

# A note on entanglement entropy for topological interfaces in RCFTs

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**ABSTRACT:** In this paper we calculate the entanglement entropy for topological interfaces in rational conformal field theories for the case where the interface lies at the boundary of the entangling interval and for the case where it is located in the center of the entangling interval. We compare the results to each other and also to the recently calculated left/right entropy of a related BCFT. We also comment of the entanglement entropies for topological interfaces for a free compactified boson and Liouville theory.

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**1 Introduction**

In two dimensional conformal field theories the construction of conformal interfaces between  $CFT_1$  and  $CFT_2$  is equivalent to the construction of conformal boundary conditions in the folded CFT which is the tensor product  $CFT_1 \otimes \overline{CFT_2}$  [1, 2]. There is a special class of interfaces which are called topological interfaces. In the folded CFT the condition for an interface to be topological is [2]

$$\left(L_n^{(1)} - \bar{L}_{-n}^{(2)}\right) | B \rangle = 0, \quad \left(L_n^{(2)} - \bar{L}_{-n}^{(1)}\right) | B \rangle = 0 \tag{1.1}$$

This condition implies that the interface is completely transmissive and these interfaces are called topological since their position and shape can be deformed without cost of energy. In addition one can define a fusion product of topological interfaces by bringing two of them close together [3]. It has been argued in [4–6] that topological interfaces can furnish spectrum generating symmetries. Topological interfaces have been constructed for a single free boson in [2, 7] and for  $n$  free bosons compactified on an  $n$  dimensional torus in [8]. Topological interfaces in orbifold theories have been studied in [9].

The entanglement entropy of a spatial region  $\mathcal{A}$  is defined by the von Neumann entropy of a reduced density matrix which is obtained by integrating out the degrees of freedom localized in the complement of  $\mathcal{A}$ . For two dimensional CFTs [10, 11] the entanglement entropy, where  $\mathcal{A}$  is a single interval of length  $L$ , is given by

$$\mathcal{S}_{\mathcal{A}} = \frac{c}{3} \ln \left( \frac{L}{\epsilon} \right) + \ln g \tag{1.2}$$

Here  $\epsilon$  is an UV cutoff and we dropped terms which vanish as  $\epsilon \rightarrow 0$ . The logarithmically divergent term only depends on the central charge of the CFT. Generically the constant term is non-universal and dependent on the definition of the cutoff since a simple rescaling of the cutoff is equivalent to adding a constant term to the entanglement entropy.

In the presence of a boundary or an interface the constant term becomes physical as one can study the difference in entanglement entropy for a system with an interface and without, and the dependence on the UV cutoff cancels in this case. In the case of a conformal interface there are two different geometric setups which one can consider. First if the interface is located at the center of the entangling region  $\mathcal{A}$  then [11] the entanglement entropy is of the form given in (1.2) and the constant term is the so called  $g$ -factor or boundary entropy [1]. A second geometry is given by choosing the interface to be located at the boundary of the entangling region  $\mathcal{A}$ . The entanglement entropy in this case has only been calculated for some specific cases, notably for generic conformal interfaces for a single free boson in [12] and the Ising model in [13]. The entanglement entropy in this case takes the form

$$\mathcal{S}_{\mathcal{A}} = \frac{c}{6} f(\mathcal{I}) \ln \left( \frac{L}{\epsilon} \right) + \ln g \quad (1.3)$$

where  $f(\mathcal{I})$  is a complicated function depending on the parameters of the conformal interface, which we schematically denote by  $\mathcal{I}$ . The entanglement entropies for Janus like interface theories [15–17] have been calculated for the symmetric geometry in [18, 19] and for the case where the interface is at the boundary of the entangling surface in [20].

The goal of this note is to calculate the entanglement entropies for both geometric setups for topological interfaces in rational CFTs. The structure of the paper is as follows: in section 2 we review the construction of topological interfaces in rational CFTs which goes back to the work of Petkova and Zuber [14]. In section 3 we adapt the calculation of Sakai and Satoh [12] to calculate the entanglement entropy for the case where the topological interface is at the boundary of the entangling surface. In section 4 we adapt the calculation of Sakai and Satoh to re-derive the entanglement entropy (1.2) for the symmetric interface and also give an argument that the location of the interface does not change the result as long as it is a finite distance away from the boundaries of the entangling interval. In section 5 we compare the entanglement entropies for various specific cases. We also include the recently computed left/right entropy [21, 22] for reference. In section 6 we compare results of entanglement entropies for compact bosons which were previously obtained in the literature. In section 7 we use the construction of topological interfaces in Liouville theory given in [23, 25] to attempt a calculation of the entanglement entropy for this system. We close with a discussion of future directions of research in section 8.

**Note added.** While this paper was finalized a paper [26] appeared, which has significant overlap with the material presented in section 3.

## 2 Topological interfaces in RCFT

In this section we consider the construction of topological interfaces in rational CFTs. A rational CFT (RCFT) [27] contains a finite number of primary states and hence a finite

number of representations of the Virasoro algebra, labeled by  $i$ , with characters  $\chi_i(q)$ . The partition function on the torus is given by

$$Z = \sum_{i\bar{j}} Z_{i\bar{j}} \chi_i(q) \chi_{\bar{j}}(\bar{q}) \tag{2.1}$$

where  $Z_{i\bar{j}}$  are positive integers which denote how many times a representation appears in the spectrum of the theory. We mostly limit ourselves to the case where  $Z_{i\bar{j}} = \delta_{i\bar{j}}$  and the theory has a diagonal spectrum.

The canonical examples for rational CFTs<sup>1</sup> are the unitary minimal models which have central charge

$$c = 1 - \frac{6}{m(m+1)}, \quad m = 3, 4, \dots \tag{2.2}$$

and the primaries are labeled by two integers  $r = 1, 2, \dots, m-1$  and  $s = 1, 2, \dots, r$  and have conformal dimension

$$h_{r,s} = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)} \tag{2.3}$$

The simplest minimal model is the Ising model which has  $m = 3$  and hence we have  $c = \frac{1}{2}$  and three primaries with  $h = 0$ ,  $h = \frac{1}{2}$  and  $h = \frac{1}{16}$ .

In [14, 28] twisted partition functions for rational CFTs were studied. They are characterized by the insertion of an operator  $I$  into the partition function, where  $I$  satisfies

$$[L_n, I] = [\bar{L}_n, I] = 0 \tag{2.4}$$

In [14] a classification of such operators was given analogous to the construction of Cardy states [29]. For the diagonal theories one finds

$$I_a = \sum_i \frac{S_{ai}}{S_{0i}} P^{i\bar{i}} \tag{2.5}$$

Here  $P^{i\bar{i}}$  is a projector on the space spanned by the  $i - th$  primary and its descendants

$$P^{i\bar{i}} = \sum_{n\bar{n}} |i, n\rangle \otimes |i, \bar{n}\rangle \langle i, n| \otimes \langle i, \bar{n}| \tag{2.6}$$

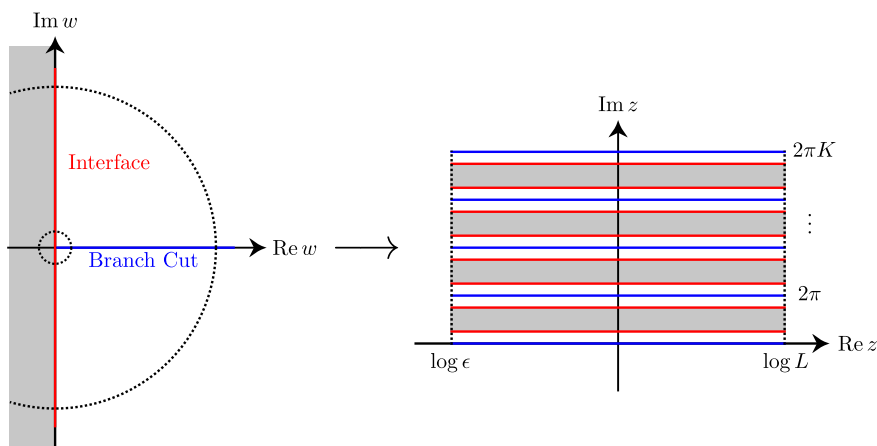
This means that in the simple diagonal case there are as many topological interfaces as there are primaries, where for simplicity we assume that each primary only appears once in the theory; a degeneracy can be easily included in the construction.<sup>2</sup> The matrix  $S$  is the modular  $S$  matrix which denotes how the characters transform under modular transformation  $q = e^{2\pi i\tau}$

$$\chi\left(-\frac{1}{\tau}\right) = \sum_j S_{ij} \chi_j(\tau) \tag{2.7}$$

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<sup>1</sup>We limit ourselves to minimal models with respect to the Virasoro algebra here, generalizing the discussion to rational CFTs with respect to extended conformal algebras would be very interesting.

<sup>2</sup>See [30] for a discussion of more general projectors including non-diagonal theories.



**Figure 1.** The logarithmic map  $z = \ln w$  maps the  $K$ -sheeted Riemann surface to the geometry on the right. The circles on the left part of the figure correspond to an UV cutoff located  $|w| = \epsilon$  and an IR cutoff located at  $|w| = L$ . This figure was adapted from [13].

The conjugate interface operator is given by

$$I_a^\dagger = \sum_i \left( \frac{S_{ai}}{S_{0i}} \right)^* P^{i\bar{i}} \tag{2.8}$$

In summary it is notable that the classification of [14, 28] of twisted partition function also provides us with a classification of topological interfaces in RCFTs.

### 3 Entanglement entropy at a topological interface

The entanglement entropy for an interval  $\mathcal{A}$  is given by the von Neumann entropy of the reduced density matrix  $\rho_{\mathcal{A}} = \text{tr}_{\bar{\mathcal{A}}} |0\rangle\langle 0|$ , where  $\bar{\mathcal{A}}$  is the complement of  $\mathcal{A}$ . The replica trick relates the entanglement entropy to the Renyi entropies as follows

$$\mathcal{S}_{\mathcal{A}} = -\frac{\partial}{\partial K} \text{tr} \rho_{\mathcal{A}}^K \Big|_{K=1} \tag{3.1}$$

The Renyi entropies are calculated by a path integral over a  $K$ -sheeted Riemann surface with branch cuts running along  $\mathcal{A}$ . The entanglement entropy can then be derived from the partition function  $Z(K)$  on the  $K$ -sheeted Riemann surface by

$$\mathcal{S}_{\mathcal{A}} = (1 - \partial_K) \ln Z(K) \Big|_{K=1} \tag{3.2}$$

In [12] it has been argued that the interface can be included in the  $K$ -sheeted partition function  $Z(K)$ . The interface gets mapped via  $z = \ln w$  to a covering space (see figure 1). Introducing an UV cutoff  $\epsilon$  and an IR cutoff  $L$  and imposing periodic boundary conditions for simplicity, the  $K$ -th replica partition function can be expressed as a trace

$$Z(K) = \text{tr} \left( I e^{-tH} I^\dagger e^{-tH} \right)^K \tag{3.3}$$

where the “time” is related to the cutoffs by

$$t = \frac{2\pi^2}{\ln(L/\epsilon)} \quad (3.4)$$

and

$$H = L_0 + \bar{L}_0 - \frac{c}{12} \quad (3.5)$$

is the Hamiltonian of the CFT. The  $K$ -th partition function with a topological interface (2.5) labeled by a primary  $a$  inserted is

$$\begin{aligned} Z_a(K) &= \text{tr} \left( \left[ I_a q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} I_a^\dagger q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right]^K \right) \\ &= \text{tr} \left( [I_a I_a^\dagger]^K q^{2K(L_0 - \frac{c}{24})} \bar{q}^{2K(\bar{L}_0 - \frac{c}{24})} \right) \\ &= \sum_i \left| \frac{S_{ai}}{S_{0i}} \right|^{2K} \chi_i(q^{2K}) \chi_{\bar{i}}(\bar{q}^{2K}) \end{aligned} \quad (3.6)$$

where we have introduced  $q = \bar{q} = e^{-t}$ . In the second line we have used (2.4) to commute  $I_a$  through the Hamiltonian and in the third line we used the fact that the  $P^{ii}$  in (2.5) are projectors to the  $i$ -th representation and the trace produces the associated character  $\chi_i$ .

Since we are interested in taking the UV cutoff  $\epsilon \rightarrow 0$  (and taking  $L \rightarrow \infty$ ), we have to evaluate (3.6) in the limit  $q \rightarrow 1$ . With the identification of a new modular parameter  $\tau'$  by

$$q^{2K} = e^{-2Kt} = e^{2\pi i \tau'} \quad (3.7)$$

with  $t$  given in (3.4), the limit can be taken by performing a modular transformation on the characters

$$\begin{aligned} \lim_{q \rightarrow 1} \chi_i(q^{2K}) \chi_{\bar{i}}(\bar{q}^{2K}) &= \lim_{\tau' \rightarrow 0} \chi_i(\tau') \chi_{\bar{i}}(\bar{\tau}') \\ &= \lim_{\tau' \rightarrow 0} \sum_{j,k} S_{ij} S_{ik}^* \chi_j(-1/\tau') \chi_k(-1/\bar{\tau}') \\ &= \sum_{j,k} S_{ij} S_{ik}^* e^{\frac{\pi^2 c}{6Kt}} e^{-\frac{2\pi^2 h_j}{Kt}} e^{-\frac{2\pi^2 h_k}{Kt}} \left( 1 + o[e^{-2\pi^2/Kt}] \right) \end{aligned} \quad (3.8)$$

In the limit  $t \rightarrow 0$  the leading contribution in (3.8) will come from the vacuum characters which have  $h_j = h_k = 0$ . In that case the partition function (3.6) becomes

$$Z_a(K) \approx \exp \left( \frac{c}{12K} \ln \frac{L}{\epsilon} \right) \sum_i |S_{ai}|^{2K} |S_{0i}|^{2-2K} + \dots \quad (3.9)$$

where the dots indicate terms which vanish as the cutoff is taken to zero. Further calculating

$$\begin{aligned} (1 - \partial_K) \ln \left( \sum_i |S_{ai}|^{2K} |S_{0i}|^{2-2K} \right) \Big|_{K=1} &= -2 \frac{\sum_i |S_{ai}|^{2K} |S_{0i}|^{2-2K} (\ln |S_{ai}| - \ln |S_{0i}|)}{\sum_j |S_{aj}|^{2K} |S_{0j}|^{2-2K}} \Big|_{K=1} \\ &= -2 \sum_i |S_{ai}|^2 \ln \left| \frac{S_{ai}}{S_{0i}} \right| \end{aligned} \quad (3.10)$$

where we have repeatedly used the fact that  $S$  is symmetric, unitary, and in particular the relation  $\sum_j |S_{aj}|^2 = 1$ . Putting everything together we arrive at the following expression for the entanglement entropy at a topological interface

$$\mathcal{S}_a = \frac{c}{6} \ln \frac{L}{\epsilon} - 2 \sum_i |S_{ai}|^2 \ln \left| \frac{S_{ai}}{S_{0i}} \right| \tag{3.11}$$

#### 4 Symmetric and left/right entanglement entropy

For an interface which is located symmetrically on the entangling interval  $\mathcal{A}$  the entanglement entropy has been calculated by [11] and is given by

$$\mathcal{S}^{\text{symm}} = \frac{c}{3} \ln \frac{L}{\epsilon} + \ln g_B \tag{4.1}$$

where  $g_B$  is the boundary g-factor which when the interface is folded into a boundary state is determined by the overlap of the boundary state corresponding to the doubled interface with the vacuum state [1, 31]

$$g_B = \langle 0 | B \rangle \tag{4.2}$$

For the topological interface (2.5) the boundary state becomes

$$|B_a\rangle = \sum_i \frac{S_{ai}}{S_{0i}} \sum_{n\bar{n}} |i, n\rangle \otimes |i, \bar{n}\rangle |i, \bar{n}\rangle \otimes |i, n\rangle \tag{4.3}$$

Consequently the  $g$  factor is given by

$$g_B = \frac{S_{a0}}{S_{00}} \tag{4.4}$$

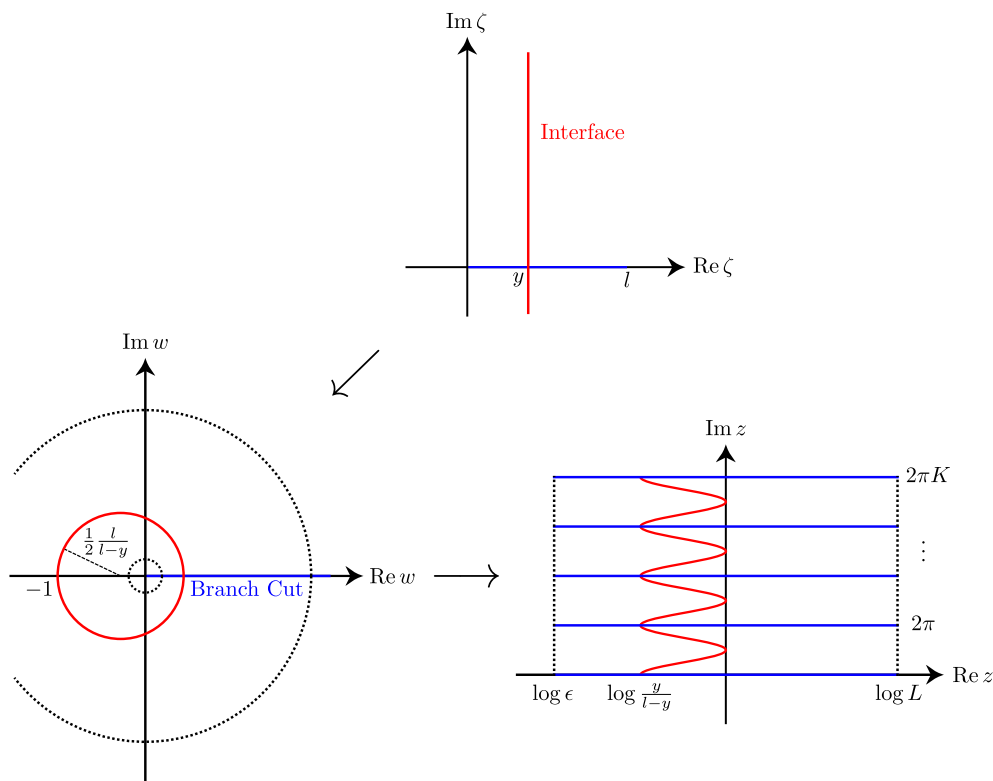
and the symmetric entanglement entropy becomes

$$\mathcal{S}_a^{\text{symm}} = \frac{c}{3} \ln \frac{L}{\epsilon} + \ln \left( \frac{S_{a0}}{S_{00}} \right) \tag{4.5}$$

Up to now we have considered the symmetric case where the interface is located at the center of the entangling interval  $\mathcal{A}$ . There is however a simple argument showing that for topological interfaces the location of the interface does not change the result as long as it is a finite distance away from the boundary of the entangling interval. We illustrate the argument in figure 2. We start in the  $\zeta$  plane with a finite interval  $\mathcal{A}$  with boundary at  $\zeta = 0$  and  $\zeta = l$ , where the interface is located along  $\zeta = y + i\xi, \xi \in \mathbb{R}$ . We map the  $\zeta$  plane into the  $w$  plane by the map

$$z = \frac{\zeta}{l - \zeta} \tag{4.6}$$

This maps the finite interval to the positive real axis and the interface  $I$  gets mapped to an off centre circle. Finally we perform the replica map to the  $z$  coordinate via  $z = \ln w$  and impose periodic boundary conditions as before at the cutoff  $z = \ln \epsilon$  and  $z = \ln L$ . This produces again a torus. Unlike the case of the interface at the boundary here the interface is mapped into a vertical curve on the torus. For a topological interface it is clear that



**Figure 2.** Mapping of a non-central interface to the replica torus.

the shape can be changed and changing the location along the real part of  $z$  corresponds to changing the original location  $y$  of the interface. This shows that partition function on the  $K$ -th sheeted Riemann surface is independent of  $y$  as long as the interface is a finite distance away from the cutoffs. We can be more specific and evaluate the partition function

$$\begin{aligned}
 Z_a(K) &= \text{tr}(I_a e^{-tH}) \\
 &= \sum_i \frac{S_{ai}}{S_{0i}} \chi_i(q) \chi_i(\bar{q})
 \end{aligned}
 \tag{4.7}$$

where  $q = \bar{q} = e^{-\frac{\pi^2}{Kt}}$  where  $t$  is again given by (3.4), hence in the limit of vanishing cutoff the sum over representations in the partition function gets projected on the vacuum character and one has

$$Z_a(K) \sim \frac{S_{a0}}{S_{00}} \exp\left(\frac{c}{12K} \ln \frac{L}{\epsilon}\right) + \dots
 \tag{4.8}$$

Applying the replica formula (3.2) one obtains

$$\mathcal{S}_a^{\text{symm}} = \frac{c}{6} \ln\left(\frac{L}{\epsilon}\right) + \ln\left(\frac{S_{a0}}{S_{00}}\right)
 \tag{4.9}$$

Comparing (4.9) with (4.5) one notices an extra factor of 1/2 in (4.9) in the  $\ln(L/\epsilon)$  term. This seeming discrepancy comes from the fact that the replica calculation leading to (4.9) calculates the entanglement entropy for a semi-infinite entangling surface (as we take  $L$  to



be very large) with only one end point, whereas the result of Cardy and Calabrese (4.5) is for an interval with two end points, which doubles the logarithmically divergent contribution according to the area law for entanglement entropy. The same remark applies when one compares (1.2) and (3.11).

Additionally it is clear that for a topological interface moving the interface along the real axis in the  $z$  coordinates does not change (4.7) as the interface operator commutes with the generator of these translations, which is the Hamiltonian. It is clear from figure 2 that the independence of the symmetric entanglement entropy from the location of the interface breaks down if the interface approaches the UV cutoff  $\epsilon$ , as part of the interface would be removed by the cutoff. This explains why the entanglement entropies (3.11) and (4.5) can be different.

A third type of entanglement entropy which takes a similar form is the so called left/right entanglement entropy [21, 22, 32]. This is defined for a boundary CFT, where the entanglement entropy is calculated with a reduced density matrix obtained by tracing over left-moving modes. Interestingly for a boundary CFT defined by a Cardy state [29] (for a single copy of the CFT, not the doubled one we are considering in the previous sections)

$$|B_a^{\text{Cardy}}\rangle = \sum_j \frac{S_a^j}{\sqrt{S_0^j}} |j\rangle\rangle \tag{4.10}$$

where  $|j\rangle\rangle$  are the Ishibashi states [33] enforcing conformal boundary conditions. We quote the result of the calculation of the left/right entanglement entropy which is also labeled by a primary  $a$  in a RCFT, obtained in [22]

$$\mathcal{S}_a^{l/r} = \frac{\pi cl}{24\epsilon} - \sum_j S_{aj}^2 \ln \left( \frac{S_{aj}^2}{S_{0j}} \right) \tag{4.11}$$

The physical interpretation of the left/right entanglement entropy (as it is non-geometrical) is not clear at this point as well as its relation to the other two entropies is not clear at the moment. The similarities of the resulting entropies might still suggest that such a relation exists. A better understanding of the relation of the cutoffs utilized may be necessary to accomplish this.

## 5 Examples of entanglement entropies

For the  $m$ -th unitary minimal models the modular  $S$  matrix is given by (see e.g. [34]).

$$S_{rs;\rho\sigma} = 2\sqrt{\frac{2}{m(m+1)}} (-1)^{1+s\rho+r\sigma} \sin\left(\pi\frac{m+1}{m}r\rho\right) \sin\left(\pi\frac{m}{m+1}s\sigma\right) \tag{5.1}$$

Using this formula it is in principle straightforward to evaluate the three entanglement entropies  $\mathcal{S}_a$  given in (3.11),  $\mathcal{S}_a^{\text{symm}}$  given in (4.5) and  $\mathcal{S}_a^{l/r}$  given in (4.11). Here we give a table for the two simplest cases, namely the Ising model with  $m = 3$  and the tri-critical Ising model with  $m = 4$ .

	$\mathcal{S}_a$	$\mathcal{S}_a^{\text{symm}}$	$\mathcal{S}_a^{1/r}$
$h = 0$	0	0	$\frac{3}{4} \ln 2$
$h = \frac{1}{2}$	0	0	$\frac{3}{4} \ln 2$
$h = \frac{1}{16}$	$-\ln 2$	$\frac{\ln 2}{2}$	0

**Table 1.** Entanglement entropies for the Ising model.

The Ising model has 3 primaries which we can label by the conformal dimension  $h = 0, \frac{1}{16}, \frac{1}{2}$  and the  $S$  matrix becomes

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \tag{5.2}$$

The entanglement entropies take the values given in table 1.

The next simplest minimal model is the tri-critical Ising model which has  $m = 4$  and has six primary states which are labelled by their conformal dimension

$$h = 0, \frac{1}{10}, \frac{3}{5}, \frac{3}{2}, \frac{3}{80}, \frac{7}{16} \tag{5.3}$$

The modular S-matrix is given by

$$S = \begin{pmatrix} s_2 & s_1 & s_1 & s_2 & \sqrt{2}s_1 & \sqrt{2}s_2 \\ s_1 & -s_2 & -s_2 & s_1 & \sqrt{2}s_2 & -\sqrt{2}s_1 \\ s_1 & -s_2 & -s_2 & s_1 & -\sqrt{2}s_2 & \sqrt{2}s_1 \\ s_2 & s_1 & s_1 & s_2 & -\sqrt{2}s_1 & -\sqrt{2}s_2 \\ \sqrt{2}s_1 & \sqrt{2}s_2 & -\sqrt{2}s_2 & -\sqrt{2}s_1 & 0 & 0 \\ \sqrt{2}s_2 & -\sqrt{2}s_1 & \sqrt{2}s_1 & -\sqrt{2}s_2 & 0 & 0 \end{pmatrix} \tag{5.4}$$

where  $s_1$  and  $s_2$  are given by

$$s_1 = \sin\left(\frac{2\pi}{5}\right), \quad s_2 = \sin\left(\frac{4\pi}{5}\right) \tag{5.5}$$

The entanglement entropies take the values given in table 2.

## 6 Entanglement entropies for a compact boson

A general class of conformal interfaces for a free boson was constructed in [2]. The interfaces are characterized by two radii  $R_1, R_2$  and two relatively prime integers  $k_1, k_2$ . Note that a compact boson is not a RCFT and the results of the previous sections cannot be directly applied. Instead we will review results obtained in the literature to contrast the entanglement entropies for this case.

a	$\mathcal{S}_a$	$\mathcal{S}_a^{\text{symm}}$	$\mathcal{S}_a^{1/r}$
$h = 0$	0	0	$\frac{-2\sqrt{5} \coth^{-1} \sqrt{5} + \ln \frac{32768}{3125}}{4}$
$h = \frac{1}{10}$	$-\sqrt{5} \coth^{-1} \left( \frac{3}{\sqrt{5}} \right)$	$\frac{1}{2} \ln \frac{3+\sqrt{5}}{2}$	$\frac{15 \ln 2 - 5 \ln 5 + \sqrt{5} \ln(9-4\sqrt{5})}{4}$
$h = \frac{3}{5}$	$-\sqrt{5} \coth^{-1} \left( \frac{3}{\sqrt{5}} \right)$	$\frac{1}{2} \ln \frac{3+\sqrt{5}}{2}$	$\frac{15 \ln 2 - 5 \ln 5 + \sqrt{5} \ln(9-4\sqrt{5})}{4}$
$h = \frac{3}{2}$	0	0	$\frac{-2\sqrt{5} \coth^{-1} \sqrt{5} + \ln \frac{32768}{3125}}{4}$
$h = \frac{3}{80}$	$\frac{(-5+\sqrt{5}) \ln(3-\sqrt{5}) - (5+\sqrt{5}) \ln(3+\sqrt{5})}{2}$	$\frac{1}{2} \ln(3 + \sqrt{5})$	$\frac{-5 \ln 5 + \sqrt{5} \ln(9-4\sqrt{5})}{4}$
$h = \frac{7}{16}$	$-5 \ln 2$	$\frac{\ln 2}{2}$	$\frac{-2\sqrt{5} \coth^{-1} \sqrt{5} - 5 \ln 5}{4}$

**Table 2.** Entanglement entropies for the tri-critical Ising model.

These correspond to an interface between two compactified bosons where the compactification radius jumps from  $R_1$  to  $R_2$  across the interface. In the doubled BCFT description the interface corresponds to a geometric D1 brane stretched on a rectangular torus with radii  $R_1$  and  $R_2$  where its one dimensional world volume wraps  $k_1$  times around the  $R_1$  circle and  $k_2$  times around the  $R_2$  circle. In general these interfaces are not topological but for special values of the radii where the following condition is satisfied

$$k_1 R_1 = k_2 R_2 \tag{6.1}$$

the defects become topological. It was shown in [12] that the complicated behavior of the logarithmically divergent term of the entanglement entropy for the interface at the boundary of the entangling surface (1.3) simplifies and the entanglement entropy becomes

$$\mathcal{S} = \frac{1}{6} \ln \left( \frac{L}{\epsilon} \right) + \ln |k_1 k_2| \tag{6.2}$$

We can contrast this explicit expression to the one of symmetric entanglement entropy, the  $g_B$  factor for a general interface is given by [2, 3]

$$g_B = \sqrt{\frac{k_1^2 R_1^2 + k_2^2 R_2^2}{2R_1 R_2}} \tag{6.3}$$

and hence the symmetric entanglement entropy for the topological interface which satisfies (6.1) becomes

$$\mathcal{S}^{\text{symm}} = \frac{1}{3} \ln \left( \frac{L}{\epsilon} \right) + \frac{1}{2} \ln |k_1 k_2| \tag{6.4}$$

Where the constant part is half the value of the constant part of  $\mathcal{S}$ . Note that if we consider the case of an interface between identical CFTs, we have  $R_1 = R_2$  and by (6.1)  $k_1 = k_2$  for topological interfaces.

## 7 Remarks on entanglement entropies for Liouville theory

In [23–25] topological interfaces for the Liouville CFT (see [35, 36] for reviews with references to the original literature) were constructed following the procedure which was used

for RCFTs. There are two types of defects which are both of the form

$$I = \int_{Q/2+iP} d\alpha D(P)\mathcal{P}^\alpha \tag{7.1}$$

where we integrate  $P$  over the positive real line, i.e.  $P \in (0, \infty)$ , and one has  $Q = b + 1/b$ , which determines the central charge as  $C = 1+6Q^2$ . Here  $\mathcal{P}$  is a projector on the continuum of primary states labeled by  $P$  and their descendants.

$$\mathcal{P}^\alpha = \sum_{M,N} |\alpha, M\rangle \otimes |\overline{\alpha}, \overline{N}\rangle \langle \alpha, M| \otimes \langle \overline{\alpha}, \overline{N}| \tag{7.2}$$

As shown in [23] one can distinguish the two  $D$  by associating them with the discrete degenerate primary states labeled by two positive integers

$$D_{m,n}(P) = \frac{\sinh(\frac{2\pi m P}{b}) \sinh(2\pi n b P)}{\sinh(\frac{2\pi P}{b}) \sinh(2\pi b P)} \tag{7.3}$$

and a non-degenerate primary state labeled by a continuous real parameter  $s$

$$D_s(P) = \frac{\cos(4\pi P s)}{2 \sinh(2\pi b P) \sinh(\frac{2\pi P}{b})} \tag{7.4}$$

We can now calculate the  $K$ -sheeted partition function (3.3) with the interface (7.1) inserted. Using the fact that the projectors satisfy

$$\mathcal{P}^\alpha \mathcal{P}^\beta = \delta(\alpha - \beta) \mathcal{P}^\alpha \tag{7.5}$$

and the fact that the interface operator  $I$  satisfies (2.4) we arrive at

$$Z(K) = \int_{Q/2+iP} dP (D(P))^{2K} \chi_P(q^{2K}) \chi_P(\bar{q}^{2K}) \tag{7.6}$$

where  $\chi_P(q)$  is the character of the non-degenerate Liouville primary field labeled by  $P$  and is given by

$$\chi_P(\tau) = \frac{q^{P^2}}{q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)} \tag{7.7}$$

where  $q = e^{2\pi i \tau}$  and we can use the following formula for the modular transformation of the character (7.7)

$$\chi_P\left(-\frac{1}{\tau}\right) = \sqrt{2} \int_{-\infty}^{\infty} dP' \chi_{P'}(\tau) e^{4\pi i P P'} \tag{7.8}$$

With the identification (3.7) and  $t$  given by (3.4) as before, the modular transformed  $K$ -sheeted partition function becomes

$$Z(K) = 2 \int_0^\infty dP (D(P))^{2K} \int_{-\infty}^\infty dP' e^{4\pi i P P'} \chi_{P'}\left(i\frac{\pi}{Kt}\right) \int_{-\infty}^\infty d\bar{P}' e^{4\pi i P \bar{P}'} \chi_{\bar{P}'}\left(i\frac{\pi}{Kt}\right) \tag{7.9}$$

In the limit  $t \rightarrow 0$  we can replace the full character  $\chi_P(q)$  by its leading term  $q^{P^2 - \frac{1}{24}}$  and perform the gaussian integrals over  $P'$  and  $\bar{P}'$  which produce the same result. Hence we arrive at

$$Z(K) = \frac{Kt}{4\pi} e^{\frac{\pi^2}{6Kt}} \int_0^\infty dP (D(P))^{2K} e^{-4P^2 Kt} + \dots \quad (7.10)$$

where the dots denote terms which vanish as  $t$  goes to zero. We would now like to use this expression to calculate the entanglement entropy using the replica formula (3.2). Note that for the case where  $D$  is labeled by a continuous parameter  $s$  and given by (7.4)  $(D(P))^{2K}$  in the integral (7.10) vanishes for large  $P$ . It is therefore legitimate to drop the exponent  $e^{-4P^2 Kt}$  in the integral and the non vanishing terms in entanglement entropy for this case are given by

$$\mathcal{S}_s = \frac{1}{6} \ln \frac{L}{\epsilon} + (1 - \partial_K) \left( \ln \frac{Kt}{4\pi} \int_0^\infty dP (D_s(P))^{2K} \right) |_{K=1} \quad (7.11)$$

We notice two curious features of this result. First, the logarithmically divergent term is multiplied by  $\frac{1}{6}$  which is what one would expect for a  $c = 1$  CFT, whereas the central charge of the Liouville theory is given by  $C_L = 1 + 6Q^2$ . A possible explanation for this behavior lies in the fact that for the interface labelled by (7.4) only the continuous primaries with conformal dimension  $\Delta = Q^2/4 + P^2$  appear. Hence the vacuum with  $\Delta = 0$  is excluded and the factor of  $1/3$  in front of (7.11) is most likely associated with a shifted effective central charge.

Second, apart from finite terms as  $t \rightarrow 0$  we also obtain an additional divergent term of the form  $\ln(\ln \frac{L}{\epsilon})$  from the second term (7.11). The significance and interpretation of this term is not clear at this point and a more careful treatment of the cutoff might be necessary. For the interfaces labeled by discrete integers  $m, n$  defined in (7.3),  $D_{mn}$  diverges for large  $P$  and the full integral has to be evaluated first in order to obtain the entanglement entropy. We leave this problem for future work.

## 8 Discussion

In this paper we have discussed entanglement entropies in the presence of topological defects in two geometric settings, namely when the interface is located at the boundary of the entangling interval  $\mathcal{A}$  and when it is in the center of the entangling interval. For topological defects in RCFTs the logarithmic part of the entanglement entropy is always universal (this is not the case for general conformal interfaces) and the constant term can be expressed in a compact form in terms of the modular matrix  $S$ . Note that the entanglement entropies have a similar form in terms of the modular matrix  $S$  as the recently obtained left/right entanglement entropy for a related BCFT, but the physical relation of the left/right entanglement entropy to the others is not clear at the moment.

There are several directions in which our results can be generalized. We have limited ourselves to RCFTs with diagonal partition functions. The construction of [14] also includes non-diagonal theories and it would be interesting to understand the entanglement entropy for this case. We also only considered CFTs which are rational with respect to the Virasoro algebra, it would also be very interesting to repeat the analysis for RCFTs with respect to extended chiral algebras.

Since the large  $m$  limit of minimal models is conjectured to approach a non-rational  $c = 1$  CFT which is different from a free boson [37] it would be interesting to study the continuation of the minimal model entanglement entropy.

It would also be interesting to understand the entanglement entropy for Liouville theory better. Apart from the calculations sketched in section 7 one might also consider semiclassical limits where  $b \rightarrow 0$  and analyze the role of topological defects in classical Liouville theory following [25, 38].

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