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A new look at instantons and large-N limit

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Abstract: We analyze instantons in the very strongly coupled large-N limit ($N \to \infty$ with g^2 fixed) of large-N gauge theories, where the effect of the instantons remains finite. By using the exact partition function of four-dimensional $\mathcal{N} = 2^*$ gauge theories as a concrete example, we demonstrate that each instanton sector in the very strongly coupled large-N limit is related to the one in the 't Hooft limit ($N \to \infty$ with g^2N fixed) through a simple analytic continuation. Furthermore we show the equivalence between the instanton partition functions of a pair of large-N gauge theories related by an orbifold projection. This can open up a new way to analyze the partition functions of low/non-supersymmetric theories. We also discuss implication of our result to gauge/gravity dualities for M-theory as well as a possible application to large-N QCD.

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Contents

1 Introduction

The 't Hooft limit of large-N gauge theories [1], $N \to \infty$ with the 't Hooft coupling $\lambda = g^2N$ fixed (where g stands for the gauge coupling), has been playing a prominent role in various fields of theoretical physics. Around the 't Hooft limit, there exists a $1/N$ expansion which rearranges the Feynman diagrams in a geometric manner. In the 't Hooft limit, only the planar diagrams survive and drastic simplification takes place. When it comes to instantons, however, it is not clear whether the 't Hooft limit is an appropriate playground, because the instanton action grows as $1/g^2 = N/\lambda \sim N$, providing an exponential suppression factor of the form e^{-N}.[1] In this paper, we demonstrate that a certain information of the instantons can be extracted from the 't Hooft limit. The starting point is to consider a more general large-N limit, in which λ scales as N^p ($p > 0$); here we call it *the very strongly coupled large-N limit*. As a special case, it contains a large-N limit with fixed g^2 ($p = 1$), where the instanton action is of order N^0 (we notice that, when we say 'order N^q' in this paper, it is in the sense of the large-N order counting, and the coefficient in front of N^q is not specified) and hence the instanton effect is not suppressed.[2] In this paper, we also call this special case simply by the very strongly coupled limit, unless otherwise stated.

In [3, 4], it has been shown that, in the zero instanton sector, various properties in the 't Hooft limit are inherited to the the very strongly coupled large-N limit. The argument is very simple for theories with gravity duals. As the simplest example, let us consider the four-dimensional $\mathcal{N} = 4$ SU(N) supersymmetric Yang-Mills theory. Through the AdS/CFT correspondence [5], this theory is dual to classical type IIB supergravity on $AdS_5 \times S^5$ background. In order for the perturbative string description to be valid, string α' correction, which is characterized by $\alpha'/R^2_{\text{AdS}} \sim 1/\sqrt{\lambda}$, and string loop correction, which is

[1]By taking quantum effect into account, the weight can behave as $e^{-f(\lambda)/g^2}$, where $f(\lambda)$ satisfies $f(\lambda) > 0$ at weak coupling. Then if $f(\lambda)$ becomes zero, the instantons can give a nontrivial contribution. See a nice review [2] for details.

[2]From field theory point of view, it is not even clear a priori whether the notion of the instanton, which is obtained by semi-classical analysis, makes sense. As we will see, such a limit can be taken safely, at least for $\mathcal{N} = 2$ supersymmetric gauge theories.

controlled by the string coupling constant $g_s \sim g^2$, must be small. In order to satisfy these conditions, usually one takes the 't Hooft limit first and then sends the 't Hooft coupling to large but still of order N^0 (i.e. λ does not scale with N). It is however not really needed for the AdS/CFT to be valid; the supergravity provides us with a good approximation in the large-N limit with $1 \ll \lambda \ll N$. Here $\lambda \ll N$ just means $g^2 \ll 1$ with $g^2 = \lambda/N$ and then g^2 can be of order N^0 but small ('small' means the coefficient in front of N^0 is small). Therefore the very strongly coupled large-N limit is also described by the classical supergravity, so far as $g^2 \ll 1$ is satisfied. In particular, we can take $p = 1$ for which g^2 does not scale with N. In other words, the correct results in the very strongly coupled large-N limit can be obtained by the analytic continuation from the 't Hooft limit. The same property holds in other theories too, even without gravity duals or sometimes even when g^2 is of order N^0 and not small or is of some positive power of N [3].

In this paper we generalize the argument of [3] to take into account the instanton effect. As we have mentioned, the instanton effect is in general exponentially suppressed in the 't Hooft limit, while it is of order N^0 when g^2 is fixed. If we consider a fixed instanton sector, however, calculations in the 't Hooft limit still make sense, because the 't Hooft expansion is allowed in this sector as in the zero-instanton sector. Below we argue that the property of the instantons in the very strongly coupled large-N limit can be extracted from such calculations.

For this purpose, we consider the free energy of four-dimensional $\mathcal{N} = 2^*$ U(N) gauge theory as a concrete example. The instanton partition function is given by Nekrasov's formula at any g and N, and we can confirm the validity of our conjecture by taking appropriate limits. It strongly suggests that various nice properties in the 't Hooft limit are smoothly extended to the very strongly coupled large-N limit, even in the sectors with non-zero instanton numbers. As an example, we show that the large-N orbifold equivalence [6–8] holds in each instanton sector. More concretely, we consider $\mathcal{N} = 2^*$ U(kN) gauge theory and $\mathcal{N} = 2$ $[\mathrm{U}(N)]^k$ necklace quiver gauge theory related by an orbifold projection, and show the matching of the contribution to the free energies from the instanton sectors. In a similar manner, we can also consider more general orbifold projections mapping $\mathcal{N} = 2$ theories to $\mathcal{N} < 2$ theories. This can allow us to analyze the instanton effects in low/non-supersymmetric theories.

This paper is organized as follows. In section 2 we introduce $\mathcal{N} = 2^*$ gauge theories and clarify the connection between the 't Hooft limit and the very strongly coupled large-N limit. In section 3 we explain the orbifold equivalence, show how it is generalized to the instanton sectors, and then confirm the validity for a specific example. Section 4 is devoted for discussions on our results and future directions.

2 From g^2N fixed to g^2 fixed

As a concrete setup, let us consider the free energy of four-dimensional $\mathcal{N} = 2^*$ U(kN) gauge theory on S^4 with a unit radius (k and N are integers). This theory is realized as a deformation of $\mathcal{N} = 4$ U(kN) supersymmetric Yang-Mills theory by adding a mass term to the $\mathcal{N}=2$ hypermultiplet part. We denote the mass parameter by m (see e.g. [9]) and fix it to be of order N^0.

The partition function of $\mathcal{N}=2^*$ U(kN) gauge theory with the gauge coupling g_p is given by the following integral expression with respect to the "eigenvalues" (or equivalently the Coulomb parameters) a_i $(i=1,2,\cdots,kN)$ [9, 10]:

$$\mathcal{Z}_{\mathcal{N}=2^*} = \int d^{kN}a \Big(\prod_{\substack{i,j=1\\ i<j}}^{kN}(a_i-a_j)^2\Big) Z^{(\mathrm{pert})}_{\mathcal{N}=2^*}(a_i,m)\big|Z^{(\mathrm{inst})}_{\mathcal{N}=2^*}(a_i,\tilde{m})\big|^2 \exp\left(-\frac{8\pi^2}{g_p^2}\sum_{i=1}^{kN}a_i^2\right). \quad (2.1)$$

Here the perturbative one-loop contribution $Z^{(\mathrm{pert})}_{\mathcal{N}=2^*}(a_i,m)$ and the instanton contribution $Z^{(\mathrm{inst})}_{\mathcal{N}=2^*}(a_i,\tilde{m})$ are given by

$$\begin{aligned}
Z^{(\mathrm{pert})}_{\mathcal{N}=2^*}(a_i,m) &= \prod_{\substack{i,j=1\\ i\neq j}}^{kN} Z^{(\mathrm{pert})}_{\mathrm{vec}}(a_i-a_j)\prod_{\substack{i,j=1\\ i\neq j}}^{kN} Z^{(\mathrm{pert})}_{\mathrm{mat}}(a_i-a_j,m)\,,\\
Z^{(\mathrm{pert})}_{\mathrm{vec}}(a_i-a_j) &= H(i(a_i-a_j))\,,\\
Z^{(\mathrm{pert})}_{\mathrm{mat}}(a_i-a_j,m) &= e^{(1+\gamma)m^2}\left[H(i(a_i-a_j+m))H(i(a_i-a_j-m))\right]^{-\frac{1}{2}}\,, \qquad (2.2)
\end{aligned}$$

and

$$\begin{aligned}
Z^{(\mathrm{inst})}_{\mathcal{N}=2^*}(a_i,\tilde{m}) &= \sum_{Y=\{Y_1,\cdot\cdot,Y_{kN}\}} e^{-\frac{8\pi^2|Y|}{g_p^2}}\prod_{i,j=1}^{kN} Z^{(\mathrm{inst})}_{\mathrm{vec}}(a_i-a_j;Y_i,Y_j)Z^{(\mathrm{inst})}_{\mathrm{mat}}(a_i-a_j;Y_i,Y_j;\tilde{m})\,,\\
Z^{(\mathrm{inst})}_{\mathrm{vec}}(a_i-a_j;Y_i,Y_j) &= \prod_{s\in Y_i}\left[E(a_i-a_j;Y_i,Y_j,s)\right]^{-1}\prod_{t\in Y_j}\left[2-E(a_j-a_i;Y_j,Y_i,t))\right]^{-1}\,, \qquad (2.3)\\
Z^{(\mathrm{inst})}_{\mathrm{mat}}(a_i-a_j;Y_i,Y_j;\tilde{m}) &= \prod_{s\in Y_i}\left(E(a_i-a_j;Y_i,Y_j,s)-\tilde{m}\right)\prod_{t\in Y_j}\left(2-E(a_j-a_i;Y_j,Y_i,t)-\tilde{m}\right)\,,
\end{aligned}$$

with

$$\begin{aligned}
H(z) &= e^{-(1+\gamma)z^2}\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)^n e^{\frac{z^2}{n}}\,, \qquad \tilde{m}=im+1\,,\\
E(a_i-a_j,Y_i,Y_j,s) &= -h_{Y_j}(s)+(v_{Y_i}(s)+1)+i(a_j-a_i)\,. \qquad (2.4)
\end{aligned}$$

Here γ is Euler's constant. Each instanton configuration is labeled by a set of Young tableaux $Y=(Y_1,\cdots,Y_{kN})$, where the total number of boxes $|Y|=\sum_{i=1}^{kN}|Y_i|$ in the tableaux corresponds to the instanton number (Y_i can simply be empty, $\emptyset$). The contributions from the instantons and anti-instantons to the partition function are given by $Z^{(\mathrm{inst})}_{\mathcal{N}=2^*}$ and its complex conjugate, respectively. The parameter $s=(s_h,s_v)$ labels the position of a box (s_h-th column and s_v-th row) in a given Young tableau Y_i. For a given s, $h_{Y_i}(s)$ and $v_{Y_i}(s)$ are defined by $h_{Y_i}(s)=\nu'_{s_v}(Y_i)-s_h$ and $v_{Y_i}(s)=\nu_{s_h}(Y_i)-s_v$, where $\nu'_{s_v}(Y_i)$ and $\nu_{s_h}(Y_i)$ are length of s_v-th row and s_h-th column in the Young tableau Y_i, respectively.

The partition function is given by the sum of the partition functions for each instanton configuration which is labeled by two sets of Young tableaux:

$$\mathcal{Z}_{\mathcal{N}=2^*} = \sum_{Y,Y'}\mathcal{Z}_{Y,Y'}\,, \qquad (2.5)$$

where Y and Y' label the instanton and anti-instanton configurations, respectively. Therefore, we can analyze each sector of the instanton configuration, separately. The free energy of each instanton sector is defined by the partition function for the sector:

$$F_{Y,Y'} = -\log \mathcal{Z}_{Y,Y'} \ . \tag{2.6}$$

It can be separated into the contributions from the perturbative part and those from the instantons:

$$F_{Y,Y'} = F^{(\text{pert})} + F_Y^{(\text{inst})} + F_{Y'}^{(\text{anti-inst})} + F_{Y,Y'}^{(\text{int})} \,, \tag{2.7}$$

where $F_{Y,Y'}^{(\text{int})}$ is contribution from the interactions between instantons and anti-instantons. Since there are no contributions from instantons in the zero-instanton sector, we have

$$F^{(\text{pert})} = -\log \mathcal{Z}_{\emptyset,\emptyset} \,, \tag{2.8}$$

and we define the other parts as

$$F_Y^{(\text{inst})} = -\log \frac{\mathcal{Z}_{Y,\emptyset}}{\mathcal{Z}_{\emptyset,\emptyset}} \,, \qquad F_{Y'}^{(\text{anti-inst})} = -\log \frac{\mathcal{Z}_{\emptyset,Y'}}{\mathcal{Z}_{\emptyset,\emptyset}} \,, \qquad F_{Y,Y'}^{(\text{int})} = -\log \frac{\mathcal{Z}_{Y,Y'}\mathcal{Z}_{\emptyset,\emptyset}}{\mathcal{Z}_{Y,\emptyset}\mathcal{Z}_{\emptyset,Y'}} \ . \tag{2.9}$$

To show the validity of the analytic continuation from the 't Hooft limit to the very strongly coupled large-N limit, we first consider the zero-instanton sector. It turns out that analysis in the 't Hooft limit [11] is straightforwardly generalized to the very strongly coupled large-N limit as long as $g_p^2 \ll 1$ or equivalently $\lambda_p = g_p^2(kN) \ll N$ (which includes g^2 of order N^q $(q<0)$ as well as of order N^0 but small), because the saddle point method used in [11] is valid there. If we further assume $\lambda_p \gg 4\pi^2 m^2/(m^2+1)$, the spectral density $\rho(a) = \lim_{N\to\infty} \sum_{i=1}^{kN} \delta(a-a_i)/(kN)$ obeys the semi-circle law given by

$$\rho(x) = \frac{2}{\pi\mu^2}\sqrt{\mu^2 - x^2} \,, \tag{2.10}$$

where $\mu = \sqrt{\lambda_p(m^2+1)}/(2\pi)$. Then the free energy at the leading order of the large-N limit is given by

$$F^{(\text{pert})} = -(kN)^2(1+m^2)\left(\frac{1}{2}\log\frac{\lambda_p(1+m^2)}{16\pi^2} + \frac{1}{4} + \gamma\right) . \tag{2.11}$$

We can thus confirm that the free energy in the zero-instanton sector takes the same expression in the 't Hooft limit and the very strongly coupled large-N limit, at least at $g_p^2 \ll 1$.[3]

As a next step, we move to the instanton part.[4] Here we consider a fixed instanton sector, in which the number of the instantons and anti-instantons is fixed. Let us denote eigenvalues with which non-empty Young tableaux are associated (i.e. eigenvalues describing instantons) by b_i, while the other eigenvalues, for which the corresponding Young tableaux are empty, by a_i. In the 't Hooft large-N limit, the instanton/anti-instanton contribution to the free energy is of order N^1, and is sub-leading compared to the perturbative

[3] In this case, there is no singularity separating two limits. For the case with singularities, see [3].
[4] Analysis in the limit $m \to \infty, \lambda \to 0$ has been done in [12].

contribution, which is of order N^2 (this is the argument for a fixed instanton sector. If we consider the sum of all sectors, the suppression factor $\sim e^{-N}$ appears compared to the zero-instanton sector). Similarly, even in the very strongly coupled limit, even if g_p^2 does not scale with N, the instanton/anti-instanton contribution is suppressed, as long as $g_p^2 \ll 1$. Hence, in both of the limits, the spectral density $\rho(a)$ is the same as the one in the zero-instanton sector at the leading order of the large-N limit. Therefore, even in the very strongly coupled large-N limit, the instanton effects can be calculated in a similar fashion to that in the 't Hooft limit. Note however that, when g_p^2 is small but of order N^0, the instanton contribution to the free energy is of order N^0 and does not completely disappear at large-N, as we will see shortly.

Let us denote a part of eigenvalues a_i's in (2.3) which are associated with instantons ($Y_i \neq \emptyset$) by using different notation, b_i. Then, the interaction between b_i's is negligible compared to the one between b_i's and a_i's, because the number of b_i's are fixed while the number of a_i's goes to infinity with N. The contribution to the free energy from the instanton $F_Y^{(\mathrm{inst})} = -\log(\mathcal{Z}_{Y,\emptyset}/\mathcal{Z}_{\emptyset,\emptyset})$ is therefore given by

$$F_Y^{(\mathrm{inst})} = -kN \sum_{b_i \in (\mathrm{inst})} \int da\, \rho(a) \log Z_Y^{(\mathrm{inst})}(b_i, a, Y_i, \emptyset) + \Delta_{(\mathrm{pert})}(b_i) \tag{2.12}$$

at the leading order at large-N. Here $Z_Y^{(\mathrm{inst})}$ stands for the contribution of the instanton configuration labeled by Y on the right hand side of the first line of (2.3), and $\Delta_{(\mathrm{pert})}(b_i)$ is the change of the perturbative part, which is zero as long as $-\mu \leq b \leq \mu$ and is positive otherwise. In the same manner, the contribution from anti-instantons, labeled by Young tableaux Y', is given by

$$F_{Y'}^{(\mathrm{anti-inst})} = -kN \sum_{b'_i \in (\mathrm{inst})} \int da\, \rho(a) \left(\log Z_{Y'}^{(\mathrm{inst})}(b'_i, a, Y'_i, \emptyset)\right)^* + \Delta_{(\mathrm{pert})}(b'_i)\,, \tag{2.13}$$

where b'_i are eigenvalues describing anti-instantons. The interaction term $F_{Y,Y'}^{(\mathrm{int})}$ has non-zero contribution when both Y_i and Y'_i are not empty for some i. In the present case, however, contributions from such configurations are sub-leading.

At $g_p^2 \ll 1$ and $\lambda_p \gg 4\pi^2 m^2/(m^2+1)$, the expression (2.13) can be simplified by substituting the semi-circle law (2.10) for the instanton contribution. Because both the 't Hooft limit and the very strongly coupled large-N limit satisfy these conditions, and because the semi-circle law takes the same form in both of these limits, one obtains the same expression for the contribution to the free energy from a fixed (finite-)instanton sector in both of the limits.

For example, one-instanton contribution to the free energy labeled by $Y = (\square, \emptyset, \cdots, \emptyset)$, is

$$\begin{aligned} F_Y^{(\mathrm{inst})} \simeq & \frac{8\pi^2}{g_p^2} - \log\left[\frac{(1-\tilde{m})^2}{(2-\tilde{m})\tilde{m}}\right] \\ & -kN \int da\, \rho(a) \log\left[\frac{(2-i(b-a)-\tilde{m})(i(b-a)-\tilde{m})}{(2-i(b-a))i(b-a)}\right] + \Delta_{(\mathrm{pert})}(b). \end{aligned} \tag{2.14}$$

If one further wants to integrate with respect to b, dominant contribution comes from $-\mu < b < \mu$, where $\Delta_{(\mathrm{pert})}(b)$ vanishes. The result of the integration is

$$\log \mathrm{Re} \int_{-\mu}^{\mu} db \exp\left(kN \int da\, \rho(a) \log\left[\frac{(2 - i(b-a) - \tilde{m})(i(b-a) - \tilde{m})}{(2 - i(b-a))i(b-a)} \right] \right) = O(\log g_p^2) \tag{2.15}$$

and hence only the first two terms on the right hand side of (2.14) survive. We note that, from the dual gravity point of view, it is natural to expect that the contribution from the higher genus in the perturbative part is less important at least when $g_p^2 \ll 1$. We also notice that, although genus one diagrams in the perturbative sector may give comparable contribution to the free energy as the one from the instantons when g_p^2 is of order N^0, they are common to all the sectors with finite instanton numbers at the leading order of the large-N limit and hence the comparison of the instanton actions still makes sense.

It is straightforward to take into account multi-instanton configurations to the free energy and sum them up at the leading order of the large-N limit. To understand it let us remind that the free energy for generic tableaux Y decomposes to a sum of contribution from each eigenvalue as $F_Y^{(\mathrm{inst})} \simeq \sum_{Y_i \neq \emptyset} F_{Y_i}^{(\mathrm{inst})}$, because the interaction between the instantons, b_i's, is negligible. (For the same reason, the interaction between instantons and anti-instantons $F_{Y,Y'}^{(\mathrm{int})}$ is negligible.) The free energy therefore becomes

$$F = -\log \mathcal{Z}_{\emptyset} + kN \log \left| 1 + \sum_{\tilde{Y}} e^{-F_{\tilde{Y}}^{(\mathrm{inst})}} \right|^2 , \tag{2.16}$$

where $\tilde{Y}$'s stand for a subset of Y's with $Y_1 \neq \emptyset$, $Y_2 = Y_3 = \cdots = Y_N = \emptyset$.

3 Orbifold equivalence in the instanton sector

In this part we consider the orbifold equivalence which relates the 'parent theory' to its 'orbifold daughter theory' obtained by an orbifold projection. The statement of the usual orbifold equivalence for the free energy is as follows; in the 't Hooft limit, by setting the 't Hooft couplings of the parent and daughter theories, $\lambda_p = g_p^2(kN)$ and $\lambda_d = g_d^2 N$ respectively, to be the same, $\lambda_p = \lambda_d \equiv \lambda$, the free energies of these two theories are related by $F_p(\lambda, N) = kF_d(\lambda, N)$. In [3, 4] it was generalized to the very strongly coupled large-N limit. Around the zero-instanton vacuum, this equivalence can be proven by matching the planar diagrams in the two theories. It is natural to expect that the same argument holds around the vacuum with a non-zero instanton number in the 't Hooft limit, and it can be extended to the very strongly coupled large-N limit. Below we demonstrate that this equivalence does hold in the instanton sectors. Our argument below can apply both to the 't Hooft limit and to the very strongly coupled large-N limit.

As a concrete example, we take four-dimensional $\mathcal{N} = 2^*$ U(kN) gauge theory as a parent theory, and relate it to the $\mathcal{N} = 2$ $[\mathrm{U}(N)]^k$ necklace quiver gauge theory by an appropriate orbifold projection. Here we consider an orbifold projection preserving 4d

$\mathcal{N}=2$ supersymmetry, so that the free energy of the daughter theory can also be calculated analytically. The projection condition for the fields in the $\mathcal{N}=2$ vector multiplet, such as a gauge field A_μ, is given by $\Omega A_\mu \Omega^{-1} = A_\mu$, while the fields in the $\mathcal{N}=2$ hypermultiplets (we denote them symbolically as Φ) are projected as $\Omega\Phi\Omega^{-1} = \omega^{-1}\Phi$, where $\Omega = \mathrm{diag}(\omega \otimes \mathbf{1}_{N\times N}, \omega^2 \otimes \mathbf{1}_{N\times N}, \cdots, \omega^k \otimes \mathbf{1}_{N\times N})$ and $\omega = \exp(2\pi i/k)$.

We denote the gauge fields of the parent and daughter theories by $\mathcal{A}_\mu$ and $A_\mu^{(\alpha)}$, respectively, where (α) is the label of the k U(N) gauge groups ($\alpha = 1, 2, \cdots, k$). Let us denote the instanton solutions of the U(kN) and $[\mathrm{U}(N)]^k$ by $\bar{\mathcal{A}}_\mu$ and $(\bar{A}_\mu^{(1)}, \cdots, \bar{A}_\mu^{(k)})$. Then, in the U($kN$) theory, as a special case we have a block-diagonal configuration $\bar{\mathcal{A}}_\mu = \mathrm{diag}(\bar{A}_\mu^{(1)}, \cdots, \bar{A}_\mu^{(k)})$, that is, the instanton moduli of the $[\mathrm{U}(N)]^k$ theory is a subset of that of the U(kN) theory. Note that this configuration is projected to $(\bar{A}_\mu^{(1)}, \cdots, \bar{A}_\mu^{(k)})$ by the orbifold projection. If the instanton number of the configuration $\bar{A}_\mu^{(\alpha)}$ is l_α, the instanton number is $\sum_{\alpha=1}^k l_\alpha$ in both theories (From here on we consider only instantons for notational simplicity, but anti-instantons can be incorporated straightforwardly).

The correspondence between the classical actions of the parent and daughter theories, S_p^{cl} and S_d^{cl} respectively, is easy to see: they are calculated as

$$S_p^{\mathrm{cl}} = \frac{8\pi^2}{g_p^2}\sum_{\alpha=1}^k l_\alpha = \frac{8\pi^2 kN}{\lambda}\sum_{\alpha=1}^k l_\alpha, \quad S_d^{\mathrm{cl}} = \frac{8\pi^2}{g_d^2}\sum_{\alpha=1}^k l_\alpha = \frac{8\pi^2 N}{\lambda}\sum_{\alpha=1}^k l_\alpha, \tag{3.1}$$

respectively and thus $S_p^{\mathrm{cl}} = kS_d^{\mathrm{cl}}$ is satisfied.

As a next step, we show the agreement of quantum corrections. The partition function of the daughter theory is [9, 10]

$$\mathcal{Z}_{[\mathrm{U}(N)]^k} = \int \Big(\prod_{\alpha=1}^k d^N a^{(\alpha)} \prod_{\substack{i,j=1\\ i<j}}^N (a_i^{(\alpha)} - a_j^{(\alpha)})^2\Big) Z^{(\mathrm{pert})}_{[\mathrm{U}(N)]^k} \Big|Z^{(\mathrm{inst})}_{[\mathrm{U}(N)]^k}\Big|^2 \exp\Big(-\frac{8\pi^2}{g_d^2}\sum_{\alpha=1}^k\sum_{i=1}^N (a_i^{(\alpha)})^2\Big), \tag{3.2}$$

where $a_i^{(k+1)} = a_i^{(1)}$. The perturbative and instanton parts, $Z^{(\mathrm{pert})}_{[\mathrm{U}(N)]^k}$ and $Z^{(\mathrm{inst})}_{[\mathrm{U}(N)]^k}$, are given by

$$Z^{(\mathrm{pert})}_{[\mathrm{U}(N)]^k} = \Big(\prod_{\alpha=1}^k \prod_{\substack{i,j=1\\ i\neq j}}^N Z^{(\mathrm{pert})}_{\mathrm{vec}}(a_i^{(\alpha)} - a_j^{(\alpha)})\Big)\Big(\prod_{\alpha=1}^k \prod_{\substack{i,j=1\\ i\neq j}}^N Z^{(\mathrm{pert})}_{\mathrm{mat}}(a_i^{(\alpha)} - a_j^{(\alpha+1)})\Big), \tag{3.3}$$

$$\begin{aligned} Z^{(\mathrm{inst})}_{[\mathrm{U}(N)]^k} = &\sum_{Y^{(1)},\cdots,Y^{(k)}} \exp\left(-\frac{8\pi^2}{g_d^2}\sum_{\alpha=1}^k |Y^{(\alpha)}|\right) \\ &\times \prod_{\alpha=1}^k \prod_{i,j=1}^N Z^{(\mathrm{inst})}_{\mathrm{vec}}(a_i^{(\alpha)} - a_j^{(\alpha)}; Y_i^{(\alpha)}, Y_j^{(\alpha)}) Z^{(\mathrm{inst})}_{\mathrm{mat}}(a_i^{(\alpha)} - a_j^{(\alpha+1)}; Y_i^{(\alpha)}, Y_j^{(\alpha+1)}; \tilde{m}). \end{aligned}$$

We can confirm the orbifold equivalence in the zero-instanton sector in the following way. At $g_p^2, g_d^2 \ll 1$, one can use the saddle point method, as we have seen before. For the daughter theory, we introduce the spectral densities for the k U(N) gauge groups,

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$\rho^{(1)}, \cdots, \rho^{(k)}$. Then the saddle point equation of the parent and daughter theories coincide by taking the 'democratic ansatz',

$$\rho^{(1)}(y) = \cdots = \rho^{(k)}(y). \tag{3.4}$$

By substituting this ansatz, one obtains $F_p^{(\mathrm{pert})}(\lambda, N) = kF_d^{(\mathrm{pert})}(\lambda, N)$ for the zero-instanton sector even without using the detail of the spectral densities.

A generalization of the orbifold equivalence to the instanton sectors goes as follows, so far as the instanton number is of order one. We compare the partition functions of the parent and daughter theories at each instanton sector. Once one fixes a sector to consider, at the leading order of the large-N limit, the eigenvalues describing instantons can be treated as probes and thus do not affect the distribution of other eigenvalues. Then the free energies of the $\mathrm{U}(kN)$ and $[\mathrm{U}(N)]^k$ theories are respectively expressed as (2.12) and

$$\begin{aligned} F_{d,Y}^{(\mathrm{inst})} = &-N \sum_{\alpha=1}^{k} \sum_{b_i \in (\mathrm{inst})} \int da\, \rho^{(\alpha)}(a) \log Z_{\mathrm{vec}}^{(\mathrm{inst})}(b_i^{(\alpha)}, a, Y_i, \emptyset) \\ &-N \sum_{\alpha=1}^{k} \sum_{b_i \in (\mathrm{inst})} \int da\, \rho^{(\alpha+1)}(a) \log Z_{\mathrm{mat}}^{(\mathrm{inst})}(b_i^{(\alpha)}, a, Y_i, \emptyset; \tilde{m}) + \sum_{\alpha=1}^{k} \sum_{b_i \in (\mathrm{inst})} \Delta_{(\mathrm{pert})}^{(\alpha)}(b_i), \end{aligned} \tag{3.5}$$

($\rho^{(k+1)}(a) = \rho^{(1)}(a)$) at the leading order in the large-N limit. By substituting the democratic ansatz to (3.5) and comparing it with (2.12) , we obtain $F_{p,Y}^{(\mathrm{inst})} = kF_{d,Y}^{(\mathrm{inst})}$ for each instanton sector.

Before closing this section, we remark on a subtle issue associated with the vacuum structure. As emphasized in [8], the orbifold equivalence requires that the vacuum structures of the parent and daughter theories be properly related. In the present case, because the numbers of instantons and anti-instantons are finite and of order one, it did not change the vacuum structure and the equivalence in the zero-instanton sector is naturally extended. When the number of instantons and anti-instantons is of order N, the vacuum structure in the large-N limit is modified and hence careful identification of the right vacua is required. One has to assign the instantons and anti-instantons in the daughter theory 'democratically' to the k nodes, so that the instanton background becomes $\mathbb{Z}_k$ invariant and the democratic ansatz for the eigenvalues holds.

4 Discussions

Although we have used $\mathcal{N} = 2^*$ gauge theory and its orbifold daughter theory preserving $\mathcal{N} = 2$ supersymmetry for explicit demonstration, our calculation can be immediately generalized to other $\mathcal{N} = 2$ theories. We note that we have considered $\mathcal{N} = 2$ theories just because the free energies are calculable analytically. As discussed in [3], supersymmetry does not seem necessary.

The very strongly coupled large-N limit we have discussed in this paper may be useful for studying M-theory through gauge/gravity duality [5]. Within the framework of string theory, gauge/gravity duality relates the classical gravity to the planar diagrams, and the $1/N$ expansion around the 't Hooft limit is identified with the string loop expansion. When it comes to M-theory, however, situation had not been clear because the 't Hooft coupling grows with N where the dual gravity description turns to the eleven-dimensional supergravity. Our proposal for this issue is simple: the eleven-dimensional supergravity is also related to the planar sector in that the very strongly coupled large-N limit simply picks it up at the leading order. As we have investigated here, it is true not only in the perturbative sector [3, 4] but also in the instanton sectors. Although we performed explicit calculation at each instanton sector only at $g^2 \ll 1$ (both in the 't Hooft limit and the very strongly coupled large-N limit with $p = 1$), the sum of all instanton sectors (2.16) would allow analytic continuation to the case in which the gauge coupling g^2 is of order N^0 and not small. Then, according to the philosophy of the gauge/gravity duality, it should be reproduced from the gravity side. We hope to report a development along this direction in near future.

In the end of this paper, let us speculate on a possible application of our result to large-N QCD. In this theory, the beta function for the 't Hooft coupling becomes of order N^0 in the 't Hooft limit, and hence the 't Hooft coupling at the UV cutoff should be taken N-independent, so that the running 't Hooft coupling at each N becomes the same up to the $1/N$ correction. It however does not necessarily mean that instantons must obey a naive counting; the coupling constant in the instanton action should be evaluated at the characteristic energy scale of the instantons, which is the inverse of the radius of instantons. Therefore, as the inverse of the radius approaches the QCD scale, the 't Hooft coupling diverges and the very strongly coupled large-N limit can be realized. Then it would be possible to study the essence of such large instantons by the analytic continuation from the 't Hooft limit. The instantons whose size corresponds to $g^2 \sim N^0$ gives finite contributions, and the effects of much larger instantons could be calculated by considering the analytic continuation to a larger value of g. It would be interesting if such treatments lead us to a solution of the infrared embarrassment problem and precise understanding of the instanton effect for the QCD phase transition.

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