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A simple solution for marginal deformations in open string field theory

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Abstract: We derive a new open string field theory solution for boundary marginal deformations generated by chiral currents with singular self-OPE. The solution is algebraically identical to the Kiermaier-Okawa-Soler solution and it is gauge equivalent to the Takahashi-Tanimoto identity-based solution. It is wedge-based and we can analytically evaluate the Ellwood invariant and the action, reproducing the expected results from BCFT. By studying the isomorphism between the states of the initial and final background a dual derivation of the Ellwood invariant is also obtained.

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1 Introduction and conclusion

A major question in Open String Field Theory (OSFT) is how the different sets of conformal boundary conditions, in a given closed string background, can be described by the gauge orbits of classical solutions. Hidden in this correspondence there is the mechanism by which OSFT is supposed to tame contact-term singularities. In the sigma-model approach one can formally move in the space of two-dimensional boundary field theories by means of boundary interactions. However, when interpreted as operator insertions in the world-sheet path integral of the starting background, such interactions have notorious contact-term problems. The advantage of Witten's cubic open string field theory, in this regard, is that contact-singularities can be naturally avoided by expanding the string field in the Fock-space basis (level truncation), thanks to the explicit "security strips" that every Fock-space state has. However, the level expansion is not well fit for analytic computations. On the other hand, with the standard wedge-based analytic methods we have today, essentially stemming from Schnabl's original work, [1], it is not known how to systematically deal with contact term divergences.

Notable progress has been achieved in the case of boundary marginal deformations, [2], in [3–6], where consistent ways have been devised to regularize and renormalize the contact divergences of boundary marginal operators, order by order in a perturbative expansion in the marginal parameter, so that an exact solution of OSFT can be defined.

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More recently, a new world-sheet mechanism for regularizing the collisions of the marginal operators has been put forward in [7] by Inatomi, Kishimoto and Takahashi. They analyzed an analytic tachyon vacuum solution in the background of an identity-based solution constructed long-ago by Takahashi and Tanimoto (TT), [8, 9]. They were able to analytically compute the observables of the tachyon vacuum solution and they reproduced the disk partition function in the marginally deformed background by computing the action, and the marginally deformed closed string tadpoles by computing the Ellwood invariant, [10]. In their construction the contact-term divergences of marginal operators are automatically resolved by analytically continuing the boundary marginal field along vertical line integrals into the bulk, something which is always possible for boundary fields coming from the chiral algebra. The spreading in the bulk of the boundary interaction is controlled by a function which, in a limit, localizes to the boundary, thus reproducing the familiar marginal deformations of [2]. This is a new, convenient way of dealing with contact term divergences, which doesn't require any subtraction or normal ordering.

Despite this remarkable construction, and other corollary arguments [11], it is not possible to directly evaluate the observables of the TT solution, because it is an identity-based string field and its action, as it stands, is not defined in a standard, known sense.

The aim of this paper is to search for a new, not identity-based, solution which realizes the above-mentioned world-sheet regularization of contact-term divergences and, at the same time, has well-defined observables. Surprisingly, by just appropriately gauge transforming the TT solution, we end up rediscovering the Kiermaier-Okawa-Soler (KOS) solution [12]. For various reasons concerning its precise world-sheet realization, [13], the KOS solution was believed to be able to describe only a limited class of marginal deformations, namely the less interesting case where the marginal operator has regular OPE with itself and therefore there is nothing to regulate. The world-sheet description of our new solution is indeed quite different from the original KOS construction, but the identical algebraic structure allows for an analytic –algebraic– computation of the observables which are precisely reduced to the tachyon vacuum observables considered and computed in [7]. We also take the opportunity of analyzing the physical fluctuations around the new solution which are explicitly constructed in terms of the degrees of freedom of the perturbative vacuum. Starting from the similarity transformation of TT, we derive a simple world-sheet transformation which can be applied to both boundary and bulk fields. The way bulk fields are affected by this transformation precisely accounts for the change in the closed string one-point function between the starting and the final background. With the assumption that the g-function doesn't change, this gives a dual derivation of the Ellwood conjecture.

Despite the very simple algebraic structure, however, the behaviour of the solution towards the identity is, still, potentially problematic since we encounter a new, previously un-noticed, singularity which occurs when negative weight fields (such as the c-ghost) are placed off the boundary on a vanishing width wedge state. We devote an appendix to a preliminary presentation of these new kind of singularities which would deserve, by themselves, further study and whose presence, if not properly tamed, can be quite dramatic. Luckily, it is possible to avoid these singularities by deforming the original solution into a one-parameter gauge orbit which is safe by construction and which reduces to our original solution in a limit. Quite remarkably, the observables of the regularized solution can be exactly shown to reduce to the difference in observables of tachyon vacuum solutions, where the regulator can be safely removed.

The solution we are proposing is quite handy (essentially as easy-to-handle as the original KOS solution) and at least for chiral marginal deformations is hopefully more advantageous than the standard approaches for singular OPE's such as the counter-terms generalizations of B-gauge solutions [3, 4] or the general method of [5, 6], which are perturbative approaches in the marginal parameter. Our construction is based on the TT solution and hence on marginal deformations, but the algebraic structure we describe is completely general. We thus hope our results can be a useful step towards the analytic construction of more general backgrounds in open string field theory, whose numerical landscape has been recently shown to be vaster than what is known analytically, [14, 15].

2 From TT to KOS

In this section we first review the needed ingredients from the Takahashi-Tanimoto (TT) solution, [8, 9], formulated in the sliver frame. Then we show that, after a gauge transformation, the TT solution is mapped to a new solution which is algebraically identical to a KOS solution [12].

2.1 TT solution

We start with a chiral current algebra

$$\jmath^a(z)\jmath^b(0) = \frac{g^{ab}}{z^2} - \frac{c^{abc}}{z}\jmath^c(0) + (reg.), \tag{2.1}$$

and its antiholomorphic counterpart

$$\bar{\jmath}_a(\bar{z})\bar{\jmath}_b(0) = \frac{g_{ab}}{\bar{z}^2} - \frac{c_{abc}}{\bar{z}}\bar{\jmath}_c(0) + (reg.), \tag{2.2}$$

with totally antisymmetric structure constant c^{abc}. Our reference BCFT$_0$ is chosen to preserve a linear combination of the two isomorphic chiral algebras, from which it is possible to define a single chiral current, defined on the whole complex plane (doubling trick)

$$j^a(z) = \jmath^a(z), \quad \operatorname{Im} z > 0 \tag{2.3}$$

$$j^a(z) = \Omega^{ab}\bar{\jmath}_b(\bar{z}), \quad \operatorname{Im} z < 0, \tag{2.4}$$

where Ω^{ab} is gluing map which is part of the data which define the starting background BCFT$_0$.

$$\jmath^a(z) = \Omega^{ab}\bar{\jmath}_b(\bar{z}), \quad \operatorname{Im} z = 0. \tag{2.5}$$

The current algebra structure (2.2) guarantees that each $j^a(z)$, when placed at the boundary of the world-sheet, generates an exactly marginal boundary deformation of BCFT$_0$, [2]. The TT identity-based solution can then be written as a state in BCFT$_0$ as

$$\Phi = \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\left(f^a(z)cj_a(z) + \frac{1}{2}f^a f_a(z)c(z)\right). \tag{2.6}$$

The $f^a(z)$ are functions defined on the imaginary axis, whose properties will be derived shortly. Here we are employing the rather formal but quite useful notation [7, 16]

$$\phi(z) \equiv e^{zK}\phi e^{-zK},$$

which allows to manipulate string fields as if they were local operators on the world-sheet.[1] For generic z, $\phi(z)$ is a formal string field which only makes sense if it is multiplied (from the correct side) by a wedge state of minimum width $|\mathrm{Re}z|$. When $\mathrm{Re}z = 0$, $\phi(z)$ is an identity based string field which can be given a Fock space expansion and which can be multiplied by wedge based states. The identity-like string field ϕ is defined as

$$\phi = \phi(0) \equiv \tilde{\phi}(1/2)I,$$

where $\tilde{\phi}(w)$ is a local vertex operator in the $\frac{2}{\pi}$ arctan-sliver frame, and I is the identity string field.

For concreteness we will specialize to a single polarization inside the current algebra (2.2), by choosing one single current

$$j(z) \equiv \frac{t^a}{\sqrt{t_a t_b g^{ab}}} j_a(z), \qquad \to \qquad f^a(z) = \frac{t^a}{\sqrt{t_a t_b g^{ab}}} f(z), \tag{2.7}$$

for constant t^a, with OPE

$$j(z)j(w) = \frac{1}{(z-w)^2} + reg, \tag{2.8}$$

although most of our results readily apply to the fully non abelian case (2.6).

With this understanding we explicitly write

$$\Phi = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \left(f(z)cj(z) + \frac{1}{2} f^2(z)c(z) \right). \tag{2.9}$$

Given a generic vertex operator $\phi(z)$ in the sliver frame, the Fock space definition of the identity-based string field Φ is given by computing a correlator on a cylinder C_L of width $L = 1$

$$\mathrm{Tr}\left[\Phi e^{-\frac{K}{2}}\phi e^{-\frac{K}{2}}\right] = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \left\langle \left(f(z)cj(z+1/2) + \frac{1}{2} f^2(z)c(z+1/2) \right) \phi(0) \right\rangle_{C_1} . \tag{2.10}$$

In order for Φ to have well-defined Fock space coefficients (2.10, the function $f(z)$ must vanish fast enough at the midpoint $\pm i\infty$, so that the dz integral will be finite. The finiteness of the first term involving $cj(z)$ gives the generic condition

$$\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} f(z)\, H(z) < \infty, \tag{2.11}$$

where

$$H(z) \equiv \langle\, cj(z)\phi(1/2)\,\rangle_{C_1} = O(1), \quad z \to \pm i\infty, \tag{2.12}$$

[1]The well known fields K, B, c are used in the conventions of [17].

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is the contraction between cj on the imaginary axis and the test state at $z = 1/2$. This condition essentially states that $f(z)$ should be integrable towards $\pm i\infty$. The finiteness of the second term involving $c(z)$ gives a much stronger constraint since the negative weight field c must be damped as it approaches the midpoint. For example, by contracting with $c\partial c(0)|0\rangle$, we get the condition

$$\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} f^2(z) \cos^2 \pi z < \infty. \tag{2.13}$$

Other contractions with ghost number two Fock states similarly imply that $f(z)$ must separately vanish at $\pm i\infty$ at least exponentially, faster than $e^{-\pi|z|}$, to make the integral convergent. We will see in the appendix that the requirement of finite contractions with generic wedge based states will further damp the behaviour of f at the midpoint.

Let's see how the equation of motion works in the sliver frame. In order to consider $Q\Phi + \Phi^2$ as a concrete thing, we need some world-sheet, since this is not provided by the solution itself. Let us then consider

$$e^{-\epsilon_1 K}(Q\Phi + \Phi^2)e^{-\epsilon_2 K}. \tag{2.14}$$

The kinetic term readily gives

$$e^{-\epsilon_1 K}(Q\Phi)e^{-\epsilon_2 K} = \frac{1}{2}\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} f^2(z)\; e^{-\epsilon_1 K} c\partial c(z) e^{-\epsilon_2 K}. \tag{2.15}$$

The interaction term gives three possible contributions

$$\begin{aligned} e^{-\epsilon_1 K}(\Phi^2)e^{-\epsilon_2 K} = & \frac{1}{2}\int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} f(w) \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} f(z) e^{-\epsilon_1 K}(cj(z)cj(w) + cj(w)cj(z))e^{-\epsilon_2 K} \\ & + \frac{1}{2}\int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} f(w) \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} f(z) e^{-\epsilon_1 K}(cj(z)c(w)f(w) + f(w)c(w)cj(z))e^{-\epsilon_2 K} \\ & + \frac{1}{8}\int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} f^2(w) \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} f^2(z) e^{-\epsilon_1 K}(c(z)c(w) + c(w)c(z))e^{-\epsilon_2 K}. \end{aligned} \tag{2.16}$$

We now demand that $f(z)$ is analytic in an infinitesimal strip containing the imaginary axis.[2] Then, since f is also suppressed at the midpoint, we can slightly shift the dz integrals on the left and on the right of the imaginary axis, respectively for the first and second terms in the parentheses (while staying on the surface thanks to the added strips of world-sheet). Then the two terms in the parentheses are equivalent to a contour integral around w[3]

$$\begin{aligned} e^{-\epsilon_1 K}(\Phi^2)e^{-\epsilon_2 K} = & \frac{1}{2}\int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} f(w) \oint_w \frac{dz}{2\pi i} f(z) e^{-\epsilon_1 K} cj(z)cj(w) e^{-\epsilon_2 K} \\ & + \frac{1}{2}\int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} f(w) \oint_w \frac{dz}{2\pi i} f(z) e^{-\epsilon_1 K} cj(z)c(w)f(w) e^{-\epsilon_2 K} \\ & + \frac{1}{8}\int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} f^2(w) \oint_w \frac{dz}{2\pi i} f^2(z) e^{-\epsilon_1 K} c(z)c(w) e^{-\epsilon_2 K}. \end{aligned} \tag{2.17}$$

[2] This is not strictly needed, but it is a fairly general simplifying assumption.

[3] Since we are dealing with string fields and not vertex operators, all products must be understood to be ordered, [7]

$$\phi_1(z)\phi_2(w) = (-1)^{|\phi_1||\phi_2|}\phi_2(w)\phi_1(z), \qquad \mathrm{Re}\, w > \mathrm{Re}\, z.$$

Only the cj-cj OPE can give a simple pole

$$cj(z)cj(w) \sim -\frac{1}{z-w}\, c\partial c(z), \tag{2.18}$$

and therefore a non vanishing result

$$\begin{aligned} e^{-\epsilon_1 K}(\Phi^2)e^{-\epsilon_2 K} &= -\frac{1}{2}\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i} f^2(z)\, e^{-\epsilon_1 K} c\partial c(z) e^{-\epsilon_2 K} \\ &= -e^{-\epsilon_1 K}(Q\Phi)e^{-\epsilon_2 K}. \end{aligned} \tag{2.19}$$

Since the solution is identity-based, it is not possible to directly compute its observables, because they would correspond to correlators on cylinders of vanishing width. To appreciate this, let's compute a possible (naive) regularization of the kinetic term by simply inserting small regulating strips, for a choice of function $f(z) = e^{z^2}$, which is well suppressed at the midpoint. We get

$$\mathrm{Tr}[\Phi e^{-\epsilon_1 K} Q\Phi e^{-\epsilon_2 K}] = \frac{(\epsilon_1+\epsilon_2)^2}{64\pi^3}\left(e^{\frac{\pi^2}{(\epsilon_1+\epsilon_2)^2}}\cos\frac{2\pi\epsilon_1}{\epsilon_1+\epsilon_2} - 1\right), \qquad f(z) = e^{z^2}. \tag{2.20}$$

Not only the limit $(\epsilon_1, \epsilon_2) \to 0$ does not exist, but it also wildly oscillates from $-\infty$ to ∞.

Despite the failure of a naive direct evaluation of the action, following the discussion in [8, 9, 19], the solution is expected to describe a marginal deformation with marginal parameter given by the reparametrization invariant (see appendix A for the relation between $f(z)$ and $F(w)$)

$$\lambda_{\mathrm{BCFT}} \equiv \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\, f(z) = \int_{C_{\mathrm{left}}}\frac{dw}{2\pi i}\, F(w). \tag{2.21}$$

This quantity is real if the reality condition (A.10) is obeyed.

As discussed in [7], it is useful to define the matter string field[4]

$$J \equiv [B,\Phi] = \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\left(f(z)j(z) + \frac{1}{2}f^2(z)\right), \tag{2.22}$$

and the deformed world-sheet hamiltonian generating horizontal translations on the cylinder C_L

$$K' \equiv K + J, \tag{2.23}$$

whose BRST variation is given by[5]

$$Q(K+J) = QJ = Q[B,\Phi] = \partial\Phi - [B, Q\Phi] = \partial\Phi + [B,\Phi^2] = [K+J,\Phi]. \tag{2.24}$$

The string field $K+J$ is exact in the cohomology of the shifted BRST operator

$$K + J = (Q + \mathrm{ad}_\Phi)B \equiv Q_{\Phi\Phi}B, \tag{2.25}$$

[4] $[\cdot,\cdot]$ is the graded commutator.
[5] $\partial \equiv \mathrm{ad}_K \equiv [K,\cdot]$.

where we have used the notation of [20] for the kinetic operator between two backgrounds A and B

$$Q_{AB}\phi \equiv Q\phi + A\phi - (-1)^{|\phi|}\phi B. \tag{2.26}$$

Generic functions of K' are thus killed by $Q_{\Phi\Phi}$

$$Q_{\Phi\Phi}F(K') = 0. \tag{2.27}$$

The string field $F(K')$, if analytic for $\mathrm{Re}\, K' \geq 0$, can be geometrically understood as a superposition of wedge-states with a path-ordered exponential integration of the chiral current, [7], in much the same way as [12, 21, 22]

$$F(K') = \int_0^\infty dt\, \mathcal{F}(t) e^{-t(K+J)} \tag{2.28}$$

$$\mathrm{Tr}[F(K')e^{-\frac{K}{2}}\phi e^{-\frac{K}{2}}] = \int_0^\infty dt\, \mathcal{F}(t) \left\langle e^{-\int_1^{t+1} ds J(s)} \phi\left(\frac{1}{2}\right)\right\rangle_{C_{t+1}} \tag{2.29}$$

$$J(s) = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i}\left(f(z)j(z+s) + \frac{1}{2}f^2(z)\right). \tag{2.30}$$

Notice however that the exponential interaction integrates the marginal current $j(z)$ over the whole bulk. This bulk (rather than boundary) integration is what naturally regularizes the contact term divergences between the j's. The more common BCFT intuition of a renormalized boundary interaction, [2], can be achieved by studying the phantom term of the solution [23], along the lines of [20, 24], essentially observing that very large deformed wedges can be reparametrized to finite size while localizing the function $f(z)$ to the boundary. Indeed, considering the scaling derivation [1]

$$L^- \equiv \frac{1}{2}\left(\mathcal{L}_0 - \mathcal{L}_0^*\right),$$

we have

$$\begin{aligned} L^- cj(z) &= z\partial_z cj(z) \\ L^- c(z) &= (z\partial_z - 1)c(z), \end{aligned} \tag{2.31}$$

and we can easily show

$$t^{-L^-}\Phi[f(z)] = \Phi[tf(tz)]. \tag{2.32}$$

For $t \to \infty$ (which is the needed rescaling to bring the sliver to finite width) the support of the function $tf(tz)$ gets localized to $\mathrm{Im}\, z = 0$ and the bulk interaction (2.29) localizes to the boundary, see also [11] for an almost equivalent mechanism.

In [7] it was also proven that (appropriately normalizing the space time volume)

$$\left\langle e^{-\int_0^t ds J(s)}\right\rangle_{C_t}^{\mathrm{matter}} = \langle 1\rangle_{C_t}^{\mathrm{matter}} = 1. \tag{2.33}$$

This correlator is a regularized expression for the marginally deformed disk partition function which should therefore coincide with the undeformed one, as it is the case.

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2.2 KOS-like solution

Using the ingredients discussed in the previous subsection, we can write down the solution[6]

$$\Psi = \frac{1}{1+K}\left(\Phi - \Phi\frac{B}{1+K'}\Phi\right). \tag{2.35}$$

Although not self-evident, this solution falls in the class of solutions studied by Kiermaier Okawa and Soler (KOS), [12]. To see this we formally write

$$\Phi = \sigma_L Q \sigma_R, \tag{2.36}$$

where the string fields $\sigma_{L,R}$'s obey the algebraic properties

$$\sigma_L\sigma_R = \sigma_R\sigma_L = 1 \tag{2.37}$$
$$[B, \sigma_{L,R}] = 0. \tag{2.38}$$

The expression (2.36) is precisely the pure gauge form of the TT solution, advocated in [8, 9]. If we assume the existence of a logarithmic chiral field $\chi(z)$ which is a 'primitive' for $j(z)$,

$$j(z) = i\partial\chi(z), \tag{2.39}$$
$$cj(z) = iQ\chi(z), \tag{2.40}$$
$$\chi(z)\chi(w) \sim -\log(z-w), \tag{2.41}$$

then we can write[7]

$$\begin{aligned}\sigma_L &= e^{-i\chi_f},\\ \sigma_R &= e^{i\chi_f},\\ \chi_f &\equiv \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}f(z)\chi(z).\end{aligned} \tag{2.42}$$

One can explicitly verify (2.36) by appropriately differentiating the operator/star exponentials defining the σ's. As elaborated in [18], we can try to trivialize the solution Φ by making χ_f an allowed state, integrating by part

$$i\chi_f = -\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}h(z)j(z) + i\left[\frac{1}{2\pi i}h(z)\chi(z)\right]_{-i\infty}^{i\infty}, \tag{2.43}$$

[6]This is obtained via the "Zeze map", [25],

$$\Phi \to \Psi \equiv F\Phi\frac{1}{1+A\Phi} = (1+A\Phi)(Q+\Phi)\frac{1}{1+A\Phi}, \tag{2.34}$$

(where $A \equiv B\frac{1-F(K)}{K}$ and, in our case, $F(K) = \frac{1}{1+K}$). Because the map is a gauge transformation it maps solutions to solutions

$$Q\Psi + \Psi^2 = F\frac{1}{1+\Phi A}(Q\Phi+\Phi^2)\frac{1}{1+A\Phi},$$

and it can be useful for turning identity-based solutions into more regular ones. It is not guaranteed, however, that the "identity-ness" can always be removed by gauge transformations, the residual solutions of [26] being a counter-example.

[7] As an explicit example one can take $j = i\sqrt{2}\partial X$ and $\chi = \sqrt{2}X$, for a free boson. Notice that the exponentials defining the σ's are not normal ordered (the contact singularities are spread in the bulk).

where, with no loss of generality, we choose $h(z)$ as

$$h(z) = \int_{-i\infty}^{z} d\xi\, f(\xi). \tag{2.44}$$

However, since $\chi(z)$ is logarithmic, the boundary term only vanishes if

$$h(i\infty) = \int_{-i\infty}^{i\infty} d\xi\, f(\xi) = 0.$$

The parameter defined in (2.21) is thus zero if and only if the solution Φ can be trivialized. Otherwise, if Φ is non trivial, the σ's are formal objects which do not belong to the state space of BCFT$_0$ (very much like bcc operators).

The use of the σ's is nevertheless quite useful to rewrite some of the objects we previously defined. In particular we have

$$J \equiv [B, \Phi] = \sigma_L[B, Q\sigma_R] = \sigma_L[K, \sigma_R] = \sigma_L \partial \sigma_R, \tag{2.45}$$

and

$$K' = \sigma_L K \sigma_R \tag{2.46}$$

$$F(K') = \sigma_L F(K) \sigma_R, \tag{2.47}$$

which allows to rewrite (2.35) precisely as a KOS solution [12]

$$\Psi = \frac{1}{1+K}\left(\sigma_L Q \sigma_R + Q\sigma_L \frac{B}{1+K} Q\sigma_R\right). \tag{2.48}$$

Notice that, differently from the original paper by KOS, the formal string fields $\sigma_{L,R}$ don't correspond to local boundary insertions of weight zero matter primaries, and their worldsheet realization is only meaningful when a pair of them appears

$$\langle (\ldots)\, \sigma_L(a)\; \sigma_R(b)\, (\ldots) \rangle_{C_L} \equiv \left\langle (\ldots)\, e^{-\int_a^b ds J(s)}\, (\ldots) \right\rangle_{C_L}, \tag{2.49}$$

where the non-local operator $J(s)$ is defined in (2.30). In the following, whenever possible, we will avoid using explicitly $\sigma_{L,R}$ and instead use the more general expression (2.35). At will, one can easily switch between the two notations, having (2.36), (2.49) in mind. In subsection 3.3 we will elaborate more on the σ's in presence of generic vertex operators. Notice also that the auxiliary derivation $B^- \equiv \frac{1}{2}(\mathcal{B}_0 - \mathcal{B}_0^*)$ doesn't annihilate Φ and therefore, contrary to the original KOS construction, the solution is not in a dressed B-gauge, [12, 17]. This matches with the expectation that a solution for marginal deformations cannot be found in a dressed B-gauge when, as is generically the case here, the marginal field has singular OPE with itself.

As a side-comment,[8] notice that given the objects, $(\sigma_{L,R}, K)$ one can also construct a Kiermaier-Okawa-like solution, [5], via the substitution of the building block

$$\left[e^{\lambda V(a,b)}\right]_r \to \sigma_L e^{-(a-b)K} \sigma_R = e^{-(a-b)(K+J)},$$

[8]This possibility has been suggested by Ted Erler.

where the λ dependence in the σ's, (2.42)(or equivalently in J) is realized by choosing $f(z) = \lambda \bar{f}(z)$, with

$$\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \bar{f}(z) = 1.$$

In this case everything is already finite and directly applies to the case of a marginal field with singular self-OPE (assuming it is local w.r.t. all the fields in the theory, which is true if it belongs to the chiral algebra).

3 Observables

3.1 Ellwood invariant

To compute the Ellwood invariant [10], and thus the boundary state [14], we use a simple but powerful trick. Writing the solution as, [13]

$$\Psi = \frac{1}{1+K}\Phi\frac{1}{1+K'} - Q\left(\frac{1}{1+K}\Phi\frac{B}{1+K'}\right) \tag{3.1}$$

the Ellwood invariant is easily evaluated by inserting the KBc-identity

$$[B, c] = 1, \tag{3.2}$$

as[9]

$$\begin{aligned} \mathrm{Tr}_V[\Psi] &= \mathrm{Tr}_V\left[\frac{1}{1+K}\Phi\frac{1}{1+K'}\right] = \mathrm{Tr}_V\left[\frac{1}{1+K}\Phi\frac{1}{1+K'}[B,c]\right] \\ &= \mathrm{Tr}_V\left[\frac{1}{1+K}[B,\Phi]\frac{1}{1+K'}c\right] = \mathrm{Tr}_V\left[\frac{1}{1+K}J\frac{1}{1+K'}c\right] \\ &= \mathrm{Tr}_V\left[\frac{1}{1+K}c\right] - \mathrm{Tr}_V\left[\frac{1}{1+K'}c\right], \end{aligned} \tag{3.4}$$

where, in going from the second to the third line, we have used the identity

$$\frac{1}{1+K}J\frac{1}{1+K'} = \frac{1}{1+K'}J\frac{1}{1+K} = \frac{1}{1+K} - \frac{1}{1+K'}. \tag{3.5}$$

What we have obtained is precisely the difference of the invariants of the Erler-Schnabl solutions in the original background and in the background expanded around Φ.

$$\mathrm{Tr}_V[\Psi] = \mathrm{Tr}_V[\Psi^{(0)}_{TV}] - \mathrm{Tr}_V[\Psi^{(\Phi)}_{TV}] \tag{3.6}$$

$$\Psi^{(0)}_{TV} = \frac{1}{1+K}\left[c + Q(Bc)\right] \tag{3.7}$$

$$\Psi^{(\Phi)}_{TV} = \frac{1}{1+K'}\left[c + Q_{\Phi\Phi}(Bc)\right]. \tag{3.8}$$

[9] The notation is as follows

$$\mathrm{Tr}_V[\Phi] \equiv \langle I|V(i,-i)|\Phi\rangle, \tag{3.3}$$

where $\langle I|$ is the bpz of the identity string field and V is a weight zero bulk operator $V = c\bar{c}V^{\mathrm{matter}}$.

The first observable has been computed in [17], while the second has been computed in [7] and shown to reproduce the closed string tadpoles of a marginally deformed BCFT at deformation parameter $\lambda_{\rm BCFT}$ given by (2.21). We will present an alternative derivation of this result in section 4.

Notice that all traces in the game involve computation of correlators on cylinders of generic finite width, by the usual Schwinger parametrization of $\frac{1}{1+K} = \int_0^\infty e^{-t(1+K)}$. In addition, our algebraic derivation is also applicable to the regularized solution (C.6) discussed in the appendix, which has the advantage of having support on wedge based states with strictly positive width, thus avoiding the potentially problematic $t \to 0$ limit in the overall Schwinger integral.

3.2 Action

Using a similar trick we can evaluate the action. Dropping the trivial BRST exact pieces in (3.1) and appropriately rotating the trace we have

$$S[\Psi] = -\frac{1}{6}\,{\rm Tr}[\Psi Q\Psi] = \frac{1}{6}\,{\rm Tr}\left[\frac{1}{1+K}\Phi\frac{1}{1+K'}\Phi\frac{1}{1+K}\Phi\frac{1}{1+K'}\right]. \tag{3.9}$$

This quantity can be in principle computed as the partition function of a wedge state with insertions and deformed/undeformed regions, with four Schwinger parameters to integrate over. This doesn't look simple at all. But let us insert $[B,c]=1$ rightmost in the trace and, as we did for the Ellwood invariant, pull out the adjoint action of B on the other string fields in the trace

$$\begin{aligned} S[\Psi] &= \frac{1}{6}\,{\rm Tr}\left[\frac{1}{1+K}\Phi\frac{1}{1+K'}\Phi\frac{1}{1+K}\Phi\frac{1}{1+K'}\right]\\ &= \frac{1}{6}\,{\rm Tr}\left[\frac{1}{1+K}\Phi\frac{1}{1+K'}\Phi\frac{1}{1+K}\Phi\frac{1}{1+K'}[B,c]\right]\\ &= \frac{1}{6}\,{\rm Tr}\left[\frac{1}{1+K}J\frac{1}{1+K'}\Phi\frac{1}{1+K}\Phi\frac{1}{1+K'}c\right]\\ &- \frac{1}{6}\,{\rm Tr}\left[\frac{1}{1+K}\Phi\frac{1}{1+K'}J\frac{1}{1+K}\Phi\frac{1}{1+K'}c\right]\\ &+ \frac{1}{6}\,{\rm Tr}\left[\frac{1}{1+K}\Phi\frac{1}{1+K'}\Phi\frac{1}{1+K}J\frac{1}{1+K'}c\right]. \end{aligned} \tag{3.10}$$

Using (3.5) three times, we get some cancellations and we end up with

$$S[\Psi] = \frac{1}{6}\,{\rm Tr}\left[\frac{1}{1+K}\Phi\frac{1}{1+K'}\Phi\frac{1}{1+K}c\right] - \frac{1}{6}\,{\rm Tr}\left[\frac{1}{1+K'}\Phi\frac{1}{1+K}\Phi\frac{1}{1+K'}c\right]. \tag{3.11}$$

Now recognize the BRST-exact quantities

$$\Phi\frac{1}{1+K'}\Phi = -Q\left(\frac{1}{1+K'}\Phi\right) \tag{3.12}$$

$$\Phi\frac{1}{1+K}\Phi = Q_{\Phi\Phi}\left(\frac{1}{1+K}\Phi\right), \tag{3.13}$$

which allow to integrate by part

$$\begin{aligned} S[\Psi] &= -\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K}Q\left(\frac{1}{1+K'}\Phi\right)\frac{1}{1+K}c\right]-\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K'}Q_{\Phi\Phi}\left(\frac{1}{1+K}\Phi\right)\frac{1}{1+K'}c\right] \\ &= -\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K}\frac{1}{1+K'}\Phi\frac{1}{1+K}Qc\right]-\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K'}\frac{1}{1+K}\Phi\frac{1}{1+K'}Q_{\Phi\Phi}c\right], \end{aligned}$$

where we have used[10]

$$Q_{\Phi\Phi}F(K') = 0.$$

Now we insert again $[B,c]=1$ and, again, integrate by part the adjoint action of B

$$\begin{aligned} S[\Psi] &= -\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K}[B,c]\frac{1}{1+K'}\Phi\frac{1}{1+K}Qc\right]-\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K'}[B,c]\frac{1}{1+K}\Phi\frac{1}{1+K'}Q_{\Phi\Phi}c\right], \\ &= -\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K}c\frac{1}{1+K'}J\frac{1}{1+K}Qc\right]+\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K}c\frac{1}{1+K'}\Phi\frac{1}{1+K}\partial c\right] \\ &\quad -\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K}J\frac{1}{1+K'}Q_{\Phi\Phi}c\right]+\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K}\Phi\frac{1}{1+K'}\partial' c\right] \\ &= -\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K}c\frac{1}{1+K}Qc\right]+\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K}c\frac{1}{1+K'}(Qc+\Phi c)\right] \\ &\quad +\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K'}Q_{\Phi\Phi}c\right]-\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K}(Q_{\Phi\Phi}c-\Phi c)\right], \end{aligned} \tag{3.14}$$

where, in the third line, we have defined

$$\partial' c \equiv \mathrm{ad}_{K+J}\, c = [B, Q_{\Phi\Phi}c],$$

and in the last two lines we have used the cyclicity of the trace, the algebraic property (3.5) as well as

$$\frac{1}{1+K}\partial c\frac{1}{1+K} = \left[c, \frac{1}{1+K}\right] \tag{3.15}$$

$$\frac{1}{1+K'}\partial' c\frac{1}{1+K'} = \left[c, \frac{1}{1+K'}\right]. \tag{3.16}$$

Using (2.26) we can therefore write

$$\begin{aligned} S[\Psi] &= -\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K}c\frac{1}{1+K}Qc\right]+\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K'}Q_{\Phi\Phi}c\right] \\ &\quad +\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K}c\frac{1}{1+K'}Q_{\Phi 0}c\right]-\frac{1}{6}\operatorname{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K}Q_{0\Phi}c\right]. \end{aligned} \tag{3.17}$$

The last line vanishes on account of the generic property

$$\operatorname{Tr}[Q_{AB}(\phi_1)\phi_2] + (-1)^{|\phi_1|}\operatorname{Tr}[\phi_1 Q_{BA}(\phi_2)] = 0. \tag{3.18}$$

[10] In the present case we have $Q_{\Phi\Phi}c = Qc = c\partial c$, however we want to keep as generic as possible, without assuming that $[\Phi, c] = 0$, so that we can use this derivation also for the regularized solution described in the appendix.

Therefore the action evaluated on the solution equals

$$
\begin{aligned}
S[\Psi] &= -\frac{1}{6}\,\mathrm{Tr}\left[\frac{1}{1+K}c\frac{1}{1+K}Qc\right]+\frac{1}{6}\,\mathrm{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K'}Q_{\Phi\Phi}c\right]\\
&= -\frac{1}{6}\,\mathrm{Tr}\left[\frac{1}{1+K}c\frac{1}{1+K}c\partial c\right]+\frac{1}{6}\,\mathrm{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K'}c\partial c\right], \qquad (3.19)
\end{aligned}
$$

where in the last line we have specialized to our precise case where $[\Phi, c] = 0$. Same as the Ellwood invariant, this is precisely the difference between the action of the Erler-Schnabl solutions (3.7), (3.8) in the original and Φ-background. Using (2.33), we see that the two actions equal each other, [7], as it is expected since the solution Ψ describe a continuos family of marginal deformations of the perturbative vacuum and must have therefore a vanishing action. It should be noted, however, that vanishing of the action is not an algebraic consequence of our derivation. There is a reason for this: given *any* solution Φ, one can always construct a gauge equivalent KOS-like solution

$$
\Psi = \frac{1}{1+K}\left(\Phi - \Phi\frac{B}{1+K+[B,\Phi]}\Phi\right), \qquad (3.20)
$$

and follow the computation of the energy we have just presented, to reduce it to the shift in the tachyon vacuum's energy. Since Φ can be a generic solution, there is no reason to expect to find a vanishing action. Therefore the algebraic form of the KOS-type solution we are discussing, can be useful for generic backgrounds, not just marginal deformations.

4 Deformed background

We can easily describe the states and the cohomology representatives in the new open string background described by the solution Ψ. In order to do so let us first address, in our formalism, the construction of the fluctuations around the TT-solution Φ itself, which was discussed in part in [8, 9, 18]. This will allow us to make some interesting connection with the standard BCFT description of a marginal deformation [2] and to perform an alternative, simpler, computation of the Ellwood invariant. Let Ξ be a Fock state around the starting background $\Psi = 0$

$$
\Xi = e^{-\frac{K}{2}}Ve^{-\frac{K}{2}}, \qquad (4.1)
$$

where $V \equiv \tilde{V}(1/2)I$ is an identity-like insertion. Since the TT solution can be written as

$$
\Phi = \sigma_L Q \sigma_R,
$$

this implies a star-algebra isomorphism between the original and the deformed states

$$
\hat{\Xi} \equiv \sigma_L \Xi \sigma_R = e^{-\frac{K'}{2}}\hat{V}e^{-\frac{K'}{2}} \qquad (4.2)
$$

$$
\hat{V} \equiv \sigma_L V \sigma_R. \qquad (4.3)
$$

Notice that if the σ's would have been allowed fields, this would just be a gauge transformation. Explicitly, using the appropriate generalization of the Leibniz rule, [20], we see

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that the cohomology problem at the TT-background is mapped to the cohomology problem at the perturbative vacuum[11]

$$Q_{\Phi\Phi}(\sigma_L \Xi \sigma_R) = (Q_{\Phi 0}\sigma_L)\,\Xi\sigma_R + \sigma_L\,(Q\Xi)\,\sigma_R + (-1)^{|\Xi|}\sigma_L\Xi\,(Q_{0\Phi}\sigma_R)$$
$$= \sigma_L\,(Q\Xi)\,\sigma_R. \tag{4.4}$$

It appears that the dressed vertex operators $\hat{V} = \sigma_L V \sigma_R$ are the only objects where a concrete definition of the σ's is needed, (2.42)

$$\hat{V}(0) = e^{-i\,\mathrm{ad}_{\chi_f}}V(0). \tag{4.5}$$

However, the $*$-commutator $[\chi_f,\cdot]$ can be rewritten using only local fields (while this is not true for left or right multiplication alone). Explicitly we can write (Rez "time ordering" is understood between the string fields χ and V)

$$-i[\chi_f, V] = -i\left(\int_{-i\infty+\epsilon}^{i\infty+\epsilon} - \int_{-i\infty-\epsilon}^{i\infty-\epsilon}\right)\frac{dz}{2\pi i}f(z)\chi(z)\,V(0). \tag{4.6}$$

The singular part of the OPE between χ and V can consist of poles or it can contain a logarithm (in case the OPE of j with V contains a single pole, as it is the case when V is a j-primary). Other cases are excluded because j belongs to the chiral algebra and it is thus local w.r.t. all bulk and boundary fields. When χ-V consists of poles, we can close the two vertical contours and shrink around 0

$$-i[\chi_f, V] = -i\oint_0 \frac{dz}{2\pi i}f(z)\chi(z)V(0). \tag{4.7}$$

Consider now a primitive for $f(z)$,

$$f(z) = i\partial g(z), \tag{4.8}$$

integrating by part the closed contour we get

$$-i[\chi_f, V] = i\oint_0 \frac{dz}{2\pi i}g(z)j(z)V(0), \qquad \chi\text{-}V = \text{pole} \tag{4.9}$$

Notice that, under the assumption we are temporarily holding (j-V contains no simple pole) the integration constant in g doesn't play any role. The constant part of g enters the game only when we transform a j-primary, so that j-V is a single pole and χ-V is a logarithm. In this case we can assume we have already diagonalized the j-primaries V in such a way that they are eigenstates under the action of j

$$j(z)V(0) \sim \frac{n_V}{z}V(0) + (reg.), \quad \rightarrow \quad \chi(z)\,V(0) \sim -in_V\,\log z\,V(0) + (reg.), \tag{4.10}$$

[11] It is important that the formal string fields $\sigma_{L,R}$ are closed but not exact, so that the states we are discussing are not trivial. Notice the difference w.r.t. the left/right gauge transformations of [20], which are instead conventional regular string fields, typically exact but not-invertible.

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and we can write[12]

$$
\begin{aligned}
&\int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{dz}{2\pi i} f(z)\chi(z)\, V(0) \\
&= -in_V \int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{dz}{2\pi i} f(z) \log z\, V(0) + (reg) \\
&\sim -in_V \int_{+\epsilon}^{i\infty+\epsilon} \frac{dz}{2\pi i} f(z) \log |z|^2\, V(0) + (reg), \quad (\epsilon \to 0)
\end{aligned}
\tag{4.11}
$$

This left vertical integral (which is finite because of the integrable singularity of the log and the fall-off of $f(z)$ at $i\infty$) precisely cancels (together with the regular parts) against the right vertical integral in (4.6). Therefore we have

$$
-i[\chi_f, V] = 0, \qquad \chi\text{-}V = \text{logarithm}. \tag{4.12}
$$

We can conveniently summarize the result as

$$
-i[\chi_f, V] = i \oint_0 \frac{dz}{2\pi i} g(z) j(z) V(0) \tag{4.13}
$$

$$
g(z) \equiv -i \int_0^z d\xi f(\xi). \tag{4.14}
$$

This can be exponentiated to give

$$
\hat{V}(0) = e^{-i\, \mathrm{ad}_{\chi_f}} V(0) = e^{i \oint_0 \frac{dz}{2\pi i} g(z) j(z)} V(0). \tag{4.15}
$$

Suppose now we want to displace $V(0)$ off the boundary

$$
V(0) \to V(ix).
$$

To compute the marginal transformation, we follow the above derivation and, again, we have to pay attention when V is a j-primary. In this case we have

$$
\begin{aligned}
&\int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{dz}{2\pi i} f(z)\chi(z)\, V(ix) \\
&= -in_V \int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{dz}{2\pi i} f(z) \log(z - ix)\, V(ix) + (reg) \\
&= -in_V \int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{dz}{2\pi i} f(z) \left(\log \frac{z - ix}{z} + \log z \right) V(ix) + (reg).
\end{aligned}
\tag{4.16}
$$

[12] We are also assuming that $f(ix) = f(-ix)$, which allows to easily deal with the unphysical cuts in the logarithm (which are an artifact of the presence of χ). This condition was also implicitly used in the first of the papers [8, 9]. Notice that a violation of $f(ix) = f(-ix)$ would not change λ_{BCFT} as defined in (2.21).

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When we add the contribution from the right vertical path as in (4.6), the part proportional to $\log z$ cancels exactly as before, but now there is in addition the term

$$\begin{aligned} -i[\chi_f, V(ix)] &= -n_V\Big(\int_{-i\infty+\epsilon}^{i\infty+\epsilon} - \int_{-i\infty-\epsilon}^{i\infty-\epsilon}\Big)\frac{dz}{2\pi i} f(z)\log\frac{z-ix}{z}\,V(ix) \\ &= -n_V\oint_{\mathrm{cut}_{(0,ix)}}\frac{dz}{2\pi i} f(z)\log\frac{z-ix}{z}\,V(ix) \qquad (4.17) \\ &= -n_V\int_0^{ix}\frac{dz}{2\pi i} f(z)(2\pi i)V(ix) \\ &= in_V\, g(ix)\, V(ix), \qquad (4.18) \end{aligned}$$

where the cut has been chosen so that the overall contribution vanishes when $x \to 0$. Therefore, also for holomorphic bulk insertions we find

$$\begin{aligned} \hat{V}(ix) &= e^{-i\,\mathrm{ad}_{\chi_f}} V(ix) = e^{i\oint_{ix}\frac{dz}{2\pi i}g(z)j(z)}V(ix) \qquad (4.19) \\ g(z) &\equiv -i\int_0^z dz f(z). \end{aligned}$$

Notice that when the pole between j and V is at least triple, the transformation will start evaluating the derivatives of $f(z)$. This is another reason to require that f is analytic around the imaginary axis. Assuming $f(z)$ can be holomorphically extended beyond the imaginary axis (which is typically the case), we can also write

$$\hat{V}(w) \equiv e^{w(K+J)}\hat{V}(0)e^{-w(K+J)} = e^{wK'}(e^{-i\,\mathrm{ad}_{\chi_f}}V(0))e^{-wK'} = e^{i\oint_w\frac{dz}{2\pi i}g(z)j(z)}V(w) \qquad (4.20)$$

As an example, we can derive how the energy momentum tensor $T(z)$ is deformed by the marginal flow induced by the solution. We have

$$\hat{T}(w) = \sigma_L T(w)\sigma_R = e^{i\oint_w\frac{dz}{2\pi i}g(z)j(z)}T(w). \qquad (4.21)$$

Using the j–T OPE

$$j(z)T(w) = \frac{j(w)}{(z-w)^2} + (reg.) \qquad (4.22)$$

we get, using $ig'(w) = f(w)$

$$\hat{T}(w) = e^{i\oint_w\frac{dz}{2\pi i}g(z)j(z)}T(w) = T(w) + f(w)j(w) + \frac{1}{2}f^2(w), \qquad (4.23)$$

which agrees with [18]. As a consistency check we can also compute $\hat{T}(w)$ by taking the deformed BRST variation of the antighost $b(w)$

$$\begin{aligned} \hat{T}(w) &= Q_{\Phi\Phi}b(w) = Qb(w) + \oint_w\frac{dz}{2\pi i}\left(f(z)cj(z) + \frac{1}{2}f^2(z)c(z)\right)b(w) \\ &= T(w) + f(w)j(w) + \frac{1}{2}f^2(w). \qquad (4.24) \end{aligned}$$

Another simple universal example is given by

$$\hat{j}(w) = j(w) + f(w), \qquad (4.25)$$

and one can easily check that, just as

$$Q(j(z)) = \partial(cj(z)),$$

we have

$$Q_{\Phi\Phi}\hat{j}(z) = \partial_z(c\hat{j}(z)) = \partial'(c\hat{j}(z)). \tag{4.26}$$

This example is also teaching us that the star algebra operator $\partial' = \mathrm{ad}_{K+J}$ acts on a deformed vertex operator $\hat{V}(z)$ precisely as ∂_z.

An important property of the $\hat{V}$'s is that, as suggested by the notation, they obey *the same* operator algebra as the original $V's$

$$V_i(z)V_j(w) = c_{ijk}(z-w)\,V_k(w), \tag{4.27}$$

$$\hat{V}_i(z)\hat{V}_j(w) = c_{ijk}(z-w)\,\hat{V}_k(w), \tag{4.28}$$

as can be directly verified from (4.20) by picking up residues in explicit examples. This also implies that traces involving deformed wedges and the $\hat{V}$'s will be (up to a possible universal constant) the same as the corresponding traces of undeformed wedges and the V's

$$\mathrm{Tr}[e^{-t_1K'}\hat{V}_1 \dots e^{-t_nK'}\hat{V}_{t_n}] = \frac{g'}{g}\,\mathrm{Tr}[e^{-t_1K}V_1 \dots e^{-t_nK}V_{t_n}]. \tag{4.29}$$

The constant $\frac{g'}{g}$ is the ratio of the traces of the deformed and undeformed wedges, which, as proven in [7], is equal to 1.

It is interesting to extend the marginal transformation (4.20) to closed-string bulk operators. In our doubling-trick notation a bulk operator will be written as

$$V_{ij}(w,\bar{w}) = V_i(w)V_j(w^*), \qquad w^* \equiv \bar{w},\ \mathrm{Im}\,w > 0, \tag{4.30}$$

where both V_i and V_j are holomorphic (but typically not chiral) fields. We thus have

$$\hat{V}_{ij}(w,\bar{w}) = \left(e^{i\oint_w \frac{dz}{2\pi i}g(z)j(z)}V_i(w)\right)\left(e^{i\oint_{w^*} \frac{dz}{2\pi i}g(z)j(z)}V_j(w^*)\right), \tag{4.31}$$

with $g(z)$ defined in (4.14). Let us now assume that both V_i and V_j are j-primaries (all boundary states obtained from BCFT$_0$ by deforming with j, will be written as a sum of Ishibashi states of j-primaries, defined with the appropriate deformation of the gluing map, [2]). With no loss of generality we can write down the OPE, [2][13]

$$j(z)V_{ij}(w,\bar{w}) \sim \left(\frac{a_i}{z-w} - \frac{b_j}{z-\bar{w}}\right)V_{ij}(w,\bar{w}), \tag{4.32}$$

from which we easily get[14]

$$\begin{aligned}\hat{V}_{ij}(ix,-ix) &= e^{i(a_i g(ix) - b_j g(-ix))}V_{ij}(ix,-ix)\\ &= e^{i(a_i+b_j)g(ix)}V_{ij}(ix,-ix).\end{aligned} \tag{4.33}$$

[13] As an example, in case of $j = i\sqrt{2}\partial X$, with Neumann boundary conditions, we have that bulk momentum modes have $a_i = -b_j$ while bulk winding modes have $a_i = b_j$. The situation is exactly opposite in case of Dirichlet boundary conditions.

[14] With our assumption $f(z) = f(-z)$ we have that $g(ix) = -g(-ix)$. Thus, with this condition, a bulk operator with $a_i = -b_j$ is not transformed by the marginal deformation. But in fact a bulk operator with $a_i = -b_j$ has a vanishing tadpole in BCFT$_0$, and this remains true by deforming with j, [2].

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Now imagine we want to compute the Ellwood invariant of the tachyon vacuum solution (3.8), as it was done in [7]. After standard string field manipulations, we end up with the following trace

$$\mathrm{Tr}_{\mathcal{V}_{ij}}[e^{-(K+J)}c] = \lim_{x\to\infty} \left\langle e^{-\int_0^1 ds\, J(s)}\, c(0)c\bar{c}V_{ij}(ix,-ix) \right\rangle_{C_1}, \tag{4.34}$$

where $\mathcal{V} \equiv c\bar{c}V$. We can follow (and generalize to finite x, still assuming $f(iy) = f(-iy)$) the explicit computation of [7] to find

$$\mathrm{Tr}_{\mathcal{V}_{ij}}[e^{-(K+J)}c] = \lim_{x\to\infty} e^{-i\pi(a_i+b_i)\lambda(x)}\, \mathrm{Tr}_{\mathcal{V}_{ij}}[e^{-K}c] \tag{4.35}$$

$$\lambda(x) \equiv \int_{-ix}^{ix} \frac{dz}{2\pi i} f(z). \tag{4.36}$$

Or we can proceed differently, (4.29)

$$\begin{aligned}\mathrm{Tr}_{\mathcal{V}_{ij}}[e^{-K'}c] &= \lim_{x\to\infty} \mathrm{Tr}[\sigma_L e^{-K}\sigma_R c\mathcal{V}_{ij}(ix,-ix)]\\ &= \frac{g'}{g}\lim_{x\to\infty} \mathrm{Tr}[e^{-K}c\,\sigma_R\mathcal{V}_{ij}(ix,-ix)\sigma_L] = \mathrm{Tr}_{\check{\mathcal{V}}_{ij}}[e^{-K}c].\end{aligned} \tag{4.37}$$

The closed string state $\check{\mathcal{V}}_{ij}$ is the inverse of the transformation (4.33)

$$\check{\mathcal{V}}_{ij}(ix,-ix) = \sigma_R\mathcal{V}_{ij}(ix,-ix)\sigma_L = e^{i\oint_{\pm ix}\frac{dz}{2\pi i}g(z)j(z)}\mathcal{V}_{ij}(ix,-ix) = e^{-i(a_i+b_j)g(ix)}\mathcal{V}_{ij}(ix,-ix).$$

Therefore we get

$$\mathrm{Tr}_{\mathcal{V}_{ij}}[e^{-K'}c] = e^{-i(a_i+b_j)g(ix)}\,\mathrm{Tr}_{\mathcal{V}_{ij}}[e^{-K}c], \tag{4.38}$$

which coincides with (4.35), remembering that we are taking $f(ix) = f(-ix)$ and

$$g(ix) = -i\int_0^{ix} dz\, f(z) = \pi\lambda(x).$$

Notice that, in this 'dual' derivation, the Ellwood invariant is precisely reduced to a deformed closed string tadpole, in the sense of [2] (see e.g. eq (3.3) there), and Ellwood conjecture is transparent. In the BCFT description of [2], the countours, encircling the bulk operator, were originally at the boundary, while in this peculiar OSFT description they originate from vertical line integrals, (4.6).

Assuming that the contour integral $\oint \frac{dz}{2\pi i} g(z)j(z)$ is well defined on local vertex operators (which is true if j belongs to the chiral algebra, but generically false if j is only self-local, [2]), the isomorphism (4.2) can be performed on the whole Fock space of BCFT$_0$ and, being a similarity transformation, it is clearly compatible with both the star product and the BRST differential.

All the above can be straightforwardly extended to the KOS-like solution Ψ (2.35), (2.48), where the previous isomorphism is dressed with the gauge parameters connecting the TT solution with the KOS solution

$$\begin{aligned} e^{-\frac{K}{2}}Ve^{-\frac{K}{2}} &\to (1+A\Phi)\,e^{-\frac{K'}{2}}\hat{V}e^{-\frac{K'}{2}}(1+A\Phi)^{-1}\\ &= \left(1+\frac{B}{1+K}\Phi\right)e^{-\frac{K'}{2}}\hat{V}e^{-\frac{K'}{2}}\left(1-\frac{B}{1+K'}\Phi\right)\\ &= \left(\sigma_L-\frac{B}{1+K}Q\sigma_L\right)e^{-\frac{K}{2}}Ve^{-\frac{K}{2}}\left(\sigma_R-\frac{B}{1+K}Q\sigma_R\right). \end{aligned} \tag{4.39}$$

Again, we can use generic states of BCFT_0, to describe the off-shell degrees of freedom around the new background Ψ, and the cohomology is again mapped in the cohomology.

Notice that, with this construction of the off-shell fluctuations, the action in the new background Ψ *coincides* with the action around the TT background Φ. Notice however that the states (4.39) are not real despite the almost reality of the solution Ψ

$$\Psi \to \sqrt{1+K}\,\Psi\frac{1}{\sqrt{1+K}} \equiv \Psi^{\rm real}.$$

Given $\Psi^{\rm real}$ one can find real cohomology elements by using the right gauge transformation [20]

$$U=\sqrt{F}\frac{1}{1+\Phi A}, \qquad F(K)=\frac{1}{1+K},\ A=B\frac{1-F}{K}$$

and the (reality-conjugate) left gauge transformation

$$U^\dagger=\frac{1}{1+A\Phi}\sqrt{F},$$

both connecting Φ with $\Psi^{\rm real}$. This gives a construction of the cohomology which is essentially the one considered in [3, 29]

$$e^{-\frac{K}{2}}cVe^{-\frac{K}{2}} \to U\,e^{-\frac{K'}{2}}c\hat{V}e^{-\frac{K'}{2}}\,U^\dagger. \tag{4.40}$$

This is a map from cohomology to cohomology but, contrary to the non-real construction (4.39), it is not a star algebra homomorphism. It is also possible, at least formally, to connect Φ with $\Psi^{\rm real}$ with a *real* gauge transformation $W=U^\dagger\frac{1}{\sqrt{UU^\dagger}}$ obeying $W^\dagger W=WW^\dagger=1$, [30] which gives a star algebra homomorphism compatible with reality. However, we do not see obvious problems in using the simpler non real deformed states (4.39).

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A TT on the upper half plane, BPZ and reality

To make contact with the original form of the TT solution, we relate the function $f(z)$ to the function $F(w)$ appearing in the work of TT [8, 9] by mapping the semi-infinite cylinder C_L of circumference $L=2$ (with coordinate z) to the upper half plane (with coordinate

w) by $w = \tan\frac{\pi z}{2}$.

$$\begin{aligned}\Phi &= \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\left(f(z)cj(z)+\frac{1}{2}f^2(z)c\,(z)\right)\\&= \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\left(f(z)\tilde{c}\tilde{j}\,(z+1/2)+\frac{1}{2}f^2(z)\tilde{c}\,(z+1/2)\right)I\\&= \int_{C_{\text{left}}}\frac{dw}{2\pi i}\left(F(w)cj(w)+\frac{1}{2}F^2(w)c(w)\right)I \qquad \text{(A.1)}\\ f(z) &= \frac{\pi}{2}\frac{F\left(\tan\frac{\pi}{2}\left(z+\frac{1}{2}\right)\right)}{\cos^2\frac{\pi}{2}\left(z+\frac{1}{2}\right)}, \qquad \text{(A.2)}\\ F(w) &= \frac{2}{\pi}\frac{f\left(\frac{2}{\pi}\tan^{-1}\frac{w-1}{w+1}\right)}{w^2+1}, \qquad \text{(A.3)}\end{aligned}$$

where C_{left} is the semicircle in the complex plane connecting $-i, 1, i$, oriented towards i.

In [8, 9] and in the papers that followed, the authors also require that $F(w)$ obey

$$F\left(-\frac{1}{w}\right) = w^2F(w),$$

which, in the sliver frame, translates into

$$f(z) = f(z-1).$$

We do not require this periodicity condition because (for example) a simple scale transformation in the sliver frame (a reparametrization generated by $L^- \equiv \frac{1}{2}\left(\mathcal{L}_0 - \mathcal{L}_0^*\right)$) would not respect it. As explained in [8, 9, 18], this property ensures that

$$\begin{aligned}\Phi_L I &\equiv \int_{C_{\text{left}}}\frac{dw}{2\pi i}\left(F(w)cj(w)+\frac{1}{2}F^2(w)c(w)\right)I\\&= \int_{C_{\text{right}}}\frac{dw}{2\pi i}\left(F(w)cj(w)+\frac{1}{2}F^2(w)c(w)\right)I \equiv \Phi_R I, \qquad \text{(A.4)}\end{aligned}$$

so that we can write

$$\Phi_L I * \Phi_L I = (-1)^{|\Phi|}\Phi_R\Phi_L I * I = \Phi_L\Phi_R I = \Phi_L^2 I, \qquad \text{(A.5)}$$

where the commutation between left and right charges holds if $F(\pm i) = 0$. However, to prove the equation of motion, as we saw in section 2, we only used that $F(w)$ vanishes at the midpoint and that it is analytic in an infinitesimal neighborhood of C_{left}. The corresponding right charge Φ_R can be defined, if needed, by the same expression (A.4) but with

$$F(w) \to \frac{1}{w^2}F(-1/w),$$

i.e.

$$\Phi_R \to (\text{bpz}\Phi_L)$$

which is a right-type charge which also vanishes at the midpoint and is analytic around $C_{\rm right}$. In this way (A.5) is still satisfied

$$\Phi_L I * \Phi_L I = (-1)^{|\Phi|}(\text{bpz}\Phi_L)\Phi_L I * I = \Phi_L(\text{bpz}\Phi_L)I = \Phi_L^2 I, \tag{A.6}$$

because we can use the generic properties

$$\Phi_L I = (\text{bpz}\Phi_L)I \tag{A.7}$$
$$A * (\Phi_L B) = (-1)^{|A|\,|\Phi|}((\text{bpz}\Phi_L)A) * B, \tag{A.8}$$

which encode the gluing conditions

$$-1 = w^{(i)}w^{(i+1)}\Big|_{|w^{(i)}|=1,(-1)^i\text{Re}w^i>0},$$

for N-strings vertices. If we like, given $F(w)$ defined on $C_{\rm left}$ $(\text{Re}\, w > 0)$ we can always extend $F(w)$ on $C_{\rm right}$ $(\text{Re}\, w < 0)$ by

$$F(w) \equiv 1/w^2 F(-1/w), \quad \text{for Re}w < 0,$$

but this isn't in general an analytic continuation.[15] Since, to define the solution, we only need to know $F(w)$ on C_L, we avoid talking about the value of $F(w)$ on C_R.

The reality condition, on the other hand, gives a real constraint on $F(w)$. The string field Φ is real (bpz=hc) if the function $F(w)$ satisfies[16]

$$F(w) = \frac{1}{w^2}F^*\left(\frac{1}{w}\right), \quad |w| = 1, \text{Re}\, w > 0 \qquad \texttt{Reality}, \tag{A.9}$$

which in the sliver frame translates into the quite intuitive

$$f(z) = f^*(-z) = f^*(z^*), \quad \text{Re}\, z = 0. \tag{A.10}$$

B A new singularity towards the identity

The simple algebraic derivation of observables we have presented in section 3 is potentially endangered by a singularity towards the identity which has to do with the c- ghost, as we now briefly explain.

To start with, it is better to specialize a bit on the function $f(z)$ which defines the solution. Because of the omnipresence of the quantity $\frac{1}{1+K}$, a basic requirement is that, when we add Φ to K, B, c, its contraction is well defined against wedge-based states of arbitrarily small width. While this was essentially guaranteed in previous enlargements of the KBc algebra, [12, 21], which dealt with boundary insertions, and even [7], where

[15] I thank Ted Erler for a useful discussion on this.

[16] The reality condition for a related identity based solution for the tachyon vacuum has been discussed in [27].

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only matter operators were allowed to enter the bulk, here the story is more delicate. To appreciate the problem consider the simple overlap

$$\mathrm{Tr}[\Phi\Omega^t c\partial c] = \frac{1}{2}\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}f^2(z)\langle c(z+t)c\partial c(0)\rangle_{C_t} \tag{B.1}$$

$$= -\frac{t^2}{2\pi^2}\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}f^2(z)\sin^2\left(\frac{\pi z}{t}\right). \tag{B.2}$$

This integral is divergent for small enough $t > 0$ unless the function $f(z)$ is suppressed at $i\infty$ more than exponentially. For example, the standard choice by Takahashi and Tanimoto [8, 9],

$$F_{\mathrm{TT}}(w) = 1 + \frac{1}{w^2} \qquad \to \qquad f_{\mathrm{TT}}(z) = \frac{2\pi}{\cos^2 \pi z} \tag{B.3}$$

does not respect this property. Indeed, although the TT-solution based on f_{TT} is finite in the Fock space, a finite L^- reparametrization of it, (2.31), appears to be singular.[17] In particular

$$\langle \mathrm{Fock}|t^{L^-}\Phi_{f_{\mathrm{TT}}}\rangle = \infty, \qquad t \le \frac{1}{2}. \tag{B.4}$$

This is certainly un-welcome for the purpose of enlarging the K, B, c algebra with Φ, as we have been doing in the previous sections. Therefore we would like to limit the choice of f in such a way that generic contractions with wedge based states and finite L^- reparametrizations give finite results. A simple example that does the job is the family of gaussians

$$f_t(z) \equiv 2\lambda\sqrt{\pi}\, t\, e^{(tz)^2}, \tag{B.5}$$

for which we have

$$\lambda = \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}f_t(z). \tag{B.6}$$

In the following we will specialize to the family of gauge equivalent solutions described by $f_t(z)$. These solutions are all related by L^--reparametrizations

$$\Phi_{f_t} = t^{-L^-}\Phi_{f_{t=1}}. \tag{B.7}$$

We can easily check that, for this choice of function, the TT solution Φ is finite in the Fock space, and also against generic wedge states with insertions whose width can be taken arbitrarily small. In particular, for example

$$\mathrm{Tr}[\Phi_{f_{t=1}}\Omega^s c\partial c] = -\frac{2\lambda^2 s^2}{\pi}\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}e^{2z^2}\sin^2\left(\frac{\pi z}{s}\right)$$

$$= \frac{\lambda^2 s^2}{2\sqrt{2}\pi^{3/2}}\left(e^{\frac{\pi^2}{2s^2}} - 1\right). \tag{B.8}$$

[17]This singularity is absent if the marginal field j has finite OPE with itself, which reflects in the absence of the c-part of the solution $\frac{1}{2}\int dz f^2(z)c(z)$

Notice however that, although the overlap is finite for $s > 0$, it nevertheless diverges super-exponentially in the identity limit $s \to 0$. Sticking to this example, this means that the following overlap is badly divergent

$$\mathrm{Tr}\left[\Phi\frac{1}{1+K}c\partial c\right] = \int_0^\infty dt e^{-t}\,\mathrm{Tr}\left[\Phi\Omega^t c\partial c\right] = \infty. \tag{B.9}$$

We may hope that further suppressing f at the midpoint could improve the situation, but in fact there is a more basic problem. When two c-ghosts have a separation with a tiny imaginary part on a cylinder of width t, the correlator always diverges in the limit $t \to 0$.

$$\lim_{t\to 0}\mathrm{Tr}[c(ix)e^{-tK}c\partial c] = \lim_{t\to 0}\frac{t^2}{\pi^2}\sinh^2\frac{\pi x}{t} = \infty, \quad \mathrm{Re}\,x \neq 0 \tag{B.10}$$

Notice that the negative scaling dimension of c would suppress the correlator, but this comes together with a rather violent exponential divergence, which only occurs when c is placed off the boundary. Therefore, even in the original K, B, c algebra we have the problem

$$\mathrm{Tr}[c(ix)\frac{1}{1+K}c\partial c] = \int_0^\infty dt e^{-t}\frac{t^2}{\pi^2}\sinh^2\frac{\pi x}{t} = \infty, \quad \mathrm{Re}\,x \neq 0. \tag{B.11}$$

This is a new kind of identity-like singularity which would be worth studying by itself. Notice in particular that naive attempts to evaluate the action of the TT solution Φ are affected by this singularity, (2.20). Notice that the singularity is much more violent than previous identity-like singularities studied in the sliver frame, [26], whose behavior is typically power law. Since we know quite little about these singularities and how they effectively cancel in the algebraic computations we have been doing in the main text, our primary aim will be to show that these singularities can be avoided by an infinitesimal deformation of our solution.

C Regularization

The singular expressions we met in the previous section are structurally quite close to the expressions that appear in the computation of observables in the main text. For example, consider the kinetic term of the string field $\chi \equiv \frac{1}{1+K}\Phi$, which is a part of our solution Ψ, (2.35). The explicit computation goes as follows

$$\mathrm{Tr}\left[\chi Q\chi\right] = \frac{1}{4}\int_{-\infty}^\infty \frac{dx\,dy}{(2\pi)^2}f^2(ix)f^2(iy)\,\mathrm{Tr}\left[\frac{1}{1+K}c(ix)\frac{1}{1+K}c\partial c(iy)\right]. \tag{C.1}$$

If we consider the kernel

$$\mathrm{Tr}\left[\frac{1}{1+K}c(ix)\frac{1}{1+K}c\partial c(iy)\right] = -\frac{1}{\pi^2}\int_0^\infty dt\,e^{-t}t\int_0^1 dq\sin^2\pi\left(q+i\frac{x-y}{t}\right), \tag{C.2}$$

we see that this is not a well defined quantity since, if we perform the dt integral first, we encounter a bad exponential singularity at $t \to 0$ of the type $e^{2\pi\frac{|x-y|}{t}}$. On the other hand, performing the q integral first, the dependence on $(x-y)$ drops and everything is finite.

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In this case, it is not difficult to realize that the 'correct' prescription for computing the above integral would be to define

$$\mathrm{Tr}\left[\frac{1}{1+K}c(ix)\frac{1}{1+K}c\partial c(iy)\right] = \lim_{\epsilon\to 0^+}\int_\epsilon^\infty dte^{-t}\int_0^t ds\,\mathrm{Tr}[\Omega^s c(ix)\Omega^{t-s}c\partial c(iy)]. \quad \text{(C.3)}$$

With this regularization of the trace (cut-off in the overall Schwinger parameter of the string field whose trace we want to compute), the algebraic derivations of the Ellwood invariant and the kinetic term, presented in section 3, are rigorously justified, as it is easy to check. However it is very difficult to understand this regularization at the level of the individual string fields before $*$-multiplication and, importantly, to understand how the equation of motion in the action is violated and how (and if) it is restored when the regulator is removed.

Our algebraic computation suggests there exists a prescription (which is consistent with the equation of motion) to correctly compute the observables. But to make this precise, we need a regularization which allows us to control the $t\to 0$ limit in the overall Schwinger integral and to maintain, at the same time, the equation of motion. Perhaps the simplest and safest approach (but other strategies might be possible) is to realize that the solution we are dealing with can be obtained as a limit of a one parameter family of gauge equivalent solutions which have, generically, a minimum fixed width and therefore, by construction, cannot have any singularity related to the identity string field. To construct such a family is easy and amounts to choosing a security strip in the Zeze map (2.34) given by (for example, other choices are of course possible)

$$F_\epsilon = \frac{\Omega^\epsilon}{1+\bar\epsilon K}, \qquad \bar\epsilon \equiv 1-\epsilon, \quad \text{(C.4)}$$

where

$$\Omega \equiv e^{-K} = |0\rangle_{SL(2,R)}. \quad \text{(C.5)}$$

The regularized solutions are given by

$$\Psi_\epsilon = \frac{\Omega^\epsilon}{1+\bar\epsilon K}\Phi\left(1-\frac{B}{1+h_\epsilon J}h_\epsilon\Phi\right), \quad \text{(C.6)}$$

where

$$h_\epsilon = \frac{1-\frac{\Omega^\epsilon}{1+\bar\epsilon K}}{K} = \frac{1}{1+\bar\epsilon K}\left(\bar\epsilon + \int_0^\epsilon dt\Omega^t\right). \quad \text{(C.7)}$$

These solutions span a gauge orbit interpolating from KOS ($\epsilon = 0$) and the generalization of the Schnabl-KORZ solution, [3, 4], ($\epsilon = 1$) which, for completeness, reads

$$\begin{aligned}\Psi_1 &= \Omega\Phi\left(1-\frac{1}{1+\frac{1-\Omega}{K}J}\frac{1-\Omega}{K}B\Phi\right)\\ &= \Omega\Phi\left[1-\sum_{n=0}^\infty\left(\int_0^1 dt\Omega^t J\right)^n\int_0^1 dt\Omega^t B\Phi\right].\end{aligned} \quad \text{(C.8)}$$

The strategy is therefore to define the observables of the KOS solution as the $\epsilon \to 0^+$ limit of the observables of the interpolating solutions (which, by gauge invariance, will be ϵ-independent). At finite ϵ it is guaranteed that no identity-singularity can affect the computation. However the price we pay for manifest regularity is that the generic solution in the orbit is much more complicated than the original KOS solution and it is not clear, at this stage, how one could compute the observables as we did in the main text. But in fact we can *rewrite* the regularized solution (C.6) again as a KOS solution, where the fields (K, B, c, Φ) have undergone the automorphism [28]

$$
\begin{aligned}
c &\to c_\epsilon = c\frac{KB}{G_\epsilon(K)}c && \text{(C.9)}\\
B &\to B_\epsilon = B\frac{G_\epsilon(K)}{K} && \text{(C.10)}\\
K &\to K_\epsilon = QB_\epsilon = G_\epsilon(K) && \text{(C.11)}\\
\Phi &\to \Phi && \text{(C.12)}\\
J &\to J_\epsilon = [B_\epsilon, \Phi]. && \text{(C.13)}\\
& && \text{(C.14)}
\end{aligned}
$$

where $G_\epsilon(K)$ is defined by[18]

$$\frac{1}{1+K_\epsilon} = \frac{1}{1+G_\epsilon(K)} = \frac{\Omega^\epsilon}{1+\bar{\epsilon}K}. \tag{C.15}$$

Notice that the TT solution remains invariant under the automorphism.

With the new variables, and some standard algebra, we can re-write (C.6) in few interesting ways

$$
\begin{aligned}
\Psi_\epsilon &= \frac{1}{1+K_\epsilon}\Phi\frac{1}{1+K_\epsilon+J_\epsilon} - Q\left(\frac{1}{1+K_\epsilon}\Phi\frac{B_\epsilon}{1+K_\epsilon+J_\epsilon}\right) && \text{(C.16)}\\
&= \frac{1}{1+K_\epsilon}\Phi\frac{1}{1+K_\epsilon+J_\epsilon} + \frac{1}{1+K_\epsilon}\Phi\frac{1}{1+K_\epsilon+J_\epsilon}(K_\epsilon+\Phi B_\epsilon)\\
&= \frac{\Omega^\epsilon}{1+\bar{\epsilon}K}\Phi\left(\frac{1}{1+h_\epsilon B\Phi}\frac{\Omega^\epsilon}{1+\bar{\epsilon}K}\frac{1}{1+\Phi B h_\epsilon}\right)\\
&\quad+\frac{\Omega^\epsilon}{1+\bar{\epsilon}K}\Phi\frac{1}{1+h_\epsilon B\Phi}\left(1-\frac{\Omega^\epsilon}{1+\bar{\epsilon}K}\frac{1}{1+\Phi B h_\epsilon}\right) && \text{(C.17)}\\
&= \frac{\Omega^\epsilon}{1+\bar{\epsilon}K}\Phi\left(\frac{1}{1+h_\epsilon B\Phi}\frac{\Omega^\epsilon}{1+\bar{\epsilon}K}\frac{1}{1+\Phi B h_\epsilon}\right)\\
&\quad-Q\left(\frac{\Omega^\epsilon}{1+\bar{\epsilon}K}\frac{\Phi h_\epsilon B}{1+\Phi h_\epsilon B}\right). && \text{(C.18)}
\end{aligned}
$$

The reader can explicitly verify that the first 'physical' term in the regularized solution (C.18) has support on wedge-based states of minimum width 2ϵ while the BRST

[18] $G_\epsilon(K) = K_\epsilon$ is a purely formal string field which is proportional to the inverse wedge $e^{\epsilon K}$. However it always appear in the combination $\frac{1}{1+K_\epsilon}$ which is fine. Similar considerations apply to B_ϵ and $J_\epsilon \equiv [B_\epsilon, \Phi]$ which are formal by themselves but always appear in the regular combinations $\frac{B_\epsilon}{1+K_\epsilon}$, $\frac{1}{1+K_\epsilon+J_\epsilon}$, $\frac{1}{1+K_\epsilon+J_\epsilon}J_\epsilon\frac{1}{1+K_\epsilon}$, $B_\epsilon c_\epsilon$, $c_\epsilon B_\epsilon$, etc...

exact one has minimum width ϵ. Notice that we have

$$\frac{1}{1+K_\epsilon+J_\epsilon}=\frac{1}{1+h_\epsilon B\Phi}\frac{\Omega^\epsilon}{1+\bar{\epsilon}K}\frac{1}{1+\Phi B h_\epsilon}, \tag{C.19}$$

which reveals that the automorphism mixes the objects in the game in a rather non trivial way. In particular

$$Q_{\Phi\Phi}\frac{1}{1+K_\epsilon+J_\epsilon} = 0 \tag{C.20}$$

$$\mathrm{ad}_{K+J}\frac{1}{1+K_\epsilon+J_\epsilon} = [Q_{\Phi\Phi},\mathrm{ad}_B]\frac{1}{1+K_\epsilon+J_\epsilon}=0, \tag{C.21}$$

which trivially descend from the automorphism, but which appear rather surprising in the original variables. Other notable quantities are given by

$$\frac{B_\epsilon}{1+K_\epsilon} = Bh_\epsilon \tag{C.22}$$

$$\frac{B_\epsilon}{1+K'_\epsilon} = Bh_\epsilon\frac{1}{1+Jh_\epsilon}=B\frac{1}{1+h_\epsilon J}h_\epsilon \tag{C.23}$$

$$c_\epsilon B_\epsilon = cB \tag{C.24}$$

$$B_\epsilon c_\epsilon = Bc \tag{C.25}$$

$$c_\epsilon K_\epsilon B_\epsilon c_\epsilon = cKBc \tag{C.26}$$

$$c_\epsilon(K_\epsilon+J_\epsilon)B_\epsilon c_\epsilon = c(K+J)Bc, \tag{C.27}$$

notice that the automorphism doesn't increase the minimum width of the above quantities, which all continue to have a non vanishing support on the identity. Since the fields $(K_\epsilon, B_\epsilon, c_\epsilon, \Phi)$ have identical properties to (K, B, c, Φ) the computations for Ψ_ϵ can be read-off from the main text by formally substituting (K, B, c, J) with $(K_\epsilon, B_\epsilon, c_\epsilon, J_\epsilon)$. By inspecting the involved correlators, we see than only well defined combinations of the deformed variables explicitly appear, so we have just to trace back how the simplifications in the deformed variables occur in the original variables. In doing this we encounter very non trivial simplifications between different structures, which would have been practically impossible to discover if not guided by the formal automorphism (C.9—C.13). For fixed $\epsilon \neq 0$ we can then precisely show that the observables of the regularized solution Ψ_ϵ reduce to the shift in the observables of the tachyon vacuum's

$$\Psi^{(0)}_{TV,\epsilon} = \frac{1}{1+K_\epsilon}(c_\epsilon+Q(B_\epsilon c_\epsilon)) \tag{C.28}$$

$$\Psi^{(\Phi)}_{TV,\epsilon} = \frac{1}{1+K'_\epsilon}(c_\epsilon+Q_{\Phi\Phi}(B_\epsilon c_\epsilon)). \tag{C.29}$$

When we take the $\epsilon \to 0$ limit these two solutions become the Erler-Schnabl solutions (3.7), (3.8), which are manifestly safe from the identity singularities we encountered in the previous section, simply because no explicit Φ enters in their definition and therefore there is no c-field going off the boundary. For completeness, it would be interesting to have an analytic computation of the observables of $\Psi^{(\Phi)}_{TV,\epsilon}$, at finite ϵ, which we leave for the future.

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