12}\,(3),\ \frac{-2}{12}\,(2)$
$\oplus = R$ $(+++)$	$\frac{-5}{12}$	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+3}{12}],0[\frac{-3}{12}]$	$\frac{+3}{12}\,(0),\ \frac{-3}{12}\,(0)$
$\oplus = R$ $(+--)$	0	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+3}{12}],0[\frac{-3}{12}]$	$\frac{+8}{12}\,(3),\ \frac{+2}{12}\,(2)$
$\oplus = R$ $(-+-)$	$\frac{+1}{12}$	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+3}{12}],0[\frac{-3}{12}]$	$\frac{+9}{12}\,(0),\ \frac{+3}{12}\,(0)$
$\oplus = R$ $(--+)$	$\frac{+4}{12}$	$\frac{+9}{12},\ \frac{-3}{12}$	$\frac{21}{48}\quad \frac{-169}{48}$	$0[\frac{+8}{12}],0[\frac{-3}{12}]$	$\frac{+12}{12}\,(4),\ \frac{+6}{12}\,(2)$

Table 5. Two index vector for V_+^3: the entries are the same as table 4, and we obtain $\Psi_R^{[AB]} + 10(\Phi_L^{[AB]} + \Phi_R^{[AB]})$.

Massless states are shown in table 4. Since the phase $\frac{21}{48} - \frac{169}{48}$ is an even integer times $\frac{1}{24}$, we choose the vector form $-p_{\text{vec}}\cdot\phi_s = -[\frac{5}{12}\times(-1) + \frac{4}{12}\times(-1) + \frac{1}{12}\times(0)] = \frac{18}{12}$, instead of the spinor form p_{spin}. So, we used $-p_{\text{vec}}\cdot\phi_s = \frac{+9}{12}$ in the table.

- Two index spinor form for V_+^3: the spinor form gives $(P+3V_0)^2 = \frac{13}{8}$ but $(P+3V_0)\cdot a_3 \neq 0$, and there is no allowed states.

- Two index vector form for V_+^3: for a vector form of P,

$$3V_+ = \left(\frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}; \frac{9}{4}\right)\left(0, 0, 0, 0, 0, \frac{4}{4}, \frac{27}{4}, \frac{-9}{4}\right)',\ V_+^2 = \frac{914}{144}$$

$$P = (\underline{2,2,1,1,1,1,1}; -1)(-1,-1,-1,-1,0,-1,-7,2)',\ P\cdot V_+ = \frac{-219}{12} = \frac{+9}{12}$$

$$P + 3V_0 = \left(\frac{3}{4}\,\frac{3}{4}\,\frac{-1}{4}\,\frac{-1}{4}\,\frac{-1}{4}\,\frac{-1}{4}\,\frac{-1}{4}; \frac{1}{4}\right)\left(0^6\,\frac{-1}{4}\,\frac{-1}{4}\right)',\ (P+3V_0)\cdot a_3 = 0,$$

$$3\phi_s = \left(\frac{5}{4}, \frac{4}{4}, \frac{1}{4}\right),\ p_{\text{vec}} = (-1,-1,0),\ p_{\text{vec}}\cdot\phi_s = \frac{-9}{12},\ p_{\text{vec}} + 3\phi_s = \left(\frac{1}{4}, 0, \frac{1}{4}\right).$$

(4.13)

Massless states are shown in table 5, which are exactly the same as those of table 4.

Chirality	$\tilde{s}$	$-\tilde{s}\cdot\phi_s$	$-p_{\text{vec}}\cdot\phi_s$, $P\cdot V_-$		$(3/2)\phi_s^2$,	$-(3/2)V_-^2$,	$\Delta_3^N[\delta^3], \Delta_3^N[-\delta^3]$	$\Theta_0\,(\mathcal{P}_3^N)$
$\ominus = L$ $(---)$	$\frac{+5}{12}$	$\frac{+9}{12},$	$\frac{-4}{12}$	$\frac{21}{48}$	$\frac{-617}{48}$	$0[\frac{+3}{12}], 0[\frac{-3}{12}]$	$\frac{+8}{12}\,(3),\ \frac{+2}{12}\,(2)$
$\ominus = L$ $(-++)$	0	$\frac{+9}{12},$	$\frac{-4}{12}$	$\frac{21}{48}$	$\frac{-617}{48}$	$0[\frac{+3}{12}], 0[\frac{-3}{12}]$	$\frac{+3}{12}\,(0),\ \frac{-3}{12}\,(0)$
$\ominus = L$ $(+-+)$	$\frac{-1}{12}$	$\frac{+9}{12},$	$\frac{-4}{12}$	$\frac{21}{48}$	$\frac{-617}{48}$	$0[\frac{+3}{12}], 0[\frac{-3}{12}]$	$\frac{+2}{12}\,(2),\ \frac{-4}{12}\,(3)$
$\ominus = L$ $(++-)$	$\frac{-4}{12}$	$\frac{+9}{12},$	$\frac{-4}{12}$	$\frac{21}{48}$	$\frac{-617}{48}$	$0[\frac{+3}{12}], 0[\frac{-3}{12}]$	$\frac{-1}{12}\,(0),\ \frac{-7}{12}\,(0)$
$\oplus = R$ $(+++)$	$\frac{-5}{12}$	$\frac{+9}{12},$	$\frac{-4}{12}$	$\frac{21}{48}$	$\frac{-617}{48}$	$0[\frac{+3}{12}], 0[\frac{-3}{12}]$	$\frac{-2}{12}\,(2),\ \frac{-8}{12}\,(3)$
$\oplus = R$ $(+--)$	0	$\frac{+9}{12},$	$\frac{-4}{12}$	$\frac{21}{48}$	$\frac{-617}{48}$	$0[\frac{+3}{12}], 0[\frac{-3}{12}]$	$\frac{+3}{12}\,(0),\ \frac{-3}{12}\,(0)$
$\oplus = R$ $(-+-)$	$\frac{+1}{12}$	$\frac{+9}{12},$	$\frac{-4}{12}$	$\frac{21}{48}$	$\frac{-617}{48}$	$0[\frac{+3}{12}], 0[\frac{-3}{12}]$	$\frac{+4}{12}\,(3),\ \frac{-2}{12}\,(2)$
$\oplus = R$ $(--+)$	$\frac{+4}{12}$	$\frac{+9}{12},$	$\frac{-4}{12}$	$\frac{21}{48}$	$\frac{-617}{48}$	$0[\frac{+8}{12}], 0[\frac{-3}{12}]$	$\frac{+7}{12}\,(0),\ \frac{+1}{12}\,(0)$

Table 6. Two index vector for V_-^3: chiralities (in the first column) and multiplicities are $10(\Phi_L^{[AB]} + \Phi_R^{[AB]})$.

- Two index spinor form for V_-^3: the spinor form gives $(P + 3V_0)^2 = \frac{13}{8}$ but $(P + 3V_0)\cdot a_3 \neq 0$, and there is no allowed states.

- Two index vector form for V_-^3: for a vector form of P,

$$3V_- = \left(\frac{-9}{4}, \frac{-9}{4}, \frac{-9}{4}, \frac{-9}{4}, \frac{-9}{4}, \frac{-9}{4}, \frac{-9}{4}; \frac{+1}{4}\right)\left(\frac{8}{4}, \frac{8}{4}, \frac{8}{4}, \frac{8}{4}, 0, \frac{4}{4}, \frac{-13}{4}, \frac{15}{4}\right)', \ V_-^2 = \frac{1234}{144}$$

$$P = (\underline{2, 2, 1, 1, 1, 1, 1}; -1)(-2, -2, -2, -2, 0, -1, 3, -4)', \ P \cdot V_- = \frac{-100}{12} = \frac{-4}{12},$$

$$P + 3V_- = \left(\underline{\frac{3}{4}\ \frac{3}{4}\ \frac{-1}{4}\ \frac{-1}{4}\ \frac{-1}{4}\ \frac{-1}{4}\ \frac{-1}{4}}; \frac{1}{4}\right)\left(0^6\ \frac{-1}{4}\ \frac{-1}{4}\right)', \ (P + 3V_0)\cdot a_3 = 0,$$

$$3\phi_s = \left(\frac{5}{4}, \frac{4}{4}, \frac{1}{4}\right), \ p_{\text{vec}} = (-1, -1, 0), \ p_{\text{vec}}\cdot\phi_s = \frac{-9}{12}, \ p_{\text{vec}} + 3\phi_s = \left(\frac{1}{4}, 0, \frac{1}{4}\right).$$
(4.14)

Massless states are shown in table 6.

The chiral spectrum we obtained for the two index tensors in T_3 is

$$T_3 : \ 2\,\Psi_{R,T_3^0}^{[AB]}.$$
(4.15)

These make up three chiral families together with $\Psi_R^{[ABC]}$ from U. The number in (4.15) is the same as the one if we treat T_3 with multiplicity 2. This multiplicity is because $3V$ is a $\mathbf{Z}_4$ twist which has two fixed points in a two-dimensional torus. Since there is no Wilson line, we could have treated only $3V$ with multiplicity 2 of $\mathbf{Z}_4$. The multiplicity 2 is accounted by T_3^0 and T_3^+. But, T_3^- produce additional vectorlike pairs, which must be fictitious. We may consider the spectra in table 6 are fictitious. In the remainder of the paper, we will not consider V_+^3 and V_-^3. We consider only V_0^3 and take into account the multiplicity 2 of T_3.

Now let us proceed to consider one index tensors in T_3. In the final result, we will multiply the overall multiplicity 2 as commented above.

- One index spinor form for V_0^3:

Chirality $\tilde{s}$	$-\tilde{s}\cdot\phi_s$	$-p_{\rm vec}\cdot\phi_s,\ P\cdot V_0$	$(3/2)\phi_s^2,\ -(3/2)V_0^2,\ \pm\Delta_3^N[\pm\delta^3]$	$\Theta_0\,(\mathcal{P}_3^N)$
$\ominus=L\ \ (---)$	$\frac{+5}{12}$	$\frac{+9}{12},\quad \frac{+4}{12}$	$\frac{21}{48}\quad \frac{-169}{48}\quad \frac{\pm3}{12}[\frac{\pm3}{12}]$	$\frac{+12}{12}(4),\ \frac{+6}{12}(2),\ \frac{+6}{12}(2),\ \frac{0}{12}(4)$
$\ominus=L\ \ (-++)$	0	$\frac{+9}{12},\quad \frac{+4}{12}$	$\frac{21}{48}\quad \frac{-169}{48}\quad \frac{\pm3}{12}[\frac{\pm3}{12}]$	$\frac{+7}{12}(0),\ \frac{+1}{12}(0),\ \frac{+1}{12}(0),\ \frac{-5}{12}(0)$
$\ominus=L\ \ (+-+)$	$\frac{-1}{12}$	$\frac{+9}{12},\quad \frac{+4}{12}$	$\frac{21}{48}\quad \frac{-165}{48}\quad \frac{\pm3}{12}[\frac{\pm3}{12}]$	$\frac{+6}{12}(2),\ \frac{0}{12}(4),\ \frac{0}{12}(4),\ \frac{-6}{12}(2)$
$\ominus=L\ \ (++-)$	$\frac{-4}{12}$	$\frac{+9}{12},\quad \frac{+4}{12}$	$\frac{21}{48}\quad \frac{-165}{48}\quad \frac{\pm3}{12}[\frac{\pm3}{12}]$	$\frac{+3}{12}(0),\ \frac{-3}{12}(0),\ \frac{-3}{12}(0),\ \frac{-9}{12}(0)$
$\oplus=R\ \ (+++)$	$\frac{-5}{12}$	$\frac{+9}{12},\quad \frac{+4}{12}$	$\frac{21}{48}\quad \frac{-165}{48}\quad \frac{\pm3}{12}[\frac{\pm3}{12}]$	$\frac{+2}{12}(2),\ \frac{-4}{12}(3),\ \frac{-4}{12}(3),\ \frac{-10}{12}(2)$
$\oplus=R\ \ (+--)$	0	$\frac{+9}{12},\quad \frac{+4}{12}$	$\frac{21}{48}\quad \frac{-165}{48}\quad \frac{\pm3}{12}[\frac{\pm3}{12}]$	$\frac{+7}{12}(0),\ \frac{+1}{12}(0),\ \frac{+1}{12}(0),\ \frac{-5}{12}(0)$
$\oplus=R\ \ (-+-)$	$\frac{+1}{12}$	$\frac{+9}{12},\quad \frac{+4}{12}$	$\frac{21}{48}\quad \frac{-165}{48}\quad \frac{\pm3}{12}[\frac{\pm3}{12}]$	$\frac{+8}{12}(3),\ \frac{+2}{12}(2),\ \frac{+2}{12}(2),\ \frac{-4}{12}(3)$
$\oplus=R\ \ (--+)$	$\frac{+4}{12}$	$\frac{+9}{12},\quad \frac{+4}{12}$	$\frac{21}{48}\quad \frac{-165}{48}\quad \frac{\pm3}{12}[\frac{\pm3}{12}]$	$\frac{+11}{12}(0),\ \frac{+5}{12}(0),\ \frac{+5}{12}(0),\ \frac{-1}{12}(0)$

Table 7. One index spinor form for V_0^3: in the multiplicity, the order of $\frac{\pm3}{12}[\frac{\pm3}{12}]$ is $\frac{+3}{12}[\frac{+3}{12}]$, $\frac{-3}{12}[\frac{+3}{12}]$, $\frac{+3}{12}[\frac{-3}{12}]$, and $\frac{-3}{12}[\frac{-3}{12}]$. Massless states are $4\left(\Psi_{[\alpha']L,1}+\Psi_{[\alpha']L,\bar{1}}\right)\oplus 20\left(\Phi_{[\alpha']L,1}+\Phi_{[\alpha']L,\bar{1}}\right)\oplus 20\left(\Phi_{[\alpha']R,1}+\Phi_{[\alpha']R,\bar{1}}\right)$, where multiplicity 2 of T_3 is taken into account.

$$3V_0=\left(\frac{-5}{4},\frac{-5}{4},\frac{-5}{4},\frac{-5}{4},\frac{-5}{4},\frac{-5}{4},\frac{-5}{4};\frac{+5}{4}\right)\left(\frac{4}{4},\frac{4}{4},\frac{4}{4},\frac{4}{4},0,\frac{4}{4},\frac{7}{4},\frac{3}{4}\right)',\ V_0^2=\frac{338}{144}$$

$$P=\left(\underline{\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2}};\frac{-3}{2}\right)(-1,-1,-1,-1,0,-1,-2,-1)',\ P\cdot V_0=\frac{-92}{12}$$

$$P+3V_0=\left(\underline{\frac{-3}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}};\frac{-1}{4}\right)\left(0^6\ \frac{-1}{4}\ \frac{-1}{4}\right)',\ (P+3V_0)\cdot a_3=0,$$

(4.16)

which make up $\frac{9}{8}$. The oscillator contributions of $2\frac{3}{12}$ are needed to satisfy the masslessness condition. Chiralities and multiplicities are tabulated in table 7.

- One index vector form for V^3: vector forms do not give massless states because $(P+3V_-)\cdot a_3\neq 0$.

4.2.2 Twisted sector $T_6\,(\delta^6=0)$

- One index spinor form for V^6: we have[7]

$$6V_0=\left(\frac{-5}{2},\frac{-5}{2},\frac{-5}{2},\frac{-5}{2},\frac{-5}{2},\frac{-5}{2},\frac{-5}{2};\frac{+5}{2}\right)\left(\frac{4}{2},\frac{4}{2},\frac{4}{2},\frac{4}{2},0,\frac{4}{2},\frac{7}{2},\frac{3}{2}\right)',\ V_0^2=\frac{338}{144}$$

$$P=(\underline{3+,5+,5+,5+,5+,5+,5+};5-)(-2,-2,-2,-2,0,-2,-3,-1)',\ \frac{-159}{12},$$

$$P+6V_0=\left(\underline{-1,0^6};0\right)\left(0^6,\frac{1}{2},\frac{1}{2}\right)',\ (P+6V_0)\cdot a_3=0,$$

(4.17)

which saturate the needed masslessness condition $\frac{3}{2}$ of T_6, which are tabulated in table 8. Note that to make the phase an integer times $\frac{1}{12}$, we choose $-p_{\rm vev}\cdot\phi_s$ as: since $-(6/2)(V_0^2-\phi_s^2)$ is even number times $1/24$, we choose a vector $p_{\rm vev}$: $-p_{\rm vev}\cdot\phi_s=-[(-2)\times\frac{5}{12}+(-2)\times\frac{4}{12}+(0)\times\frac{1}{12}]=\frac{18}{12}$. So, we used $\frac{18}{12}$ in the table.

- One index vector form for V_0^6: vector type forms cannot satisfy the masslessness condition.

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[7]Its CTP conjugate is provided by $P=(\underline{7+,5+,5+,5+,5+,5+,5+};5-)(-2,-2,-2,-2,0,-2,-4,-2)'$.

Chirality	$\tilde{s}$	$-\tilde{s}\cdot\phi_s$	$-p_{\text{vec}}\cdot\phi_s,\ P\cdot V_0$		$(6/2)\phi_s^2,$	$-(6/2)V_0^2,$	$\Delta_6^N[\delta^6]$	$\Theta_0\,(\mathcal{P}_6^N)$
$\ominus = L$ $(---)$		$\frac{+5}{12}$	$\frac{+18}{12},$	$\frac{-3}{12}$	$\frac{21}{24}$	$\frac{-169}{24}$	$0[0]$	$\frac{+6}{12}$ (2)
$\ominus = L$ $(-++)$		0	$\frac{+18}{12},$	$\frac{-3}{12}$	$\frac{21}{24}$	$\frac{-169}{24}$	$0[0]$	$\frac{+1}{12}$ (0)
$\ominus = L$ $(+-+)$		$\frac{-1}{12}$	$\frac{+18}{12},$	$\frac{-3}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$0[0]$	$\frac{0}{12}$ (4)
$\ominus = L$ $(++-)$		$\frac{-4}{12}$	$\frac{+18}{12},$	$\frac{-3}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$0[0]$	$\frac{-3}{12}$ (0)
$\oplus = R$ $(+++)$		$\frac{-5}{12}$	$\frac{+18}{12},$	$\frac{-3}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$0[0]$	$\frac{-4}{12}$ (3)
$\oplus = R$ $(+--)$		0	$\frac{+18}{12},$	$\frac{-3}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$0[0]$	$\frac{+1}{12}$ (0)
$\oplus = R$ $(-+-)$		$\frac{+1}{12}$	$\frac{+18}{12},$	$\frac{-3}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$0[0]$	$\frac{+2}{12}$ (2)
$\oplus = R$ $(--+)$		$\frac{+4}{12}$	$\frac{+18}{12},$	$\frac{-3}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$0[0]$	$\frac{+5}{12}$ (0)

Table 8. One index spinor from V_0^6: chiralities and multiplicities, $\Psi_{[A]L} \oplus 5\left(\Phi_{[A]L} + \Phi_{[A]R}\right)$.

The remaining SU(7) non-singlet massless particles together with SU(4)′ nonsinglets are presented in appendix. The SU(7) and SU(4)′ indices are represented by A and α', respectively. Therefore, twisted sectors T_3, T_6, and T_9 may be guessed that they are not affected by Wilson lines. However, T_3 and T_9 are affected by Wilson lines because in the calculation of the phase Δ_k^0 there is an additional factor $\frac{k}{2}$ (viz. eq. (3.9)). Indeed, the inclusion of this factor $\frac{k}{2}$ correctly produces a combination of an anomaly free set.

Let us comment on the multiplicities in T_3 and T_6. In T_3, the multiplicity is 2 as mentioned in subsubsection T_3. In T_6, we note that it is a $\mathbf{Z}_2$ shift which is in fact an untwisted sector. The multiplicity of $\mathbf{Z}_2$ untwisted sector is 2, but it must include antiparticles also. Thus, the multiplicity of $\mathbf{Z}_2$ untwisted sector is 1 [27]. It is taken into account in the twisted sector $\mathbf{Z}_6$.

The SU(7) and SU(4)′ non-singlet massless states are summarized as R-handed fields in table 9. The matter fields are denoted as Ψ and vectorlike representations are represented by Φ. Some of Φ fields develop VEVs. The Φ fields can be removed at the GUT scale if correct combinations of sectors and oscillators are satisfied. The chiral fields of table 9 are

$$\Psi_R^{[ABC]} \oplus 2\,\Psi_R^{[AB]} \oplus 8\,\Psi_{[A]R} \tag{4.18}$$

which do not have the SU(7) nonabelian anomaly. In the untwisted sector, there is no $[\bar{1}]$. Thus, the family $\Psi^{[ABC]}$ from U has more suppressed $Q_{\text{em}} = \frac{2}{3}$ quark Yukawa coupling and $\Psi^{[ABC]}$ is interpreted to include the 1st family members. All $\Psi_{[A]R}$'s appear in twisted sectors. Two chiral fields, (e) and one combination from (c), form a vectorlike pair and removed at a high energy scale. The field in (d) and the remaining 7 fields from (c) are the needed 8 fields for $\Psi_{[A]R}$. $\Psi_{[A]R}$ from (e), i.e. from T_5^+, is interpreted as u^c because it can lead to the smallest Yukawa coupling among $Q_{\text{em}} = \frac{2}{3}$ quarks. The other 7 fields $\Psi_{[A]R}$ from T_3 have the same fate. Then, note that t^c and c^c are located in T_3.

Since both $\Psi^{[AB]}$ and t^c (from $\Psi_{[A]}$) arise at T_3, the cubic Yukawa coupling is possible from the BEH boson from T_6. There are many possibilities for assigning H_u and H_d of the MSSM in T_3 and T_6. We will choose a specific one in subsection 4.4.

	$\mathcal{P}\times$(rep.)	Sector	Weight	V_a^k	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
(a)	$\Psi_R^{[ABC]}$	U_1	$(\underline{----+++};+)(0^8)'$	0	$\frac{-6}{12}$	$\frac{6}{12}$	0	0	0	0	0
(b)	$2\,\Psi_R^{[AB]}$	T_3	$\left(\underline{\frac{3}{4}\frac{3}{4}\frac{-1}{4}\frac{-1}{4}\frac{-1}{4}\frac{-1}{4}\frac{-1}{4}};\frac{1}{4}\right)\left(0^6\frac{-1}{4}\frac{-1}{4}\right)'$	V_0^3	$\frac{3}{12}$	$\frac{3}{12}$	0	0	0	$\frac{-3}{12}$	$\frac{-3}{12}$
(c)	$8\,\Psi_{[A]R}$	T_3	$\left(\underline{\frac{-3}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}};\frac{-1}{4}\right)\left(0^6\frac{-1}{4}\frac{-1}{4}\right)'$	V_0^3	$\frac{9}{12}$	$\frac{-3}{12}$	0	0	0	$\frac{-3}{12}$	$\frac{-3}{12}$
(d)	$\Psi_{[A]R}$	T_5^+	$\left(\frac{11}{12}\frac{-1}{12}\frac{-1}{12}\frac{-1}{12}\frac{-1}{12}\frac{-1}{12}\frac{-1}{12};\frac{-3}{12}\right)\left(00000\frac{4}{12}\frac{-3}{12}\frac{-3}{12}\right)'$	V_+^5	$\frac{5}{12}$	$\frac{-3}{12}$	0	0	$\frac{4}{12}$	$\frac{-3}{12}$	$\frac{-3}{12}$
(e)	$\Psi_R^{[A]}$	T_6	$(\underline{-1,0^6};0)\left(0^6,\frac{1}{2},\frac{1}{2}\right)'$	V_0^6	$\frac{-12}{12}$	0	0	0	0	$\frac{6}{12}$	$\frac{6}{12}$
(f)	$40\left(\Phi_{[A]R}+\Phi_R^{[A]}\right)$	T_3	$\left(\underline{\frac{-3}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}};\frac{-1}{4}\right)\left(0^6\frac{-1}{4}\frac{-1}{4}\right)'\oplus$ H.c.	V_0^3	0	0	0	0	0	0	0
(g)	$5\left(\Phi_{[A]R}+\Phi_R^{[A]}\right)$	T_6	$(\underline{-1\,0^6}\,0)\left(0^6\,\frac{1}{2}\,\frac{1}{2}\right)'\oplus$ H.c.	V_0^6	0	0	0	0	0	0	0
(h)	$10\left(\Phi_{[A]R}+\Phi_R^{[A]}\right)$	T_5^+	$\left(\frac{11}{12}\frac{-1}{12}\frac{-1}{12}\frac{-1}{12}\frac{-1}{12}\frac{-1}{12}\frac{-1}{12};\frac{-3}{12}\right)\left(00000\frac{4}{12}\frac{-3}{12}\frac{-3}{12}\right)'\oplus$ H.c.	V_+^5	0	0	0	0	0	0	0
				$\sum_i$	$\frac{35}{12}$	$\frac{63}{12}$	0	0	$\frac{4}{12}$	$\frac{-51}{12}$	$\frac{-51}{12}$
(a')	$\Psi_{[\alpha']R}$	U_3	$(0^8)(\underline{-+++};-+++)'$	0	0	0	$\frac{1}{12}$	$\frac{-1/2}{12}$	$\frac{1/2}{12}$	$\frac{1/2}{12}$	$\frac{1/2}{12}$
(b')	$\Psi_R^{[\alpha']}$	T_1^0	$\left(\left(\frac{1}{12}\right)^7;\frac{-1}{12}\right)\left(\frac{10}{12}\frac{-2}{12}\frac{-2}{12}\frac{-2}{12};\frac{-6}{12}\frac{-2}{12}\frac{1}{12};\frac{-3}{12}\right)'$	V_0^1	$\frac{7}{12}$	$\frac{-1}{12}$	$\frac{4}{12}$	$\frac{-6}{12}$	$\frac{-2}{12}$	$\frac{1}{12}$	$\frac{-3}{12}$
(c')	$\Psi_{[\alpha']R}$	T_4^0	$\left(\left(\frac{-1}{6}\right)^7;\frac{1}{6}\right)\left(\frac{-1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{3};0\frac{1}{3}\frac{1}{3}0\right)'$	V_0^4	$\frac{-14}{12}$	$\frac{2}{12}$	$\frac{8}{12}$	0	$\frac{4}{12}$	$\frac{4}{12}$	0
(d')	$\Psi_R^{[\alpha']}$	T_5^0	$\left(\left(\frac{1}{12}\right)^7;\frac{-1}{12}\right)\left(\frac{10}{12}\frac{-2}{12}\frac{-2}{12}\frac{-2}{12};\frac{-6}{12}\frac{-2}{12}\frac{-5}{12}\frac{3}{12}\right)'$	V_0^5	$\frac{7}{12}$	$\frac{-1}{12}$	$\frac{4}{12}$	$\frac{-6}{12}$	$\frac{-2}{12}$	$\frac{-5}{12}$	$\frac{3}{12}$
(e')	$10\left(\Phi_{[\alpha']R}+\Phi_R^{[\alpha']}\right)$	T_1^0	H.c.$\oplus\left(\left(\frac{1}{12}\right)^7;\frac{-1}{12}\right)\left(\frac{10}{12}\frac{-2}{12}\frac{-2}{12}\frac{-2}{12};\frac{-6}{12}\frac{-2}{12}\frac{1}{12};\frac{-3}{12}\right)'$	V_0^1	0	0	0	0	0	0	0
(f')	$5\left(\Phi_{[\alpha']R}+\Phi_R^{[\alpha']}\right)$	T_4^0	$\left(\left(\frac{-1}{6}\right)^7;\frac{1}{6}\right)\left(\frac{-1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{3};0\frac{1}{3}\frac{1}{3}0\right)'\oplus$ H.c.	V_0^4	0	0	0	0	0	0	0
(g')	$10\left(\Phi_{[\alpha']R}+\Phi_R^{[\alpha']}\right)$	T_5^0	H.c.$\oplus\left(\left(\frac{1}{12}\right)^7;\frac{-1}{12}\right)\left(\frac{10}{12}\frac{-2}{12}\frac{-2}{12}\frac{-2}{12};\frac{-6}{12}\frac{-2}{12}\frac{-5}{12}\frac{3}{12}\right)'$	V_0^5	0	0	0	0	0	0	0
(h')	$7\left(\Phi_{[\alpha']R}+\Phi_R^{[\alpha']}\right)$	T_6	$(0^8)(\underline{1\,0\,0\,0};0\,0\,\frac{-1}{2}\frac{-1}{2})'\oplus$ H.c.	V_0^4	0	0	0	0	0	0	0
				$\sum_i$	0	0	$\frac{17}{12}$	$\frac{-25/2}{12}$	$\frac{1/2}{12}$	$\frac{1/2}{12}$	$\frac{1/2}{12}$

Table 9. Non-singlet $SU(7)_{\mathrm{anti2}}$ spectra represented as R-handed chiral fields. H.c. means the opposite numbers of those in the same site.

4.3 U(1) charges and anomalous U(1)

We use the normalization that the index ℓ for fundamental representation $\mathbf{N}$ of SU(N) is 1. Then, the indices of some representations are [28, 29],

$$\text{SU}(N): \ \ell(\mathbf{N}) = 1, \ \ell([2]) = N - 2, \ \ell([3]) = \frac{(N-2)(N-3)}{2},$$

$$\ell(\text{Adj.}) = 2N, \ \ell(\{2\}) = N + 2,$$

$$\ell(\{3\}) = \frac{(N+2)(N+3)}{2}, \tag{4.19}$$

$$\text{U}(1)_{\text{em}}: \ \ell(Q_{\text{em}}) = 2Q_{\text{em}}^2.$$

where [2] means the dimension $\left(\frac{\mathbf{N(N-1)}}{\mathbf{2!}}\right)$ with two antisymmetric indices, {2} means the dimension $\left(\frac{\mathbf{N(N+1)}}{\mathbf{2!}}\right)$ with two symmetric indices, [3] means the dimension $\left(\frac{\mathbf{N(N-1)(N-2)}}{\mathbf{3!}}\right)$ with three antisymmetric indices, etc. For SU(7), the index of $\Psi^{[ABC]}$ is 10 and the index of $\Psi^{[AB]}$ is 5. We need these numbers for the contribution of $\Psi^{[ABC]}$ and $\Psi^{[AB]}$ to the U(1)-SU(7)^2 anomalies.

We choose the following seven U(1) directions, in terms of Q_i [26],

$$Q_1 = (1\,1\,1\,1\,1\,1\,1\,0)\,\left(0^8\right)'$$

$$Q_2 = (0\,0\,0\,0\,0\,0\,0\,1)\,\left(0^8\right)'; \quad Q_2' = \frac{103}{35}Q_1 + Q_6 + Q_7,$$

$$Q_3 = \left(0^8\right)(1\,1\,1\,1\,0\,0\,0\,0)'; \quad Q_3' = \frac{103}{63}Q_2 + Q_6 + Q_7,$$

$$Q_4 = \left(0^8\right)(0\,0\,0\,0\,1\,0\,0\,0)'; \quad Q_4' = \frac{7}{34}Q_3 + Q_5, \tag{4.20}$$

$$Q_5 = \left(0^8\right)(0\,0\,0\,0\,0\,1\,0\,0)'; \quad Q_5' = -\frac{7}{25}Q_4 + Q_5,$$

$$Q_6 = \left(0^8\right)(0\,0\,0\,0\,0\,0\,1\,0)'; \quad Q_6' = \frac{11}{255}Q_4 + Q_5 + \frac{2}{51}(Q_6 + Q_7),$$

$$Q_7 = \left(0^8\right)(0\,0\,0\,0\,0\,0\,0\,1)',$$

where the redefined primed U(1) combinations give the identical sum for the SU(7) and SU(4)$'$ anomalies. Note that Q_6' itself is anomaly free. In terms of $Q_2', \cdots, Q_5'$, we can redefine anomaly free combinations. Six nonabelian-anomaly-free U(1) combinations are denoted with tilde,

$$\tilde{Q}_1 = Q_2' - Q_3',$$
$$\tilde{Q}_2 = 2(Q_2' + Q_3') - Q_4',$$
$$\tilde{Q}_3 = Q_4' - Q_5',$$
$$\tilde{Q}_4 = 2Q_2' + \frac{1}{2}Q_4' - Q_5', \tag{4.21}$$
$$\tilde{Q}_5 = Q_6' = \frac{11}{255}Q_4 + Q_5 + \frac{2}{51}(Q_6 + Q_7),$$
$$\tilde{Q}_6 = Q_6 - Q_7.$$

The remaining U(1) must carry anomaly, which can be represented as

$$Q_a = Q_2' + aQ_3' + bQ_4' + cQ_5'. \tag{4.22}$$

Parameters a, b and c are determined by how one breaks the SU(7)$_{\text{anti2}}$, which defines the electroweak hypercharges or the electromagnetic charges of the SM particles. Note that SU(5)$_{\text{GG}}$ subgroup of SO(10) use the U(1) direction, or equivalently the $B - L$ direction $(1\,1\,1\,0\,0\,0\,0; 0)(0^8)'$. The SU(5)$_{\text{flip}}$ subgroup of SO(10) use instead the U(1) direction $(1\,1\,1\,1\,1\,0\,0; 0)(0^8)'$. SU(7)$_{\text{anti2}}$ uses $(1\,1\,1\,1\,1\,0\,0; 0)(0^8)'$ and in addition $(1\,1\,1\,0\,0\,1\,1; 0)(0^8)'$. These two directions of SU(7)$_{\text{anti2}}$ can be fixed only after SU(7)$_{\text{anti2}}$ is broken down to the SM gauge group. The U(1)$_X$ of SU(7)$_{\text{anti2}}$ is given by

$$Q_X = Q_1 + Q_2 - \frac{1}{3}(Q_3 + Q_4) + Q_5 + Q_6 + Q_7, \tag{4.23}$$

which is anomaly-free. The orthogonalities of Q_a with Q_X and the above two SU(7)$_{\text{anti2}}$ directions determine three parameters of (4.22).

4.4 Yukawa couplings

For Yukawa couplings, we must satisfy all the symmetries of low energy effective fields and the selection rules in the orbifold compactification. For the fields from twisted sectors, the Yukawa coupling structure is simpler than those involving the untwisted sector fields. Consider for example a vectorlike set from T_6 in table 8. For the coupling, $\Phi_R^{[A]} \cdot \Phi_{[A]R}$, we must satisfy the selection rules for the right-mover and for the left-mover conditions. For the right-mover condition, 36 times $p_{\text{vec}} \cdot \phi_s$ is 0 mod. integer. It is satisfied for the coupling $\Phi_R^{[A]} \cdot \Phi_{[A]R}$. For the left-mover condition, 36 times $P \cdot V$ is 0 mod. integer. It is also satisfied for the coupling $\Phi_R^{[A]} \cdot \Phi_{[A]R}$.

As commented above, t, c, t^c and c^c quarks are located T_3. On the other hand, u is located in U and u^c is located in T_5^+. Order 1 Yukawa coupling of the form $\mathbf{21}(T_3) \times \overline{\mathbf{7}}(T_3) \times \overline{\mathbf{7}}(T_6)$ is possible if H_u in T_6 is not removed at the GUT scale. This requires a hierarchy of scales,

$$M_s \ll M_3 \tag{4.24}$$

where M_3 is a vacuum expectation value of a singlet $\mathbf{1}$ in T_3,

$$\langle \mathbf{1}(T_3) \rangle = M_3. \tag{4.25}$$

Eight $\Psi_{[A]}$'s and five $(\Phi^{[A]} + \Phi_{[A]})$'s have the following Yukawa couplings

$$\mathbf{1}(T_3)\Psi_{[A]}^i \Phi_\mu^{[A]}, \ M_s \Phi_{[A]}^i \Phi_\mu^{[A]}; \ i = 1, \cdots, 8, \ \mu = 1, \cdots, 5. \tag{4.26}$$

Due to the hierarchy (4.25), five $\Phi^{[A]}$'s of T_6 are paired with five $\Psi_{[A]}$'s from T_3. Three $\Psi_{[A]}$'s of T_3 and five $\Phi_{[A]}$'s of T_6 remain light at this stage. Introducing an angle $\tan\theta = \frac{M_s}{M_3}$, five BEH fields

$$\cos\theta \Phi_{[A]}^\mu - c_{\mu i} \sin\theta \Psi_{[A]}^i. \tag{4.27}$$

obtain mass of order $M_3 \sin\theta$. Because of the democracy of couplings, four out of five $\Phi_{[A]}$'s of T_6 remain light. Collecting light $\overline{\bf 7}$'s up to this stage, we have

$$\text{Three } \Psi_{[A]}(T_3), \ \Psi_{[A]}(T_5^+), \ \text{Four } \Phi_{[A]}(T_6). \tag{4.28}$$

At the $\mathrm{SU}(7)_{\mathrm{anti2}}$ level, still we have eight light $\overline{\bf 7}$'s. Thus, the BEH fields are located at T_6. Depending on θ, the BEH fields contain small components from T_3, viz. eq. (4.27). We interpret this angle as the ratio $m_c/m_t = \tan\theta$. The t-quark Yukawa coupling in $\mathrm{SU}(5)_{\mathrm{anti2}}$ is

$$T_3^{\bf 21} T_3^{\overline{\bf 7}} T_{6,\mathrm{BEH}}^{\overline{\bf 7}} \ (t \text{ mass}). \tag{4.29}$$

The BEH fields giving mass to the b-quark are located in T_3. There are 40 $\Phi^{[A]}$ fields in (f) of table 9. Because of the mass democracy, there can remain some light fields. Most of them will be removed when $\mathrm{SU}(7)_{\mathrm{anti2}}$ is broken, but we need one $\Phi^{[A]}$ for the $Q_{\mathrm{em}} = -\frac{1}{3}$ quark masses. For the b-quark mass, we need the coupling

$$\sim \frac{1}{M_s} T_3^{\bf 21} T_3^{\bf 21} T_{3,\mathrm{BEH}}^{\bf 21} T_{3,\mathrm{BEH}}^{\bf 7}. \tag{4.30}$$

Thus, the b-quark mass is expected to be much smaller than the t-quark mass, $O(\langle T_{3,\mathrm{BEH}}^{\bf 21} \rangle \langle T_{3,\mathrm{BEH}}^{\bf 7} \rangle / M_s \langle T_{6,\mathrm{BEH}}^{\bf 7} \rangle)$, where $\langle T_{3,\mathrm{BEH}}^{\bf 21} \rangle$ is the SU(5) splitting VEV $\langle \Phi^{[67]} \rangle$. Thus, we expect $m_b/m_t \sim \frac{\langle \Phi^{[67]} \rangle}{M_s \tan\beta}$. Even if $\tan\beta = O(1)$, we can fit m_b/m_t to the observed value by appropriately tuning $\langle \Phi^{[67]} \rangle$. A similar suppression occurs for the second family members.

For the 1st family members, the story is different. This is because d^c appears in $\bf 35$ of $\mathrm{SU}(7)_{\mathrm{anti2}}$, appearing in U. The d-quark mass may arise from

$$\sim \frac{1}{M_s^2} {\bf 35}_{U_1} {\bf 35}_{U_1} {\bf 7}_{T_3,\mathrm{BEH}} \langle {\bf 1}_{T_5^0,\mathrm{BEH}} \rangle \langle {\bf 1}_{T_6,\mathrm{BEH}} \rangle. \tag{4.31}$$

Let us check whether this coupling is present. $p_{\mathrm{spin}} \cdot \phi_s = \frac{10}{12}$ for ${\bf 35}_{U_1} {\bf 35}_{U_1}$. From table 7, we note $p_{\mathrm{vec}} \cdot \phi_s = \frac{-9}{12}$ for ${\bf 7}_{T,BEH}$. We need the remaining singlet combinations to provide $\frac{-1}{12}$. Since we do not list singlets here, it cannot be shown at this stage, but there are numerous singlets and we assume that it is possible. For the left-mover conditions which are the gauge invariance conditions, the above coupling satisfies the condition. For the u-quark mass, we must consider a higher dimensional operator than eq. (4.31),

$$\sim \frac{1}{M_s^3} {\bf 35}_{U_1} \overline{\bf 7}_{T_5^+} \overline{\bf 7}_{T_6,\mathrm{BEH}} \langle {\bf 1}_{T_4^-,\mathrm{BEH}} \rangle \langle \overline{\bf 7}_{T_3,\mathrm{BEH}} \rangle \langle {\bf 1}_{T_1^0,\mathrm{BEH}} \rangle. \tag{4.32}$$

It is because u appears in U, requiring another VEV $\langle {\bf 7} \rangle$ and also a field carrying another Wilson shift $-$ to remove the Wilsone shift $+$. Thus, there exists a possibility that $m_u < m_d$. It is a new mechanism for the inverted 1st family quark mass structure.

4.5 Missing partner mechanism

At the $\mathrm{SU}(7)_{\mathrm{anti2}}$ level, we assume that one pair $\Phi_{[A]} = \overline{\bf 7}$ (giving mass to the t quark) and $\Phi^{[A]} = {\bf 7}$ (giving mass to the b quark) are surviving down to low energy. The missing partner mechanism is discussed in this setup.

In a sense, the absence of $\mathbf{7}_{\mathrm{BEH}} \cdot \overline{\mathbf{7}}_{\mathrm{BEH}}$ is not guaranteed at field theory level. In the MSSM, it is related to the μ problem, "Why there does not exists $H_u H_d$ at the GUT scale" [30]. Some interesting solutions with hidden-sector quarks exist [31, 32]. These solutions are based on the Peccei-Quinn (PQ) symmetry with the very light axion [33]. In the effective SUSY framework language, the superpotential W should not allow $\mu H_u H_d$ by assigning a nonvanishing PQ quantum number to the combination $H_u H_d$. But, the global PQ symmetry is spoiled by gravity [34–37]. We may resort to some discrete subgroup, e.g. matter parity [38], of a U(1) gauge symmetry [39]. Suppose assigning the mother gauge charges of H_u and H_d as $Q(H_u) = Q(H_d) = 1$ such that the matter parity forbids $H_u H_d$ at the GUT scale. But we must allow the t-quark mass at the cubic order. It means, t and t^c carry the mother gauge charge, $Q(t) = Q(t^c) = -\frac{1}{2}$ for example. In string compactification, we do not worry the gravity spoil of global symmetries. Just string selection rules are enough to consider the coupling. It has been noted that some string compactifications do not lead to quadratic term in W as in $\mathbf{Z}_3$ [32], but in non-prime orbifolds the absence of $\overline{\mathbf{7}} \cdot \mathbf{7}$ must be studied case by case. In our example discussed above, the coupling $\mathbf{7}_{\mathrm{BEH}} \cdot \overline{\mathbf{7}}_{\mathrm{BEH}}$ is not allowed because $\mathbf{7}_{\mathrm{BEH}}$ is located in T_3, and $\overline{\mathbf{7}}_{\mathrm{BEH}}$ is located in T_6. But, GUTs need the doublet-triplet splitting that in the same GUT scale BEH multiplet the colored fields are superheavy while H_u and H_d remain light. In the absence of the coupling $\mathbf{7}_{\mathrm{BEH}} \cdot \overline{\mathbf{7}}_{\mathrm{BEH}}$, as discussed above for $\mathbf{7}_{\mathrm{BEH}} \in T_3$ and $\overline{\mathbf{7}}_{\mathrm{BEH}} \in T_6$, the missing partner mechanism of $\mathrm{SU}(7)_{\mathrm{anti2}}$ is realized. Consider the coupling,

$$
\begin{aligned}
&\frac{1}{M_s}\, \epsilon^{ABCDEFG}\, \Phi_{[AB]} \Phi_{[CD]} \Phi_{[EF]} \Phi_{[G]}, \text{ and/or} \\
&\frac{1}{M_s^2}\, \epsilon^{ABCDEFG}\, \Phi_{[AB]} \Phi_{[CD]} \Phi_{[E]} \langle \Phi'_{[F]} \rangle \langle \Phi''_{[G]} \rangle,
\end{aligned}
\tag{4.33}
$$

where Φ' and Φ'' obtain string scale VEVs, and $\Phi_{[AB]} = \Phi_{[45]}$ of eq. (4.33) are essential for separating the color and weak parts. The BEH bosons H_u and H_d are in $\Phi_{[A]}$ and $\Phi^{[A]}$, respectively. Equation (4.33) makes colored scalars heavy, viz.

$$
\begin{aligned}
&\Phi^{[23]} \Phi^{[1]} \langle \Phi^{[45]} \rangle \langle \Phi^{[67]} \rangle, \\
&\Phi^{[23]} \Phi^{[1]} \langle \Phi^{[45]} \rangle \langle \Phi'^{[6]} \rangle \langle \Phi''^{[7]} \rangle,
\end{aligned}
\tag{4.34}
$$

where $\Phi^{[1]}$ is $Q_{\mathrm{em}} = -\frac{1}{3}$ colored boson whose partner is $\Phi^{[23]}$. The color-weak separating VEV $\langle \Phi^{[45]} \rangle \approx M_{\mathrm{GUT}} \sim M_s$ is the key making the colored scalar heavy. In the same multiplet $\Phi^{[A]}$, the BEH doublet H_d is present at $\Phi^{[4]}$ and $\Phi^{[5]}$. But, the indices 4 and 5 are already used for the GUT scale VEV, hence H_d does not find a partner in $\Phi^{[AB]}$. This is the missing partner mechanism we realize in $\mathrm{SU}(7)_{\mathrm{anti2}}$.

5 Conclusion

In this paper, we proposed unification of families in string compactification. Here, we suggested anti-SU(N) scheme [7] where the adjoint representation of SU(N) is not needed for breaking the GUT group down to $\mathrm{SU}(3)_c \times \mathrm{U}(1)_{\mathrm{em}}$. It is pointed out that the anti-SU(N) scheme has a merit in string compactification and can even save the SO(32) heterotic

string theory for many phenomenological purposes. The minimal model for UGUTF is SU(7) GUT with the representation $[3] + 2[2] + 8[\bar{1}]$. We show it explicitly that this representation is realized in the $\mathbf{Z}_{12-I}$ orbifold compactification of $E_8 \times E_8'$ heterotic string.

The large top quark mass is possible in $\text{SU}(7)_{\text{anti2}}$, where t^c is in $\overline{\mathbf{7}} = [\bar{1}]$. In the example discussed, $\overline{\mathbf{7}}$'s in T_3 contain t^c and c^c. The cubic coupling $\Psi^{[AB]}_{R,T_3} \overline{\Psi}_{[A]R,T_3} \Phi_{[B]R,T_6,BEH}$ gives a dimension-3 superpotential for the t mass. This cubic coupling is the only possible dimension 3 superpotential in our model and hence only m_t is expected to be of order the electroweak scale. Other fermion masses are much smaller than m_t. We also presented an argument why there is an inverted mass ratio in the u-quark family. It is because u is located in U which requires an additional VEV for $\langle \overline{7}_{\text{BEH}} \rangle$. Finally, we presented the missing partner mechanism in string compactification. The key assumption, the absence of the coupling $7_{\text{BEH}} \cdot \overline{7}_{\text{BEH}}$ in the superpotential, is achieved here by locating 7_{BEH} and $\overline{7}_{\text{BEH}}$ separately in T_3 and T_6. Then, it is shown that the missing partner mechanism works for H_u and H_d in $\overline{7}_{\text{BEH}}$ and 7_{BEH}. The colored particles in $\overline{7}_{\text{BEH}}$ and 7_{BEH} find their partners in $\overline{21}_{\text{BEH}}$ and 21_{BEH} and obtain superheavy masses.

Here, we neglected the details of singlet vacuum expectation values, toward removing vectorlike representations, though the singlet VEVs have been widely used in other string compactification papers toward the MSSM [41–44]. In this sense, the $\text{SU}(7)_{\text{anti2}}$ presented in this paper may be an aethetic choice toward a desirable UGUTF. Other physics implications such as the quark and lepton mass textures, dark matter, and very light axions, including SU(7) singlet representations, will be presented elsewhere.

Acknowledgments

A   SU(7)$_{\text{anti2}}$GUT in Z$_{12-I}$

In this appendix, we list up the remaining non-singlet states not included in section 4 where U, T_3, and T_6 are discussed. We only show the sectors containing SU(7) or SU(4)$'$ nonsinglets. They are listed up in the order of T_4, T_1, T_2, and T_5.

A.1   SU(7)$_{\text{anti 2}}$ spectra

Here, we list $\text{SU}(7)_{\text{anti 2}}$ spectra not listed in section 4.

A.1.1   Twisted sector T_1 $\left(\delta^1 = 1/12 \right)$

The masslessness condition for $2c_1$ requires $(p_{\text{vec}} + \phi_s)^2 = \frac{66}{144}$−(oscillator contributions). Oscillator conribution from the right mover is $2\delta^1 = \frac{24}{144} = 2 \cdot \frac{1}{12}$.

- One index spinor form for V_0^1: for the spinor,

Chirality	$\tilde{s}$	$-\tilde{s}\cdot\phi_s$	$-p_{\text{vec}}\cdot\phi_s,\ P\cdot V_0$	$(1/2)\phi_s^2,\ -(1/2)V_0^2$	$\Delta_1^N[\delta^1],\Delta_1^N[-\delta^1]$	$\Theta_0\,(\mathcal{P}_4^N)$
$\ominus = L$ $(---)$	$\frac{+5}{12}$	$\frac{0}{12},\ \frac{+6}{12}$	$\frac{21}{72}\quad\frac{-457}{72}$	$\frac{-2/3}{12}[\frac{+1}{12}],\ \frac{-2/3}{12}[\frac{-1}{12}]$	$\frac{+12}{12}\,(4),\ \frac{+10}{12}\,(2)$
$\ominus = L$ $(-++)$	0	$\frac{0}{12},\ \frac{+6}{12}$	$\frac{21}{72}\quad\frac{-169}{72}$	$\frac{+2/3}{12}[\frac{+1}{12}],\ \frac{+2/3}{12}[\frac{-1}{12}]$	$\frac{+7}{12}\,(0),\ \frac{+5}{12}\,(0)$
$\ominus = L$ $(+-+)$	$\frac{-1}{12}$	$\frac{0}{12},\ \frac{+6}{12}$	$\frac{21}{72}\quad\frac{-457}{72}$	$\frac{-2/3}{12}[\frac{+1}{12}],\ \frac{-2/3}{12}[\frac{-1}{12}]$	$\frac{+6}{12}\,(2),\ \frac{+4}{12}\,(3)$
$\ominus = L$ $(++-)$	$\frac{-4}{12}$	$\frac{0}{12},\ \frac{+6}{12}$	$\frac{21}{72}\quad\frac{-457}{72}$	$\frac{-2/3}{12}[\frac{+1}{12}],\ \frac{-2/3}{12}[\frac{-1}{12}]$	$\frac{+3}{12}\,(0),\ \frac{+1}{12}\,(0)$
$\oplus = L$ $(+++)$	$\frac{-5}{12}$	$\frac{0}{12},\ \frac{+6}{12}$	$\frac{21}{72}\quad\frac{-457}{72}$	$\frac{-2/3}{12}[\frac{+1}{12}],\ \frac{-2/3}{12}[\frac{-1}{12}]$	$\frac{+2}{12}\,(2),\ \frac{0}{12}\,(4)$
$\oplus = L$ $(+--)$	0	$\frac{0}{12},\ \frac{+6}{12}$	$\frac{21}{72}\quad\frac{-457}{72}$	$\frac{-2/3}{12}[\frac{+1}{12}],\ \frac{-2/3}{12}[\frac{-1}{12}]$	$\frac{+7}{12}\,(0),\ \frac{+5}{12}\,(0)$
$\oplus = L$ $(-+-)$	$\frac{+1}{12}$	$\frac{0}{12},\ \frac{+6}{12}$	$\frac{21}{72}\quad\frac{-457}{72}$	$\frac{-2/3}{12}[\frac{+1}{12}],\ \frac{-2/3}{12}[\frac{-1}{12}]$	$\frac{+8}{12}\,(3),\ \frac{+6}{12}\,(2)$
$\oplus = L$ $(--+)$	$\frac{+4}{12}$	$\frac{0}{12},\ \frac{+6}{12}$	$\frac{21}{72}\quad\frac{-457}{72}$	$\frac{-2/3}{12}[\frac{+1}{12}],\ \frac{-2/3}{12}[\frac{-1}{12}]$	$\frac{+11}{12}\,(0),\ \frac{+9}{12}\,(0)$

Table 10. One index spinor from V_0^1: chiralities and multiplicities are $11\left(\Phi_{[A]L}+\Phi_{[A]R}\right)$.

$$V_0 = \left(\frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}; \frac{+5}{12}\right)\left(\frac{4}{12}, \frac{4}{12}, \frac{4}{12}, \frac{4}{12}, 0, \frac{4}{12}, \frac{7}{12}, \frac{3}{12}\right)',\ V_0^2 = \frac{338}{144}$$

$$P = (\underline{-+++++}+;-)(--------)',\ P\cdot V_0 = \frac{-30}{12},$$

$$P + V_0 = \left(\frac{-11}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}; \frac{-1}{12}\right)\left(\frac{-2}{12}, \frac{-2}{12}, \frac{-2}{12}, \frac{-2}{12}, \frac{-6}{12}, \frac{-2}{12}, \frac{1}{12}, \frac{-3}{12}\right)',$$

$$(P + V_0)\cdot a_3 \neq 0,$$

$$(A.1)$$

we have $(P + V_0)^2 = \frac{194}{144}$. The masslessness condition for $2\tilde{c}_1$ requires $(P + V^a)^2 = \frac{210}{144}-$(oscillator contributions). Note that $(1/2)\phi_s^2 - (1/2)V_0^2 = -\frac{148}{12} = -13 + \frac{+2/3}{12}$. The oscillator contribution is $\frac{16}{144} = 2\cdot\frac{2/3}{12}$. So we need $-\Delta_4^N$ to cancel $\frac{+2/3}{12}$ in $(1/2)\phi_s^2 - (1/2)V_0^2$. These are shown in table 10.

- One index vector form for V_0^2: we have

$$2V_0 = \left(\frac{-5}{6}, \frac{-5}{6}, \frac{-5}{6}, \frac{-5}{6}, \frac{-5}{6}, \frac{-5}{6}, \frac{-5}{6}; \frac{+5}{6}\right)\left(\frac{4}{6}, \frac{4}{6}, \frac{4}{6}, \frac{4}{6}, 0, \frac{4}{6}, \frac{7}{6}, \frac{3}{6}\right)',\ V_0^2 = \frac{338}{144},$$

$$P = (1,1,1,1,1,1,1;-1)(\underline{0,-1,-1,-1},0,-1,-1,-1)',\ P\cdot V_0 = \frac{-65}{12}$$

$$P + 2V_0 = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}; \frac{-1}{6}\right)\left(\underline{\frac{2}{3}, \frac{-1}{3}, \frac{-1}{3}, \frac{-1}{3}}, 0, \frac{-1}{3}, \frac{1}{6}, \frac{-1}{2}\right)',\ (P_2 + 2V_0)\cdot a_3 \neq 0.$$

$$(A.2)$$

Note that $(P + 2V_0)^2 = \frac{50}{36} = \frac{216}{144} - \frac{16}{144}$, which needs $\frac{16}{144} = 2\cdot\frac{2/3}{12}$ as an oscillator contribution with $\Delta_2^N = \frac{2/3}{12}$. We have $(2/2)\phi_s^2 - (2/2)V_0^2 = -\frac{296}{144} = -2 - \frac{2/3}{12}$. So, we need $+\Delta_2^N$. Massless states are presented in table 11.

A.1.2 Twisted sector T_5 $(\delta^5 = 1/12)$

- One index spinor form for V_+^5:

Chirality		$\tilde{s}$	$-\tilde{s}\cdot\phi_s$	$-p_{\mathrm{vec}}\cdot\phi_s,\ P\cdot V_0$	$(2/2)\phi_s^2,\ -(2/2)V_0^2,$	$\Delta_2^N[\delta^2],\Delta_2^N[-\delta^2]$	$\Theta_0\,(\mathcal{P}_2^N)$
$\ominus=L$	$(---)$	$\frac{+5}{12}$	$\frac{+9}{12},\frac{-3}{12}$	$\frac{21}{24}$ $\frac{-169}{24}$	$\frac{+2/3}{12}[\frac{+2}{12}],\frac{+2/3}{12}[\frac{-2}{12}]$	$\frac{+1}{12}(0),\frac{+9}{12}(0)$
$\ominus=L$	$(-++)$	0	$\frac{+9}{12},\frac{-3}{12}$	$\frac{21}{24}$ $\frac{-169}{24}$	$\frac{+2/3}{12}[\frac{+2}{12}],\frac{+2/3}{12}[\frac{-2}{12}]$	$\frac{+8}{12}(3),\frac{+4}{12}(3)$
$\ominus=L$	$(+-+)$	$\frac{-1}{12}$	$\frac{+9}{12},\frac{-3}{12}$	$\frac{21}{24}$ $\frac{-165}{24}$	$\frac{+2/3}{12}[\frac{+2}{12}],\frac{+2/3}{12}[\frac{-2}{12}]$	$\frac{+7}{12}(0),\frac{+3}{12}(0)$
$\ominus=L$	$(++-)$	$\frac{-4}{12}$	$\frac{+9}{12},\frac{-3}{12}$	$\frac{21}{24}$ $\frac{-165}{24}$	$\frac{+2/3}{12}[\frac{+2}{12}],\frac{+2/3}{12}[\frac{-2}{12}]$	$\frac{+4}{12}(3),\frac{0}{12}(4)$
$\oplus=R$	$(+++)$	$\frac{-5}{12}$	$\frac{+9}{12},\frac{-3}{12}$	$\frac{21}{24}$ $\frac{-165}{24}$	$\frac{+2/3}{12}[\frac{+2}{12}],\frac{+2/3}{12}[\frac{-2}{12}]$	$\frac{+3}{12}(0),\frac{-1}{12}(0)$
$\oplus=R$	$(+--)$	0	$\frac{+9}{12},\frac{-3}{12}$	$\frac{21}{24}$ $\frac{-165}{24}$	$\frac{+2/3}{12}[\frac{+2}{12}],\frac{+2/3}{12}[\frac{-2}{12}]$	$\frac{+8}{12}(3),\frac{+4}{12}(3)$
$\oplus=R$	$(-+-)$	$\frac{+1}{12}$	$\frac{+9}{12},\frac{-3}{12}$	$\frac{21}{24}$ $\frac{-165}{24}$	$\frac{+2/3}{12}[\frac{+2}{12}],\frac{+2/3}{12}[\frac{-2}{12}]$	$\frac{+9}{12}(0),\frac{+5}{12}(0)$
$\oplus=R$	$(--+)$	$\frac{+4}{12}$	$\frac{+9}{12},\frac{-3}{12}$	$\frac{21}{24}$ $\frac{-165}{24}$	$\frac{+2/3}{12}[\frac{+2}{12}],\frac{+2/3}{12}[\frac{-2}{12}]$	$\frac{+12}{12}(4),\frac{+8}{12}(3)$

Table 11. One index vector from V_0^2: chiralities and multiplicities, $13\left(\Phi_{[A]L}+\Phi_{[A]R}\right)$.

Chirality		$\tilde{s}$	$-\tilde{s}\cdot\phi_s$	$p_{\mathrm{vec}}\cdot\phi_s,\ P\cdot V_+$	$(5/2)\phi_s^2,\ -(5/2)V_+^2,$	$\Delta_4^N[\delta^5],\Delta_5^N[-\delta^5]$	$\Theta_0\,(\mathcal{P}_5^N)$
$\ominus=L$	$(---)$	$\frac{+5}{12}$	$\frac{+18}{12},\frac{-3}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{+9}{12}(0),\frac{+7}{12}(0)$
$\ominus=L$	$(-++)$	0	$\frac{+18}{12},\frac{-3}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{+4}{12}(3),\frac{+2}{12}(2)$
$\ominus=L$	$(+-+)$	$\frac{-1}{12}$	$\frac{+18}{12},\frac{-3}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{+3}{12}(0),\frac{+1}{12}(0)$
$\ominus=L$	$(++-)$	$\frac{-4}{12}$	$\frac{+18}{12},\frac{-3}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{0}{12}(4),\frac{-2}{12}(2)$
$\oplus=R$	$(+++)$	$\frac{-5}{12}$	$\frac{+18}{12},\frac{-3}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{-1}{12}(0),\frac{-3}{12}(0)$
$\oplus=R$	$(+--)$	0	$\frac{+18}{12},\frac{-3}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{+4}{12}(3),\frac{+2}{12}(2)$
$\oplus=R$	$(-+-)$	$\frac{+1}{12}$	$\frac{+18}{12},\frac{-3}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{+5}{12}(0),\frac{+3}{12}(0)$
$\oplus=R$	$(--+)$	$\frac{+4}{12}$	$\frac{+18}{12},\frac{-3}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{+8}{12}(3),\frac{+6}{12}(2)$

Table 12. One index spinor from V_+^5: chiralities and multiplicities, $\Psi_{L,1}^{[A]}\oplus 10\left(\Phi_{L,1}^{[A]}+\Phi_{R,1}^{[A]}\right)$.

$$5V_+ = \left(\frac{-5}{12},\frac{-5}{12},\frac{-5}{12},\frac{-5}{12},\frac{-5}{12},\frac{-5}{12},\frac{-5}{12};\frac{45}{12}\right)\left(0,0,0,0,0,\frac{20}{12},\frac{135}{12},\frac{-45}{12}\right)',\ V_+^2 = \frac{914}{144},$$

$$P = (\underline{-++++++};7-)(0,0,0,0,0,-2,-11,4)',\ P\cdot V_+ = \frac{-375}{12} = \frac{-3}{12},$$

$$P+5V_+ = \left(\frac{-11}{12},\frac{1}{12},\frac{1}{12},\frac{1}{12},\frac{1}{12},\frac{1}{12},\frac{1}{12};\frac{3}{12}\right)\left(0,0,0,0,0,\frac{-4}{12},\frac{3}{12},\frac{3}{12}\right)',\ (P+5V_0)\cdot a_3 \neq 0,$$

$$\tag{A.3}$$

which gives $(P+5V_+)^2 = \frac{170}{144}$, and the oscillator contribution of $\frac{40}{144} = 2\frac{5/3}{12}$ is needed. Note that $(5/2)\phi_s^2 - (5/2)V_+^2 = \frac{-18-4/3}{12}$, which means we select $\Delta_5^N[\delta^5]$ and $\Delta_5^N[-\delta^5]$. The total gauge shift is even but it is even due to (odd shift) from E_8 and (odd shift) from E_8'. Thus, the gauge quantum numbers must be the opposite of eq. (A.8). These are shown in table 12.

A.2 SU(4)$'$ spectra from twisted sectors T

A.2.1 Twisted sector $T_6\,(\delta^6=0)$

- Hidden index vector form for V_0^6: we have

Chirality	$\tilde{s}$	$-\tilde{s}\cdot\phi_s$	$-p_{\text{vec}}\cdot\phi_s,\ P\cdot V_0$	$(6/2)\phi_s^2,$	$-(6/2)V_0^2,$	$\Delta_6^N[\delta^6]$	$\Theta_0\left(\mathcal{P}_6^N\right)$
$\ominus=L$ $(---)$	$\frac{+5}{12}$	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-169}{24}$	$0[0]$	$\frac{+5}{12}\ (0)$
$\ominus=L$ $(-++)$	0	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-169}{24}$	$0[0]$	$\frac{0}{12}\ (4)$
$\ominus=L$ $(+-+)$	$\frac{-1}{12}$	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$0[0]$	$\frac{-1}{12}\ (0)$
$\ominus=L$ $(++-)$	$\frac{-4}{12}$	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$0[0]$	$\frac{-4}{12}\ (3)$
$\oplus=R$ $(+++)$	$\frac{-5}{12}$	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$0[0]$	$\frac{-5}{12}\ (0)$
$\oplus=R$ $(+--)$	0	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$0[0]$	$\frac{0}{12}\ (4)$
$\oplus=R$ $(-+-)$	$\frac{+1}{12}$	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$0[0]$	$\frac{+1}{12}\ (0)$
$\oplus=R$ $(--+)$	$\frac{+4}{12}$	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$0[0]$	$\frac{+4}{12}\ (3)$

Table 13. Hidden index spinor from V_0^6: chiralities and multiplicities, $7\left(\Phi_L^{[\alpha']}+\Phi_R^{[\alpha']}\right)$.

$$6V_0 = \left(\frac{-5}{2},\frac{-5}{2},\frac{-5}{2},\frac{-5}{2},\frac{-5}{2},\frac{-5}{2},\frac{-5}{2};\frac{+5}{2}\right)\left(\frac{4}{2},\frac{4}{2},\frac{4}{2},\frac{4}{2},0,\frac{4}{2},\frac{7}{2},\frac{3}{2}\right)',\ V_0^2=\frac{338}{144}$$

$$P = (5+,5+,5+,5+,5+,5+,5+;5-)(\underline{-1,-2,-2,-2},0,-2,-3,-1)',\ P\cdot V_0=\frac{-160}{12},\ \text{(A.4)}$$

$$P+6V_0 = \left(0^8\right)\left(\underline{1,0,0,0},0,0,\frac{1}{2},\frac{1}{2}\right)',\ (P+6V_0)\cdot a_3=0,$$

which saturates the needed masslessness condition $\frac{3}{2}$ of T_6, which are tabulated in table 13. P is odd under both E_8 and E_8', we complex conjugate the E_8' quantum numbers.

A.2.2 Twisted sector $T_4\ (\delta^4=0)$

- Hidden index vector form for V_0^4:

$$4V_0 = \left(\frac{-5}{3},\frac{-5}{3},\frac{-5}{3},\frac{-5}{3},\frac{-5}{3},\frac{-5}{3},\frac{-5}{3};\frac{5}{3}\right)\left(\frac{4}{3},\frac{4}{3},\frac{4}{3},\frac{4}{3},0,\frac{4}{3},\frac{7}{3},\frac{3}{3}\right)',\ V_0^2=\frac{338}{144},$$

$$P = \left(\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2};\frac{-3}{2}\right)\left(\underline{-2,-1,-1,-1},0,-1,-2,-1\right)',\ P_2\cdot V_0=\frac{-101}{12}=\frac{-5}{12},$$

$$P+4V_0 = \left(\frac{-1}{6},\frac{-1}{6},\frac{-1}{6},\frac{-1}{6},\frac{-1}{6},\frac{-1}{6},\frac{-1}{6};\frac{1}{6}\right)\left(\underline{\frac{-2}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}},0,\frac{1}{3},\frac{1}{3},0\right)',\ (P_2+4V_0)\cdot a_3\neq0.$$
$$\text{(A.5)}$$

Since the shift is (odd) $\times$ (odd) under $E_8\times E_8'$, we interchange the $SU(4)'$ gauge quantum numbers. Note $(P_2+4V_0)^2=\frac{44}{36}=\frac{48}{36}-\frac{4}{36}$. The oscillator contribution of $2\cdot\frac{2/3}{12}$ is needed. Using $-(4/2)(V_0^2-\phi_s^2)=\frac{-49-1/3}{12}$, we choose $-\Delta_4^N$.

Table 14.

A.2.3 Twisted sector $T_1\ (\delta^1=1/12)$

- Hidden index spinor form for V_0^1: for the spinor,

Chirality	$\tilde{s}$	$-\tilde{s}\cdot\phi_s$	$-p_{\text{vec}}\cdot\phi_s, P\cdot V_0$		$(4/2)\phi_s^2,$	$-(4/2)V_0^2,$	$-\Delta_4^N[\delta^4]$	$\Theta_0(\mathcal{P}_6^N)$
$\ominus = L$ $(---)$		$\frac{+5}{12}$	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-169}{24}$	$\frac{-2/3}{12}[0]$	$\frac{+12}{12}$ (4)
$\ominus = L$ $(-++)$		0	$\frac{+14}{12},$	$\frac{-5}{12}$	$\frac{21}{36}$	$\frac{-169}{36}$	$\frac{-2/3}{12}[0]$	$\frac{+7}{12}$ (0)
$\ominus = L$ $(+-+)$		$\frac{-1}{12}$	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$\frac{-2/3}{12}[0]$	$\frac{+6}{12}$ (2)
$\ominus = L$ $(++-)$		$\frac{-4}{12}$	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$\frac{-2/3}{12}[0]$	$\frac{+3}{12}$ (0)
$\oplus = R$ $(+++)$		$\frac{-5}{12}$	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$\frac{-2/3}{12}[0]$	$\frac{+2}{12}$ (2)
$\oplus = R$ $(+--)$		0	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$\frac{-2/3}{12}[0]$	$\frac{+7}{12}$ (0)
$\oplus = R$ $(-+-)$		$\frac{+1}{12}$	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$\frac{-2/3}{12}[0]$	$\frac{+8}{12}$ (3)
$\oplus = R$ $(--+)$		$\frac{+4}{12}$	$\frac{+18}{12},$	$\frac{-4}{12}$	$\frac{21}{24}$	$\frac{-165}{24}$	$\frac{-2/3}{12}[0]$	$\frac{+11}{12}$ (0)

Table 14. Hidden sector vector from V_0^4: chiralities and multiplicities, $\Psi_L^{[\alpha']} \oplus 5\left(\Phi_L^{[\alpha']} + \Phi_R^{[\alpha']}\right)$.

Chirality	$\tilde{s}$	$-\tilde{s}\cdot\phi_s$	$-p_{\text{vec}}\cdot\phi_s, P\cdot V_0$		$(1/2)\phi_s^2,$	$-(1/2)V_0^2,$	$\Delta_1^N[\delta^1], \Delta_1^N[-\delta^1]$	$\Theta_0(\mathcal{P}_4^N)$
$\ominus = L$ $(---)$		$\frac{+5}{12}$	$\frac{0}{12},$	$\frac{+5}{12}$	$\frac{21}{72}$	$\frac{-169}{72}$	$\frac{-2/3}{12}[\frac{+1}{12}], \frac{-2/3}{12}[\frac{-1}{12}]$	$\frac{+9}{12}$ (0), $\frac{+7}{12}$ (0)
$\ominus = L$ $(-++)$		0	$\frac{0}{12},$	$\frac{+5}{12}$	$\frac{21}{72}$	$\frac{-169}{72}$	$\frac{-5/3}{12}[\frac{+1}{12}], \frac{-5/3}{12}[\frac{-1}{12}]$	$\frac{+4}{12}$ (3), $\frac{+2}{12}$ (2)
$\ominus = L$ $(+-+)$		$\frac{-1}{12}$	$\frac{0}{12},$	$\frac{+5}{12}$	$\frac{21}{72}$	$\frac{-169}{72}$	$\frac{-2/3}{12}[\frac{+1}{12}], \frac{-2/3}{12}[\frac{-1}{12}]$	$\frac{+3}{12}$ (0), $\frac{+1}{12}$ (0)
$\ominus = L$ $(++-)$		$\frac{-4}{12}$	$\frac{0}{12},$	$\frac{+5}{12}$	$\frac{21}{72}$	$\frac{-169}{72}$	$\frac{-2/3}{12}[\frac{+1}{12}], \frac{-2/3}{12}[\frac{-1}{12}]$	$\frac{0}{12}$ (4), $\frac{-2}{12}$ (2)
$\oplus = L$ $(+++)$		$\frac{-5}{12}$	$\frac{0}{12},$	$\frac{+5}{12}$	$\frac{21}{72}$	$\frac{-169}{72}$	$\frac{-2/3}{12}[\frac{+1}{12}], \frac{-2/3}{12}[\frac{-1}{12}]$	$\frac{-1}{12}$ (0), $\frac{-3}{12}$ (0)
$\oplus = L$ $(+--)$		0	$\frac{0}{12},$	$\frac{+5}{12}$	$\frac{21}{72}$	$\frac{-169}{72}$	$\frac{-2/3}{12}[\frac{+1}{12}], \frac{-2/3}{12}[\frac{-1}{12}]$	$\frac{+4}{12}$ (3), $\frac{+2}{12}$ (2)
$\oplus = L$ $(-+-)$		$\frac{+1}{12}$	$\frac{0}{12},$	$\frac{+5}{12}$	$\frac{21}{72}$	$\frac{-169}{72}$	$\frac{-2/3}{12}[\frac{+1}{12}], \frac{-2/3}{12}[\frac{-1}{12}]$	$\frac{+5}{12}$ (0), $\frac{+3}{12}$ (0)
$\oplus = L$ $(--+)$		$\frac{+4}{12}$	$\frac{0}{12},$	$\frac{+5}{12}$	$\frac{21}{72}$	$\frac{-169}{72}$	$\frac{-2/3}{12}[\frac{+1}{12}], \frac{-2/3}{12}[\frac{-1}{12}]$	$\frac{+8}{12}$ (3), $\frac{+6}{12}$ (2)

Table 15. Hidden index spinor from V_0^1: chiralities and multiplicities are $\Psi_{[\alpha']L} \oplus 10\left(\Phi_{[\alpha']L} + \Phi_{[\alpha']R}\right)$.

$$V_0 = \left(\frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}; \frac{+5}{12}\right)\left(\frac{4}{12}, \frac{4}{12}, \frac{4}{12}, \frac{4}{12}, 0, \frac{4}{12}, \frac{7}{12}, \frac{3}{12}\right)', \ V_0^2 = \frac{338}{144}$$

$$P = (+++++++;-)(\underline{+---}, ----)', \ P\cdot V_0 = \frac{-31}{12} = \frac{+5}{12},$$

$$P + V_0 = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}; \frac{-1}{12}\right)\left(\underline{\frac{10}{12}, \frac{-2}{12}, \frac{-2}{12}, \frac{-2}{12}}, \frac{-6}{12}, \frac{-2}{12}, \frac{1}{12}, \frac{-3}{12}\right)',$$

$$(P + V_0)\cdot a_3 \neq 0,$$

(A.6)

we have $(P+V_0)^2 = \frac{170}{144} = \frac{210}{144} - \frac{40}{144}$. Reverse the gauge quantum numbers. Note that $(1/2)\phi_s^2 - (1/2)V_0^2 = -\frac{148}{12} = -12 + \frac{-1/3}{12}$. The oscillator contribution is $\frac{40}{144} = 2 \cdot \frac{5/3}{12}$. So we need $-\Delta_4^N = \frac{-5/3}{12}$ to cancel $\frac{+2/3}{12}$ in $(1/2)\phi_s^2 - (1/2)V_0^2$. These are shown in table 15.

A.2.4 Twisted sector T_5 ($\delta^5 = 1/12$)

- Hidden index vector form for V_0^5: we have

Chirality		$\tilde{s}$	$-\tilde{s}\cdot\phi_s$	$p_{\mathrm{vec}}\cdot\phi_s, P\cdot V_0$	$(5/2)\phi_s^2,\ -(5/2)V_0^2,$	$\Delta_4^N[\delta^5],\Delta_5^N[-\delta^5]$	$\Theta_0\,(\mathcal{P}_5^N)$
$\ominus=L$	$(---)$	$\frac{+5}{12}$	$\frac{+18}{12},\ \frac{0}{12}$	$\frac{105}{144}$ $\frac{-845}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{+11}{12}\,(0),\frac{+9}{12}\,(0)$
$\ominus=L$	$(-++)$	0	$\frac{+18}{12},\ \frac{0}{12}$	$\frac{105}{144}$ $\frac{-845}{144}$	$\frac{+2/3}{12}[\frac{+1}{12}],\frac{+2/3}{12}[\frac{-1}{12}]$	$\frac{+6}{12}\,(2),\frac{+4}{12}\,(3)$
$\ominus=L$	$(+-+)$	$\frac{-1}{12}$	$\frac{+18}{12},\ \frac{0}{12}$	$\frac{105}{144}$ $\frac{-845}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{+5}{12}\,(0),\frac{+3}{12}\,(0)$
$\ominus=L$	$(++-)$	$\frac{-4}{12}$	$\frac{+18}{12},\ \frac{0}{12}$	$\frac{105}{144}$ $\frac{-845}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{2}{12}\,(2),\frac{0}{12}\,(4)$
$\oplus=R$	$(+++)$	$\frac{-5}{12}$	$\frac{+18}{12},\ \frac{0}{12}$	$\frac{105}{144}$ $\frac{-845}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{+1}{12}\,(0),\frac{-1}{12}\,(0)$
$\oplus=R$	$(+--)$	0	$\frac{+18}{12},\ \frac{0}{12}$	$\frac{105}{144}$ $\frac{-845}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{+6}{12}\,(2),\frac{+4}{12}\,(3)$
$\oplus=R$	$(-+-)$	$\frac{+1}{12}$	$\frac{+18}{12},\ \frac{0}{12}$	$\frac{105}{144}$ $\frac{-845}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{+7}{12}\,(0),\frac{+5}{12}\,(0)$
$\oplus=R$	$(--+)$	$\frac{+4}{12}$	$\frac{+18}{12},\ \frac{0}{12}$	$\frac{105}{144}$ $\frac{-845}{144}$	$\frac{+5/3}{12}[\frac{+1}{12}],\frac{+5/3}{12}[\frac{-1}{12}]$	$\frac{+10}{12}\,(2),\frac{+8}{12}\,(3)$

Table 16. One index spinor from V_0^5: chiralities and multiplicities, $\Psi_{[\alpha']L}\oplus 10\left(\Phi_{[\alpha']L}+\Phi_{[\alpha']R}\right)$.

Chirality		$\tilde{s}$	$-\tilde{s}\cdot\phi_s$	$p_{\mathrm{vec}}\cdot\phi_s, P\cdot V_+$	$(5/2)\phi_s^2,\ -(5/2)V_+^2,$	$\Delta_4^N[\delta^5],\Delta_5^N[-\delta^5]$	$\Theta_0\,(\mathcal{P}_5^N)$
$\ominus=L$	$(---)$	$\frac{+5}{12}$	$\frac{+18}{12},\frac{-4}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+2/3}{12}[\frac{+1}{12}],\frac{+2/3}{12}[\frac{-1}{12}]$	$\frac{+7}{12}\,(0),\frac{+5}{12}\,(0)$
$\ominus=L$	$(-++)$	0	$\frac{+18}{12},\frac{-4}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+2/3}{12}[\frac{+1}{12}],\frac{+2/3}{12}[\frac{-1}{12}]$	$\frac{+2}{12}\,(2),\frac{-2}{12}\,(2)$
$\ominus=L$	$(+-+)$	$\frac{-1}{12}$	$\frac{+18}{12},\frac{-4}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+2/3}{12}[\frac{+1}{12}],\frac{+2/3}{12}[\frac{-1}{12}]$	$\frac{+1}{12}\,(0),\frac{-3}{12}\,(0)$
$\ominus=L$	$(++-)$	$\frac{-4}{12}$	$\frac{+18}{12},\frac{-4}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+2/3}{12}[\frac{+1}{12}],\frac{+2/3}{12}[\frac{-1}{12}]$	$\frac{-2}{12}\,(2),\frac{-6}{12}\,(2)$
$\oplus=R$	$(+++)$	$\frac{-5}{12}$	$\frac{+18}{12},\frac{-4}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+2/3}{12}[\frac{+1}{12}],\frac{+2/3}{12}[\frac{-1}{12}]$	$\frac{-3}{12}\,(0),\frac{-7}{12}\,(0)$
$\oplus=R$	$(+--)$	0	$\frac{+18}{12},\frac{-4}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+2/3}{12}[\frac{+1}{12}],\frac{+2/3}{12}[\frac{-1}{12}]$	$\frac{+2}{12}\,(2),\frac{-2}{12}\,(2)$
$\oplus=R$	$(-+-)$	$\frac{+1}{12}$	$\frac{+18}{12},\frac{-4}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+2/3}{12}[\frac{+1}{12}],\frac{+2/3}{12}[\frac{-1}{12}]$	$\frac{+3}{12}\,(0),\frac{-1}{12}\,(0)$
$\oplus=R$	$(--+)$	$\frac{+4}{12}$	$\frac{+18}{12},\frac{-4}{12}$	$\frac{105}{144}$ $\frac{-2285}{144}$	$\frac{+2/3}{12}[\frac{+1}{12}],\frac{+2/3}{12}[\frac{-1}{12}]$	$\frac{+6}{12}\,(2),\frac{+2}{12}\,(2)$

Table 17. Hidden sector V_+^5: massless states are $8\left(\Phi_L^{[\alpha']}+\Phi_R^{[\alpha']}\right)$.

$$5V_0=\left(\frac{-25}{12},\frac{-25}{12},\frac{-25}{12},\frac{-25}{12},\frac{-25}{12},\frac{-25}{12},\frac{-25}{12};\frac{+25}{12}\right)\left(\frac{20}{12},\frac{20}{12},\frac{20}{12},\frac{20}{12},0,\frac{20}{12},\frac{35}{12},\frac{15}{12}\right)',$$

$$V_0^2=\frac{338}{144}$$

$$P_2=(2,2,2,2,2,2,2;-2)(\underline{5-,3-,3-,3-},+,3-,5-,3-)',\ P\cdot V_0=\frac{-456}{12}=\frac{0}{12},$$

$$P_2+5V_0=\left(\frac{-1}{12},\frac{-1}{12},\frac{-1}{12},\frac{-1}{12},\frac{-1}{12},\frac{-1}{12},\frac{-1}{12};\frac{+1}{12}\right)\left(\frac{-10}{12},\frac{2}{12},\frac{2}{12},\frac{2}{12},\frac{6}{12},\frac{2}{12},\frac{5}{12},\frac{-3}{12}\right)',$$

(A.7)

which gives $(P_1+5V_0)^2=\frac{194}{144}=\frac{210}{144}-\frac{16}{144}$. The oscillator contribution of $2\cdot\frac{2/3}{12}$ is needed. Massless states are shown in table 16.

- Hidden index spinor form for V_+^5:

$$5V_+ = \left(\frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}, \frac{-5}{12}; \frac{45}{12} \right) \left(0, 0, 0, 0, 0, \frac{20}{12}, \frac{135}{12}, \frac{-45}{12} \right)', \ V_+^2 = \frac{914}{144},$$

$$P = (+ + + + + + +; 7-)(\underline{1, 0, 0, 0}, 0, -2, -11, 4)', \ P \cdot V_+ = \frac{-376}{12} = \frac{-4}{12},$$

$$P + 5V_+ = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}; \frac{3}{12} \right) \left(\underline{1, 0, 0, 0}, 0, \frac{-4}{12}, \frac{3}{12}, \frac{3}{12} \right)', \ (P + 5V_0) \cdot a_3 \neq 0$$

$$\text{(A.8)}$$

which gives $(P + 5V_+)^2 = \frac{194}{144} = \frac{210}{144} - \frac{16}{144}$. The oscillator contribution of $2 \cdot \frac{2/3}{12}$ is needed. Note that $(5/2)\phi_s^2 - (5/2)V_+^2 = \frac{-181 - 2/3}{12}$, which means we select $\Delta_5^N[\delta^5]$ and $\Delta_5^N[-\delta^5]$. Massless states are shown in table 17.

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