Minimal length, Friedmann equations and maximum density

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ABSTRACT: Inspired by Jacobson's thermodynamic approach [4], Cai et al. [5, 6] have shown the emergence of Friedmann equations from the first law of thermodynamics. We extend Akbar-Cai derivation [6] of Friedmann equations to accommodate a general entropy-area law. Studying the resulted Friedmann equations using a specific entropy-area law, which is motivated by the generalized uncertainty principle (GUP), reveals the existence of a maximum energy density closed to Planck density. Allowing for a general continuous pressure $p(\rho, a)$ leads to bounded curvature invariants and a general nonsingular evolution. In this case, the maximum energy density is reached in a finite time and there is no cosmological evolution beyond this point which leaves the big bang singularity inaccessible from a spacetime prospective. The existence of maximum energy density and a general nonsingular evolution is independent of the equation of state and the spacial curvature k. As an example we study the evolution of the equation of state $p = \omega\rho$ through its phase-space diagram to show the existence of a maximum energy which is reachable in a finite time.

Contents

1 Introduction

A genuine connection between the laws of black holes mechanics and those of thermodynamics is widely believed today. It started out with a mere analogy connecting these two sets of laws which was first pointed out by Bardeen, Carter and Hawking [1]. Only after the discovery of Hawking radiation [2], it has been realized that these thermodynamic relations describe the thermal properties of a black hole. In this work Hawking was able to show that a black hole behaves quantum mechanically as a black body radiator, with a temperature proportional to its surface gravity and entropy proportional to its horizon area [3].

Based on the above connection between entropy and horizon area, and that of temperature and surface gravity, Jacobson [4] found that Einstein field equations can follow exactly from the fundamental thermodynamic relation between heat, entropy, and temperature $dQ = TdS$, i.e., Clausius relation. The argument is based on demanding that Clausius relation holds for all the local Rindler causal horizons through each spacetime point. Jacobson interpreted dQ as the energy flux and T as Unruh temperature seen by an accelerated observer just inside the horizon. Jacobson's work can be stated as follows; from a thermodynamic point of view, Einstein field equations are nothing but an equation of state for the spacetime under consideration. Inspired by Jacobson approach, Cai et al. [5] derived Friedman equations of $(n + 1)$-dimensional Friedman-Robertson-Walker (FRW) universe, from the Clausius relation $(TdS = -dE)$ with the apparent horizon of FRW universe, assuming that the entropy is proportional to the area of apparent horizon [5]. The derivation of Cai et al. was based on assuming that the apparent horizon has

an entropy S, and a temperature T;

$$S = \frac{A}{4G}, \tag{1.1}$$

$$T = \frac{1}{2\pi\tilde{r}_A} \tag{1.2}$$

Notice here that the entropy is proportional to the area, but the temperature is not proportional to the surface gravity κ, since

$$\kappa = -\frac{1}{\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2\,H\,\tilde{r}_A}\right). \tag{1.3}$$

As a result, the first law in this context is an approximation, where $\frac{\dot{\tilde{r}}_A}{2\,H\,\tilde{r}_A}\big|_{\text{horizon}} = \frac{\dot{H}}{2H^2} \ll 1$, has to be satisfied. This has been noticed and discussed by the authors in [5]. The drawback of this approximation is that it constrains possible equations of state (EoS) to be $p \simeq -\rho$, i.e., EoS has to be close enough to that of the vacuum energy or de Sitter spacetime. This can be shown clearly through the cosmological equation $\dot{H} = -\frac{3H^2}{2}\,(p/\rho + 1)$ for a non-vanishing H. To improve this thermodynamic description of FRW cosmology, Akbar and Cai [6] used the Misner-Sharp energy relation inside a sphere of radius $\tilde{r}_A$ of an apparent horizon and rewrite the dynamical Friedman equation in the form of the first law of thermodynamics with a work term.

$$dE = T\,dS + W\,dV, \tag{1.4}$$

where the work density W is given in terms of the energy density ρ and the pressure of matter in the universe p as follows:

$$W = \frac{1}{2}\,(\rho - p). \tag{1.5}$$

In addition, they considered the Misner-Sharp energy to be the total energy of the matter existing inside the apparent horizon which is given by $E = \rho V$, where V is the volume of the apparent horizon.

Motivated again, by black hole physics, there is an another interesting connection between the Hawking radiation and the uncertainty principle [7–13] where the black hole can be modeled as n-dimensional sphere of size equal to twice of Schwarzschild radius. Since the Hawking radiation is a quantum process, one could assume that the emitted particles should obey the Heisenberg uncertainty relation. From this, one can derive exactly the Hawking temperature and the thermodynamic properties of the black hole [8–10]. From the black hole thermodynamics, it has been realized that; as the black hole approaches zero mass, its temperature approaches infinity with infinite radiation rate which is considered a *catastrophic* evaporation of the black hole.

A generalized uncertainty principle(GUP) was proposed by different approaches to quantum gravity such as string theory and black hole physics [14–25] in which Planck length plays an important role. The existence of Planck length as a minimal observable length l_p is a universal feature among all approaches of QG [14–22]. This minimal length

works as a natural cutoff, which is expected to have a crucial role in resolving curvature singularities in general relativity. By employing this GUP, it was found that the black hole thermodynamics quantities, such as temperature, and entropy are completely modified such that the end-point of Hawking radiation is not catastrophic anymore. This is because the GUP implies the existence of black hole *remnants* at which the specific heat vanishes and, therefore, the black hole cannot exchange heat with the surrounding space [7–13]. This means that GUP prevents the black hole from evaporating completely, just like the standard uncertainty principle prevents the hydrogen atom from collapsing. It is worth mentioning that the GUP modifies, significantly, the entropy-area law as has been pointed out by several authors (see for example refs. [7, 11–13]).

In this work we study exact modifications of the entropy-area law due to GUP, which is used to modifying Friedmann equations derived by Akbar and Cai in [6] at very high densities. It is worth mentioning that GUP has been studied with Friedmann equations through different frameworks in [26, 27]. In this paper, we first derive the modified Friedmann equations for a general form of the entropy as a function of area using the first law of thermodynamics ($dE = T \ dS + W \ dV$). This generalization can host all possible correction to the entropy-area law like the logarithmic correction and power law corrections which are motivated by different approaches to quantum gravity such as string theory and loop quantum gravity [28–30]. We then investigate an important entropy-area law which is motivated by the GUP and study the modified Friedmann equations derived through the first law approach introduced by Akbar and Cai in [6]. Studying the resulted Friedmann equations reveals the existence of a maximum energy density closed to Planck density. Assuming a general continuous pressure $p(\rho, a)$ leads to bounded curvature invariants and a general nonsingular evolution. As a result, the maximum energy density is reached in a finite time and there is no cosmological evolution beyond this point from a spacetime prospective. The existence of maximum energy density and a general nonsingular evolution is independent of the equation of state and the spacial curvature k. As an example we study the evolution of the equation of state $p = w\rho$, using a phase-space diagram [31], to show the existence of a maximum energy density and a finite time to reach it. Our results reveal that the big bang singularity is not accessible in this description since the spacetime itself can not be extended beyond Planck density as a result of the GUP and the thermodynamic approach to gravity which modifies Friedmann equations.

The paper is organized as follows, in section 2, we review the derivation of Friedmann equations from the first law of thermodynamics due to Akbar and Cai [6], assuming that the entropy is proportional to the area of apparent horizon. In section 3, we extend this procedure for an arbitrary entropy, which could host various possible correction to entropy-area law, to obtain a set of modified Friedmann equations. In section 4, we review the generalized uncertainty principle and calculate the exact entropy-area law from the first law of thermodynamics. In section 5, we calculate the modified Friedmann equations due to the exact entropy-area law obtained from GUP using the first law of thermodynamics. In section 6, we discuss direct implications of the modified Friedmann equations arguing for the existence of a maximum energy density closed to Planck density. Assuming a continuous pressure we show that all curvature invariants are finite and the previous two

features are independent of the equation of state and the spacial curvature. As an example we study the evolution of the equation of state $p = w\rho$, using a phase-space diagram, to show the existence of a maximum energy density and a finite time to reach it. At the end we conclude by showing the general implications of the modified Friedmann equations that we found due to the exact entropy-area law obtained from the GUP.

2 Friedmann equations from the first law of thermodynamics

In this section, we review the derivation of Friedmann equations from the first law of thermodynamics relation with the apparent horizon of FRW universe with assuming that the entropy is proportional to the area of the apparent horizon [4, 5]. The $(n + 1)$-dimensional Friedmann-Robertson-Walker (FRW) universe is described by the following metric:

$$ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2 d\Omega_{n-1}^2, \tag{2.1}$$

where $\tilde{r} = a(t)r$, $x^a = (t, r), h_{ab} = (-1, a^2/(1 - kr^2))$, $d\Omega_{n-1}^2$ is the metric of $(n - 1)$-dimensional sphere, $a, b = 0, 1$ and the spatial curvature constant k takes the values $0, 1, -1$ for a flat, closed and open universe, respectively. The dynamical apparent horizon is determined by the relation $h^{ab}\partial_a \tilde{r}\partial_b \tilde{r} = 0$, which would give the radius of the apparent horizon to be [5]:

$$\tilde{r}_A = a\ r = \frac{1}{\sqrt{H^2 + k/a^2}}, \tag{2.2}$$

where $H = \dot{a}/a$ is the Hubble parameter. By assuming that the matter which occupy the FRW universe forms a perfect fluid, so the energy-momentum tensor would be:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}. \tag{2.3}$$

where u_μ is the four velocity of the fluid. The energy conservation law ($T^{\mu\nu}_{;\nu} = 0$) leads to the continuity equation

$$\dot{\rho} + nH(\rho + p) = 0. \tag{2.4}$$

Based on the arguments of [32], one can define work density W as follows

$$W = -\frac{1}{2}T^{ab}h_{ab}, \tag{2.5}$$

where T_{ab} is the projection of the energy-momentum tensor $T_{\mu\nu}$ in the normal direction. For FRW universe with perfect fluid, the work density will be

$$W = \frac{1}{2}(\rho - p). \tag{2.6}$$

Now, we give a brief review for the procedure that has been followed by Akbar-Cai in [6]. Considering the first law of thermodynamics as follows:

$$dE = T\ dS + W\ dV. \tag{2.7}$$

Let us calculate term by term in eq. (2.7). We start with dE. The term dE represents the infinitesimal change in the total energy during small interval of time dt. Since E introduces

the total energy of matter inside the apparent horizon (Misner-Sharp energy), so it can take the following form

$$E = \rho \, V, \tag{2.8}$$

where $V = \Omega_n \tilde{r}_A^n$ is the volume of n-dimensional sphere with radius $\tilde{r}_A$ and $\Omega_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$. Hence, the differential element of E would be

$$\begin{aligned} dE &= \rho dV + V \, d\rho \\ &= n\Omega_n \tilde{r}_A^{n-1} \rho d\tilde{r}_A + \Omega_n \tilde{r}_A^n d\rho. \end{aligned} \tag{2.9}$$

By using the continuity equation of eq. (2.4) in eq. (2.9), we get;

$$dE = n\Omega_n \tilde{r}_A^{n-1} \rho d\tilde{r}_A - n\Omega_n \tilde{r}_A^n (\rho + p) H dt \tag{2.10}$$

Turning to the other term $W \, dV$, it can be written as follows:

$$W \, dV = \frac{1}{2} n\Omega_n \tilde{r}_A^{n-1} (\rho - p) \, d\tilde{r}_A. \tag{2.11}$$

For the term $T \, dS$, we should use definition of Hawking temperature of eq. (1.2) as well as the entropy-area law. We start with Hawking temperature which is defined in terms of surface gravity as follows:

$$T = \frac{\kappa}{2\pi} \tag{2.12}$$

where κ introduces the surface gravity for the metric of eq. (2.1) and is defined as

$$\begin{aligned} \kappa &= \frac{1}{2\sqrt{-h}} \partial_a(\sqrt{-h} h^{ab} \partial_b \tilde{r}) \\ &= -\frac{1}{\tilde{r}_A} \left(1 - \frac{\dot{\tilde{r}}_A}{2 \, H \, \tilde{r}_A} \right) \end{aligned} \tag{2.13}$$

Besides, we use the entropy-area law of eq. (1.1) and the expression of area for n-dimensional sphere $A = n\Omega_n \tilde{r}_A^{n-1}$. This yields at the end the following expression

$$\begin{aligned} T \, dS &= \frac{\kappa}{2\pi} d\left(\frac{n\Omega_n \tilde{r}_A^{n-1}}{4G} \right) \\ &= -\frac{1}{2\pi \tilde{r}_A} \left[1 - \frac{\dot{\tilde{r}}_A}{2 \, H \, \tilde{r}_A} \right] \left(\frac{n(n-1)\Omega_n}{4G} \tilde{r}_A^{n-2} \right) \end{aligned} \tag{2.14}$$

By substituting eqs. (2.10), (2.11) and (2.14) into the first law of thermodynamics of eq. (2.7), we get

$$\frac{d\tilde{r}_A}{\tilde{r}_A^3} = \frac{8\pi G}{n-1} (\rho + p) \, H dt. \tag{2.15}$$

Using eq. (2.2) which yields $d\tilde{r}_A = -H\tilde{r}_A^3 \left(\dot{H} - k/a^2 \right) dt$ [5], one simply finds that eq. (2.15) introduces the dynamical Friedman equation.

$$\dot{H} - \frac{k}{a^2} = -\frac{8\pi G}{n-1}(\rho + p) \tag{2.16}$$

Using the continuity equation (2.4) and integrating eq. (2.16), one simply gets

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{n(n-1)}\rho, \tag{2.17}$$

which is the Friedmann equation for $(n+1)$-dimensional FRW universe.

3 Modified Friedmann equation for a general form of the entropy

In this section, we derive the general modified Friedmann equation for a general expression of the entropy as a function of area through the apparent horizon approach and first law of thermodynamics. Suppose that the general expression for the entropy-area relation takes the following form:

$$S = \frac{f(A)}{4G}, \tag{3.1}$$

$$\frac{dS}{dA} = \frac{f'(A)}{4G}, \tag{3.2}$$

where $f'(A) = df(A)/dA$. Using the first law of thermodynamics $dE = T\, dS + W\, dV$, and following the same procedure that we reviewed in section 2, we get

$$f'(A)\frac{d\tilde{r}_A}{\tilde{r}_A^3} = \frac{8\pi G}{n-1}(\rho + p)\ H\ dt \tag{3.3}$$

Again, using eq. (2.2) which yields [5],

$$d\tilde{r}_A = -H\tilde{r}_A^3\left(\dot{H} - k/a^2\right)dt \tag{3.4}$$

one simply finds that eq. (2.15) introduces the dynamical Friedman equation.

$$\left(\dot{H} - \frac{k}{a^2}\right)f'(A) = -\frac{8\pi G}{n-1}(\rho - p) \tag{3.5}$$

Using the continuity equation of eq. (2.4), and with few calculations, we can rearrange eq. (3.3) as follows

$$f'(A)\frac{(n\Omega_n)^{\frac{n+1}{n-1}}}{n(n-1)\Omega_n}\frac{dA}{A^{\frac{n+1}{n-1}}} = -\frac{8\pi G}{n(n-1)}d\rho \tag{3.6}$$

The last equation can be integrated to give the general modified Friedmann equation for $(n+1)$-dimensional FRW universe due to a general form of the entropy using the first law of thermodynamics.

$$-\frac{8\pi G}{n(n-1)}\rho = \int \frac{(n\Omega_n)^{\frac{n+1}{n-1}}}{n(n-1)\Omega_n}f'(A)\frac{dA}{A^{\frac{n+1}{n-1}}} \tag{3.7}$$

Again, if we set $f(A) = A$, the first Friedmann equation will be satisfied. These modified equations was gotten first in [38] using the Clausius relation $dE = T\, dS$. We are deriving them here with considering the contribution of the work W in the first law of thermodynamics.

4 The generalized uncertainty principle and entropy-area law

We first review briefly the Generalized uncertainty principle (GUP) [14–21] and secondly we review its effect on the area-entropy law [7–13]. The existence of a minimum measurable length originates as an intriguing prediction of various frameworks of quantum gravity such as string theory [14] and black hole physics [15–21]. This implies a direct modification of the standard uncertainty principle [14–25]:

$$\Delta x \gtrsim \frac{\hbar}{\Delta p} \left[1 + \frac{\beta \, \ell_P^2}{\hbar^2} (\Delta p)^2 \right],\tag{4.1}$$

where ℓ_P is the Planck length and β is a dimensionless constant which depends on the quantum gravity theory. The new correction term in eq. (4.1) becomes effective when the momentum and length scales are of order the Planck mass and of the Planck length, respectively. It was straightforward to find that eq. (4.1) implies the existence of minimal measurable length scale as follows:

$$\Delta x \gtrsim \Delta x_{\min} = 2\beta \, \ell_P\tag{4.2}$$

By some manipulations, GUP can be represented by another form as follows:

$$\frac{\Delta p}{\hbar} \gtrsim \frac{\Delta x}{2\beta \, \ell_P^2} \left[1 - \sqrt{1 - \frac{4\beta \, \ell_P^2}{\Delta x^2}} \right]\tag{4.3}$$

There has been investigations devoted to study the impact of GUP on the black hole thermodynamics and to the Bekenstein-Hawking (black hole) entropy (e.g., [7–13]). These studies are based on the argument that Hawking radiation is a quantum process and it should respect the uncertainty principle.

Here we review the analysis of [7] and use the arguments in [33, 34] which say that the uncertainty principle $\Delta p \geq 1/\Delta x$ can be represented by the lower bound $E \geq 1/\Delta x$, so one can get for the GUP case:

$$E \gtrsim \frac{\Delta x}{2\beta \ell_P^2} \left[1 - \sqrt{1 - \frac{4\beta \, \ell_P^2}{\Delta x^2}} \right].\tag{4.4}$$

For any black hole absorbing or emitting a quantum particle whose energy E and size R, the area of the black hole would change by an amount [35, 36].

$$\Delta A \geq 8\pi \, \ell_p^2 \, E \, R,\tag{4.5}$$

The quantum particle itself implies the existence of finite bound given by

$$\Delta A_{\min} \geq 8\pi \, \ell_p^2 \, E \, \Delta x.\tag{4.6}$$

Using GUP by substituting (4.4) into (4.6), we get

$$\Delta A_{\min} \gtrsim \frac{8\pi \Delta x^2}{2\beta} \left[1 - \sqrt{1 - \frac{4\beta \, \ell_P^2}{\Delta x^2}} \right].\tag{4.7}$$

The value of Δx in this analysis is set to be the inverse of surface gravity $\Delta x = \kappa^{-1} = 2r_s$ where r_s is the Schwarzschild radius, where this is probably the most sensible choice of length scale in the context of near-horizon geometry [7–10]. This implies the following identity:

$$\Delta x^2 = \frac{A}{\pi}. \tag{4.8}$$

Substituting eq. (4.8) into eq. (4.7), we get

$$\Delta A_{\min} \gtrsim \frac{8\,A}{2\beta} \left[1 - \sqrt{1 - \frac{4\beta\,\ell_P^2\pi}{A}} \right]. \tag{4.9}$$

The area change is then determined as:

$$\Delta A_{\min} \simeq \lambda \frac{8\,A}{2\beta} \left[1 - \sqrt{1 - \frac{4\beta\,\ell_P^2\pi}{A}} \right], \tag{4.10}$$

where the parameter λ will be fixed from the Bekenstein-Hawking entropy formula. According to [1–3], the black hole's entropy is conjectured to depend on the horizon's area. From the information theory [37], it has been stated that the minimal increase of entropy should be independent of the area. It is just one bit of information which is $\Delta S_{\min} = b = \ln(2)$.

$$\frac{dS}{dA} = \frac{\Delta S_{\min}}{\Delta A_{\min}} = \frac{b}{\lambda \frac{8\,A}{2\beta} \left[1 - \sqrt{1 - \frac{4\beta\,\ell_P^2\pi}{A}} \right]}. \tag{4.11}$$

According to [7], the Bekenstein-Hawking entropy formula has been used to calibrate the the constants $b/\lambda = 2\pi$, so the we have

$$\frac{dS}{dA} = \frac{\Delta S_{\min}}{\Delta A_{\min}} = \frac{\pi}{\frac{2\,A}{\beta} \left[1 - \sqrt{1 - \frac{4\beta\,\ell_P^2\pi}{A}} \right]} \tag{4.12}$$

To simplify the expression of eq. (4.12), we set $\alpha = 4\beta\ell_P^2\pi$, so we get

$$\frac{dS}{dA} = \frac{\alpha}{8\ell_P^2} \frac{1}{A\left[1 - \sqrt{1 - \frac{\alpha}{A}}\ \right]}. \tag{4.13}$$

In this paper, we are interested in the *exact* form of the entropy instead of the approximated one that was used in [7], so we will not make any approximation for the expression of dS/dA. The exact form of eq. (4.13) would enable us to study the behavior of a general solution of the modified Friedmann equations (using its fixed points) due to GUP through the first law of thermodynamics. To get the exact expression of the entropy, we integrate eq. (4.13) to yield:

$$S = \frac{1}{8\ell_P^2} \left[A + \sqrt{A^2 - A\alpha} - \frac{\alpha}{2} \ln \left(A + \sqrt{A^2 - A\alpha} - \frac{\alpha}{2} \right) \right] + S_0 \tag{4.14}$$

where S_0 is an integration constant. We find that eq. (4.14) modifies Bekenstein-Hawking entropy as a result of a minimum length scale or the GUP considered above. In the next section, we will calculate the modified Friedmann equations due to the modified entropy, eq. (4.14), of the apparent horizon of FRW universe.

5 Modified Friedmann equations due to GUP

In this section, we implement the modified entropy of eq. (4.14) in the first law of thermodynamics relation $dE = TdS + WdV$, and derive the modified Friedmann equations with the apparent horizon approach [4–6]. Now, we consider our current case of modified entropy due to GUP in eq. (4.13) and eq. (4.14). By substituting eq. (4.13) into the general modified Friedmann equations eq. (3.5) and eq. (3.7) with $n = 3$, we get the following expression:

$$\left(\dot{H} - \frac{k}{a^2} \right) \frac{\alpha}{2} \frac{1}{A - \sqrt{A^2 - A\,\alpha}} = -4\pi G(\rho + p), \tag{5.1}$$

$$\frac{8\pi G}{3}\rho = -16\pi \ell_P^2 \int \frac{1}{A^2} \frac{\alpha}{8\ell_P^2} \frac{1}{A - \sqrt{A^2 - \alpha A}} dA,$$

$$= 2\pi \left(\frac{1}{A} - \frac{2(A^2 - \alpha A)^{3/2}}{3\alpha A^3} \right) + C, \tag{5.2}$$

where C is a constant of integration and it can be fixed from the initial conditions in eq. (5.2). As the universe expands, the area of apparent horizon is supposed to go to infinity with the density having vacuum energy density $\rho_{\text{vac}} = \Lambda$, where Λ is the cosmological constant. From this argument, C takes the following value:

$$C = \frac{8\pi G}{3}\Lambda + \frac{4\pi}{3\alpha} = \frac{8\pi G}{3}\left(\Lambda + \frac{1}{2G\alpha} \right). \tag{5.3}$$

The area of the apparent horizon is given in section (2) as: [5]

$$A = 4\pi \tilde{r}_A^2 = \frac{4\pi}{H^2 + \frac{k}{a^2}}. \tag{5.4}$$

Accordingly, the modified Friedmann equations (5.1) and (5.2) due to GUP will be

$$\frac{8\pi G}{3}(\rho - \Lambda) = \frac{1}{2}\left(H^2 + \frac{k}{a^2} \right) + \frac{4\pi}{3\alpha}\left[1 - \left(1 - \frac{\alpha}{4\pi}\left(H^2 + \frac{k}{a^2} \right) \right)^{\frac{3}{2}} \right], \tag{5.5}$$

$$-4\pi G(\rho + p) = \left(\dot{H} - \frac{k}{a^2} \right) \frac{\alpha}{8\pi} \frac{(H^2 + \frac{k}{a^2})}{\left[1 - \left(1 - \frac{\alpha}{4\pi}\left(H^2 + \frac{k}{a^2} \right) \right)^{\frac{1}{2}} \right]}. \tag{5.6}$$

We find that eq. (5.5) and eq. (5.6) give the exact modified Friedmann equations due to GUP using Akbar-Cai approach [6]. In the next section we are going to discuss consequences of eq. (5.5) and eq. (5.6) on the behavior of the FRW cosmology.

6 Maximum density and curvature singularities

It is interesting to notice that eq. (5.5) leads to a bounded energy density ρ since the inequality $H^2 + \frac{k}{a^2} \leq 4\pi/\alpha$ must be satisfied, otherwise, the density is complex. Let us analyze this more closely for cases with $k = 0$, and $k = 1$. In order to keep the density

real the previous inequality requires "a" to have a minimum and "H" to have a maximum. This leads to a maximum energy density $\rho_{\max} = \Lambda + \frac{5}{4G\alpha} \sim \rho_p$ (since Λ is tiny), where ρ_p is Planck's density.

Having a maximum value for H (or ρ since they are related) does not necessarily means a finite curvature or nonsingular behavior. The reason is that curvature invariants such as Ricci scalar, Riemann tensor squared, etc., depend not only on H but also on $\dot{H}$. But the behavior of $\dot{H}$ is controlled by both ρ and p as it is clear from eq. (5.6). To check whether p has a diverging behavior or not we don't have to know the equation of state, it is enough to know some generally properties of pressure. Here we will follow the discussion in ref. [31] on general properties of pressure which leaves the mathematical problem addressed by eq. (5.5) well defined. Here we assume that the pressure, $p(H,a)$ or $p(\rho,a)$ is a continues of function of its arguments, in order to have a well defined mathematical evolution for H or ρ. This ensures existence and uniqueness of the general solution of the continuity equation or eq. (5.6). In addition, having a discontinues pressure in H or ρ could imply a noncausal evolution since it leads to a divergent $dp/d\rho$, or an unbounded speed of sound! Therefore, it is natural to assume a continues $p(H,a)$ or $p(\rho,a)$. As a result, one observes that if a has a minimum value and H is bounded, then $\dot{H}$ must be bounded given eq. (5.6). Therefore, all curvature invariants of the Friedmann-Robertson-Walker metric are finite since they can be expressed as functions of H and $\dot{H}$.

But what about the $k = -1$ case, which potentially could still leads to a singular solution. Here we need to satisfy the inequality $H^2 + \frac{k}{a^2} \leq 4\pi/\alpha$, for this case too. For a general continues pressure $p(H,a)$, such that $H^2 \neq a^{-2}$ as $a \to 0$, the previous inequality leads to a minimum value for a and maximum value for H. But for the specific case where $H^2 = a^{-2} + c_1$ as $a \to 0$ the scale factor a can go to zero and the Hubble rate to infinity without violating the inequality. In this case, it is interesting to notice that both density and pressure are finite. A finite pressure can be shown through taking the derivative of H with respect to time using $H = \sqrt{a^{-2} + c_1}$ and calculating p using eq. (5.6). Analyzing this case we found that it leads to a specific EoS, namely $p = -\rho$, which gives a de Sitter evolution as $a \to 0$. Our conclusion is that the above modified Friedmann equation leads to a bounded energy density with a maximum value given by the Planck's density for any equation of state and all values k. In addition, by assuming a continues pressure $p(\rho,a)$ one can show that curvatures invariants are finite as a result of maximum density ρ and minimum scale factor a, therefore, the solution is nonsingular. Our results reveals the limitation or the breaking down of the spacetime description of gravity near Planck scale energies which was not evident from general relativity. As an example we will discuss next the EoS $p = \omega\rho$ and its Raychaudhuri equation which describes a nonsingular universe which starts from a finite time with a Planck density.

Now to show some of the features discussed above let us consider the usual equation of state $p = \omega\rho$, in the light of this modified Friedmann equation and check the behavior of the universe in early times. To do that one can study Raychaudhuri equation or eq. (5.6) on the following form [31]

$$\dot{H} = F(H) \tag{6.1}$$
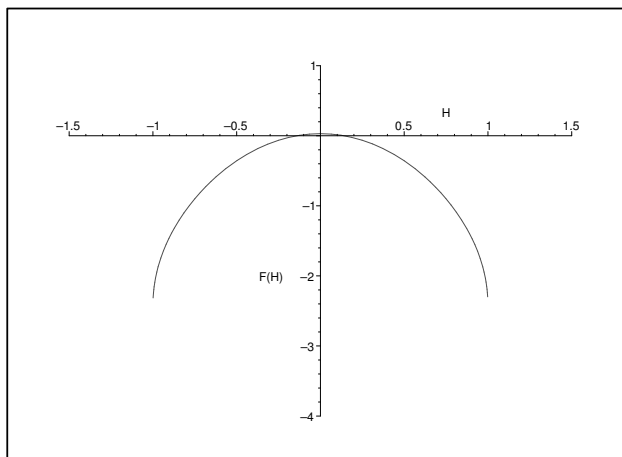
Figure 1. $\dot{H} = F(H)$ versus H, $4\pi/\alpha = 1$. H has a maximum value $H_{\max} \sim \sqrt{\rho_p/3}$.

Using the modified Friedmann equations of eq. (5.5) and eq. (5.6), and the above equation of state, we get the following form of Raychaudhuri equation.

$$
\left(\dot{H} - \frac{k}{a^2} \right) = -\frac{3}{2}(1+\omega) \left[\frac{1}{2} \left(H^2 + \frac{k}{a^2} \right) - \frac{4\pi}{3\alpha} \left(1 - \frac{\alpha}{4\pi} \left(H^2 + \frac{k}{a^2} \right) \right)^{\frac{3}{2}} + C \right]
$$
$$
\times \left(\frac{8\pi}{\alpha} \frac{\left(1 - \left(1 - \frac{\alpha}{4\pi} \left(H^2 + \frac{k}{a^2} \right) \right)^{\frac{1}{2}} \right)}{\left(H^2 + \frac{k}{a^2} \right)} \right).
\tag{6.2}
$$

For the flat case with $k = 0$ the Raychaudhuri equation takes the following form

$$
\dot{H} = -\frac{3}{2}(1+\omega) \left[\frac{H^2}{2} - \frac{4\pi}{3\alpha} \left(1 - \frac{\alpha}{4\pi} H^2 \right)^{\frac{3}{2}} + \frac{8\pi G}{3}\Lambda + \frac{4\pi}{3\alpha} \right]
$$
$$
\times \left[\frac{8\pi}{\alpha} \frac{\left(1 - \left(1 - \frac{\alpha}{4\pi} H^2 \right)^{\frac{1}{2}} \right)}{H^2} \right]
\tag{6.3}
$$

Using the general analysis of the eq. (6.1) in [31] for FRW cosmology. The above first-order system is well studied in dynamical system in cosmological context, see e.g., [31], or see [39] for more general applications. Knowing $F(H)$ fixed points (i.e., its zeros. Let us call them H_i) and its asymptotic behavior enables one to qualitatively describe the behavior of the solution without actually solving the system [31]. Drawing the phase-space diagram for For eq. (6.3) for $\omega = 1/3$ (i.e., plotting $\dot{H}$ versus H) we get the following behavior in figure 1.

As one can notice we have a future fixed point at $H_f = \sqrt{\Lambda/3}$, which we can provide the asymptotic behavior of our universe at late times. In addition this fixed point is reached in an infinite time since by integrating eq. (6.3) we get

$$
t = \int_{H_0}^{H_f} \frac{dH}{F(H)} = \infty,
\tag{6.4}
$$

where H_0 is the initial value of the Hubble rate and F(H) is

$$F(H) = -\frac{24\pi(1+\omega)}{2\alpha H^2}\left[\frac{H^2}{2} - \frac{4\pi}{3\alpha}\left(1 - \frac{\alpha}{4\pi}H^2\right)^{\frac{3}{2}} + \frac{8\pi G}{3}\Lambda + \frac{4\pi}{3\alpha}\right]\left(1 - \left(1 - \frac{\alpha}{4\pi}H^2\right)^{\frac{1}{2}}\right)$$
(6.5)

The interesting observation in this diagram is the existence of a maximum value for the Hubble rate H, beyond which there is no evolution allowed. We have calculated the time to reach the maximum Hubble rate $H_{\max}$

$$t_m = -\int_{H_0}^{H_{\max}} \frac{dH}{F(H)} = \text{finite},$$
(6.6)

Since all curvature invariants of the FRW metric are functions of the Hubble rate and it first-time-derivative, it is straight forward to show that they are all finite as a result of a maximum density and the EoS $p = w\rho$ given the above modified Friedmann equations. It is interesting to observe that the big bang singularity is not accessible in this description since the spacetime itself can not be extended beyond Planck density as a result of the minimum length or the GUP and the thermodynamic approach to gravity provided which modifies Friedmann equations.

7 Conclusions

We generalize Akbar-Cai derivation [6] of Friedmann equations from the first law of thermodynamics $dE = TdS + WdV$, to include an arbitrary entropy-area law which could include possible corrections arise from different approaches to quantum gravity. Studying the resulted Friedmann equations for an entropy-area law motivated by the generalized uncertainty principle (GUP) revealed the existence of a maximum energy density with a value around Planck density. Allowing for a general continuous pressure $p(\rho, a)$ lead to bounded curvature invariants and a general nonsingular evolution. In this case, the maximum energy density is reached in a finite time and there is no cosmological evolution beyond this point from a spacetime prospective. The existence of maximum energy density and a general nonsingular evolution is independent of the equation of state and the spacial curvature k. As an example we study the evolution of the equation of state $p = w\rho$, using a phase-space diagram, to show the existence of a maximum energy density and a finite time to reach it. Our results reveal that the big bang singularity is not accessible in this description since the spacetime itself can not be extended beyond Planck density as a result of the minimum length or the GUP and the thermodynamic approach to gravity which modifies Friedmann equations.

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