A minimal supersymmetric E_6 unified theory

K.S. Babu,[a] **Borut Bajc**[b] **and Vasja Sušič**[b]

[a]*Department of Physics, Oklahoma State University,*
Stillwater, OK, 74078, U.S.A.

[b]*J. Stefan Institute,*
1000 Ljubljana, Slovenia

E-mail: babu@okstate.edu, borut.bajc@ijs.si, vasja.susic@ijs.si

ABSTRACT: We show explicitly that supersymmetric E_6 Grand Unified Theory with a Higgs sector consisting of $\{27 + \overline{27} + 351' + \overline{351'} + 78\}$ fields provides a realistic scenario for symmetry breaking and fermion mass generation. While gauge symmetry breaking can be achieved without the 78 field, its presence is critical for a successful doublet-triplet mass splitting. The Yukawa sector of the model consists of only two symmetric matrices describing all of quark, lepton and neutrino masses and mixings. The fermion mass matrices are computed at low energy and a fit to the second and third generation masses and mixings is performed. We find a good numerical fit to the low-energy data. Thus, this model, having 11 superpotential parameters, alongside the two symmetric Yukawa matrices, seems to be the best realistic candidate for a minimal renormalizable supersymmetric E_6 unified theory.

Contents

1 Introduction

There are three types of unified groups (simple groups containing the Standard Model gauge group) that admit chiral representations for fermions: $SU(N)$ for $N \geq 3$ [1], $SO(4N+2)$ for $N \geq 2$ and E_6 [2]. The first two have been thoroughly studied in many versions, renormalizable as well as non-renormalizable, with and without supersymmetry. Strange enough, E_6, the only exceptional group which contains chiral representations, has been mainly ignored over the last forty years. Apart from a few exceptions where symmetry breaking was analyzed (via renormalizable potential with 78 and two 27s in ref. [3] and via renormalizable superpotential with 78, 27 and $\overline{27}$ in ref. [4]), only the Yukawa sectors have been touched upon without using explicit constraints arising from symmetry breaking [5–10].

Recently, two of us have tried to fill this gap by suggesting a possible minimal renormalizable supersymmetric E_6 model [11]. The Higgs sector is composed of pairs of fundamental $27 + \overline{27}$ and the two-index symmetric $351' + \overline{351'}$ representations. Much to our

surprise we found that, although successful in the symmetry breaking pattern to the Standard Model (SM) gauge group, this sector is unable to provide two light Higgs doublets of the minimal supersymmetric Standard Model (MSSM), or technically said, to perform the doublet-triplet (DT) splitting. The impossibility of DT splitting was due to the identical nature of the expressions for a massless doublet and for a massless triplet; this result was obtained by an explicit computation, and we are not aware of a shortcut explanation of this unusual feature.[1] To put this result in perspective, consider the minimal renormalizable supersymmetric breaking sectors (able to break to the SM group) for other groups. In the minimal renormalizable SU(5) superpotential (consisting of the adjoint 24), there are no SM doublets. In the corresponding minimal SO(10) case (made out of 210, 126 and $\overline{126}$) one has in principle enough doublets, but only one (in $\overline{126}$) is coupled to fermions. In a minimal E$_6$ case with $27 + \overline{27} + 351' + \overline{351'}$, there are many more doublets which couple to fermions, yet the naive DT splitting fails (a completely different approach to DT splitting in E$_6$ can be found in [12]). We thus proposed in [11] to add another $27 + \overline{27}$ pair. In this way, the Yukawa sector consisted of three Yukawa matrices: the matter fields (3 copies of 27_F) have one coupling to the $\overline{351'}$ and a coupling to each of the two Higgs-like 27s. Though we did not perform it explicitly, we strongly believe that in this model a realistic fit of the Yukawa sector is possible to obtain, but the model is essentially not predictive due to the large number of free parameters in the three Yukawa matrices.

The purpose of this paper is to find an E$_6$ theory with only two Yukawa matrices and thus a simpler Yukawa sector than the model in [11], while keeping the symmetry breaking sector as simple as possible. The idea is to add to the minimal breaking sector (instead of the extra $27 + \overline{27}$ pair) another multiplet, which preferably has the following properties:

1. It does not couple to the matter bilinears (and so is not a 27, $\overline{351}$ or $\overline{351'}$).

2. It contributes to the symmetry breaking vacuum.

3. It increases the size of the mass matrices for weak doublets and/or color triplets.

The minimal such multiplet is the adjoint representation 78, for which all of the above properties hold true. We will see that its inclusion is enough to allow the doublet-triplet splitting in the theory, thus leading to a theory with the correct (SM) vacuum and the correct (MSSM) low energy field theory.

The Yukawa sector in this model consists of two matrices only. Assuming 3 generations, the down quark and charged lepton sectors have 3 extra vector-like fields, while the neutrino sector has 3 extra vector-like lepton doublet-antidoublet pairs, as well as 6 SM singlets, all coming from the extra fields in the decomposition of the fundamental 27 into the SM subgroup. The projection of all these fields into the usual 3 light generations will be performed explicitly and shown, in the simplified case of only two generations, to provide a realistic fit of the masses and mixing angles.

Special attention needs to be paid to the overall neutrino mass scale. In minimal SO(10) [13–15] this scale poses a serious problem and eventually rules out the low-energy

---

[1]The sparsity of SU(5) breaking vacuum expectation values in the respective mass matrices is likely to be the fundamental origin of this problem.

supersymmetry scenario [16–19]: a too low $SU(2)_R$ breaking is disallowed by unification constraints and/or $D = 6$ proton decay [20], while a too large breaking scale makes the right-handed neutrinos too heavy and thus the seesaw mechanism ineffective to produce a large enough scale for the light neutrinos. In our model this conclusion is avoidable because there are many possible fields that can break $SU(2)_R$. While the neutrino mass scale can be adjusted by choosing the vacuum expectation value (VEV) of the $(1, 3, 10)$ of the Pati-Salam (PS) $SU(2)_L \times SU(2)_R \times SU(4)_C$, located in the $\overline{126}$ of SO(10), which in turn is found in the $\overline{351'}$ of E_6, all other fields' masses depend also on several other VEVs. We can thus arrange a lighter VEV of this $(1, 3, 10)$ still avoiding dramatic changes in the renormalization group (RG) gauge running.

We arrange the paper in the following way: we define the model and specify the terms in the superpotential in section 2. We find a suitable vacuum solution, which breaks E_6 into the SM group in section 3, perform the doublet-triplet splitting in section 4 and compute the mass matrices in the Yukawa sector and identify the low energy content of the theory in section 5. We then use the obtained results to perform a 2 generation fit of the masses and mixing angles in section 6. We finish with a discussion in section 7. Five different appendices collect various definitions and technical results. All explicit computation in E_6 has been performed using methods from [21] and [11]; also, [22–24] might also prove useful to the reader.

For ease of use we stick to the following color convention in this paper: red denotes VEVs coming from the spontaneous symmetry breaking of the unified theory at the scale M_{GUT}, while blue denotes VEVs coming from electroweak (EW) symmetry breaking at the scale m_{EW}.

2 Defining the model

The renormalizable E_6 SUSY GUT that we consider here is an alternative to the model in [11]. It is motivated by the fact that the minimal Higgs sector in a renormalizable SUSY E_6 model, which can break E_6 to the Standard Model, is $351' + \overline{351'} + 27 + \overline{27}$. This minimal breaking sector cannot accommodate doublet-triplet splitting, however, and therefore needs to be extended to get a realistic model. One possible extension is an addition of a $27 + \overline{27}$ pair, which was analyzed in [11], while an alternative, taken in this paper, is to extend it by the representation 78 instead. The Yukawa sector in the present model will consist of only two matrices, unlike that in [11], which has three matrices.

Our renormalizable E_6 model thus contains the following:

- The "fermionic sector" of three copies of a chiral supermultiplet 27, denoted by 27_F^i, with $i = 1, 2, 3$. Also, we assume a $\mathbb{Z}_2$ *matter parity*, under which the 27_F are -1, and the remaining chiral superfields are $+1$. With this symmetry, the ansatz $\langle 27_F \rangle = 0$ is consistent with the equations of motion, which we shall adopt.

- The "breaking sector" consisting of $351' + \overline{351'} + 27 + \overline{27} + 78$.

The model under consideration is supersymmetric. The problem of SUSY breaking will not be considered, since it is (usually) an orthogonal problem to breaking the gauge

group. One can imagine however, that whatever the detailed mechanism of SUSY breaking might be, we get soft SUSY breaking terms at energies not far above m_{EW}, such that we get automatic unification of gauge couplings at M_{GUT} assuming no extra particle states up to the unification scale. The effective low energy theory of our model will thus be MSSM. The soft SUSY breaking terms do not give contributions to the fermion masses at tree level. A fit to fermion masses and mixings can then be done without specifying the details of SUSY breaking, although the RG evolution of these parameters does depend on the SUSY scale (which we assume is of order few TeV).

The full superpotential of our model is

$$
\begin{aligned}
W_{\mathrm{full}} = {} & m_{27}\, 27\, \overline{27}\ +\ m_{351'}\, 351'\, \overline{351'}\ +\ m_{78}\, 78\, 78 \\
& +\ \lambda_1\, 351'^3\ +\ \lambda_2\, \overline{351'}^3\ +\ \lambda_3\, 27^2\, \overline{351'}\ +\ \lambda_4\, \overline{27}^2\, 351' \\
& +\ \lambda_5\, 27^3\ +\ \lambda_6\, \overline{27}^3\ +\ \lambda_7\, \overline{27}\, 78\, 27\ +\ \lambda_8\, \overline{351'}\, 78\, 351' \\
& +\ \frac{1}{2} Y_{27}^{ij}\, 27_F^i\, 27_F^j\, 27\ +\ \frac{1}{2} Y_{351'}^{ij}\, 27_F^i\, 27_F^j\, \overline{351'}.
\end{aligned}
\tag{2.1}
$$

Note that the definitions of $351'$ and $\overline{351'}$ used here are switched compared to Slansky's defintions [25].

3 Spontaneous symmetry breaking

3.1 Equations of motion

The Higgs sector $351' + \overline{351'} + 27 + \overline{27} + 78$ forms a realistic Higgs sector, which is able to break the gauge group from E_6 to $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. We provide just such a vacuum below.

First, note the decompositions of 27, 78 and $351'$ under $\mathrm{SO}(10) \times \mathrm{U}(1)$:

$$
27 = 16(1) + 10(-2) + 1(4),
\tag{3.1}
$$

$$
78 = 45(0) + 16(-3) + \overline{16}(3) + 1(0),
\tag{3.2}
$$

$$
351' = 1(8) + 10(2) + 16(5) + 54(-4) + 126(2) + \overline{144}(-1).
\tag{3.3}
$$

The representation $351'$ contains 5 SM singlets, 3 of which are $\mathrm{SU}(5)$ singlets (in 1, 16 and 126 of $\mathrm{SO}(10)$), and 2 are part of a 24 under $\mathrm{SU}(5)$ (in 54 and $\overline{144}$ of $\mathrm{SO}(10)$). Similarly, the representation 78 also has 5 singlets, with one being a 24 under $\mathrm{SU}(5)$ (the one in 45 of $\mathrm{SO}(10)$), while the remaining ones are singlets under $\mathrm{SU}(5)$ (the 1, 16 and $\overline{16}$ of $\mathrm{SO}(10)$, as well as another one in the 45). The Higgs sector $351' + \overline{351'} + 27 + \overline{27} + 78$ therefore contains $5 + 5 + 2 + 2 + 5 = 19$ singlets in total. We list their VEVs[2] in table 1.

---

[2]Notice that the Standard Model singlet VEVs have been denoted by u_1, u_2, v, w and y. The notation from [4] is changed due to the states now being those which have well defined transformation properties under the $\mathrm{SU}(5)$ and $\mathrm{SO}(10)$ subgroups of E_6. The connection between the two notations is $u_1 \propto a_1$, $u_2 \propto a_2$, $w \propto a_3 - b_3$, $v \propto -\sqrt{3}a_3 + 2a_4 - \sqrt{3}b_3$, $y \propto a_3 + \sqrt{3}a_4 + b_3$, with the usual normalization $\langle 78^i{}_j 78^{*j}_i \rangle = |u_1|^2 + |u_2|^2 + |v|^2 + |w|^2 + |y|^2$.

label	$\subseteq$ PS	$\subseteq$ SU(5)	$\subseteq$ SO(10)	$\subseteq$ E$_6$	label	$\subseteq$ PS	$\subseteq$ SU(5)	$\subseteq$ SO(10)	$\subseteq$ E$_6$
c_1	$(1,1,1)$	1	1	27	d_1	$(1,1,1)$	1	1	$\overline{27}$
c_2	$(1,2,\overline{4})$	1	16	27	d_2	$(1,2,4)$	1	$\overline{16}$	$\overline{27}$
e_1	$(1,3,10)$	1	126	$351'$	f_1	$(1,3,\overline{10})$	1	$\overline{126}$	$\overline{351'}$
e_2	$(1,2,\overline{4})$	1	16	$351'$	f_2	$(1,2,4)$	1	$\overline{16}$	$\overline{351'}$
e_3	$(1,1,1)$	1	1	$351'$	f_3	$(1,1,1)$	1	1	$\overline{351'}$
e_4	$(1,1,1)$	24	54	$351'$	f_4	$(1,1,1)$	24	54	$\overline{351'}$
e_5	$(1,2,4)$	24	$\overline{144}$	$351'$	f_5	$(1,2,\overline{4})$	24	144	$\overline{351'}$
u_1	$(1,2,\overline{4})$	1	16	78					
u_2	$(1,2,4)$	1	$\overline{16}$	78					
w	$(1,1,1)$	1	1	78					
v	/	1	45	78					
y	/	24	45	78					

Table 1. SM singlet VEVs in our Higgs sector.

With the ansatz $\langle 27_F \rangle = 0$, the breaking part of the superpotential is

$$
\begin{aligned}
W = {} & m_{351'}\, I_{351' \times \overline{351'}} + m_{27}\, I_{27 \times \overline{27}} + m_{78}\, I_{78 \times 78} \\
& + \lambda_1\, I_{351'^3} + \lambda_2\, I_{\overline{351'}^3} + \lambda_3\, I_{27^2 \times \overline{351'}} + \lambda_4\, I_{\overline{27}^2 \times 351'} \\
& + \lambda_5\, I_{27^3} + \lambda_6\, I_{\overline{27}^3} + \lambda_7\, I_{27 \times 78 \times \overline{27}} + \lambda_8\, I_{351' \times 78 \times \overline{351'}}.
\end{aligned}
\tag{3.4}
$$

Explicit computation yields the following expressions for the superpotential invariants (VEV terms only):

$$
I_{351' \times \overline{351'}} = \overline{351'}_{\mu\nu}\, 351'^{\mu\nu} = e_1 f_1 + e_2 f_2 + e_3 f_3 + e_4 f_4 + e_5 f_5,
\tag{3.5}
$$

$$
I_{27 \times \overline{27}} = \overline{27}_\mu\, 27^\mu = c_1 d_1 + c_2 d_2,
\tag{3.6}
$$

$$
I_{78 \times 78} = 78^\mu{}_\nu\, 78^\nu{}_\mu = 2 u_1 u_2 + w^2 + v^2 + y^2,
\tag{3.7}
$$

$$
I_{351'^3} = 351'^{\mu\alpha}\, 351'^{\nu\beta}\, 351'^{\lambda\gamma}\, d_{\alpha\beta\gamma} d_{\mu\nu\lambda} = 3\left(e_3 e_4{}^2 + e_1 e_5{}^2 - \sqrt{2} e_2 e_4 e_5 \right),
\tag{3.8}
$$

$$
I_{\overline{351'}^3} = \overline{351'}_{\mu\alpha}\, \overline{351'}_{\nu\beta}\, \overline{351'}_{\lambda\gamma}\, d^{\alpha\beta\gamma}\, d^{\mu\nu\lambda} = 3\left(f_3 f_4{}^2 + f_1 f_5{}^2 - \sqrt{2} f_2 f_4 f_5 \right),
\tag{3.9}
$$

$$
I_{27^2 \times \overline{351'}} = \overline{351'}_{\mu\nu}\, 27^\mu\, 27^\nu = c_2{}^2 f_1 + \sqrt{2} c_1 c_2 f_2 + c_1{}^2 f_3,
\tag{3.10}
$$

$$
I_{\overline{27}^2 \times 351'} = 351'^{\mu\nu}\, \overline{27}_\mu\, \overline{27}_\nu = d_2{}^2 e_1 + \sqrt{2} d_1 d_2 e_2 + d_1{}^2 e_3,
\tag{3.11}
$$

$$
I_{27^3} = 27^\mu\, 27^\nu\, 27^\lambda\, d_{\mu\nu\lambda} = 0,
\tag{3.12}
$$

$$
I_{\overline{27}^3} = \overline{27}_\mu\, \overline{27}_\nu\, \overline{27}_\lambda\, d^{\mu\nu\lambda} = 0,
\tag{3.13}
$$

$$
\begin{aligned}
I_{27 \times 78 \times \overline{27}} = {} & \overline{27}_\mu\, 78^\mu{}_\nu\, 27^\nu \\
= {} & \frac{1}{\sqrt{6}} u_1 c_1 d_2 + \frac{1}{\sqrt{6}} u_2 c_2 d_1 - \frac{1}{6\sqrt{2}} w\left(4 c_1 d_1 + c_2 d_2\right) - \frac{1}{2}\sqrt{\frac{5}{6}} v c_2 d_2,
\end{aligned}
\tag{3.14}
$$

$$I_{351'\times 78\times \overline{351'}} = \overline{351'}_{\mu\nu}\, 78^{\nu}{}_{\kappa}\, 351'^{\kappa\mu} =$$
$$= \frac{1}{2\sqrt{6}}u_1\left(\sqrt{2}e_2f_1 + \sqrt{2}e_3f_2 + e_5f_4\right) + \frac{1}{2\sqrt{6}}u_2\left(\sqrt{2}e_1f_2 + \sqrt{2}e_2f_3 + e_4f_5\right)$$
$$+ \frac{1}{12\sqrt{2}}w\left(-2e_1f_1 - 5e_2f_2 - 8e_3f_3 + 4e_4f_4 + e_5f_5\right)$$
$$- \frac{1}{4}\sqrt{\frac{5}{6}}v\left(2e_1f_1 + e_2f_2 - e_5f_5\right). \tag{3.15}$$

The general formula for D-terms in our case yields

$$D^A = (27^\dagger)_\mu\,(\hat{t}^A\,27)^\mu + (\overline{27}^\dagger)^\mu\,(\hat{t}^A\,\overline{27})_\mu + (78^\dagger)^\nu{}_\mu\,(\hat{t}^A\,78)^\mu{}_\nu$$
$$+ (351'^\dagger)_{\mu\nu}\,(\hat{t}^A\,351')^{\mu\nu} + (\overline{351'}^\dagger)^{\mu\nu}\,(\hat{t}^A\,\overline{351'})_{\mu\nu}. \tag{3.16}$$

Of the 78 D-terms, 5 vanish non-trivially, corresponding to the following generators of the $\mathrm{SU}(3)_C \times \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ subgroup of E_6: t^8_L, t^3_R, t^6_R, t^7_R, t^8_R, given explicitly by

$$D^8_L = \frac{1}{\sqrt{3}}\left(|c_1|^2 + |c_2|^2 + 2|e_1|^2 + 2|e_2|^2 + 2|e_3|^2 - |e_4|^2 - |e_5|^2\right.$$
$$\left. -|d_1|^2 - |d_2|^2 - 2|f_1|^2 - 2|f_2|^2 - 2|f_3|^2 + |f_4|^2 + |f_5|^2\right), \tag{3.17}$$

$$D^3_R = \frac{1}{6}\left(-3|c_2|^2 - 6|e_1|^2 - 3|e_2|^2 + 3|e_5|^2 - |u_1|^2\right.$$
$$\left. +3|d_2|^2 + 6|f_1|^2 + 3|f_2|^2 - 3|f_5|^2 + |u_2|^2\right), \tag{3.18}$$

$$D^6_R = \frac{1}{12}\left(6c_2c_1{}^* + 6c_1c_2{}^* - \sqrt{3}u_1w^* - \sqrt{3}wu_1{}^* + \sqrt{5}u_1v^* + \sqrt{5}vu_1{}^*\right.$$
$$-6d_2d_1{}^* - 6d_1d_2{}^* + \sqrt{3}u_2w^* + \sqrt{3}wu_2{}^* - \sqrt{5}u_2v^* - \sqrt{5}vu_2{}^*$$
$$+6\sqrt{2}e_1e_2{}^* + 6\sqrt{2}e_2e_1{}^* + 6\sqrt{2}e_2e_3{}^* + 6\sqrt{2}e_3e_2{}^* + 6e_4e_5{}^* + 6e_5e_4{}^*$$
$$\left. -6\sqrt{2}f_1f_2{}^* - 6\sqrt{2}f_2f_1{}^* - 6\sqrt{2}f_2f_3{}^* - 6\sqrt{2}f_3f_2{}^* - 6f_4f_5{}^* - 6f_5f_4{}^*\right), \tag{3.19}$$

$$D^7_R = \frac{i}{12}\left(6c_2c_1{}^* - 6c_1c_2{}^* - \sqrt{3}u_1w^* + \sqrt{3}wu_1{}^* + \sqrt{5}u_1v^* - \sqrt{5}vu_1{}^*\right.$$
$$+6d_2d_1{}^* - 6d_1d_2{}^* - \sqrt{3}u_2w^* + \sqrt{3}wu_2{}^* + \sqrt{5}u_2v^* - \sqrt{5}vu_2{}^*$$
$$+6\sqrt{2}e_1e_2{}^* - 6\sqrt{2}e_2e_1{}^* + 6\sqrt{2}e_2e_3{}^* - 6\sqrt{2}e_3e_2{}^* + 6e_4e_5{}^* - 6e_5e_4{}^*$$
$$\left. +6\sqrt{2}f_1f_2{}^* - 6\sqrt{2}f_2f_1{}^* + 6\sqrt{2}f_2f_3{}^* - 6\sqrt{2}f_3f_2{}^* + 6f_4f_5{}^* - 6f_5f_4{}^*\right), \tag{3.20}$$

$$D^8_R = \frac{1}{2\sqrt{3}}\left(-2|c_1|^2 + |c_2|^2 + 2|e_1|^2 - |e_2|^2 - 4|e_3|^2 + 2|e_4|^2 - |e_5|^2 + |u_1|^2\right.$$
$$\left. +2|d_1|^2 - |d_2|^2 - 2|f_1|^2 + |f_2|^2 + 4|f_3|^2 - 2|f_4|^2 + |f_5|^2 - |u_2|^2\right). \tag{3.21}$$

They can be rewritten into 3 independent D-terms:

$$D^I = |c_1|^2 - |d_1|^2 + |e_2|^2 - |f_2|^2 + 2|e_3|^2 - 2|f_3|^2 - |e_4|^2 + |f_4|^2 - \frac{1}{3}|u_1|^2 + \frac{1}{3}|u_2|^2, \tag{3.22}$$

$$D^{II} = |c_2|^2 - |d_2|^2 + |e_2|^2 - |f_2|^2 + 2|e_1|^2 - 2|f_1|^2 - |e_5|^2 + |f_5|^2 - \frac{1}{3}|u_2|^2 + \frac{1}{3}|u_1|^2, \tag{3.23}$$

$$D^{III} = +c_1c_2{}^* - \frac{\sqrt{3}}{6}wu_1{}^* + \frac{\sqrt{5}}{6}vu_1{}^* + \sqrt{2}e_2e_1{}^* + \sqrt{2}e_3e_2{}^* + e_5e_4{}^*$$
$$- d_2d_1{}^* + \frac{\sqrt{3}}{6}u_2w^* - \frac{\sqrt{5}}{6}u_2v^* - \sqrt{2}f_1f_2{}^* - \sqrt{2}f_2f_3{}^* - f_4f_5{}^*, \tag{3.24}$$

via the definitions $D^I := \sqrt{3}D_L^8 + 2D_R^3$, $D^{II} := -2D_R^3$ and $D^{III} := D_R^6 + iD_R^7$, where D^{III} now forms a complex equation. The other independent combination $D^Y = D_L^8 + \sqrt{3}D_R^3 + D_R^8$ is trivially zero, as it should be, since this D-term corresponds to the unbroken generator of $U(1)_Y$.

3.2 A specific vacuum solution

In this section, we obtain a vacuum solution which breaks the gauge group to the Standard Model group $SU(3)_C \times SU(2)_L \times U(1)_Y$. Due to the complexity of the equations we are unable to provide a full classification of vacua; a short discussion on alternative vacua can be found in appendix D.

The equations of motion in SUSY models are

$$D^a = 0, \tag{3.25}$$

$$F_\phi = 0, \tag{3.26}$$

with the usual definition of the F-term:

$$F_\phi := \frac{\partial W}{\partial \phi}. \tag{3.27}$$

In our case, there are 19 Standard Model singlets in the Higgs sector, giving 19 non-trivial F-terms, which can be easily reconstructed from the superpotential in equation (3.4) and the all-singlet terms of the invariants given by equations (3.5)–(3.15). The non-trivial D-terms are given by equations (3.22)–(3.24).

To obtain a vacuum solution, we perform the following steps:

- First, we notice that F_y leads directly to $y = 0$ (y is present only in the mass term 78^2). By taking the self-consistent ansatz[3]

$$c_1 = d_1 = f_5 = e_5 = 0, \tag{3.28}$$

$$u_1 = u_2 = e_2 = f_2 = 0, \tag{3.29}$$

the system of equations is greatly simplified: $F_{c_1}, F_{d_1}, F_{e_2}, F_{f_2}, F_{e_5}, F_{f_5}, F_{u_1}, F_{u_2}, F_y$ and D^{III} are solved automatically.

- Solve F_{e_3} and F_{f_3} for f_3 and e_3, respectively to get

$$e_3 = -\frac{9\lambda_2 f_4{}^2}{3m_{351'} - \sqrt{2}\lambda_8 w}, \qquad f_3 = -\frac{9\lambda_1 e_4{}^2}{3m_{351'} - \sqrt{2}\lambda_8 w}. \tag{3.30}$$

- Solve F_{c_2} and F_{d_2} for f_1 and e_1, respectively to get

$$e_1 = \frac{c_2}{24\lambda_4 d_2} \left(\sqrt{2}\lambda_7 \left(\sqrt{15}v + w \right) - 12m_{27} \right), \tag{3.31}$$

$$f_1 = \frac{d_2}{24\lambda_3 c_2} \left(\sqrt{2}\lambda_7 \left(\sqrt{15}v + w \right) - 12m_{27} \right). \tag{3.32}$$

---

[3]This ansatz is motivated by the ansatz used in the model with the 78 omitted [11], where a classification of vacua is known. See appendix D for further discussion.

- Simultaneously solve F_{e_4} and F_{f_4} for f_4:

$$f_4 = \frac{(3\sqrt{2}m_{351'} - 2\lambda_8 w)(3\sqrt{2}m_{351'} + \lambda_8 w)}{324 e_4 \lambda_1 \lambda_2}. \tag{3.33}$$

- Simultaneously solve F_{e_1} and F_{f_1} for d_2:

$$d_2 = \frac{\left(6\sqrt{2}m_{27} - \lambda_7(\sqrt{15}v + w)\right)\left(6\sqrt{2}m_{351'} - \lambda_8(\sqrt{15}v + w)\right)}{144\lambda_3\lambda_4 c_2}. \tag{3.34}$$

- It is now convenient to define a new quantity $A := \sqrt{15}v + w$. We can now solve F_v for v as a linear equation:

$$v = \frac{\sqrt{10}A^2\lambda_7^2\lambda_8 - 8\sqrt{5}\lambda_7 A(m_{351'}\lambda_7 + 2m_{27}\lambda_8) + 24\sqrt{10}m_{27}(2m_{351'}\lambda_7 + m_{27}\lambda_8)}{768\sqrt{3}m_{78}\lambda_3\lambda_4}. \tag{3.35}$$

- Three variables remain to be determined: A, c_2 and e_4. We are left with only one unsolved F-term F_w, which is a polynomial in A:

$$0 = P_0 + P_1 A + P_2 A^2 + P_3 A^3 + P_4 A^4, \tag{3.36}$$

$$P_0 = -576m_{27}\left(2m_{351'}\lambda_7 + m_{27}\lambda_8\right)\Big(25m_{27}\left(2m_{351'}\lambda_7 + m_{27}\lambda_8\right)\lambda_8^3$$
$$- 480m_{351'}m_{78}\lambda_3\lambda_4\lambda_8^2 + 110592m_{78}^2\lambda_1\lambda_2\lambda_3\lambda_4\Big), \tag{3.37}$$

$$P_1 = 192\sqrt{2}\Big(995328\lambda_1\lambda_2\lambda_3^2\lambda_4^2 m_{78}^3$$
$$+ 4608\lambda_3\lambda_4\left(24m_{27}\lambda_1\lambda_2\lambda_7\lambda_8 + m_{351'}\left(12\lambda_1\lambda_2\lambda_7^2 - \lambda_3\lambda_4\lambda_8^2\right)\right)m_{78}^2$$
$$+ 240\lambda_3\lambda_4\lambda_8^2\left(-m_{351'}^2\lambda_7^2 + 2m_{351'}m_{27}\lambda_8\lambda_7 + 2m_{27}^2\lambda_8^2\right)m_{78}$$
$$+ 25m_{27}\lambda_7\lambda_8^3\left(2m_{351'}^2\lambda_7^2 + 5m_{351'}m_{27}\lambda_8\lambda_7 + 2m_{27}^2\lambda_8^2\right)\Big), \tag{3.38}$$

$$P_2 = -16\Big(\lambda_8\left(18432\lambda_3\lambda_4\left(9\lambda_1\lambda_2\lambda_7^2 + \lambda_3\lambda_4\lambda_8^2\right)m_{78}^2\right.$$
$$+ 240\lambda_3\lambda_4\lambda_7\lambda_8^2\left(5m_{351'}\lambda_7 + 16m_{27}\lambda_8\right)m_{78}$$
$$\left.+ 25\lambda_7^2\lambda_8^2\left(2m_{351'}^2\lambda_7^2 + 14m_{351'}m_{27}\lambda_8\lambda_7 + 11m_{27}^2\lambda_8^2\right)\right)\Big), \tag{3.39}$$

$$P_3 = 40\sqrt{2}\lambda_7^2\lambda_8^4\left(96m_{78}\lambda_3\lambda_4 + 5\lambda_7\left(m_{351'}\lambda_7 + 2m_{27}\lambda_8\right)\right), \tag{3.40}$$

$$P_4 = -25\lambda_7^4\lambda_8^5. \tag{3.41}$$

Note that the coefficients P_i depend only on the Lagrangian parameters; choosing those, we can determine A numerically.

Finally, we solve the remaining D-terms and determine c_2 and e_4 from

$$0 = D^I = 2|e_3|^2 - |e_4|^2 - 2|f_3|^2 + |f_4|^2, \tag{3.42}$$

$$0 = D^{II} = |c_2|^2 - |d_2|^2 + 2|e_1|^2 - 2|f_1|^2. \tag{3.43}$$

Note that v and w are determined, once A is fixed. We can therefore see that $f_4 \propto 1/e_4$, $e_3 \propto f_4^2 \propto 1/e_4^2$ and $f_3 \propto e_4^2$. Therefore D^I can be written as a quartic polynomial in $|e_4|^2$; the constant term has a positive coefficient (the e_3 term), while the highest order term in $|e_4|^2$ has a negative coefficient, a solution for a real $e_4 > 0$ will always exist. Similarly, $d_2 \propto 1/c_2$, $e_1 \propto c_2/d_2 \propto c_2^2$ and $f_1 \propto d_2/c_2 \propto 1/c_2^2$, D^{II} is a quartic polynomial in c_2 independent of e_4; the constant coefficient will be negative (the f_1 term), while the highest order coefficient in $|c_2|^2$ is positive (the e_1 term), which again guarantees a real solution $c_2 > 0$.

The initial Lagrangian parameters constitute of the 3 masses m_{27}, $m_{351'}$, m_{78} and 8 massless parameters λ_i, $i = 1, \ldots, 8$. The simplest order in which to compute the given vacuum solution with these parameters is given below:

1. Take the ansatz for some of the VEVs (symmetric under conjugation symmetry [11])

$$c_1 = 0, \qquad\qquad d_1 = 0,$$
$$e_2 = 0, \qquad\qquad f_2 = 0,$$
$$e_5 = 0, \qquad\qquad f_5 = 0,$$
$$u_1 = 0, \qquad\qquad u_2 = 0,$$
$$y = 0. \tag{3.44}$$

2. A is determined through the polynomial, and then the VEVs v and w are determined by

$$v = \frac{\sqrt{10}A^2\lambda_7^2\lambda_8 - 8\sqrt{5}\lambda_7 A(m_{351'}\lambda_7 + 2m_{27}\lambda_8) + 24\sqrt{10}m_{27}(2m_{351'}\lambda_7 + m_{27}\lambda_8)}{768\sqrt{3}m_{78}\lambda_3\lambda_4}, \tag{3.45}$$

$$w = A + \frac{5}{768m_{78}\lambda_3\lambda_4}\left(-24\sqrt{2}m_{27}(2m_{351'}\lambda_7 + m_{27}\lambda_8)\right.$$
$$\left. +\lambda_7 A(8m_{351'}\lambda_7 + 16m_{27}\lambda_8 - \sqrt{2}A\lambda_7\lambda_8)\right). \tag{3.46}$$

3. e_4 and c_2 are determined through D^I and D^{II}, respectively, using eqs. (3.47)–(3.52).

4. The remaining nonvanishing VEVs are $d_2, e_1, f_1, e_3, f_3, f_4$, and they can be computed in terms of A, w, c_2, e_4:

$$d_2 = \frac{1}{144\lambda_3\lambda_4 c_2}\left(6\sqrt{2}m_{27} - \lambda_7 A\right)\left(6\sqrt{2}m_{351'} - \lambda_8 A\right), \tag{3.47}$$

$$e_1 = \frac{12\lambda_3 c_2^2}{\sqrt{2}A\lambda_8 - 12m_{351'}}, \tag{3.48}$$

$$f_1 = \frac{\sqrt{2}A\lambda_7 - 12m_{27}}{3456\lambda_3^2\lambda_4 c_2^2}\left(6\sqrt{2}m_{27} - \lambda_7 A\right)\left(6\sqrt{2}m_{351'} - \lambda_8 A\right), \tag{3.49}$$

$$e_3 = \frac{\left(3\sqrt{2}m_{351'} - 2\lambda_8 w\right)\left(3\sqrt{2}m_{351'} + \lambda_8 w\right)^2}{5832\sqrt{2}e_4^2\lambda_1^2\lambda_2}, \tag{3.50}$$

$$f_3 = -\frac{9e_4{}^2\lambda_1}{3m_{351'} - \sqrt{2}w\lambda_8}, \tag{3.51}$$

$$f_4 = \frac{1}{324e_4\lambda_1\lambda_2}\left(3\sqrt{2}m_{351'} - 2\lambda_8 w\right)\left(3\sqrt{2}m_{351'} + \lambda_8 w\right). \tag{3.52}$$

The solution above does indeed break E_6 into the Standard Model group. This can be checked by explicitly computing the gauge boson masses, found in table 5 of appendix A. It is possible to further illuminate this result by considering that under the standard embeddings of the E_6 subgroups from table 1, the $SU(5)$ breaking in our solution is solely due to non-vanishing VEVs e_4 and f_4.

4 Doublet-triplet splitting

We tackle now the issue of doublet-triplet splitting. We denote the doublets and anti-doublets by $D \sim (1,2,+1/2)$ and $\overline{D} \sim (1,2,-1/2)$, while the triplets and antitriplets are denoted by $T \sim (3,1,-1/3)$ and $\overline{T} \sim (\bar{3},1,+1/3)$. The detailed labels of these states are given in table 6 of appendix B.

In the Higgs sector of our model, there are 12 doublet-antidoublet pairs and 13 triplet-antitriplet pairs. This is one extra pair of each compared to the renormalizable model with the Higgs sector $351' + \overline{351'} + 27 + \overline{27}$, in which doublet-triplet splitting surprisingly fails (see [11] for details). The extra states come from the added 78, and are labeled by the index 0, i.e. $D_0, \overline{D}_0, T_0, \overline{T}_0$. We shall see that this extra row and column, together with a new vacuum compared to the model without 78 now enable doublet-triplet splitting in the usual way (by fine-tuning). Note that all the doublets and triplets are located in a 5 or 45 (or their conjugates) of $SU(5)$, except for one extra triplet in the 50 of $SU(5)$.

The mass terms for the doublets and triplets are written as

$$\left(D_0 \cdots D_{11}\right)\mathcal{M}_{\text{doublets}}\begin{pmatrix}\overline{D}_1\\\vdots\\\overline{D}_{11}\end{pmatrix} + \left(T_0 \cdots T_{12}\right)\mathcal{M}_{\text{triplets}}\begin{pmatrix}\overline{T}_1\\\vdots\\\overline{T}_{12}\end{pmatrix}. \tag{4.1}$$

The matrices $\mathcal{M}_{\text{doublets}}$ and $\mathcal{M}_{\text{triplets}}$ are similar; we can compactly write a 13×13 matrix $\mathcal{M}$ with block form

$$\mathcal{M} = \begin{pmatrix}M_{11} & M_{12}\\M_{21} & M_{22}\end{pmatrix}, \tag{4.2}$$

with the diagonal blocks defined by

$$M_{11} = \begin{pmatrix}
m_{78} & 0 & \frac{\lambda_7 d_2}{2\sqrt{3}} & 0 & 0 & 0 \\
0 & m_{27} + \frac{\lambda_7(\sqrt{3}v + \sqrt{5}w)}{3\sqrt{10}} & \frac{\alpha\lambda_3 f_4}{\sqrt{15}} & 6\lambda_5 c_2 & 0 & 0 \\
\frac{\lambda_7 c_2}{2\sqrt{3}} & \frac{\alpha\lambda_4 e_4}{\sqrt{15}} & m_{27} + \frac{\lambda_7(\sqrt{5}w - \sqrt{3}v)}{3\sqrt{10}} & 0 & 0 & 0 \\
0 & 6\lambda_6 d_2 & 0 & m_{27} + \frac{\lambda_7(3\sqrt{3}v - \sqrt{5}w)}{6\sqrt{10}} & 0 & -\frac{\lambda_4 d_2}{\sqrt{10}} \\
0 & 0 & 0 & 0 & m_{351'} + \frac{\lambda_8 v}{2\sqrt{30}} - \frac{\lambda_8 w}{6\sqrt{2}} & \sqrt{\frac{3}{5}}\alpha\lambda_1 e_4 \\
0 & 0 & 0 & -\frac{\lambda_3 c_2}{\sqrt{10}} & \sqrt{\frac{3}{5}}\alpha\lambda_2 f_4 & m_{351'} - \frac{\lambda_8 v}{2\sqrt{30}} - \frac{\lambda_8 w}{6\sqrt{2}}
\end{pmatrix},$$

$$\tag{4.3}$$

$$M_{22} = \begin{pmatrix}
m_{351'}+\frac{\sqrt{3}}{4\sqrt{10}}\lambda_8 v-\frac{5\lambda_8 w}{12\sqrt{2}} & 0 & 0 & \frac{1}{2}\sqrt{3}\alpha\lambda_2 f_4 \\
0 & m_{351'}-\frac{\lambda_8 v}{2\sqrt{30}}-\frac{\lambda_8 w}{6\sqrt{2}} & \sqrt{5}\beta\lambda_2 f_4 & 0 \\
0 & \sqrt{5}\beta\lambda_1 e_4 & m_{351'}+\frac{\lambda_8 v}{2\sqrt{30}}-\frac{\lambda_8 w}{6\sqrt{2}} & 0 \\
\frac{1}{2}\sqrt{3}\alpha\lambda_1 e_4 & 0 & 0 & m_{351'}-\frac{\sqrt{3}}{4\sqrt{10}}\lambda_8 v+\frac{\lambda_8 w}{12\sqrt{2}}, \\
0 & 0 & 0 & 0 \\
\frac{1}{2}\sqrt{15}\beta\lambda_1 e_4 & 0 & 0 & 0 \\
0 & 0 & 2\sqrt{10}\lambda_2 f_4 & 0
\end{pmatrix}$$

$$\begin{pmatrix}
0 & \frac{1}{2}\sqrt{15}\beta\lambda_2 f_4 & 0 \\
0 & 0 & 0 \\
0 & 0 & 2\sqrt{10}\lambda_1 e_4 \\
0 & 0 & 0 \\
m_{351'}-\frac{7\lambda_8 v}{4\sqrt{30}}+\frac{\lambda_8 w}{12\sqrt{2}} & 0 & 0 \\
0 & m_{351'}-\frac{\sqrt{3}}{4\sqrt{10}}\lambda_8 v+\frac{\lambda_8 w}{12\sqrt{2}} & 0 \\
0 & 0 & m_{351'}-\frac{\lambda_8 v}{2\sqrt{30}}-\frac{\lambda_8 w}{6\sqrt{2}}
\end{pmatrix} \qquad (4.4)$$

and the off-diagonal blocks defined by

$$M_{12} = \begin{pmatrix}
-\frac{\lambda_8 f_3}{2\sqrt{6}} & 0 & 0 & \frac{\alpha\lambda_8 e_4}{24\sqrt{2}} & -\frac{\lambda_8 f_1}{2\sqrt{6}} & \frac{\sqrt{5}\beta\lambda_8 e_4}{24\sqrt{2}} & 0 \\
0 & 0 & 0 & 2\sqrt{\frac{2}{5}}\lambda_3 c_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{2}\lambda_4 d_2 & 0 & 0 \\
0 & -\sqrt{\frac{3}{2}}\lambda_4 d_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \qquad (4.5)$$

$$M_{21} = \begin{pmatrix}
-\frac{\lambda_8 e_3}{2\sqrt{6}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{\frac{3}{2}}\lambda_3 c_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\alpha\lambda_8 f_4}{24\sqrt{2}} & 2\sqrt{\frac{2}{5}}\lambda_4 d_2 & 0 & 0 & 0 & 0 \\
-\frac{\lambda_8 e_1}{2\sqrt{6}} & 0 & -\sqrt{2}\lambda_3 c_2 & 0 & 0 & 0 \\
\frac{\sqrt{5}\beta\lambda_8 f_4}{24\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \qquad (4.6)$$

The separation of the matrix $\mathcal{M}$ into 4 blocks is arbitrary and is used above merely as a simple way of presenting a large matrix. The matrix $\mathcal{M}_{\text{doublets}}$ is obtained out of $\mathcal{M}$ by removing the last row and column, and taking $\alpha = -3$, $\beta = -\sqrt{3}$, while the matrix $\mathcal{M}_{\text{triplets}}$ is obtained by taking $\alpha = \beta = 2$. Note that the matrices were already simplified by taking the vacuum ansatz of vanishing VEVs

$$c_1 = d_1 = e_2 = f_2 = e_5 = f_5 = u_1 = u_2 = y = 0, \qquad (4.7)$$

while the rest of the vacuum solution was not plugged-in. Notice also that the coefficients α and β are located as factors in front of e_4 or f_4; this is expected, since e_4 and f_4 are the SU(5)-breaking VEVs, so they control the difference between the doublets and triplets. Also, the coefficients α are -3 and 2 for the doublets, respectively, which come from the VEV $\langle 24 \rangle$ in the terms $5 \cdot \langle 24 \rangle \cdot \bar{5}$ of SU(5). The coefficients β are $-\sqrt{3}$ and 2 for the

doublets and triplets respectively, which are the Clebsch-Gordan coefficients coming from the terms $5 \cdot \langle 24 \rangle \cdot \overline{45}$ or $\bar{5} \cdot \langle 24 \rangle \cdot 45$ of SU(5). One can check that this is indeed the case by locating the position of α and β coefficients in $\mathcal{M}$ and checking, to which states in table 6 these matrix entries correspond to.

The procedure for fine-tuning is now similar to the one attempted in the $351' + \overline{351'} + 27 + \overline{27}$ case in [11]. Once the vacuum solution is plugged-in, the determinants of the two matrices become zero:

$$\det(\mathcal{M}_{\text{doublets}})\Big|_{\text{vacuum}} = 0, \tag{4.8}$$

$$\det(\mathcal{M}_{\text{triplets}})\Big|_{\text{vacuum}} = 0. \tag{4.9}$$

The massless doublet-antidoublet and triplet-antitriplet pairs are simply the unphysical would-be Goldstone bosons from the breaking of E_6. They correspond to the broken generators with the same quantum numbers, which can be found in SO(10) language in the 16 and $\overline{16}$ parts of the adjoint 78. The doublet-triplet splitting condition then requires another doublet-antidoublet pair to be massless, which imposes the following condition:

$$\text{Cond}(\mathcal{M}) := \text{pmin}_1 \mathcal{M}, \tag{4.10}$$

where pmin_i denotes the i-th principal minor of rank 1 (the subdeterminant when the i-th row and column are removed). In our specific case, we chose to remove the first row and column, which correspond to the fields D_0 and $\overline{D}_0$. We chose these due to simplicity, but note that this choice is valid only if the Goldstone modes of the given vacuum have nonzero D_0 and $\overline{D}_0$ components. This can be checked later on via equations (4.18) and (4.19). The logic behind the whole method is explained in appendix C.

Given the notation above, DT-splitting can be performed by a fine-tuning:

$$\text{Cond}(\mathcal{M}_{\text{doublets}}) = 0, \tag{4.11}$$

$$\text{Cond}(\mathcal{M}_{\text{triplets}}) \neq 0. \tag{4.12}$$

The above conditions can in principle be computed analytically, but they are too complicated to be of any practical use. The viability of the splitting conditions can be shown numerically though, as well as with some careful considerations. The most convenient way to perform the DT splitting is to make use of the dimensionless parameters λ_5 and λ_6 from the superpotential (see (3.4)). Since the invariants 27^3 and $\overline{27}^3$ have no all-singlet terms, λ_5 and λ_6 are not involved in the computation of the vacuum. Moreover, the matrix $\mathcal{M}$ contains only a single entry with the parameter λ_5 and a single entry with the parameter λ_6. The condition in equations (4.11) and (4.12) can be written as

$$K_1 - K_2\,\lambda_5\,\lambda_6 = 0, \tag{4.13}$$

$$K_1' - K_2'\,\lambda_5\,\lambda_6 \neq 0, \tag{4.14}$$

where K_1, K_2, K_1', K_2' depend only on the other parameters in the Lagrangian ($m_{351'}$, m_{78}, m_{27}, λ_1, λ_2, λ_3, λ_4, λ_7, λ_8) and the vacuum itself, all of which can be computed independently from the parameters λ_5 and λ_6. The analytic forms of Ks are not very illuminating, but they can easily be evaluated numerically for any values of the parameters

$m_{351'}, m_{78}, m_{27}, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_7, \lambda_8$. One can then take

$$\lambda_5 = \frac{K_1}{K_2 \, \lambda_6}, \tag{4.15}$$

with λ_6 arbitrary, and get an extra massless doublet mode. Substituting into condition (4.14), we can numerically check that indeed

$$K_1' K_2 \neq K_2' K_1. \tag{4.16}$$

We therefore conclude that the addition of the 78 in the Higgs sector, both with the new vacuum and enlarged matrices $\mathcal{M}_{\text{doublets}}$ and $\mathcal{M}_{\text{triplets}}$, now allows for a DT splitting in the model. Without the 78, a similar attempt at fine-tuning is not possible, since in that case we get $K_2 = K_2' = 0$ after inserting the vacuum and fine-tuning in λ_5 or λ_6 is not possible. Note that this tree-level fine-tuning is stable under quantum corrections due to the non-renormalization theorem for the superpotential.

The linear combinations of Ds and $\bar{D}$s, which correspond to the MSSM Higgses H_u and H_d, can be found by computing the new left- and right- null-eigenvectors of the fine-tuned matrix $\mathcal{M}_{\text{doublets}}$. In practice, the physical MSSM Higgses can most easily be extracted by computing the (left and right) null-eigenspace of the fine-tuned matrix $\mathcal{M}_{\text{doublets}}$, which is 2-dimensional. Given any basis of the null-eigenspace, the Higgs will always be orthogonal to the would-be Goldstone boson. The would-be Goldstone itself can be easily identified by the fact that it has no component in the directions D_i (or $\overline{D}_i$) for $i = 1, 3, 4, 5, 7, 8$. This absence of some doublets in the Goldstone can be deduced from the mass matrix in equation (4.2), but we also confirmed this by noting that the Goldstone components ϕ_i are the ones which have couplings of their derivatives to the the gauge field through the following type of expression (originating from the kinetic terms of scalar fields):

$$-ig \, A_\mu^a \, (\partial^\mu \phi_i^\dagger) \, (\hat{t}^a \langle \phi \rangle)^i. \tag{4.17}$$

Choosing a to be the doublet/antidoublet broken generators and using our vacuum, we identify the doublet components in $\phi_i^\dagger$ to which there is no coupling as indeed those listed above. Explicit computation identifies that the prevailing cause of some components not being present in the would-be Goldstone mode is our ansatz of vanishing VEVs. More precisely, without the vacuum ansatz, the terms in expression (4.17) can be schematically written as

$$\begin{aligned}
\partial D_G^* \propto {}&+ \frac{3\sqrt{5}v + 5\sqrt{3}w - \sqrt{30}y}{10\sqrt{2}} \, \partial D_0^* - \frac{d_2}{2} \, \partial D_2^* - \frac{d_1}{2} \, \partial D_3^* + \frac{e_5}{4}\sqrt{\frac{3}{2}} \, \partial D_4^* \\
&+ \frac{\sqrt{5}f_2}{4} \, \partial D_5^* + \frac{f_3}{\sqrt{2}} \, \partial D_6^* + \frac{\sqrt{3}f_2}{4} \partial D_7^* + \frac{e_5}{4}\sqrt{\frac{5}{2}} \, \partial D_8^* \\
&- \frac{e_4}{4}\sqrt{\frac{3}{2}} \, \partial D_9^* + \frac{f_1}{\sqrt{2}} \, \partial D_{10}^* - \frac{e_4}{4}\sqrt{\frac{5}{2}} \, \partial D_{11}^*,
\end{aligned} \tag{4.18}$$

$$\partial \overline{D}_G^* \propto -\frac{3\sqrt{5}v + 5\sqrt{3}w - \sqrt{30}y}{10\sqrt{2}} \, \partial \overline{D}_0^* + \frac{c_2}{2} \, \partial \overline{D}_2^* + \frac{c_1}{2} \partial \overline{D}_3^* - \frac{f_5}{4}\sqrt{\frac{3}{2}} \, \partial \overline{D}_4^*$$
$$- \frac{\sqrt{5}e_2}{4} \, \partial \overline{D}_5^* - \frac{e_3}{\sqrt{2}} \, \partial \overline{D}_6^* - \frac{\sqrt{3}e_2}{4} \, \partial \overline{D}_7^* - \frac{f_5}{4}\sqrt{\frac{5}{2}} \, \partial \overline{D}_8^*$$
$$+ \frac{f_4}{4}\sqrt{\frac{3}{2}} \, \partial \overline{D}_9^* - \frac{e_1}{\sqrt{2}} \, \partial \overline{D}_{10}^* + \frac{f_4}{4}\sqrt{\frac{5}{2}} \, \partial \overline{D}_{11}^*. \tag{4.19}$$

We see that the absence of components D_i and $\overline{D}_i$ for $i = 3, 4, 5, 7, 8$ are directly related to the vacuum ansatz with vanishing VEVs. D_1 and $\overline{D}_1$, however, are absent.[4]

The ratios of the various components of the Higgs, and hence the ratios of the EW VEVs of these component, are computed from the null-eigenspaces, while the magnitudes of the VEVs are given by the VEVs $v_u = \langle H_u \rangle$ and $v_d = \langle H_d \rangle$. We have

$$v_u^2 = \sum_{i=0}^{11} |v_i|^2, \tag{4.20}$$

$$v_d^2 = \sum_{i=0}^{11} |\bar{v}_i|^2, \tag{4.21}$$

where $v_i = \langle D_i \rangle$ and $\bar{v}_i = \langle \overline{D}_i \rangle$, and additionally the following MSSM relations hold:

$$v_u^2 + v_d^2 = (246 \, \text{GeV})^2, \tag{4.22}$$

$$v_u/v_d = \tan \beta. \tag{4.23}$$

5 Yukawa sector

The Yukawa sector comes from the Yukawa part of the superpotential after inserting the vacuum solution:

$$W_{\text{Yukawa}} = 27_F^i \, 27_F^j \left(Y_{27}^{ij} \langle 27 \rangle + Y_{351'}^{ij} \langle \overline{351'} \rangle \right). \tag{5.1}$$

In addition to the GUT scale VEVs in the representations $\overline{351'}$ and 27, the EW Higgses H_u and H_d also need to be present in both of these representations. The low-energy MSSM Higgses H_u and H_d come from a linear combination of the doublets of type $D \sim (1, 2, +1/2)$ and antidoublets of the type $\overline{D} \sim (1, 2, -1/2)$, respectively. These states are identified in table 6, and their EW VEVs are labeled by $v_i := \langle D_i \rangle$ and $\bar{v}_i := \langle \overline{D}_i \rangle$, where $i = 0, 1, \ldots, 11$. As discussed in section 4, we generically expect $v_i \neq 0$ and $\bar{v}_i \neq 0$.

Each generation of fermions is found in a fundamental representation 27 of E_6. The matter content of each generation, subdivided into SO(10) representations, is the following:

- The 16 of SO(10) contains the SM particles[5] and the right-handed neutrino ν^c.

---

[4]The 27 contains 2 singlets, but 3 doublets, so the singlets cannot be pushed to all the doublets with a t^a generator, and thus one doublet component is missing. The representation 351' has 5 singlets and 8 doublets, but the projection relation $d_{ijk} 351'^{jk} = 0$ reshuffles the definitions of singlets and doublets, so there is no loss of doublet components.

[5]We use the standard notation Q, L, u^c, d^c and e^c for the representations $(3, 2, +1/6)$, $(1, 2, -1/2)$, $(\bar{3}, 1, -2/3)$, $(\bar{3}, 1, 1/3)$ and $(1, 1, 1)$, respectively. The lepton doublet contains the electron e and the neutrino ν.

- The 10 of SO(10) contains a vector-like pair of down-type quarks $(3, 1, -1/3) + (\bar{3}, 1, +1/3)$, as well as a vector-like pair of lepton doublets $(1, 2, -1/2) + (1, 2, 1/2)$. These exotic are labeled d', d'^c, L' and L'^c, respectively.

- The SO(10) singlet 1 is denoted by s is sterile, and has a role analogous to the right-handed neutrino.

Suppressing generation indices, the mass terms are computed to be

$$u^T \left(-v_1 Y_{27} + \frac{1}{2\sqrt{10}} v_5 Y_{\overline{351'}} - \frac{1}{2\sqrt{6}} v_7 Y_{\overline{351'}} \right) u^c$$

$$+ \begin{pmatrix} d^{cT} & d'^{cT} \end{pmatrix} \begin{pmatrix} \bar{v}_2 Y_{27} + \frac{1}{2\sqrt{10}} \bar{v}_4 Y_{351'} + \frac{1}{2\sqrt{6}} \bar{v}_8 Y_{351'} & c_2 Y_{27} + \frac{1}{\sqrt{15}} f_5 Y_{351'} \\ -\bar{v}_3 Y_{27} - \frac{1}{2\sqrt{10}} \bar{v}_9 Y_{351'} - \frac{1}{2\sqrt{6}} \bar{v}_{11} Y_{351'} & -c_1 Y_{27} + \frac{1}{\sqrt{15}} f_4 Y_{351'} \end{pmatrix} \begin{pmatrix} d \\ d' \end{pmatrix}$$

$$+ \begin{pmatrix} e^T & e'^T \end{pmatrix} \begin{pmatrix} -\bar{v}_2 Y_{27} - \frac{1}{2\sqrt{10}} \bar{v}_4 Y_{351'} + \sqrt{\frac{3}{8}} \bar{v}_8 Y_{351'} & c_2 Y_{27} - \frac{3}{2} \frac{1}{\sqrt{15}} f_5 Y_{351'} \\ \bar{v}_3 Y_{27} + \frac{1}{2\sqrt{10}} \bar{v}_9 Y_{351'} - \sqrt{\frac{3}{8}} \bar{v}_{11} Y_{351'} & -c_1 Y_{27} - \frac{3}{2} \frac{1}{\sqrt{15}} f_4 Y_{351'} \end{pmatrix} \begin{pmatrix} e^c \\ e'^c \end{pmatrix}$$

$$+ \begin{pmatrix} \nu^T & \nu'^T \end{pmatrix} \begin{pmatrix} v_1 Y_{27} - \frac{1}{2\sqrt{10}} v_5 Y_{351'} - \sqrt{\frac{3}{8}} v_7 Y_{351'} & -\frac{1}{\sqrt{2}} v_6 Y_{351'} & c_2 Y_{27} - \frac{3}{2} \frac{1}{\sqrt{15}} f_5 Y_{351'} \\ -\frac{1}{\sqrt{2}} v_{10} Y_{351'} & -v_1 Y_{27} - \sqrt{\frac{2}{5}} v_5 Y_{351'} & -c_1 Y_{27} - \frac{3}{2} \frac{1}{\sqrt{15}} f_4 Y_{351'} \end{pmatrix} \begin{pmatrix} \nu^c \\ s \\ \nu'^c \end{pmatrix}$$

$$+ \frac{1}{2} \begin{pmatrix} \nu^{cT} & s^T & \nu'^{cT} \end{pmatrix} \begin{pmatrix} f_1 Y_{\overline{351'}} & \frac{1}{\sqrt{2}} f_2 Y_{\overline{351'}} & -\bar{v}_3 Y_{27} + \sqrt{\frac{2}{5}} \bar{v}_9 Y_{351'} \\ \frac{1}{\sqrt{2}} f_2 Y_{\overline{351'}} & f_3 Y_{\overline{351'}} & \bar{v}_2 Y_{27} - \sqrt{\frac{2}{5}} \bar{v}_4 Y_{351'} \\ -\bar{v}_3 Y_{27} + \sqrt{\frac{2}{5}} \bar{v}_9 Y_{351'} & \bar{v}_2 Y_{27} - \sqrt{\frac{2}{5}} \bar{v}_4 Y_{351'} & 0 \end{pmatrix} \begin{pmatrix} \nu^c \\ s \\ \nu'^c \end{pmatrix}$$

$$+ \frac{1}{2} \begin{pmatrix} \nu^T & \nu'^T \end{pmatrix} \begin{pmatrix} \Delta_1 Y_{\overline{351'}} & \frac{1}{\sqrt{2}} \Delta_2 Y_{\overline{351'}} \\ \frac{1}{\sqrt{2}} \Delta_2 Y_{\overline{351'}} & \Delta_3 Y_{\overline{351'}} \end{pmatrix} \begin{pmatrix} \nu \\ \nu' \end{pmatrix}. \tag{5.2}$$

Notice the different Clebsch-Gordan coefficients in front of $\bar{v}_i$ ($i = 2, 3, 4, 9$) that come from the couplings with the $\bar{5}$ of SU(5) and in front of $\bar{v}_{8,11}$ that come from the $\overline{45}$, while $c_{1,2}$ originate from a SU(5) singlet and $f_{4,5}$ from a SU(5) adjoint 24.

The Δ_i correspond to VEVs induced in the electrically neutral component of weak triplets of type $(1, 3, +1)$, while $\overline{\Delta}_i$ correspond to VEVs induced in weak triplets $(1, 3, -1)$. The Δ_i and $\overline{\Delta}_i$ are found only in the representations $351'$ and $\overline{351'}$, as shown in the definitions of table 7. The mass matrix M_Δ has contributions from the terms $351' \cdot \overline{351'}$ and $351' \cdot \langle 78 \rangle \cdot \overline{351'}$. Its explicit form is computed to be

$$M_\Delta = \tag{5.3}$$

$$\begin{pmatrix} m_{351'} - \lambda_8 \left(\frac{w}{6\sqrt{2}} - \frac{1}{2} \sqrt{\frac{3}{10}} v + \frac{y}{2\sqrt{5}} \right) & \frac{u_1 \lambda_8}{2\sqrt{3}} & 0 & 6\lambda_1 e_1 \\ \frac{\lambda_8}{2\sqrt{3}} u_2 & m_{351'} + \lambda_8 \left(\frac{w}{12\sqrt{2}} + \frac{v}{4\sqrt{30}} - \frac{y}{2\sqrt{5}} \right) & \frac{\lambda_8}{2\sqrt{3}} u_1 & -6\lambda_1 e_2 \\ 0 & \frac{\lambda_8}{2\sqrt{3}} u_2 & m_{351'} + \lambda_8 \left(\frac{w}{3\sqrt{2}} - \frac{v}{\sqrt{30}} - \frac{y}{2\sqrt{5}} \right) & 6\lambda_1 e_3 \\ 6\lambda_2 f_1 & -6\lambda_2 f_2 & 6\lambda_2 f_3 & m_{351'} + \lambda_8 \left(\frac{w}{3\sqrt{2}} + \frac{v}{\sqrt{30}} + \frac{y}{2\sqrt{5}} \right) \end{pmatrix}.$$

Integrating out the heavy weak triplets and inserting the F-term ansatz of vanishing VEVs, we get

$$
\begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{pmatrix} = \begin{pmatrix} m_{351'} - \lambda_8\left(\frac{w}{6\sqrt{2}} - \frac{1}{2}\sqrt{\frac{3}{10}}v\right) & 0 & 0 & 6\lambda_1 e_1 \\ 0 & m_{351'} + \lambda_8\left(\frac{w}{12\sqrt{2}} + \frac{v}{4\sqrt{30}}\right) & 0 & 0 \\ 0 & 0 & m_{351'} + \lambda_8\left(\frac{w}{3\sqrt{2}} - \frac{v}{\sqrt{30}}\right) & 6\lambda_1 e_3 \\ 6\lambda_2 f_1 & 0 & 6\lambda_2 f_3 & m_{351'} + \lambda_8\left(\frac{w}{3\sqrt{2}} + \frac{v}{\sqrt{30}}\right) \end{pmatrix}^{-1}
$$
$$
\times \begin{pmatrix} \lambda_4 v_3{}^2 \\ \lambda_4 \sqrt{2} v_2 v_3 \\ \lambda_4 v_2{}^2 \\ \lambda_3 v_1{}^2 \end{pmatrix}.
\tag{5.4}
$$

After integrating out the heavy vector-like states from equation (5.2), and using the ansatz $c_1 = f_5 = 0$, we get the matrices for the low energy states:

$$
M_U = -v_1 Y_{27} + \left(\frac{1}{2\sqrt{10}}v_5 - \frac{1}{2\sqrt{6}}v_7\right) Y_{\overline{351'}},
\tag{5.5}
$$

$$
M_D^T = \left(1 + (9/4)XX^\dagger\right)^{-1/2} \left(\left(\bar{v}_2 - \frac{3}{2}\bar{v}_3 X\right) Y_{27}\right.
$$
$$
\left. + \left(\frac{1}{2\sqrt{10}}(\bar{v}_4 - \frac{3}{2}\bar{v}_9 X) + \frac{1}{2\sqrt{6}}(\bar{v}_8 - \frac{3}{2}\bar{v}_{11} X)\right) Y_{\overline{351'}}\right),
\tag{5.6}
$$

$$
M_E = \left(1 + XX^\dagger\right)^{-1/2} \left((-\bar{v}_2 - \bar{v}_3 X) Y_{27} + \left(-\frac{1}{2\sqrt{10}}(\bar{v}_4 + \bar{v}_9 X) + \sqrt{\frac{3}{8}}(\bar{v}_8 + \bar{v}_{11} X)\right) Y_{\overline{351'}}\right),
\tag{5.7}
$$

$$
M_N = -\left(1 + XX^\dagger\right)^{-1/2} \left(\left(-\frac{1}{\sqrt{10}}\frac{v_1 v_5}{f_1} - \sqrt{\frac{3}{2}}\frac{v_1 v_7}{f_1} + \frac{1}{\sqrt{3}}\frac{v_5 v_{10}}{f_1}\frac{c_2}{f_4} + \sqrt{5}\frac{v_7 v_{10}}{f_1}\frac{c_2}{f_4}\right.\right.
$$
$$
+ \frac{4}{\sqrt{3}}\frac{v_5 v_6}{f_3}\frac{c_2}{f_4} - 2\sqrt{\frac{10}{3}}\Delta_2\frac{c_2}{f_4}\bigg)Y_{27}
$$
$$
+ \left(\frac{1}{40}\frac{v_5{}^2}{f_1} + \sqrt{\frac{3}{80}}\frac{v_7 v_5}{f_1} + \frac{3}{8}\frac{v_7{}^2}{f_1} + \frac{1}{2}\frac{v_6{}^2}{f_3} - \Delta_1\right) Y_{\overline{351'}}
$$
$$
+ \left(\frac{v_1{}^2}{f_1} - 2\sqrt{\frac{10}{3}}\frac{v_1 v_{10}}{f_1}\frac{c_2}{f_4} + \frac{10}{3}\frac{v_{10}{}^2}{f_1}\frac{c_2{}^2}{f_4{}^2} + 2\sqrt{\frac{10}{3}}\frac{v_1 v_6}{f_3}\frac{c_2}{f_4}\right.
$$
$$
+ \frac{8}{3}\frac{v_5{}^2}{f_3}\frac{c_2{}^2}{f_4{}^2} - \frac{20}{3}\Delta_3\frac{c_2{}^2}{f_4{}^2}\bigg)Y_{27}Y_{\overline{351'}}^{-1}Y_{27}
$$
$$
+ \left(\frac{8\sqrt{10}}{3}\frac{v_1 v_5}{f_3}\frac{c_2{}^2}{f_4{}^2}\right) Y_{27}Y_{\overline{351'}}^{-1}Y_{27}Y_{\overline{351'}}^{-1}Y_{27}
$$
$$
+ \left(\frac{20}{3}\frac{v_1{}^2}{f_3}\frac{c_2{}^2}{f_4{}^2}\right) Y_{27}Y_{\overline{351'}}^{-1}Y_{27}Y_{\overline{351'}}^{-1}Y_{27}Y_{\overline{351'}}^{-1}Y_{27}\bigg)
$$
$$
\left(1 + X^*X^T\right)^{-1/2},
\tag{5.8}
$$

where

$$X = -2\sqrt{\frac{5}{3}} \frac{c_2}{f_4} Y_{27} Y_{\overline{351'}}^{-1}. \tag{5.9}$$

Notice that the main factor in the expressions is a linear combination of the following matrices:

$$Y_{27}, \quad Y_{\overline{351'}}, \quad Y_{27}Y_{\overline{351'}}^{-1}Y_{27}, \quad Y_{27}Y_{\overline{351'}}^{-1}Y_{27}Y_{\overline{351'}}^{-1}Y_{27}, \quad Y_{27}Y_{\overline{351'}}^{-1}Y_{27}Y_{\overline{351'}}^{-1}Y_{27}Y_{\overline{351'}}^{-1}Y_{27}, \tag{5.10}$$

which are all symmetric, since Y_{27} and $Y_{\overline{351'}}$ are symmetric, as are M_U and M_N. The matrices M_D and M_E are not symmetric, though, due to the projection factor onto the light families.

As usual with vector-like states, the expressions for the low energy masses in equations (5.5)–(5.8) are nonlinear, which complicates the analysis of the masses. We comment more on the low-energy part of the Yukawa sector in section 6, in which we also do a numeric fit in the 2-generation case.

6 Numeric fit of the Yukawa sector

The presented model has 3 masses and 8 dimensionless parameters λ_i in the breaking sector, as well as two symmetric 3×3 Yukawa matrices in the Yukawa sector, which is easily seen from the superpotential in equation (2.1). Note also that the product $\lambda_5\lambda_6$ is fixed by the fine-tuning of the EW Higgs mass. Also, a rotation in family-space can bring one of the Yukawa matrices to be diagonal. We shall limit ourselves to the case where all parameters of the Lagrangian are real. The independent number of real parameters in the breaking sector is thus $3 + 8 - 1 = 10$, while the 3-family Yukawa sector has $6 + 3 = 9$ real parameters. Since the number of independent real parameters is 19, while there are only 17 real numbers to be measured (3 masses in the up, down, charged lepton sector each, the two differences of masses-squared in the neutrino sector, as well as 3 angles in each of the CKM and PMNS matrices; we neglect the CP-phases), the general expectation is that a fit is possible to perform. There may exist, however, non-obvious mass relations concealed due to the complexity of the low energy expressions in equations (5.5)–(5.8), which are not respected by the experimental values; these fears can be alleviated by finding points in parameter space, which give a good fit for to the masses and mixing angles.

In this section, we perform a fit in the simplified case of 2 families. Here, there are again 10 real parameters in the breaking sector, while the Yukawa sector has $3 + 2 = 5$ real parameters. The fit is performed for the results of the quark masses m_t, m_c, m_b, m_s, the charged lepton masses m_τ and m_μ, the difference of the squared neutrino masses $m_{\nu_3}^2 - m_{\nu_2}^2$, and the mixing angles θ_{cb} and θ_{23} in the CKM and PMNS matrices, respectively.

The most convenient way to perform the fit is to take some of the GUT scale VEVs as the parameters in the fit, instead of the initial parameters in the Lagrangian. This is advantageous since the equations of motion are linear in the Lagrangian parameters. Taking the ansatz $c_1 = d_1 = e_2 = f_2 = e_5 = f_5 = u_1 = u_2 = y = 0$, we need 12 parameters;

only 10 are independent, while 2 are determined through the D-terms. It is convenient, for example, to take the following quantities as independent parameters:

$$m_{351'}, \quad c_2, \quad d_2, \quad f_1, \quad f_3, \quad f_4, \quad e_4, \quad v, \quad w, \quad \lambda_5. \tag{6.1}$$

We can then use the equations of motion to determine the remaining initial parameters and VEVs. The D-terms are solved by taking

$$e_1 = \pm\sqrt{|d_2|^2 - |c_2|^2 + 2|f_1|^2}/\sqrt{2}, \tag{6.2}$$

$$e_3 = \pm\sqrt{|e_4|^2 - |f_4|^2 + 2|f_3|^2}/\sqrt{2}. \tag{6.3}$$

Since we want e_1 and e_3 to be real numbers, the arguments in the square root need to be positive, which limits the space of parameters in equation (6.1). Alternatively, we could also take the 6 independent parameters, for example,[6] to be the VEVs d_2, f_1, e_3, f_4 (now with no restriction) and two angles $\theta_1, \theta_2 \in (0, 2\pi)$, with the remaining quantities computed as

$$f_1 = \sin\theta_1 \, \sqrt{|c_2|^2 + 2|e_1|^2}/\sqrt{2}, \tag{6.4}$$

$$d_2 = \cos\theta_1 \, \sqrt{|c_2|^2 + 2|e_1|^2}, \tag{6.5}$$

$$f_3 = \sin\theta_2 \, \sqrt{|f_4|^2 + 2|e_3|^2}/\sqrt{2}, \tag{6.6}$$

$$e_4 = \cos\theta_2 \, \sqrt{|f_4|^2 + 2|e_3|^2}. \tag{6.7}$$

The F-terms then yield

$$m_{27} = \frac{m_{351'}}{2w\left(\sqrt{15}w - v\right)c_2 d_2 \left(4e_3 f_3 + e_4 f_4\right)} \Big(2v(w + \sqrt{15}v)\left(e_4 f_4 - 2e_3 f_3\right)^2$$
$$+ e_1 f_1(v - \sqrt{15}w)\left(2e_3 f_3(3\sqrt{15}v - 5w) - e_4 f_4(3\sqrt{15}v + 7w)\right)\Big), \tag{6.8}$$

$$m_{78} = \frac{\sqrt{15}m_{351'}\left(e_4 f_4 - 2e_3 f_3\right)^2}{2w\left(\sqrt{15}w - v\right)\left(4e_3 f_3 + e_4 f_4\right)}, \tag{6.9}$$

$$\lambda_1 = -\frac{f_3 f_4 m_{351'}}{e_4(e_4 f_4 + 4e_3 f_3)}, \tag{6.10}$$

$$\lambda_2 = -\frac{e_3 e_4 m_{351'}}{f_4(e_4 f_4 + 4e_3 f_3)}, \tag{6.11}$$

$$\lambda_3 = -\frac{e_1 m_{351'}\left(\sqrt{15}v\left(e_4 f_4 - 2e_3 f_3\right) + 3w\left(2e_3 f_3 + e_4 f_4\right)\right)}{2w c_2^2 \left(4e_3 f_3 + e_4 f_4\right)}, \tag{6.12}$$

$$\lambda_4 = -\frac{f_1 m_{351'}\left(\sqrt{15}v\left(e_4 f_4 - 2e_3 f_3\right) + 3w\left(2e_3 f_3 + e_4 f_4\right)\right)}{2w d_2^2 \left(4e_3 f_3 + e_4 f_4\right)}, \tag{6.13}$$

$$\lambda_7 = \frac{3\sqrt{2}m_{351'}\left(2e_3 f_3 - e_4 f_4\right)\left(\left(v - \sqrt{15}w\right)e_1 f_1 + 2v\left(2e_3 f_3 - e_4 f_4\right)\right)}{w\left(\sqrt{15}w - v\right)c_2 d_2 \left(4e_3 f_3 + e_4 f_4\right)}, \tag{6.14}$$

$$\lambda_8 = \frac{3\sqrt{2}m_{351'}\left(2e_3 f_3 - e_4 f_4\right)}{w\left(e_4 f_4 + 4e_3 f_3\right)}. \tag{6.15}$$

---

[6]In both parametrizations of encoding the D-terms, we wrote them so that we retained control to make f_1 and f_3 potentially small; this will be important for the neutrino sector, as described later in this section.

Using these values, λ_6 can then be determined by fine-tuning in the doublet mass matrix (see section 4), from which also the EW VEVs v_i and $\bar{v}_i$ are computed. One can then use these to compute the mass matrices from equations (5.5)–(5.8), and ultimately the masses and the mixing angles. For our numeric fit, we assume:

- the simplified case of second and third generation only;

- all parameters real;

- all errors in measured quantities at the 10% level; we believe that at the present stage such a choice is a good compromise between the realistic case and the simplicity of the analysis;

- the values for the masses and mixing angles at the GUT scale $\sim 10^{16}$ GeV as shown in table 4 [26], valid for $\tan \beta = 10$;

- positive signs in eqs. (6.2)–(6.3).

We performed the fit by minimizing the chi-squared function

$$\chi^2 := \sum_i \frac{(f_i(x) - y_i)}{\sigma_i^2}, \tag{6.16}$$

with respect to the initial parameters x. The experimental values are denoted by y_i, the values computed using our model are $f_i(x)$, and σ_i are the 1-sigma deviations from the values y_i. For our fit, we have 9 different measured quantities, thus $i = 1, \ldots, 9$. The errors are taken to be $\sigma_i = 0.1 \, y_i$, giving

$$\chi^2 = 100 \sum_i \left(\frac{f_i(x) - y_i}{y_i} \right)^2. \tag{6.17}$$

We define the convenient measure $\hat{\chi}^2 = \chi^2/9$, which tells us the average σ^2 deviation per measured value. Also, we define the pulls $\chi_i = (f_i(x) - y_i)/\sigma_i$, which tell us how many sigma a certain quantity deviates from the measured one.

Due to the large number of parameters, the χ^2 function will have many local minima in the parameter space. We give below two such points in the parameter space, corresponding to the best fits that were found and which we deem sufficiently good (with $\hat{\chi}^2 \lesssim 1$). The results are given in tables 2, 3, 4; the points in the parameter space are given in terms of the independent parameters best suited to a numeric search (as discussed in this section) in table 2 and in terms of the original Lagrangian parameters in table 3, while table 4 shows the obtained results for the masses and mixing angles.

Notice from tables 2 and 3 that of the two Yukawa matrices Y_{27} and $Y_{\overline{351'}}$, $Y_{\overline{351'}}$ was chosen to be the diagonal one. Furthermore, the original Lagrangian parameters in table 3 are given so that it can be checked they roughly fall into the perturbative regime. A possible exception could be the value λ_8 of the first solution.

Note that the given two points are merely the best ones we found. Due to the high dimensionality of the parameter space, we suspect there are likely many more points which

parameter	point 1	point 2
$m_{351'}$ [GeV]	1.17×10^{16}	-4.17×10^{15}
c_2 [GeV]	6.68×10^{15}	3.98×10^{15}
d_2 [GeV]	-6.78×10^{15}	-4.90×10^{15}
f_1 [GeV]	4.12×10^{11}	-5.52×10^{12}
f_3 [GeV]	-1.84×10^{16}	1.38×10^{16}
f_4 [GeV]	1.61×10^{16}	1.49×10^{16}
e_4 [GeV]	5.27×10^{15}	-1.69×10^{16}
v [GeV]	-7.07×10^{13}	8.44×10^{14}
w [GeV]	6.13×10^{15}	-1.78×10^{16}
λ_5	-1.58×10^{-3}	1.50×10^{-1}
$(Y_{27})_{11}$	-0.723	1.93
$(Y_{27})_{12}$	0.703	-1.19
$(Y_{27})_{22}$	-0.676	0.730
$(Y_{\overline{351'}})_{11}$	-0.371	0.733
$(Y_{\overline{351'}})_{22}$	0.363	-0.287

Table 2. Two example points in the parameter space, written in terms of the independent parameters suitable for a search.

give a comparable or a better fit. There are a number of observations that can be made about these points in general, however, by deducing them from the formulae, comparing the two parameter points given in the tables and through experience obtained by performing the fit:

- *The parameter points are not necessarily close to each other, which also holds true for any specific single parameter.* We can see in table 2 that it is not necessary for any parameter to be at a very specific value to obtain a good fit. The suitable areas of parameter space thus form many disconnected regions, and no specific value can be claimed for any parameter. In this sense, the mass formulae of the theory are not very predictive of the original parameters.

- *There is no specific mass or mixing angle, where one would consistently be getting tension.* As observed from table 4, while a specific solution might have most tension with observation coming from a single mass or mixing angle, there are other points, which also give a good fit and where this parameter is predicted better. In this sense we cannot claim any tendencies in tensions of the observables.

- Imagine that in table 2 or 3 we rescale all the mass parameters with a common factor. Such a rescaling would have no influence on the masses of the quarks and charged leptons, which are controlled only by the EW VEVs, as can be seen from equations (5.5)–(5.7) and equation (5.9). Rescaling would influence the neutrino masses, however, due to the seesaw mechanism, confirmed by equation (5.8). In principle, the rescaling factor can always be adjusted, so that the fit of the mass-square difference of the neutrino masses $m_{\nu_3}^2 - m_{\nu_2}^2$ is exact (provided this does not

parameter	point 1	point 2
$m_{351'}$ [GeV]	1.17×10^{16}	-4.17×10^{15}
m_{27} [GeV]	-2.92×10^{14}	-1.64×10^{15}
m_{78} [GeV]	-6.19×10^{16}	-5.04×10^{15}
λ_1	-6.49×10^{-1}	-8.91×10^{-2}
λ_2	5.66×10^{-2}	-1.24×10^{-1}
λ_3	-1.5×10^{-1}	2.78×10^{-1}
λ_4	-7.38×10^{-5}	-5.04×10^{-4}
λ_5	-1.58×10^{-3}	1.50×10^{-1}
λ_6	-7.59×10^{-3}	3.06×10^{-2}
λ_7	-4.23×10^{-1}	9.55×10^{-1}
λ_8	5.08	1.16
$(Y_{27})_{11}$	-7.23×10^{-1}	1.87
$(Y_{27})_{12}$	7.03×10^{-1}	-1.09
$(Y_{27})_{22}$	-6.76×10^{-1}	6.30×10^{-1}
$(Y_{\overline{351'}})_{11}$	-3.71×10^{-1}	7.39×10^{-1}
$(Y_{\overline{351'}})_{12}$	0	0
$(Y_{\overline{351'}})_{22}$	3.63×10^{-1}	-2.58×10^{-1}

Table 3. Two example points in the parameter space presented in terms of the original parameters in the superpotential.

spoil the GUT scale, or the upper bounds on neutrino masses). *Thus only neutrinos are actually sensitive to the GUT scale.*

- The mass parameters are chosen to be at approximately the GUT scale, say at the order of 10^{15-16} GeV. If this is true for all the mass parameters, there might be a problem with the neutrino masses. It is a well known fact that this GUT scale is a few orders of magnitude too large compared to the seesaw scale for sufficiently large mass differences in the neutrino sector. One can cure this problem by having the spontaneous symmetry breaking occur in multiple stages, with a mass hierarchy between different VEVs and the seesaw scale corresponding to one of the intermediate stages. The gauge coupling unification in such a scenario could then be spoilt by the particles appearing at these intermediate mass scales.

In our case, however, the number of different VEVs is large enough, so that having one or two of the VEVs at a smaller scale does not disturb the breaking pattern. To see this, note that to obtain sufficiently large neutrino masses, it is enough for one of the terms in equation (5.8) to be of the proper scale, which can be achieved by simply taking f_1 or f_3 to be several orders of magnitude smaller than the GUT scale (the seesaw type I contributions). In the solutions given in table 2, we achieved sufficiently high neutrino masses by taking the parameter f_1 (and also v) to be a few orders of magnitude below the GUT scale. As seen from the gauge boson masses in table 5, small f_1 and v do not spoil the one-stage breaking scenario; since the intermediate

quantity	experiment	parameter point 1		parameter point 2	
	y_i	$f_i(x)$	χ_i	$f_i(x)$	χ_i
m_c	0.236	0.226	-0.432	0.205	-1.30
m_t	92.2	94.0	$+0.193$	105	$+1.38$
θ_{cb}	0.0409	0.0358	-1.24	0.0378	-0.757
m_s	0.013	0.0144	$+1.06$	0.146	$+1.20$
m_b	0.79	0.79	$+0.0021$	0.781	-0.112
m_μ	0.0599	0.0613	$+0.241$	0.0664	$+1.08$
m_τ	1.02	0.867	-1.51	1.03	$+0.056$
m_{ν_3}		0.135		0.0824	
m_{ν_2}		0.126		0.0659	
$(m_{\nu_3}^2 - m_{\nu_2}^2)10^3$	2.32	2.45	$+0.568$	2.24	-0.325
$\sin^2\theta_{23}$	0.386	0.343	-1.12	0.327	-1.52
$\hat{\chi}^2$		0.76		1.03	

Table 4. Table of predictions for two example point in parameter space. All charge fermion masses are in units of GeV, while the neutrino masses are in units eV.

scales are not associated to an intermediate symmetry breaking at that scale, but are instead purely accidental due to a carefully chosen parameter point, we do not expect too many relevant states (apart from the singlets ν^c with f_1 Majorana mass) to be found at the scale f_1. We noticed however the appearance of a lighter color triplet-antitriplet pair, possibly due to the similarities between the doublet and triplet mass matrices. Although this influences the running of the gauge couplings, we will neglect it in view of the (presumably) large threshold uncertainties present anyway.

- A final comment on the neutrino masses: although only the difference of masses-squared needs to be fitted, one still needs to check that the neutrino masses themselves are $< 1\,\text{eV}$ [27]. As seen in table 4, this condition holds true for both of our parameter points.

7 Discussion

What we presented here is a model we believe is a good candidate for a minimal supersymmetric renormalizable E_6 GUT. Let's see why this model can be considered as more minimal than our previous candidate [11]. Although the total number of degrees of freedom is now larger ($78 > 27 + 27$), what really counts are the number of multiplets and, even more important, the total number of free parameters. The number of parameters in the Higgs sector is now 11 complex minus 5 phases due to field redefinitions. Yukawa sector adds another 3 real and 6 complex parameters. Together with one real gauge coupling we have thus a total of 33 real parameters. This is 7 more than in the minimal SO(10) [13–15], but still 16 less than even in the simplified truncated version of E_6 in [11].

Is there any possible low-energy signature of this model? It is often said that E_6 could have possible light extra generations, coming from remnants of the three copies of the 27_F. Light vector-like fermions of the SM group could indeed emerge out of E_6 if they were associated with anomaly cancellation in a TeV-scale extra U(1). In such a setup, the extra U(1) would be a linear combination of the two U(1)s present in the rank 6 group E_6. Phenomenology of such TeV scale E_6 motivated models has been extensively studied, see for eg. [28]. In the present setup, however, there are no intermediate U(1)s and any light vector-like states would be accidental. In fact if we try to get such light states from our solution, we find them hard to obtain. The question is, does the matrix for (as an example) down quarks in eq. (5.2) allow 4 or more zero eigenvalues once we limit all $\bar{v}$'s to zero? This can in principle be obtained either by putting $c_2, f_4 \to 0$ (remember that our solution already has $c_1 = f_5 = 0$), or by imposing a vanishing determinant constraint to Yukawa matrices. The first case points towards an SU(5) invariant vacuum,[7] while the second one constrains the Yukawa parameters and so a worse fit to data is expected. So we conclude that such an extreme, albeit interesting situation is unlikely, at least in the given vacuum solution. Although we cannot make the same conclusions in general, any possibility of light states, if available, will occur due to fine-tuning in the superpotential parameters.

Another possibility for having light states could be to have flat directions. We checked by explicit computation that no such states are present in our solution. Although we omit the details of this computation here, the interested reader can reconstruct the mass matrix of the SM singlets via the all-singlet terms in the superpotential, which are given in equations (3.5)–(3.15). If the vacuum solution is then plugged-in, one discovers 4 massless singlet states, all of which are in fact would-be Goldstone bosons. Note that the adjoint 78 of E_6 contains 5 SM singlets, while only 1 remains unbroken among the SM generators, so the 4 massless singlet states are the would-be Goldstone bosons eaten up by the 4 broken singlet generators. We thus conclude there are no physical massless singlet states. Other vacua could in principle be possible: one of them is described in appendix D.

Proton decay is, as usual, quite hidden by details of superpartners' spectra: it is hard to disentangle the GUT and SUSY breaking information from it. Once however colliders will (hopefully) tell us more about the low energy spectrum, this E_6 theory as well as other grand unified theories could be tested better. For the sake of completeness the forms of the low-energy $D = 5$ operators are given in appendix E.

The model we presented here is the minimal known E_6, although for a more convincing proof we should satisfy three more checks.

- First, there is a possibility to redefine the charges under matter parity, so that the $27 + \overline{27}$ parity is now transferred to the fermionic sector. This means that the fields $351' + \overline{351'} + 78$ should alone break to the SM gauge group as well as allow for DT splitting. The fermionic sector would now consists of 27_F^a, $a = 1, \ldots, 4$ and $\overline{27}_F$. The

---

[7]This can be easily seen from the expression for λ_2 in (6.11): if $f_4 \to 0$, we need either $e_3 \to 0$ or $e_4 \to 0$; both lead to an SU(5) invariant vacuum with $e_4, f_4 \to 0$, see (6.3).

Yukawa terms could be written schematically as

$$\begin{pmatrix} 27_F^a \\ \overline{27}_F \end{pmatrix}^T \begin{pmatrix} Y_{351'}^{ab}\langle\overline{351'}\rangle & m_{27}^a + \lambda^a\langle 78 \rangle \\ m_{27}^b + \lambda^b\langle 78 \rangle & y_{351'}\langle 351' \rangle \end{pmatrix} \begin{pmatrix} 27_F^b \\ \overline{27}_F \end{pmatrix}. \tag{7.1}$$

The total number of parameters is now the following: 5 complex parameters come from the Higgs superpotential, 3 phases of which can be rotated away by Higgs field redefinitions; 4 real diagonal components are given by the only Yukawa matrix, and 8 complex and one real parameters are the off-diagonal terms; finally, 1 real gauge coupling sums to a total of 29 real free parameters. This would be 4 real parameters less than the model in this paper. Obviously there is no guarantee that such a model is realistic. We plan to come back to this issue in the future.

- Second, we should study the complete three generation case, not only its two generation subsystem. Although the number of parameters seems naively large enough, it is far from obvious that a successful fit is possible. In fact already in the two generation case considered here we could not find a solution with vanishing χ^2, in spite of enough free parameters. However the (at least partially) successful fit of the the minimal SO(10) analogous case [19, 20, 29–39] with 10 and $\overline{126}$ (instead of 27 and $\overline{351'}$) Yukawa couplings make us feel optimistic. Notice also that the usual obstruction of the neutrino mass either too low or unification violated [16–19] is here avoided as shown in section 6.

- Third, the theory is not asymptotically free and has a huge gauge coupling beta function, more precisely 159. This means, similarly as in our previous E_6 model, or the minimal renormalizable supersymmetric SO(10), that a Landau pole is close to the GUT scale and so the theory itself may be already in the non-perturbative regime. An indication of problems being present already at the matching scale M_{GUT} can be found in the calculation of the threshold corrections [40, 41]. The general problem of having large numbers of degrees of freedom (in our case through large representations) and the associated non-perturbativity is a problem, which is far from easy to solve and well beyond the purpose of this paper, although some progress has been made recently [42, 43] based on previous works on Seiberg dualities. We hope to come back to this very interesting issue soon.

But even in the case the model presented here is the minimal one, other vacua could still be realistic with in principle different predictions.

Acknowledgments

A Gauge boson masses

We compute in this appendix the expression for the gauge boson masses, so that we can confirm that solutions really break into the SM group. We write the mass terms in the Lagrangian as

$$\mathcal{L}_{\text{mass}} = g^2 A_\mu^a M^{ab} A^{\mu\,b}, \tag{A.1}$$

where g is the E_6 coupling constant and the matrix M^{ab} is computed via

$$M^{ab} := \sum_i \text{Tr}\left(\left(\hat{t}^a \langle \phi_i \rangle \right)^\dagger \left(\hat{t}^b \langle \phi_i \rangle \right) \right). \tag{A.2}$$

The sum is over all representations containing VEVs (contributions come from the representations of the breaking sector), while $\hat{t}^a$ denotes the action of the a-th generator on the representation ϕ_i. The mass matrix becomes block diagonal if we choose an appropriate basis (indices a, b), so that the basis gauge bosons have well defined transformation properties under the SM group (note that some of these states are complex). We omit the details of this calculation and only give the results, collected in table 5.

B Particle identification

In this appendix, we write the definitions of various states needed in the paper. The SM singlet VEVs were already defined in table 1. We supplement the list of definitions with tables 6 and 7. In table 6 we define the doublets $(1, 2, +1/2)$, antidoublets $(1, 2, -1/2)$, triplets $(3, 1, -1/3)$ and antitriplets $(\bar{3}, 1, +1/3)$; these definitions are needed for DT splitting. In table 7, we define the weak triplets $(1, 3, \pm 1)$ relevant for type II seesaw. All the states in the two-index $351'$ can be specified by writing the basis states of this representation by using two labels of the fundamental 27. In this notation, both labels are SM representation in the 27, assumed to have all the color and weak indices contracted in the correct manner to obtain the desired SM state in the $351'$. More details on this notation can be found in [11].

C DT splitting and Goldstone modes

Analysis of the DT splitting in the group E_6 is complicated by the fact that a vacuum breaking to the SM group will automatically cause a doublet and a triplet mode to be massless. These massless modes are would-be Goldstone bosons: $E_6 \to$ SM causes the breaking of $78 - 12 = 66$ generators, with a doublet-antidoublet and triplet-antitriplet pair among them. One possible procedure to compute the condition for an extra massless mode in a matrix M, for which $\det M = 0$, is to take

$$\frac{\lim_{\varepsilon \to 0}\left(\det(M - \varepsilon I)/\varepsilon \right)}{\langle e | f \rangle} = 0, \tag{C.1}$$

where e and f are the already present left and right null-eigenvectors of M, respectively. We present below, however, a simplified procedure of computing the conditions of DT splitting

SO(10) $\supset$	SU(5) $\supset$	SM $\supset$	$(\text{mass})^2/g^2$
45	24	$(8,1,0)$	0
45	24	$(1,3,0)$	0
45	24	$(1,1,0)$	0
45	24	$(3,2,+\frac{5}{6})$ $(\overline{3},2,-\frac{5}{6})$	$\frac{5}{6}\left(|e_4|^2 + |f_4|^2\right)$
45	10 $\overline{10}$	$(3,2,+\frac{1}{6})$ $(\overline{3},2,-\frac{1}{6})$	$\frac{4}{15}|v|^2 + \frac{1}{2}|c_2|^2 + \frac{1}{2}|d_2|^2$ $+|e_1|^2 + |f_1|^2 + \frac{5}{6}|e_4|^2 + \frac{5}{6}|f_4|^2$
45	10 $\overline{10}$	$(\overline{3},1,-\frac{2}{3})$ $(3,1,+\frac{2}{3})$	$\frac{4}{15}|v|^2 + \frac{1}{2}|c_2|^2 + \frac{1}{2}|d_2|^2 + |e_1|^2 + |f_1|^2$
45	10 $\overline{10}$	$(1,1,+1)$ $(1,1,-1)$	$\frac{4}{15}|v|^2 + \frac{1}{2}|c_2|^2 + \frac{1}{2}|d_2|^2 + |e_1|^2 + |f_1|^2$
16 $\overline{16}$	10 $\overline{10}$	$(3,2,+\frac{1}{6})$ $(\overline{3},2,-\frac{1}{6})$	$\frac{1}{60}|\sqrt{15}w-v|^2+|e_3|^2+|f_3|^2+\frac{1}{2}|e_4|^2+\frac{1}{2}|f_4|^2$
16 $\overline{16}$	10 $\overline{10}$	$(\overline{3},1,-\frac{2}{3})$ $(3,1,+\frac{2}{3})$	$\frac{1}{60}|\sqrt{15}w-v|^2+|e_3|^2+|f_3|^2+\frac{1}{2}|e_4|^2+\frac{1}{2}|f_4|^2$
16 $\overline{16}$	10 $\overline{10}$	$(1,1,+1)$ $(1,1,-1)$	$\frac{1}{60}|\sqrt{15}w-v|^2+|e_3|^2+|f_3|^2+\frac{1}{2}|e_4|^2+\frac{1}{2}|f_4|^2$
16 $\overline{16}$	$\overline{5}$ 5	$(\overline{3},1,+\frac{1}{3})$ $(3,1,-\frac{1}{3})$	$\frac{1}{4}|w + \sqrt{3/5}\,v|^2 + \frac{1}{2}|c_2|^2 + \frac{1}{2}|d_2|^2 + |e_1|^2 + |f_1|^2+$ $+|e_3|^2 + |f_3|^2 + \frac{1}{2}|e_4|^2 + \frac{1}{2}|f_4|^2$
16 $\overline{16}$	$\overline{5}$ 5	$(1,2,-\frac{1}{2})$ $(1,2,+\frac{1}{2})$	$\frac{1}{4}|w + \sqrt{3/5}\,v|^2 + \frac{1}{2}|c_2|^2 + \frac{1}{2}|d_2|^2 + |e_1|^2 + |f_1|^2+$ $+|e_3|^2 + |f_3|^2 + \frac{1}{2}|e_4|^2 + \frac{1}{2}|f_4|^2$
45 1	1 1	$(1,1,0)$ $(1,1,0)$	They mix: $\frac{2}{3}\left((A + B) \pm \sqrt{(A + B)^2 - \frac{15}{4}AB}\right),$ $A \equiv 4|e_3|^2 + 4|f_3|^2 + |e_4|^2 + |f_4|^2$ $B \equiv 4|e_1|^2 + 4|f_1|^2 + |c_2|^2 + |d_2|^2$
16 $\overline{16}$	1 1	$(1,1,0)$ $(1,1,0)$	They mix: $\frac{1}{2}\left((C+D+|F|^2)\pm\sqrt{(C - D)^2 + 16|E|^2}\right),$ $C \equiv |c_2|^2 + 2|e_1|^2 + 2|f_3|^2 + |e_4|^2$ $D \equiv |d_2|^2 + 2|f_1|^2 + 2|e_3|^2 + |f_4|^2$ $E \equiv e_1 e_3{}^* + f_1{}^* f_3$ $F \equiv \sqrt{\frac{5}{6}}v - \sqrt{\frac{1}{2}}w$

Table 5. Masses-squared of gauge bosons in SM representations using the ansatz $c_1 = d_1 = e_2 = f_2 = e_5 = f_5 = u_1 = u_2 = y = 0$.

label	$E_6 \supseteq SO(10) \supseteq SU(5)$	label	$E_6 \supseteq SO(10) \supseteq SU(5)$	doublet / triplet
$\overline{D}_0, \overline{T}_0$	$78 \supseteq 16 \supseteq \overline{5}$	D_0, T_0	$78 \supseteq \overline{16} \supseteq 5$	$\frac{1}{\sqrt{12}}(t_L^6 \pm i\, t_L^7)$
				$\frac{1}{\sqrt{12}}\bar{t}_\alpha{}^{31}, \frac{1}{\sqrt{12}}t^\alpha{}_{31}$
D_1, T_1	$27 \supseteq 10 \supseteq 5$	$\overline{D}_1, \overline{T}_1$	$\overline{27} \supseteq 10 \supseteq \overline{5}$	L'^c
				d'
$\overline{D}_2, \overline{T}_2$	$27 \supseteq 10 \supseteq \overline{5}$	D_2, T_2	$\overline{27} \supseteq 10 \supseteq 5$	L'
				d'^c
$\overline{D}_3, \overline{T}_3$	$27 \supseteq 16 \supseteq \overline{5}$	D_3, T_3	$\overline{27} \supseteq \overline{16} \supseteq 5$	L
				d^c
D_4, T_4	$351' \supseteq 10 \supseteq 5$	$\overline{D}_4, \overline{T}_4$	$\overline{351'} \supseteq 10 \supseteq \overline{5}$	$Qd^c - Le^c - 4L'^c\nu^c$
				$QL - u^c d^c - 4d's$
$\overline{D}_5, \overline{T}_5$	$351' \supseteq 10 \supseteq \overline{5}$	D_5, T_5	$\overline{351'} \supseteq 10 \supseteq 5$	$Qu^c - L\nu^c - 4L's$
				$u^c e^c - d^c\nu^c + QQ - 4d'^c s$
$\overline{D}_6, \overline{T}_6$	$351' \supseteq 16 \supseteq \overline{5}$	D_6, T_6	$\overline{351'} \supseteq \overline{16} \supseteq 5$	$-Ls$
				$-d^c s$
$\overline{D}_7, \overline{T}_7$	$351' \supseteq 126 \supseteq \overline{5}$	D_7, T_7	$\overline{351'} \supseteq \overline{126} \supseteq 5$	$-Qu^c - 3L\nu^c$
				$-u^c e^c - 3d^c\nu^c - QQ$
D_8, T_8	$351' \supseteq 126 \supseteq 45$	$\overline{D}_8, \overline{T}_8$	$\overline{351'} \supseteq \overline{126} \supseteq \overline{45}$	$Qd^c + 3Le^c$
				$QL + u^c d^c$
D_9, T_9	$351' \supseteq \overline{144} \supseteq 5$	$\overline{D}_9, \overline{T}_9$	$\overline{351'} \supseteq 144 \supseteq \overline{5}$	$-Qd'^c + 4L'^c\nu^c + L'e^c$
				$-QL' + u^c d'^c + 4d'\nu^c$
$\overline{D}_{10}, \overline{T}_{10}$	$351' \supseteq \overline{144} \supseteq \overline{5}$	D_{10}, T_{10}	$\overline{351'} \supseteq 144 \supseteq 5$	$-L'\nu^c$
				$-d'^c\nu^c$
D_{11}, T_{11}	$351' \supseteq \overline{144} \supseteq 45$	$\overline{D}_{11}, \overline{T}_{11}$	$\overline{351'} \supseteq 144 \supseteq \overline{45}$	$-dd'^c - 3e'e^c$
				$-QL' - u^c d'^c$
$\overline{T}_{12}$	$351' \supseteq 126 \supseteq \overline{50}$	T_{12}	$\overline{351'} \supseteq \overline{126} \supseteq 50$	/
				$2u^c e^c - QQ$

Table 6. Identification of doublets and triplets in the representations of the Higgs sector.

label	$E_6 \supseteq SO(10) \supseteq SU(5)$	p.n.	label	$E_6 \supseteq SO(10) \supseteq SU(5)$	state
$\overline{\Delta}_1$	$351' \supseteq 126 \supseteq \overline{15}$	$L\,L$	Δ_1	$\overline{351'} \supseteq \overline{126} \supseteq 15$	$\bar{L}\,\bar{L}$
$\overline{\Delta}_2$	$351' \supseteq \overline{144} \supseteq \overline{15}$	$L\,L'$	Δ_2	$\overline{351'} \supseteq 144 \supseteq 15$	$\bar{L}\,\bar{L}'$
$\overline{\Delta}_3$	$351' \supseteq 54 \supseteq \overline{15}$	$L'L'$	Δ_3	$\overline{351'} \supseteq 54 \supseteq 15$	$\bar{L}'\bar{L}'$
Δ_4	$351' \supseteq 54 \supseteq 15$	$L'^c L'^c$	$\overline{\Delta}_4$	$\overline{351'} \supseteq 54 \supseteq \overline{15}$	$\bar{L}'^c\bar{L}'^c$

Table 7. Induced VEVs in weak triplet scalars $(1, 3, \pm 1)$ leading to seesaw type II.

in the presence of a Goldstone mode. Its advantages are that it is computationally less intensive and that no issues with singularities, such as $\langle e|f \rangle = 0$, arise in the procedure.

Suppose we use a generic label A for an $n \times n$ complex matrix. Although the true scalar mass-squared matrix is in fact the hermitian and positive definite matrix $A^\dagger A$, it is more efficient to work with A. A might not necessarily be diagonalizable, but it has a

singular value decomposition. The presence of a zero eigenmode in $A^\dagger A$ implies

$$\det A = 0. \tag{C.2}$$

We will rotate this matrix into a basis, where the left and right Goldstone modes correspond to the first basis vector of the rows and columns, respectively. We first write A in $(1 + (n-1)) \times (1 + (n-1))$ block form:

$$A = \begin{pmatrix} m & \mathbf{m_R}^\dagger \\ \mathbf{m_L} & \mathbf{M} \end{pmatrix}, \tag{C.3}$$

where boldface small letters denote $n-1$ column vectors, and boldface capital letters denote matrices. Since A has a massless mode, there exist left and right null-eigenvectors $\vec{e}_0$ and $\vec{f}_0$, respectively:

$$A^\dagger \vec{e}_0 = 0, \qquad\qquad A\vec{f}_0 = 0. \tag{C.4}$$

Providing we choose the phases such that the first components of $\vec{e}_0$ and $\vec{f}_0$ are positive, we define $(n-1)$ column vectors $\mathbf{e}$ and $\mathbf{f}$ via

$$\frac{\vec{e}_0}{|\vec{e}_0|} =: \begin{pmatrix} \sqrt{1 - \mathbf{e}^\dagger\mathbf{e}} \\ \mathbf{e} \end{pmatrix} \qquad\qquad \frac{\vec{f}_0}{|\vec{f}_0|} =: \begin{pmatrix} \sqrt{1 - \mathbf{f}^\dagger\mathbf{f}} \\ \mathbf{f} \end{pmatrix}. \tag{C.5}$$

Writing the null-eigenvector conditions in equations (C.4) in block form, we get 3 independent equations (two $(n-1)$-vector, 1 scalar):

$$\sqrt{1 - \mathbf{e}^\dagger\mathbf{e}}\, \mathbf{m_R} + \mathbf{M}^\dagger\, \mathbf{e} = 0, \tag{C.6}$$

$$\sqrt{1 - \mathbf{f}^\dagger\mathbf{f}}\, \mathbf{m_L} + \mathbf{M}\, \mathbf{f} = 0, \tag{C.7}$$

$$m\sqrt{1 - \mathbf{e}^\dagger\mathbf{e}}\sqrt{1 - \mathbf{f}^\dagger\mathbf{f}} - \mathbf{e}^\dagger\, \mathbf{M}\, \mathbf{f} = 0 \tag{C.8}$$

These can for example be used to define m, $\mathbf{m_L}$ and $\mathbf{m_R}$ in terms of $\mathbf{M}$, $\mathbf{e}$ and $\mathbf{f}$. We now define a $n \times n$ unitary matrix $U(\mathbf{x})$, whose form will be useful for rotating the basis of A: in block form, U is written as

$$U(\mathbf{x}) := \begin{pmatrix} \sqrt{1 - \mathbf{x}^\dagger\mathbf{x}} & \mathbf{x}^\dagger \\ -\mathbf{x} & \Lambda(\mathbf{x}) \end{pmatrix}, \tag{C.9}$$

where

$$\Lambda(\mathbf{x}) := I - \frac{\mathbf{x}\,\mathbf{x}^\dagger}{1 + \sqrt{1 - \mathbf{x}^\dagger\mathbf{x}}}. \tag{C.10}$$

It is possible to check explicitly that U is indeed unitary and that the following relations hold:

$$U(\mathbf{x})\,U(\mathbf{x})^\dagger = I, \tag{C.11}$$

$$U(\mathbf{x})^{-1} = U(\mathbf{x})^\dagger = U(-\mathbf{x}), \tag{C.12}$$

$$\Lambda(\mathbf{x})^\dagger = \Lambda(\mathbf{x}) = \Lambda(-\mathbf{x}), \tag{C.13}$$

$$\Lambda(\mathbf{x})^{-1} = I + \frac{\mathbf{x}\,\mathbf{x}^\dagger}{\sqrt{1 - \mathbf{x}^\dagger\mathbf{x}}\,(1 + \sqrt{1 - \mathbf{x}^\dagger\mathbf{x}})}, \tag{C.14}$$

$$\det\Lambda(\mathbf{x}) = 1 - \frac{\mathbf{x}^\dagger\mathbf{x}}{1 + \sqrt{1 - \mathbf{x}^\dagger\mathbf{x}}}. \tag{C.15}$$

The properly rotated matrix A, denoted by A', is then by explicit computation equal to

$$A' := U(\mathbf{e})\,A\,U(\mathbf{f})^{-1} = \begin{pmatrix} 0 & \mathbf{0}^\dagger \\ \mathbf{0} & \mathbf{M_{rot}} \end{pmatrix}, \tag{C.16}$$

where

$$\mathbf{M_{rot}} = \Lambda(\mathbf{e})^{-1}\mathbf{M}\,\Lambda(\mathbf{f})^{-1}. \tag{C.17}$$

We have indeed rotated into a basis, where the first column and row correspond to the zero eigenmodes. Crucially, the formula for the rotated $(n-1) \times (n-1)$ block has only one term, where a simple biunitary rotation is performed on the original block $\mathbf{M}$; another zero-eigenmode can now be simply imposed by taking $\det\mathbf{M_{rot}} = 0$. But since we now have

$$\det\mathbf{M} = \det\mathbf{M_{rot}}\,\det\big(\Lambda(\mathbf{e})\Lambda(\mathbf{f})\big). \tag{C.18}$$

it is sufficient to impose $\det\mathbf{M} = 0$. The only possible caveat is the possibility that either $\det\Lambda(\mathbf{e}) = 0$ or $\det\Lambda(\mathbf{f}) = 0$; considering equation (C.15) and that $\mathbf{e}$ and $\mathbf{f}$ are parts of normalized vectors, this can happen only if $\mathbf{e}^\dagger\mathbf{e} = 1$ or $\mathbf{f}^\dagger\mathbf{f} = 1$, which would imply that the zero modes of A have a zero component in the direction of the first vector of the original basis. But since the eigenvalues of $A^\dagger A$ (the determinant of $A^\dagger A - \lambda I$) do not change if we rearrange the rows or the columns of A, we can always rearrange the original basis so that we take the i-th row and the j-th column to be the preferred one for the left and right null-eigenmodes, respectively.

The main result thus states the following: if an $n \times n$ complex matrix A has a zero mode present, an additional zero mode is obtained by demanding the (i,j)-th minor of A to vanish (the subdeterminant of A, when the i-th row and j-th column are removed), where the indices i and j can be arbitrarily chosen, as long as the left and right null-eigenvectors of A have respectively a nonzero i-th and j-th component.

D An alternative vacuum

The Yukawa fit presented in this paper is based on the vacuum computed in section 3. Since we have not been able to obtain (yet) a full classification of all vacua in this E_6 model, we cannot conclude much about the broader possibilities of suitable vacua (especially the Yukawa sector), but we were able to find one other alternative Standard Model vacuum.

We can obtain it by taking the following ansatz of vanishing VEVs:

$$c_2 = d_2 = e_5 = f_5 = e_2 = f_2 = u_1 = u_2 = y = 0. \tag{D.1}$$

This ansatz is similar to the ansatz of the original vacuum, but the vanishing of e_5, f_5 is now paired up with the vanishing of c_2, d_2 instead of c_1, d_1. For the purposes of this appendix, we omit the specific form of the solutions, as well as other details, such as the check that the unbroken group is indeed that of the Standard Model.

Looking at the Yukawa terms in equation (5.2), we see that the alternative vacuum with $c_2 = f_5 = 0$ decouples the 16_F from the 10_F of SO(10) (in leading order of m_{EW}/M_{GUT}). The heavy vector-like exotics (in the down-quark sector and charged lepton sector) in the fermionic 27_F are thus purely in the 10_F part. The low-energy mass matrices are thus simply those for the 16_F, the analysis of the Yukawa sector becomes linear, and numerically one can make use of the fit for the minimal supersymmetric SO(10) model [19, 20, 29–39] with the Higgs in the 10 and $\overline{126}$ coupling to fermion pairs in 16_F^2.

The intriguing possibility of this alternative vacuum, which recovers the SO(10) limit, is somewhat marred by issues in DT splitting. Given the ansatz in equation (D.1), the doublet and triplet mass matrices become block diagonal with the following block form (with the basis of the barred states rearranged in the same order as for unbarred):

$$\begin{bmatrix} D_0 \ D_3 \ D_6 \ D_9 \ D_{10} \ D_{11} \end{bmatrix}, \qquad \begin{bmatrix} D_1 \ D_2 \ D_4 \ D_5 \end{bmatrix}, \qquad \begin{bmatrix} D_7 \ D_8 \end{bmatrix}, \tag{D.2}$$

$$\begin{bmatrix} T_0 \ T_3 \ T_6 \ T_9 \ T_{10} \ T_{11} \end{bmatrix}, \qquad \begin{bmatrix} T_1 \ T_2 \ T_4 \ T_5 \end{bmatrix}, \qquad \begin{bmatrix} T_7 \ T_8 \ T_{12} \end{bmatrix}. \tag{D.3}$$

The Goldstone modes, for example, turn out to be in the first block. The problem now is, however, that a fine-tuning is block specific: the light Higgs lives only in one of the blocks. We see from equation (5.2) that the low-energy fermionic mass matrices M_D^T and M_E are now controlled solely by the EW VEVs $\bar{v}_2$, $\bar{v}_4$ and $\bar{v}_8$. In the $\bar{v}_8 = 0$ case, we get the unwanted mass relation $M_D^T = M_E$, but $\overline{D}_8$ is in a separate block compared to $\overline{D}_2$ and $\overline{D}_4$. A realistic pattern of fermion masses would thus require a double fine-tuning: one in the second block and one in the third block. Due to this feature, we consider this vacuum to be of less interest: beside the aesthetically unpleasing extra fine-tuning, the additional light Higgs pair $H'_u + H'_d$ pair gives large threshold corrections in the running of the gauge couplings, possibly spoiling unification. The model contains many heavy states though, so the situation regarding the RGE is not clear-cut. For determining the feasibility of this vacuum, further investigation would be necessary.

We conclude this section of the appendix with a brief elaboration on which kind of ansatz is suitable for a good vacuum, i.e. we motivate equations (3.28)–(3.29) and (D.1). The considerations will be very similar to the ones in the E_6 breaking sector, where the 78 is omitted [11] and a full classification of vacua is known. Suppose we look for a specific vacuum solution: we want it to be as simple as possible (it has as many vanishing VEVs as possible), yet it needs to be able to break E_6 to the SM group. Due to the D-terms in SUSY, we assume a conjugate-symmetric ansatz, where a vanishing VEV in 27 or 351′ implies that the corresponding (conjugate) VEV in the $\overline{27}$ or $\overline{351'}$ also vanishes, and vice versa. First, we identify the SU(5) breaking VEVs from table 1: e_4, e_5, f_4, f_5 and y. The

F_y equation of motion automatically implies $y = 0$. Since SU(5) needs to be broken, either the pair e_4, f_4 is non-vanishing, or the pair e_5, f_5. Considerations of alignment symmetry in [11] imply that the choice is irrelevant, since these pairs are exchanged if one changes the embedding of SO(10) in E$_6$, such that the two $\bar{5}$s of SU(5) in the 27 are exchanged. Therefore one pair needs to be necessarily non-zero, while we can try a vanishing ansatz for the other pair. In the $27 + \overline{27}$ part of the breaking sector, the pairs c_1, d_1 and c_2, d_2 also get exchanged under alignment symmetry; we assume one pair to be non-zero (such that the $27 + \overline{27}$ part does indeed contribute to the symmetry breaking), but we can again try setting the other pair to vanish in the simple ansatz. The choice of the vanishing pair now points to either the scenario of the main vacuum of this paper, or the the alternative vacuum presented in this appendix. The remaining part of the ansatz, $e_2 = f_2 = 0$ (which transforms into itself under alignment symmetry), is suggested from the solutions in the model without the 78, while $u_1 = u_2 = 0$ then follows as a consequence of the F-terms. We finish the ansatz discussion with the following points:

- The main vacuum solution of the paper follows from the ansatz, which is a direct extension of the solution ansatz in the absence of the 78 [11]. There, the EOMs were simpler, and this ansatz actually represented the most general SM solution once gauge freedom and the F-term equations were accounted for. Note that only the ansatz can be extended, the main solution itself is not merely an extension of the solution when the 78 is omitted.

- The alternative ansatz leads to a SM vacuum only after the 78 was included; there is no such option if the 78 is omitted.

- A complete classification would tell us, whether still other nonequivalent vacua exist, where the VEVs are non-vanishing. If the 78 is omitted, such vacua did not exist, but we expect this situation to change due to more terms and more VEVs in the EOM.

E Proton decay

For completeness let's summarize the analysis of $D = 5$ proton decay in this model, similar to the analysis done in [11], obtaining analogous results. The low-energy operators in the superpotential, which are relevant for proton decay, are

$$W\big|_{\text{proton}} = -\Big[\big(\overline{C}_1^{inA} - \overline{C}_1'^{imA}(X^T)_m{}^n\big)\big[(1 + X^*X^T)^{-1/2}\big]_n{}^j \, (\hat{\mathcal{M}}_T^{-1})_{AB} \, C_1^{klB}\Big] \, Q_i \hat{L}_j Q_k Q_l$$
$$\quad - \Big[\big(\overline{C}_2^{njA} + \frac{2}{3}\overline{C}_2'^{mjA}(X^T)_m{}^n\big)\big[(1 + \frac{4}{9}X^*X^T)^{-1/2}\big]_n{}^i \, (\hat{\mathcal{M}}_T^{-1})_{AB} \, C_2^{klB}\Big] \, \hat{d}_i^c u_j^c u_k^c e_l^c,$$
(E.1)

with the X defined in equation (5.9) and the mass matrix of $\mathcal{M}_{\text{T}}$ as already defined in equation (4.2) of section 4. Note that the $\mathcal{M}_T$ has a zero eigenmode corresponding to the would-be Goldstone, so its inverse cannot be directly computed; we instead write $\hat{\mathcal{M}}_T^{-1} = \lim_{M\to\infty}(\mathcal{M}_T + M\,e\,f^T)^{-1}$, with e and f being the left and right column-eigenvectors of $\mathcal{M}_T$, respectively.

The C coefficients are computed to be

$$2\,C_1^{ijA} = -Y_{27}^{ij}\,\delta^A{}_1 + \frac{1}{2\sqrt{10}}\,Y_{351'}^{ij}\,\delta^A{}_5 - \frac{1}{2\sqrt{6}}\,Y_{351'}^{ij}\,\delta^A{}_7 - \frac{1}{2\sqrt{3}}\,Y_{351'}^{ij}\,\delta^A{}_{12}, \tag{E.2}$$

$$2\,C_2^{ijA} = -Y_{27}^{ij}\,\delta^A{}_1 + \frac{1}{2\sqrt{10}}\,Y_{351'}^{ij}\,\delta^A{}_5 - \frac{1}{2\sqrt{6}}\,Y_{351'}^{ij}\,\delta^A{}_7 + \frac{2}{2\sqrt{3}}\,Y_{351'}^{ij}\,\delta^A{}_{12}, \tag{E.3}$$

$$2\,\overline{C}_1^{ijA} = -Y_{27}^{ij}\,\delta^A{}_2 + \frac{1}{2\sqrt{10}}\,Y_{351'}^{ij}\,\delta^A{}_4 + \frac{1}{2\sqrt{2}}\,Y_{351'}^{ij}\,\delta^A{}_8, \tag{E.4}$$

$$2\,\overline{C}_1'^{ijA} = \;\;\;Y_{27}^{ij}\,\delta^A{}_3 - \frac{1}{2\sqrt{10}}\,Y_{351'}^{ij}\,\delta^A{}_9 - \frac{1}{2\sqrt{2}}\,Y_{351'}^{ij}\,\delta^A{}_{11}, \tag{E.5}$$

$$2\,\overline{C}_2^{ijA} = -Y_{27}^{ij}\,\delta^A{}_2 + \frac{1}{2\sqrt{10}}\,Y_{351'}^{ij}\,\delta^A{}_4 - \frac{1}{2\sqrt{2}}\,Y_{351'}^{ij}\,\delta^A{}_8, \tag{E.6}$$

$$2\,\overline{C}_2'^{ijA} = \;\;\;Y_{27}^{ij}\,\delta^A{}_3 - \frac{1}{2\sqrt{10}}\,Y_{351'}^{ij}\,\delta^A{}_9 + \frac{1}{2\sqrt{2}}\,Y_{351'}^{ij}\,\delta^A{}_{11}. \tag{E.7}$$

We see that the C-coefficients are the same as the coefficients in [11], if we cross out the contributions from the extra Yukawa term in that model. More specifically, notice that there are no $A = 0$ contributions from triplets/antitriplets in the new representation 78, since the 78 is not present in the Yukawa sector (it does not couple to two 27_F's).

Once the model parameters are fit to the experimental values of the fermion masses and mixings, the four-fermion amplitude mediating proton decay is fixed and thus potentially dangerous, but it can always be suppressed by implementing a split supersymmetric scenario without changing any other conclusion.

References

[1] H. Georgi and S.L. Glashow, *Unity of All Elementary Particle Forces*, *Phys. Rev. Lett.* **32** (1974) 438 [INSPIRE].

[2] F. Gürsey, P. Ramond and P. Sikivie, *A Universal Gauge Theory Model Based on E6*, *Phys. Lett.* **B 60** (1976) 177 [INSPIRE].

[3] O.K. Kalashnikov and S.E. Konshtein, *Hierarchy of Interactions and Mass Relations in Asymptotically Free E6 Model of Unified Interaction*, *Nucl. Phys.* **B 166** (1980) 507 [INSPIRE].

[4] F. Buccella and G. Miele, SO(10) *From Supersymmetric E₆*, *Phys. Lett.* **B 189** (1987) 115 [INSPIRE].

[5] F. Gürsey and M. Serdaroglu, *Basic Fermion Masses and Mixings in the E₆ Model*, *Lett. Nuovo Cim.* **21** (1978) 28 [INSPIRE].

[6] Y. Achiman and B. Stech, *Quark Lepton Symmetry and Mass Scales in an E6 Unified Gauge Model*, *Phys. Lett.* **B 77** (1978) 389 [INSPIRE].

[7] Q. Shafi, *E₆ as a Unifying Gauge Symmetry*, *Phys. Lett.* **B 79** (1978) 301 [INSPIRE].

[8] H. Ruegg and T. Schucker, *Masses for Fermions in a Unified Gauge Model Based on E6*, *Nucl. Phys.* **B 161** (1979) 388 [INSPIRE].

[9] R. Barbieri and D.V. Nanopoulos, *An Exceptional Model for Grand Unification*, *Phys. Lett.* **B 91** (1980) 369 [INSPIRE].

[10] G.L. Shaw and R. Slansky, *A E_6 Model With Composite Muon and Tau Families*, *Phys. Rev.* **D 22** (1980) 1760 [INSPIRE].

[11] B. Bajc and V. Sušič, *Towards the minimal renormalizable supersymmetric E_6 model*, *JHEP* **02** (2014) 058 [arXiv:1311.0775] [INSPIRE].

[12] N. Maekawa and T. Yamashita, *E_6 unification, doublet triplet splitting and anomalous* U(1)(A) *symmetry*, *Prog. Theor. Phys.* **107** (2002) 1201 [hep-ph/0202050] [INSPIRE].

[13] T.E. Clark, T.-K. Kuo and N. Nakagawa, *A* SO(10) *supersymmetric grand unified theory*, *Phys. Lett.* **B 115** (1982) 26 [INSPIRE].

[14] C.S. Aulakh and R.N. Mohapatra, *Implications of Supersymmetric* SO(10) *Grand Unification*, *Phys. Rev.* **D 28** (1983) 217 [INSPIRE].

[15] C.S. Aulakh, B. Bajc, A. Melfo, G. Senjanović and F. Vissani, *The Minimal supersymmetric grand unified theory*, *Phys. Lett.* **B 588** (2004) 196 [hep-ph/0306242] [INSPIRE].

[16] C.S. Aulakh, *MSGUTs from germ to bloom: Towards falsifiability and beyond*, hep-ph/0506291 [INSPIRE].

[17] B. Bajc, A. Melfo, G. Senjanović and F. Vissani, *Fermion mass relations in a supersymmetric* SO(10) *theory*, *Phys. Lett.* **B 634** (2006) 272 [hep-ph/0511352] [INSPIRE].

[18] C.S. Aulakh and S.K. Garg, *MSGUT: From bloom to doom*, *Nucl. Phys.* **B 757** (2006) 47 [hep-ph/0512224] [INSPIRE].

[19] S. Bertolini, T. Schwetz and M. Malinsky, *Fermion masses and mixings in* SO(10) *models and the neutrino challenge to SUSY GUTs*, *Phys. Rev.* **D 73** (2006) 115012 [hep-ph/0605006] [INSPIRE].

[20] B. Bajc, I. Doršner and M. Nemevšek, *Minimal* SO(10) *splits supersymmetry*, *JHEP* **11** (2008) 007 [arXiv:0809.1069] [INSPIRE].

[21] T.W. Kephart and M.T. Vaughn, *Tensor Methods for the Exceptional Group E6*, *Annals Phys.* **145** (1983) 162 [INSPIRE].

[22] G.W. Anderson and T. Blazek, *E_6 unification model building. 1. Clebsch-Gordan coefficients of* $27 \times \overline{27}$, *J. Math. Phys.* **41** (2000) 4808 [hep-ph/9912365] [INSPIRE].

[23] G.W. Anderson and T. Blazek, *E_6 unification model building. 2. Clebsch-Gordan coefficients of* 78×78, *J. Math. Phys.* **41** (2000) 8170 [hep-ph/0006017] [INSPIRE].

[24] G.W. Anderson and T. Blazek, *E_6 unification model building. 3. Clebsch-Gordan coefficients in E_6 tensor products of the 27 with higher dimensional representations*, hep-ph/0101349 [INSPIRE].

[25] R. Slansky, *Group Theory for Unified Model Building*, *Phys. Rept.* **79** (1981) 1 [INSPIRE].

[26] Z.-z. Xing, H. Zhang and S. Zhou, *Updated Values of Running Quark and Lepton Masses*, *Phys. Rev.* **D 77** (2008) 113016 [arXiv:0712.1419] [INSPIRE].

[27] Particle Data Group collaboration, K.A. Olive et al., *Review of Particle Physics*, *Chin. Phys.* **C 38** (2014) 090001.

[28] J.L. Hewett and T.G. Rizzo, *Low-Energy Phenomenology of Superstring Inspired E_6 Models*, *Phys. Rept.* **183** (1989) 193 [ɪɴSPIRE].

[29] K.S. Babu and R.N. Mohapatra, *Predictive neutrino spectrum in minimal* SO(10) *grand unification*, *Phys. Rev. Lett.* **70** (1993) 2845 [hep-ph/9209215] [ɪɴSPIRE].

[30] B. Bajc, G. Senjanović and F. Vissani, *How neutrino and charged fermion masses are connected within minimal supersymmetric* SO(10), PoS(HEP2001)198 [hep-ph/0110310] [ɪɴSPIRE].

[31] T. Fukuyama and N. Okada, *Neutrino oscillation data versus minimal supersymmetric* SO(10) *model*, *JHEP* **11** (2002) 011 [hep-ph/0205066] [ɪɴSPIRE].

[32] B. Bajc, G. Senjanović and F. Vissani, $b - \tau$ *unification and large atmospheric mixing: A Case for noncanonical seesaw*, *Phys. Rev. Lett.* **90** (2003) 051802 [hep-ph/0210207] [ɪɴSPIRE].

[33] H.S. Goh, R.N. Mohapatra and S.-P. Ng, *Minimal SUSY* SO(10)*, $b\,\tau$ unification and large neutrino mixings*, *Phys. Lett.* **B 570** (2003) 215 [hep-ph/0303055] [ɪɴSPIRE].

[34] H.S. Goh, R.N. Mohapatra and S.-P. Ng, *Minimal SUSY* SO(10) *model and predictions for neutrino mixings and leptonic CP-violation*, *Phys. Rev.* **D 68** (2003) 115008 [hep-ph/0308197] [ɪɴSPIRE].

[35] S. Bertolini, M. Frigerio and M. Malinsky, *Fermion masses in SUSY* SO(10) *with type-II seesaw: A Non-minimal predictive scenario*, *Phys. Rev.* **D 70** (2004) 095002 [hep-ph/0406117] [ɪɴSPIRE].

[36] K.S. Babu and C. Macesanu, *Neutrino masses and mixings in a minimal* SO(10) *model*, *Phys. Rev.* **D 72** (2005) 115003 [hep-ph/0505200] [ɪɴSPIRE].

[37] A.S. Joshipura and K.M. Patel, *Fermion Masses in* SO(10) *Models*, *Phys. Rev.* **D 83** (2011) 095002 [arXiv:1102.5148] [ɪɴSPIRE].

[38] G. Altarelli and D. Meloni, *A non supersymmetric* SO(10) *grand unified model for all the physics below* M_{GUT}, *JHEP* **08** (2013) 021 [arXiv:1305.1001] [ɪɴSPIRE].

[39] A. Dueck and W. Rodejohann, *Fits to* SO(10) *Grand Unified Models*, *JHEP* **09** (2013) 024 [arXiv:1306.4468] [ɪɴSPIRE].

[40] C.S. Aulakh, *NMSGUT-III: Grand Unification upended*, arXiv:1107.2963 [ɪɴSPIRE].

[41] C.S. Aulakh, I. Garg and C.K. Khosa, *Baryon stability on the Higgs dissolution edge: threshold corrections and suppression of baryon violation in the NMSGUT*, *Nucl. Phys.* **B 882** (2014) 397 [arXiv:1311.6100] [ɪɴSPIRE].

[42] S. Abel and V.V. Khoze, *Direct Mediation, Duality and Unification*, *JHEP* **11** (2008) 024 [arXiv:0809.5262] [ɪɴSPIRE].

[43] S. Abel and V.V. Khoze, *Dual unified* SU(5), *JHEP* **01** (2010) 006 [arXiv:0909.4105] [ɪɴSPIRE].