

Second order effect of twist deformations in the D1D5 CFT

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ABSTRACT: Thermalization in the D1D5 CFT should occur via interactions caused by the twist operator, which deforms the theory off its free orbifold point. Earlier studies investigating this deformation at first order did not show any definite evidence of thermalization. In this paper we study the deformation to second order, where we do expect to see the effects that should give thermalization. We compute the effect of two twist operators on an initial vacuum state, which generates a squeezed state analogous to the case for a single twist. We obtain expressions for the Bogoliubov coefficients in this 2-twist case.

KEYWORDS: AdS-CFT Correspondence, Black Holes, Black Holes in String Theory, Conformal Field Models in String Theory

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1 Introduction

What happens when matter collapses to make a black hole? The details of this process are unclear, and potentially of great importance in obtaining a full understanding of quantum gravity. In string theory, this gravitational process can be put in a setting where we have a CFT dual [1–3], and we can try to study the collapse process in its dual field theory description.

The CFT dual is itself difficult to study, since the gravitational theory is dual to a strongly coupled CFT. Remarkably, one obtains many interesting results about black holes by using the CFT at its ‘free’ or ‘orbifold’ point [4–19]. At this coupling the CFT is given by several symmetrized copies of a free CFT. This symmetrization introduces twist operators σ_k into the theory. The free theory reproduces the entropy and greybody factors of near-extremal black holes [20–24], but it cannot describe their formation. The reason is that black hole formation is expected to be dual to a thermalization in the CFT, and in a free theory an initial excitation will not in general thermalize.

We are therefore led to explore the deformation of the theory away from the orbifold point. This deformation is given by the operator

$$\hat{O}_{\dot{A}\dot{B}}(w_0, \bar{w}_0) = \left[\frac{1}{2\pi i} \int_{w_0} dw G_{\dot{A}}^-(w) \right] \left[\frac{1}{2\pi i} \int_{\bar{w}_0} d\bar{w} \bar{G}_{\dot{B}}^-(\bar{w}) \right] \sigma_2^{++}(w_0, \bar{w}_0) \quad (1.1)$$

The index notations are explained in appendix A, but we note here the general structure of this operator. There is a twist σ_2 , which takes two copies of the free CFT and joins them together. If these copies were defined on circles of length $2\pi R$, the twist will give one copy of the CFT on a circle of length $4\pi R$. In addition to this twist we have the application of left and right supercharges G .

As explained in [25, 26], the G operators can be removed from the σ_2 by writing them as contours surrounding the σ_2 , and then stretching these contours away till they act on the initial or final states of the process. Thus a core part of the effect of the deformation operator is given by the action of the ‘bare twist’ σ_2 . In [26] it was found that if we apply one operator $\sigma_2(w)$ to the vacuum $|0\rangle$, the resulting state is a squeezed state, with the schematic form

$$e^{\gamma_{mn}^B \alpha_{-m} \alpha_{-n} + \gamma_{rs}^F d_{-r} d_{-s}} |0\rangle \quad (1.2)$$

The coefficients γ^B, γ^F were computed in [26] for the case where the initial copies of the CFT lived on singly twisted circles. In [27] they were computed by an alternative method that made their basic properties manifest. In [28] they were computed for the case where the initial strings had arbitrary order twists M, N , but where $m, n \gg 1$. In [29, 30] they were computed for arbitrary M, N and arbitrary m, n . In [31, 32] the mixing of operators under the twist deformations was analyzed. All of these computations addressed the case of a single σ_2 insertion, but it was not clear that thermalization can be observed at this order.

In the present paper we perform a 1-loop computation, where we have *two* twist insertions: the first twist joins two untwisted copies of the CFT to a copy living on a double circle, and the second twist returns us to the two untwisted copies. We argue that starting from the vacuum state $|0\rangle$ we again get a squeezed state of the form (1.2), and we compute the required γ^B, γ^F . We find that these coefficients can be expressed as a finite sum over a bilinear product of hypergeometric functions.

One apparently puzzling feature of our computations is the role of sectors with new periodicities for fermions. We start with both copies of the CFT in the Ramond (R) sector. After the application of both σ_2 factors we are left with a contribution where both CFTs are in the R sector, but we can also obtain a nonzero result for the amplitude if we let both the final copies of the CFT be in the NS sector. As we will explain later, we believe that the NS sectors should not be included as possible final states. We still compute the amplitude to transition to these sectors, but we will drop this contribution from our final analysis.

The plan of this paper is as follows. In section 2, we introduce the orbifold CFT. In section 3, we lay out the calculation for determining the state $|\chi(w_1, w_2)\rangle$. This state contains three major components. In section 4 we calculate the coefficients in the bosonic component. In section 5 we calculate the coefficients for the fermionic component in the NS sector. In section 6 we calculate the coefficients for the fermionic component in the Ramond sector. In section 7, we argue that the contribution from the NS sector is nonphysical. In section 8.2, we perform some numerical analyses on the R sector results to obtain some simplifying limits.

2 The D1D5 CFT at the orbifold point

In this section we summarize some properties of the D1D5 CFT at the orbifold point and the deformation operator that we will use to perturb away from the orbifold point. For more details, see [26].

2.1 The D1D5 CFT

Consider type IIB string theory, compactified as

$$M_{9,1} \rightarrow M_{4,1} \times S^1 \times T^4. \tag{2.1}$$

Wrap N_1 D1 branes on S^1 , and N_5 D5 branes on $S^1 \times T^4$. We think of the S^1 as being large compared to the T^4 , so that at low energies we look for excitations only in the direction S^1 . This low energy limit gives a conformal field theory (CFT) on the circle S^1 .

We can now vary the moduli of string theory (the string coupling g , the shape and size of the torus, the values of flat connections for gauge fields etc.). These changes move us to different points in the moduli space of the CFT. It has been conjectured that we can move to a point called the ‘orbifold point’ where the CFT is particularly simple [10, 11]. At this orbifold point the CFT is a 1+1 dimensional sigma model. We will work in the Euclidean theory, where the base space is a cylinder spanned by the coordinates

$$\tau, \sigma : \quad 0 \leq \sigma < 2\pi, \quad -\infty < \tau < \infty \tag{2.2}$$

The target space of this sigma model is the symmetrized product of $N_1 N_5$ copies of T^4 ,

$$(T_4)^{N_1 N_5} / S_{N_1 N_5}, \tag{2.3}$$

with each copy of T^4 giving 4 bosonic excitations X^1, X^2, X^3, X^4 . It also gives 4 fermionic excitations, which we call $\psi^1, \psi^2, \psi^3, \psi^4$ for the left movers, and $\bar{\psi}^1, \bar{\psi}^2, \bar{\psi}^3, \bar{\psi}^4$ for the right movers. The central charge of the theory with fields $X^i, \psi^i, i = 1 \dots 4$ is $c = 6$. The total central charge of the entire system is thus $6N_1 N_5$.

In [26] it was noted that we can write the deformation operator as

$$\hat{O}_{\dot{A}\dot{B}}(w_0) = \left[\frac{1}{2\pi i} \int_{w_0} dw G_{\dot{A}}^-(w) \right] \left[\frac{1}{2\pi i} \int_{\bar{w}_0} d\bar{w} \bar{G}_{\dot{B}}^-(\bar{w}) \right] \sigma_2^{++}(w_0). \tag{2.4}$$

We will, however, work only with the left-moving part of (2.4). The right-moving sector is fully analogous.

2.2 NS and R vacua

Consider a single copy of the $c = 6$ CFT. The lowest energy state of the left-moving sector is the NS vacuum,

$$|0_{NS}\rangle, \quad h = 0, \quad m = 0 \tag{2.5}$$

where h denotes the L_0 eigenvalue. However, we will for the most part study the CFT in the R sector. In particular we are interested in the R vacua denoted by¹

$$|0_R^\pm\rangle, \quad h = \frac{1}{4}, \quad m = \pm \frac{1}{2}. \tag{2.6}$$

We can relate the NS and R sectors using spectral flow [33]. In particular, spectral flow by $\alpha = \pm 1$ in the left-moving sector produces the transformations

$$\begin{aligned} \alpha = 1 : \quad & |0_R^-\rangle \rightarrow |0_{NS}\rangle, \quad |0_{NS}\rangle \rightarrow |0_R^+\rangle \\ \alpha = -1 : \quad & |0_R^+\rangle \rightarrow |0_{NS}\rangle, \quad |0_{NS}\rangle \rightarrow |0_R^-\rangle. \end{aligned} \tag{2.7}$$

¹There are two other left-moving R ground states, which may be obtained by acting with fermion zero modes. Including right-movers then gives a total of 16 R-R ground states. These R-R ground states are described in detail in appendix B.

3 Outline of the calculations

The deformation operator is composed of a supercharge contour acting on a twist operator. The action of the supercharge contour can be split off from the main computation, which involves the effects of the twist operators. In this paper we will focus of the effect of the twist operators alone.

Since we are looking at deformations to second order, we will have two twist-2 operators acting on our initial state. If n copies of the CFT are twisted together, we say that we have an n -times wound ‘component string’. We will start with the simplest case of two singly wound component strings. The twist from the first deformation operator will change this to a single component string with winding 2, while the second twist will take this new component string back to two component strings of winding 1 each. Thus we are looking at a ‘1-loop’ process in the interacting CFT. This process is depicted in figure 1. We believe that such second order effects will lead to the thermalization that we seek.

Each of the initial component strings are taken to be in the negative Ramond vacuum state $|0_R^-\rangle$. The final state after the application of two twists will be denoted by $|\chi(w_1)(w_2)\rangle$. Thus:

$$|\chi(w_1, w_2)\rangle \equiv \sigma_2^+(w_2)\sigma_2^+(w_1)|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)}. \tag{3.1}$$

Here w is a coordinate on the cylinder, with

$$w = \tau + i\sigma, \tag{3.2}$$

where σ is an angular coordinate for the compact spacial dimension and τ is a Euclideanized time coordinate. In (3.1), we assume that $\tau_2 > \tau_1$.

In the region $\tau_1 < \tau < \tau_2$, the two component strings are joined together to form a doubly wound componnet string. The bosonic fields are now periodic only after an interval $\Delta\sigma = 4\pi$. The fermionic fields, a priori, can be periodic or antiperiodic after $\Delta\sigma = 4\pi$. Outside the above interval of τ we have two singly wound component strings. We will now map this configuration to a double cover of the cylinder. To do this, we first map the cylinder into the complex plain with coordinate z and then map the complex plane to a double cover of itself.

Around any point z on the plane, a fermionic field can have either periodic or antiperiodic boundary conditions. If we insert only local operators at z , then the fermion will be periodic (i.e. it will have NS boundary conditions). If we wish to have instead the anti-periodic boundary condition (i.e., the R boundary condition) then we must insert a spin field S^\pm at the point z . The spin field carries a charge $j = \pm\frac{1}{2}$ and has dimension $h = \frac{1}{4}$.

The initial state (the ‘in’ state) has two singly wound component strings, each in the R sector. This brings in two spin fields, one from each component string. We write

$$|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} = S^{(1)-}(\tau = -\infty)S^{(2)-}(\tau = -\infty)|0_{NS}\rangle^{(1)}|0_{NS}\rangle^{(2)}. \tag{3.3}$$

In addition, the twist operator σ_2^+ contains a spin field:

$$\sigma_2^+(w) = S^+(w)\sigma_2(w). \tag{3.4}$$

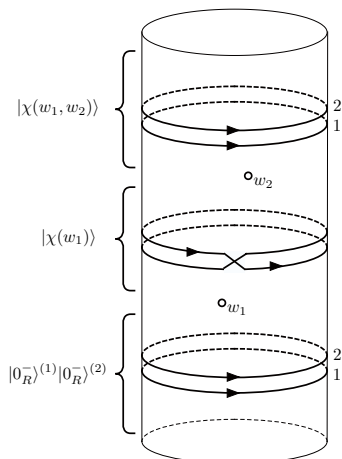


Figure 1. The cylinder with twist insertions at w_1 and w_2 . Below the first twist we have the negative Ramond vacuum on each component string. Above both twists we have the state $|\chi(w_1, w_2)\rangle$, which we will compute. In the intermediate regions we have a single doubly-wound component string in the state $|\chi(w_1, w_2)\rangle$. This state was computed in [25] and is not used here.

We will compute an amplitude by taking the inner product of the state we generate after the twists with some state of our choosing. This state with which we ‘cap’ the cylinder may then bring additional spin fields of its own.

Each of the above mentioned spin fields will have some position in our double cover. We can remove a spin field by performing a spectral flow around that point, since a spectral flow can map an R sector state to an NS sector state. We will thus perform a series of spectral flows at various points in the double cover in order to remove these spin field insertions. Once all the spin fields have been removed, the locations where they were inserted will have just the local NS vacuum inserted there, and we can close the corresponding puncture at that location with no insertions. Any contour can then be smoothly deformed through such a location. It is through these smooth deformations that we will be able to map in states to out states and determine the nature of $|\chi(w_1, w_2)\rangle$

We now divide the remainder of this section into four parts. In the first part, we outline the coordinate changes used to map the cylinder into a double cover of the complex plane and identify the images of all critical points. In the second part, we introduce the mode operators on the cylinder. In the third part, we present the general form of the state $|\chi(w_1, w_2)\rangle$ in terms of these mode operators. This general form motivates capping with certain types of states, which allows us to derive expressions for the parameters of our ansatz. Finally, we determine the spectral flows needed for each type of capping state based on the images of the spin fields of those states.

3.1 Coordinate maps

The effects of our coordinate maps are illustrated in figure 2. First we map the cylinder to the complex plane through the map:

$$z = e^w \equiv e^{\tau+i\sigma}. \tag{3.5}$$

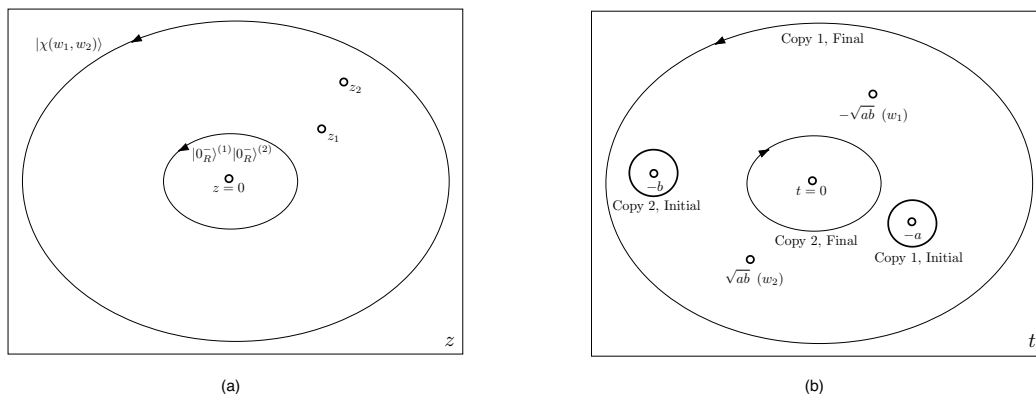


Figure 2. The z plane (a) and t plane (b) with all the relevant image points labeled. The intermediate state $|\chi(w_1)\rangle$ is not depicted. In the z plane the τ coordinate maps to the radial coordinate, while the σ coordinate maps to the phase. In the t plane there are no simple directions corresponding to τ and σ .

Here the in states at $\tau = -\infty$ map to the origin of the complex plane, while the out states at $\tau = +\infty$ map to $z = \infty$. Our fields are still double-valued near these points, as there are two component strings in both the in and out states. The timelike direction τ becomes the radial direction, while the spacelike σ becomes the phase of the z plane. Because $\tau_2 > \tau_1$, we have:

$$|z_2| = e^{\tau_2} > |z_1| = e^{\tau_1}. \tag{3.6}$$

Now we map the z plane into a double cover of itself. We need to ensure that this map will separate out the two distinct copies of our fields for the in and out states. Thus we require that $z = 0$ and $z = \infty$ both have two distinct image points. It is also useful to ensure that near these points we have leading order relations of the form $z \sim t^{\pm 1}$. Such a map can in general be written as:

$$z = \frac{(t+a)(t+b)}{t}. \tag{3.7}$$

Here, the point $z = \infty$ maps to both $t = \infty$ and $t = 0$. Near $t = \infty$ we have the leading order behavior $z \sim t$, while near $t = 0$ we have the behavior $z \sim t^{-1}$. The point $z = 0$ also has two images, at $t = -a$ and $t = -b$. This allows us to split up the images of the two copies for our in and out states, thus leaving us with only single-valued fields in the t plane.

At this point, we have some choice of which component strings map to which regions in the t plane. The in state has images at $t = -a$ and $t = -b$, but the physics is independent of which component string we call copy 1 and which we call copy 2. Indeed, the map (3.7) is symmetric under the interchange $a \leftrightarrow b$. Similarly, we may choose which component string of the out state maps to $t = \infty$ and which component string maps to $t = 0$. Here

we present the conventions used in this paper:

$$\begin{aligned}
 \text{Copy 1 In State} &\rightarrow t = -a \\
 \text{Copy 2 In State} &\rightarrow t = -b \\
 \text{Copy 1 Out State} &\rightarrow t = \infty \\
 \text{Copy 2 Out State} &\rightarrow t = 0.
 \end{aligned}
 \tag{3.8}$$

We also need to know the images of the twist insertions at w_1 and w_2 . These insertions create bifurcation points in the double cover, so we find the images by solving for these bifurcation points.

$$\frac{dz}{dt} = 1 - \frac{ab}{t^2} = 0.
 \tag{3.9}$$

This relation has two solutions, corresponding to our two twist insertions.

$$\begin{aligned}
 t_1 &= -\sqrt{ab} \\
 t_2 &= \sqrt{ab}.
 \end{aligned}
 \tag{3.10}$$

This in turn gives us a relationship between the double cover map parameters a, b and our twist insertion points w_1, w_2 :

$$\begin{aligned}
 z_1 &= a + b - 2\sqrt{ab} = e^{w_1} \\
 z_2 &= a + b + 2\sqrt{ab} = e^{w_2},
 \end{aligned}
 \tag{3.11}$$

where we choose the branch of \sqrt{ab} so as to maintain $|z_2| > |z_1|$. As expected, this relationship is unaffected by the interchange of a and b .

3.2 Modes on the cylinder

In order to determine the final state $|\chi(w_1, w_2)\rangle$ we first introduce the bosonic and fermionic modes that live on the cylinder in the region $\tau > \tau_2$. For the bosons, we have:

$$\alpha_{A\dot{A},n}^{(i)f} = \frac{1}{2\pi} \int_{\sigma=0}^{2\pi} X_{A\dot{A}}^{(i)}(w) e^{nw} dw,
 \tag{3.12}$$

where f stands for final modes, indicating that we are working in the region above both twists. The commutation relations are

$$\left[\alpha_{A\dot{A},n}^{(i)f}, \alpha_{B\dot{B},m}^{(j)f} \right] = -n \epsilon_{AB} \epsilon_{\dot{A}\dot{B}} \delta^{ij} \delta_{n+m,0}.
 \tag{3.13}$$

For fermions, we can have two different types of modes depending on which sector we are in, NS or R. In the NS sector, the modes are indexed by half-integers r :

$$d_r^{(i)f,\alpha A} = \frac{1}{2\pi i} \int_{\sigma=0}^{2\pi} \psi^{(i)\alpha A}(w) e^{rw} dw,
 \tag{3.14}$$

with anticommutation relations

$$\left\{ d_r^{(i)f,\alpha A}, d_s^{(j)f,\beta B} \right\} = -\epsilon^{\alpha\beta} \epsilon^{AB} \delta^{(i)(j)} \delta_{r+s,0} . \quad (3.15)$$

Similarly, the modes in the Ramond sector are:

$$d_n^{(i)f,\alpha A} = \frac{1}{2\pi i} \int_{\sigma=0}^{2\pi} \psi^{(i)\alpha A}(w) e^{nw} dw, \quad (3.16)$$

with integer n . These modes have anticommutation relations

$$\left\{ d_n^{(i)f,\alpha A}, d_m^{(j)f,\beta B} \right\} = -\epsilon^{\alpha\beta} \epsilon^{AB} \delta^{(i)(j)} \delta_{n+m,0} . \quad (3.17)$$

In the Ramond sector there are fermion zero modes. While bosonic zero modes annihilate all vacua, the fermionic zero modes do not always annihilate the vacuum. Instead, one finds:

$$d_0^{(i)f,+A} |0_R^+\rangle^{(i)} = d_0^{(i)f,-A} |0_R^-\rangle^{(i)} = 0 \quad (3.18)$$

$$d_0^{(i)f,+A} |0_R^-\rangle^{(i)} \neq 0, \quad d_0^{(i)f,-A} |0_R^+\rangle^{(i)} \neq 0, \quad (3.19)$$

where the copy index (i) is not summed over. For more details about the behavior of these zero modes, see appendix B.

One can of course construct modes which live before the two twist insertions, as well as modes which live between the two twists. Since we begin with the vacuum state as our in state before the twists and we do not need the state between the two twists in an explicit way, we will not write the modes in these regions. We do however require modes natural to the NS vacuum in the t plane. These are

$$\tilde{\alpha}_{A\dot{A},n} = \frac{1}{2\pi} \oint_{t=0} \partial_t X_{A\dot{A}}(t) t^n dt \quad (3.20)$$

$$\tilde{d}_r^{\alpha A} = \frac{1}{2\pi i} \oint_{t=0} \psi^{\alpha A}(t) t^{r-\frac{1}{2}} dt, \quad (3.21)$$

with commutation relations

$$\left[\tilde{\alpha}_{A\dot{A},m}, \tilde{\alpha}_{A\dot{A},n} \right] = -\epsilon_{AB} \epsilon_{\dot{A}\dot{B}} m \delta_{n+m,0} \quad (3.22)$$

$$\left\{ \tilde{d}_m^{\alpha A}, \tilde{d}_n^{\alpha A} \right\} = -\epsilon^{\alpha\beta} \epsilon^{AB} \delta_{n+m,0} . \quad (3.23)$$

As a final remark, there is a subtle concern that needs to be addressed when we wish to map the cylinder modes into the t plane. Since we have chosen the final copy 1 to map to large t , copy 1 modes must always come to the left of copy 2 modes in the t plane. As such, when we apply annihilation modes to probe the deformed state we will always choose to place all copy 1 modes to the left of all copy 2 modes. This has no effect on the bosonic calculation because bosonic annihilators commute, but it is important to account for this convention when working with fermions.

3.3 The general form

Looking at the $SU(2)$ charges of the twist operators and the initial vacuum state, it is clear that $|\chi(w_1, w_2)\rangle$ is overall neutral. One may also expect the final state to contain only pairs of excitations in the form of an exponential similar to the one-twist result of [25]. However, this cursory analysis seems to indicate that part of $|\chi(w_1, w_2)\rangle$ could live in the NS sector, as the vacuum $|0_{NS}\rangle^{(1)}|0_{NS}\rangle^{(2)}$ is neutral. We will perform the computations for both R and NS sectors in the final state, but argue at the end that only the R sector amplitudes are relevant for the physical system under study.

In [25] it was noted that the state produced after the twist had an exponential form, and a general argument was given for why this must be the case for the bosons of the theory. This proof extends trivially to the use of arbitrarily many twists. In appendix C we present a similar proof for the fermionic contribution, which again extends to arbitrarily many twists and also applies to both the NS and Ramond sectors. Keeping both sectors in the final state for the moment, we note that $|\chi(w_1, w_2)\rangle$ can be written as:

$$\begin{aligned}
 |\chi(w_1, w_2)\rangle &= C_{NS}(w_1, w_2) \exp \left[\sum_{(i),(j)} \sum_{k,l>0} \gamma_{kl}^{B(i),(j)} \left(-\alpha_{++,-k}^{(i)f} \alpha_{--,-l}^{(j)f} + \alpha_{+,-,-k}^{(i)f} \alpha_{-+,-l}^{(j)f} \right) \right] \\
 &\times \exp \left[\sum_{(i),(j)} \sum_{r,s>0} \gamma_{NS,rs}^{F(i),(j)} \left(d_{-r}^{(i)f,++} d_{-s}^{(j)f,--} - d_{-r}^{(i)f,+} d_{-s}^{(j)f,-} \right) \right] |0_{NS}\rangle^{(1)} |0_{NS}\rangle^{(2)} \\
 &+ C_{R+-}(w_1, w_2) \exp \left[\sum_{(i),(j)} \sum_{k,l>0} \gamma_{kl}^{B(i),(j)} \left(-\alpha_{++,-k}^{(i)f} \alpha_{--,-l}^{(j)f} + \alpha_{+,-,-k}^{(i)f} \alpha_{-+,-l}^{(j)f} \right) \right] \\
 &\times \exp \left[\sum_{(i),(j)} \sum_{k,l\geq 0} \gamma_{R+-,kl}^{F(i),(j)} \left(d_{-k}^{(i)f,++} d_{-l}^{(j)f,--} - d_{-k}^{(i)f,+} d_{-l}^{(j)f,-} \right) \right] |0_R^+\rangle^{(1)} |0_R^-\rangle^{(2)},
 \end{aligned} \tag{3.24}$$

where the sums over the mode indices for the fermions in the Ramond sector include zero modes only when those modes do not annihilate the vacuum $|0_R^+\rangle^{(1)}|0_R^-\rangle^{(2)}$. Note that the various γ coefficients also depend on the twist insertion points w_1 and w_2 .

From here it is fairly straightforward to apply particular capping states with only a single pair of modes so as to pick out each specific γ coefficient from $|\chi(w_1, w_2)\rangle$. Taking a ratio of amplitudes, we can isolate these coefficients in a manner that is independent of other factors, such as the overall coefficients C . We thus find:

$$\gamma_{mn}^{B(i)(1)} = -\frac{1}{mn} \frac{{}^{(1)}\langle 0_{NS} | {}^{(2)}\langle 0_{NS} | \alpha_{++,-n}^{(1)f} \alpha_{--,-m}^{(i)f} |\chi(w_1, w_2)\rangle}{{}^{(1)}\langle 0_{NS} | {}^{(2)}\langle 0_{NS} | \chi(w_1, w_2)\rangle} \tag{3.25}$$

$$\gamma_{mn}^{B(i)(2)} = -\frac{1}{mn} \frac{{}^{(1)}\langle 0_{NS} | {}^{(2)}\langle 0_{NS} | \alpha_{--,-m}^{(i)f} \alpha_{++,-n}^{(2)f} |\chi(w_1, w_2)\rangle}{{}^{(1)}\langle 0_{NS} | {}^{(2)}\langle 0_{NS} | \chi(w_1, w_2)\rangle} \tag{3.26}$$

$$\gamma_{NS,rs}^{F(i)(1)} = \frac{{}^{(1)}\langle 0_{NS} | {}^{(2)}\langle 0_{NS} | d_r^{(1)f,++} d_s^{(i)f,--} |\chi(w_1, w_2)\rangle}{{}^{(1)}\langle 0_{NS} | {}^{(2)}\langle 0_{NS} | \chi(w_1, w_2)\rangle} \tag{3.27}$$

$$\gamma_{NS,rs}^{F(i)(2)} = -\frac{{}^{(1)}\langle 0_{NS} | {}^{(2)}\langle 0_{NS} | d_s^{(i)f,-} d_r^{(2)f,+} | \chi(w_1, w_2) \rangle}{({}^{(1)}\langle 0_{NS} | {}^{(2)}\langle 0_{NS} | \chi(w_1, w_2) \rangle)} \quad (3.28)$$

$$\gamma_{R+-,mn}^{F(i)(1)} = \frac{{}^{(1)}\langle 0_{R,+} | {}^{(2)}\langle 0_{R,-} | d_n^{(1)f,+} d_m^{(i)f,-} | \chi(w_1, w_2) \rangle}{({}^{(1)}\langle 0_{R,+} | {}^{(2)}\langle 0_{R,-} | \chi(w_1, w_2) \rangle)} \quad (3.29)$$

$$\gamma_{R+-,mn}^{F(i)(2)} = -\frac{{}^{(1)}\langle 0_{R,+} | {}^{(2)}\langle 0_{R,-} | d_m^{(i)f,-} d_n^{(2)f,+} | \chi(w_1, w_2) \rangle}{({}^{(1)}\langle 0_{R,+} | {}^{(2)}\langle 0_{R,-} | \chi(w_1, w_2) \rangle)}, \quad (3.30)$$

where the minus signs in the fourth and sixth lines come from the fact that the fermion annihilators anticommute. Note that the bosonic coefficients do not depend on the vacuum the state is built upon. This is because the bosonic modes are unaffected by spectral flow, so they behave identically in both sectors. Indeed, one can calculate these same bosonic coefficients by capping with a Ramond state instead.

From these expressions, we must perform the series of coordinate maps outlined in section 3.1. This procedure removes all of the multivalued regions, though the Ramond vacua and twist insertions still bring spin fields. We deal with these spin fields in the next subsection.

3.4 Spectral flows

We have two sectors we wish to cap with, the NS sector and the R sector. The behavior of the bosonic modes is independent of the sector they act on, and as such we do not have to worry about which spectral flows we perform when it comes to the bosonic coefficients. Thus we need only apply the coordinate maps to the right side of (3.25) and (3.26). Doing so yields:

$$\gamma_{mn}^{B(i)(1)} = -\frac{1}{mn} \frac{{}_t\langle 0_{NS} | \alpha'_{++n} \alpha'_{--m} | 0_{NS} \rangle_t}{{}_t\langle 0_{NS} | 0_{NS} \rangle} \quad (3.31)$$

$$\gamma_{mn}^{B(i)(2)} = -\frac{1}{mn} \frac{{}_t\langle 0_{NS} | \alpha'_{--m} \alpha'_{++n} | 0_{NS} \rangle_t}{{}_t\langle 0_{NS} | 0_{NS} \rangle}, \quad (3.32)$$

where the primes denote that the modes have been altered by the coordinate shifts.

For the fermionic coefficients, it is important to specify which sector we are building upon because the Ramond sector brings in additional spin fields. Both cases are illustrated in figure 3.

When building upon the NS sector, the t -plane contains the following spin fields:

$$\begin{aligned} S^-(-a) & \text{ from } |0_R^- \rangle^{(1)} \\ S^-(-b) & \text{ from } |0_R^- \rangle^{(2)} \\ S^+(-\sqrt{ab}) & \text{ from } \sigma_2^+(w_1) \\ S^+(\sqrt{ab}) & \text{ from } \sigma_2^+(w_2). \end{aligned} \quad (3.33)$$

To remove these spin fields we must perform spectral flows at each of these points. We spectral flow by $\alpha = +1$ at the locations of the S^- spin fields, and by $\alpha = -1$ at the locations of the S^+ spin fields. Since this combination contains two spectral flows in each

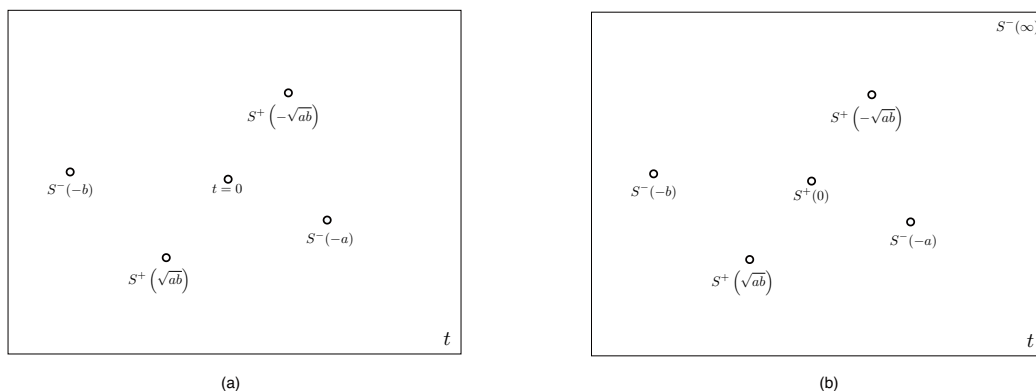


Figure 3. Here we illustrate the insertions of all the spin fields in the t plane when capping with the NS (a) and Ramond (b) sectors.

direction all at finite points, there is no net effect at infinity. This is important as we do not have any insertions at infinity. After these spectral flows we can close all punctures in the t plane, allowing us to smoothly deform the contours of the fermion modes. Recalling (3.27) and (3.28), we find:

$$\gamma_{NS,rs}^{F(i)(1)} = \frac{{}_t\langle 0_{NS} | d_s'^{(1)f,++} d_r'^{(i)f,--} | 0_{NS} \rangle_t}{{}_t\langle 0_{NS} | 0_{NS} \rangle} \quad (3.34)$$

$$\gamma_{NS,rs}^{F(i)(2)} = -\frac{{}_t\langle 0_{NS} | d_r'^{(i)f,--} d_s'^{(2)f,++} | 0_{NS} \rangle_t}{{}_t\langle 0_{NS} | 0_{NS} \rangle}, \quad (3.35)$$

where here the primes denote that the modes have been altered by both the coordinate shifts and the four spectral flows given in (3.33).

When building upon the Ramond sector, we also have spin fields from the capping state ${}^{(1)}\langle 0_{R,+} | {}^{(2)}\langle 0_{R,-} |$. Recall that for out states, copy 1 maps to $t = \infty$ while copy 2 maps to the origin. Also note that the lower-index charge indicated in the bra denotes that it is the conjugate of the ket of that charge, and thus has the opposite charge. We now see that in addition to the spin fields encountered when building upon the NS vacuum, we also have:

$$\begin{aligned} S^-(\infty) & \text{ from } {}^{(1)}\langle 0_{R,+} | \\ S^+(0) & \text{ from } {}^{(2)}\langle 0_{R,-} | \end{aligned} \quad (3.36)$$

Both of these spin fields can be removed with a single spectral flow by $\alpha = -1$ units at the origin of the t plane. Recalling (3.29) and (3.30), we find:

$$\gamma_{R+-,mn}^{F(i)(1)} = \frac{{}_t\langle 0_{NS} | \hat{d}_n'^{(1)f,++} \hat{d}_m'^{(i)f,--} | 0_{NS} \rangle_t}{{}_t\langle 0_{NS} | 0_{NS} \rangle} \quad (3.37)$$

$$\gamma_{R+-,mn}^{F(i)(2)} = -\frac{{}_t\langle 0_{NS} | \hat{d}_m'^{(i)f,--} \hat{d}_n'^{(2)f,++} | 0_{NS} \rangle_t}{{}_t\langle 0_{NS} | 0_{NS} \rangle}, \quad (3.38)$$

where the hats indicate that we have performed all of the coordinate maps and spectral flows required by (3.33) along with an additional spectral flow by $\alpha = -1$ units at the origin of the t plane.

We now have useful expressions for the various γ coefficients. By applying the required coordinate shifts and spectral flows to the bosonic and fermionic modes and expanding the results in terms of modes natural to the t plane, we can use the known commutation relations to evaluate equations (3.31) through (3.38). We perform these calculations in the following sections.

4 Computation of $\gamma_{mn}^{B(i)(j)}$

Here we calculate the bosonic γ coefficients using the relations (3.31) and (3.32). To use these expressions, we first need to compute the α' modes in the t plane. Since the bosonic modes are unaffected by spectral flow, we need only apply the coordinate maps to the original modes on the cylinder. We thus organize this section into three parts. In the first part, we will apply the coordinate maps outlined in section 3.1 to determine the expressions for the α' modes. In the second part, we will expand the α' modes in terms of the modes natural to the t plane. In the final part, we will use these expressions in (3.31) and (3.32) to compute the bosonic γ coefficients.

4.1 Mapping the boson modes to the t plane

We wish to apply the various coordinate maps presented in section 3.1 to the post-twist modes on the cylinder given in (3.12). Under a general coordinate transformation $w \rightarrow w'$, the bosonic field X transforms as:

$$X_{AA}^{(i)}(w) \rightarrow \frac{dw'}{dw} X_{AA}^{(i)}(w'). \tag{4.1}$$

However, the jacobian brings a factor:

$$dw \rightarrow \left(\frac{dw'}{dw}\right)^{-1} dw'. \tag{4.2}$$

Thus the combination of the two behaves as:

$$X_{AA}^{(i)}(w) dw \rightarrow X_{AA}^{(i)}(w') dw'. \tag{4.3}$$

We first map the cylinder with coordinate w to the plane with coordinate z via

$$z = e^w. \tag{4.4}$$

Since we do not make use of the initial or intermediate modes in this paper, we just write the modes after the second twist insertion ($|z| > e^{\tau_2}$). Here we have a contour circling $z = \infty$:

$$\alpha_{AA,n}^{(1)f} \rightarrow \frac{1}{2\pi} \oint_{z=\infty} X_{AA}^{(1)f}(z) z^n dz \tag{4.5}$$

$$\alpha_{AA,n}^{(2)f} \rightarrow \frac{1}{2\pi} \oint_{z=\infty} X_{AA}^{(2)f}(z) z^n dz. \tag{4.6}$$

We now proceed to the covering space t , where X_{AA} will be single-valued. Using the map defined earlier,

$$z = \frac{(t+a)(t+b)}{t}, \tag{4.7}$$

and again only considering the modes after the second twist insertion, we have

$$\alpha_{AA,n}^{(1)f} \rightarrow \alpha'_{AA,n}{}^{(1)f} = \frac{1}{2\pi} \oint_{t=\infty} X_{AA}(t) \left(\frac{(t+a)(t+b)}{t} \right)^n dt \quad (4.8)$$

$$\alpha_{AA,n}^{(2)f} \rightarrow \alpha'_{AA,n}{}^{(2)f} = -\frac{1}{2\pi} \oint_{t=0} X_{AA}(t) \left(\frac{(t+a)(t+b)}{t} \right)^n dt. \quad (4.9)$$

Note the minus sign in (4.9). As mentioned earlier, the copy 2 values at $z = \infty$ map to $t = 0$. This map has the leading-order behavior $z \sim \frac{ab}{t}$, which means $arg(z) \sim -arg(t)$. Thus the contour changes direction when we map to the origin of the t plane. This is the source of the minus sign.

4.2 Expanding the bosonic modes

In order to expand our α' modes in terms of $\tilde{\alpha}$, it suffices to expand the integrand of the former in powers of t . This expansion must be performed in the region where the mode lives, $t \sim \infty$ for the copy 1 modes and $t \sim 0$ for the copy 2 modes. We perform the relevant expansions for each copy below.

Copy 1, $t \sim \infty$

$$\begin{aligned} \left(\frac{(t+a)(t+b)}{t} \right)^n &= t^n (1+at^{-1})^n (1+bt^{-1})^n \\ &= t^n \sum_{j,j' \geq 0} {}^n C_j {}^n C_{j'} a^j b^{j'} t^{-j-j'} \\ &= \sum_{j,j' \geq 0} {}^n C_j {}^n C_{j'} a^j b^{j'} t^{n-j-j'}. \end{aligned} \quad (4.10)$$

Copy 2, $t \sim 0$

$$\begin{aligned} \left(\frac{(t+a)(t+b)}{t} \right)^n &= t^{-n} a^n b^n (1+ta^{-1})^n (1+tb^{-1})^n \\ &= t^{-n} a^n b^n \sum_{j,j' \geq 0} {}^n C_j {}^n C_{j'} a^{-j} b^{-j'} t^{j+j'} \\ &= \sum_{j,j' \geq 0} {}^n C_j {}^n C_{j'} a^{n-j} b^{n-j'} t^{j+j'-n}. \end{aligned} \quad (4.11)$$

Thus (4.8) and (4.9) become

$$\begin{aligned} \alpha'_{AA,n}{}^{(1)f} &= \sum_{j,j' \geq 0} {}^n C_j {}^n C_{j'} a^j b^{j'} \frac{1}{2\pi} \oint_{t=\infty} X_{AA}(t) t^{n-j-j'} dt \\ &= \sum_{j,j' \geq 0} {}^n C_j {}^n C_{j'} a^j b^{j'} \tilde{\alpha}_{AA,n-j-j'} \end{aligned} \quad (4.12)$$

$$\begin{aligned} \alpha'_{AA,n}{}^{(2)f} &= - \sum_{j,j' \geq 0} {}^n C_j {}^n C_{j'} a^{n-j} b^{n-j'} \frac{1}{2\pi} \oint_{t=\infty} X_{AA}(t) t^{j+j'-n} dt \\ &= - \sum_{j,j' \geq 0} {}^n C_j {}^n C_{j'} a^{n-j} b^{n-j'} \tilde{\alpha}_{AA,j+j'-n}. \end{aligned} \quad (4.13)$$

4.3 Computing $\gamma_{mn}^{B(i)(j)}$

In this section we compute the $\gamma_{mn}^{B(i)(j)}$ coefficients. From (3.25) and (4.12) we find:

$$\gamma_{mn}^{B(1)(1)} = -\frac{1}{mn} \sum_{j,j',k,k' \geq 0} {}^n C_j^n {}^n C_{j'}^m {}^m C_k^m {}^m C_{k'}^m a^{j+k} b^{j'+k'} \frac{t \langle 0_{NS} | \tilde{\alpha}_{++}^{n-j-j'} \tilde{\alpha}_{--}^{m-k-k'} | 0_{NS} \rangle_t}{t \langle 0_{NS} | 0_{NS} \rangle}. \quad (4.14)$$

It is clear that the summand vanishes except when

$$\begin{aligned} m+n-j-j'-k-k' = 0 &\implies k' = m+n-j-j'-k \\ n-j-j' > 0 &\implies n-j > j'. \end{aligned} \quad (4.15)$$

Furthermore,

$$\begin{aligned} j' \geq 0 &\implies n > j \\ k' \geq 0 &\implies m+n-j-j' \geq k. \end{aligned} \quad (4.16)$$

We can thus rewrite (4.14) as

$$\gamma_{mn}^{B(1)(1)} = \frac{1}{mn} \sum_{j=0}^{n-1} \sum_{j'=0}^{n-j-1} \sum_{k=0}^{m+n-j-j'} (n-j-j')^n {}^n C_j^n {}^n C_{j'}^m {}^m C_k^m {}^m C_{m+n-j-j'-k}^m a^{j+k} b^{m+n-j-k}. \quad (4.17)$$

Following these same steps for the other cases, we find:

$$\gamma_{mn}^{B(1)(2)} = -\frac{1}{mn} \sum_{j=0}^{n-1} \sum_{j'=0}^{n-j-1} \sum_{k=0}^{m-n+j+j'} (n-j-j')^n {}^n C_j^n {}^n C_{j'}^m {}^m C_k^m {}^m C_{m-n+j+j'-k}^m a^{n-j+k} b^{m+j-k} \quad (4.18)$$

$$\gamma_{mn}^{B(2)(1)} = -\frac{1}{mn} \sum_{j=0}^{n-1} \sum_{j'=0}^{n-j-1} \sum_{k=0}^{m-n+j+j'} (n-j-j')^n {}^n C_j^n {}^n C_{j'}^m {}^m C_k^m {}^m C_{m-n+j+j'-k}^m a^{m+j-k} b^{n-j+k} \quad (4.19)$$

$$\gamma_{mn}^{B(2)(2)} = \frac{1}{mn} \sum_{j=0}^{n-1} \sum_{j'=0}^{n-j-1} \sum_{k=0}^{m+n-j-j'} (n-j-j')^n {}^n C_k^n {}^n C_{m+n-j-j'-k}^m {}^m C_j^m {}^m C_{j'}^m a^{m+n-j-k} b^{j+k}. \quad (4.20)$$

The physics here is symmetric under the interchange $a \leftrightarrow b$. One can explicitly check that this symmetry holds in the above expressions, though this fact is not immediately manifest. Using this symmetry, we find

$$\gamma_{mn}^{B(2)(2)} = \frac{1}{mn} \sum_{j=0}^{n-1} \sum_{j'=0}^{n-j-1} \sum_{k=0}^{m+n-j-j'} (n-j-j')^n {}^n C_j^n {}^n C_{j'}^m {}^m C_k^m {}^m C_{m+n-j-j'-k}^m a^{j+k} b^{m+n-j-k}, \quad (4.21)$$

and therefore

$$\gamma_{mn}^{B(2)(2)} = \gamma_{mn}^{B(1)(1)}. \quad (4.22)$$

Similarly,

$$\gamma_{mn}^{B(2)(1)} = \gamma_{mn}^{B(1)(2)}. \tag{4.23}$$

We will now bring the $\gamma_{mn}^{B(i)(j)}$ coefficients into a form that involves only one summation over a pair of hypergeometric functions, binomial coefficients, and a and b terms raised to the appropriate powers. We achieve this through a re-definition of the summation indices. We present the computation for $\gamma_{mn}^{B(1)(1)}$ and simply skip to the results for the remaining cases.

For $\gamma_{mn}^{B(1)(1)}$, we begin with:

$$\gamma_{mn}^{B(1)(1)} = \frac{1}{mn} \sum_{j=0}^{n-1} \sum_{j'=0}^{n-j-1} \sum_{k=0}^{m+n-j-j'} (n-j-j')^n C_j^n C_{j'}^m C_k^m C_{m+n-j-j'-k}^{j+k} a^{j+k} b^{m+n-j-k}. \tag{4.24}$$

We now define a new summation index:

$$l = n - j - j' \implies j' = n - j - l, \tag{4.25}$$

where (4.15) and (4.16) now give

$$\begin{aligned} n - j - j' > 0 &\implies l > 0 \\ j' \geq 0 &\implies n - l \geq j \\ j, j' \geq 0 &\implies n \geq l. \end{aligned} \tag{4.26}$$

Applying these constraints, we find:

$$\gamma_{mn}^{B(1)(1)} = \frac{1}{mn} \sum_{l=1}^n l b^{m+n} \left(\sum_{j=0}^{n-l} {}^n C_j^n C_{n-j-l}^j a^j b^{-j} \right) \left(\sum_{k=0}^{m+l} {}^m C_k^m C_{m+l-k}^k a^k b^{-k} \right). \tag{4.27}$$

The k and j sums can be evaluated in *Mathematica*,² resulting in:³

$$\gamma_{mn}^{B(1)(1)} = \frac{1}{mn} \sum_{l=1}^n l^n C_l^m C_l^n a^l b^{m+n-l} {}_2F_1 \left(-n, l - n; l + 1; \frac{a}{b} \right) {}_2F_1 \left(-m, l - m; l + 1; \frac{a}{b} \right). \tag{4.28}$$

The other $\gamma^{B(i)(j)}$ coefficients are calculated in the same manner, using the same index re-definition. We present the results here.

$$\begin{aligned} \gamma_{mn}^{B(1)(2)} &= -\frac{1}{mn} \sum_{l=1}^n l^m C_l^n C_l^m a^n b^m {}_2F_1 \left(-m, l - m; l + 1; \frac{a}{b} \right) {}_2F_1 \left(-n, l - n; l + 1; \frac{b}{a} \right) \\ \gamma_{mn}^{B(2)(1)} &= -\frac{1}{mn} \sum_{l=1}^n l^m C_l^n C_l^m a^m b^n {}_2F_1 \left(-m, l - m; l + 1; \frac{b}{a} \right) {}_2F_1 \left(-n, l - n; l + 1; \frac{a}{b} \right) \\ \gamma_{mn}^{B(2)(2)} &= \frac{1}{mn} \sum_{l=1}^n l^n C_l^m C_l^m a^{m+n-l} b^l {}_2F_1 \left(-m, l - m; l + 1; \frac{b}{a} \right) {}_2F_1 \left(-n, l - n; l + 1; \frac{b}{a} \right). \end{aligned} \tag{4.29}$$

²Mathematica sometimes has trouble dealing with the fact that some of the binomial coefficients vanish when the second argument is negative. It is occasionally necessary to alter the range of the j sum to explicitly remove these vanishing terms in order to obtain the full result.

³While it is not immediately apparent, these expressions are symmetric under $a \leftrightarrow b$, as expected.

In addition to (4.22) and (4.23), the above forms allow us to show the additional relation:

$$\gamma_{mn}^{B(1)(1)} = -\gamma_{mn}^{B(1)(2)}. \tag{4.30}$$

This relationship is shown explicitly in appendix E. We then have only a single linearly independent bosonic coefficient.

5 Computation of $\gamma_{NS,mn}^{F(i)(j)}$

Here we perform the computation of the non-physical $\gamma_{NS,mn}^{F(i)(j)}$ coefficients, where the NS indicates that we are capping the squeezed state with the non-physical NS vacuum for both copy 1 and copy 2. We proceed in a manner identical to the boson case, except that the fermion modes are affected by the spectral flows which were needed to remove the spin fields in (3.33).

5.1 Mapping the fermion modes to the t plane

The NS sector fermion modes on the cylinder were given in (3.14). Here we map these modes to the t plane by way of the z plane.

Under a general coordinate transformation $w \rightarrow w'$, the fermion field ψ transforms as:

$$\psi^{(i),\alpha A}(w) \rightarrow \left(\frac{dw'}{dw}\right)^{\frac{1}{2}} \psi^{(i),\alpha A}(w'), \tag{5.1}$$

while the Jacobian again brings the factor described in (4.2). Thus the two together transform as:

$$\psi^{(i),\alpha A}(w) dw \rightarrow \left(\frac{dw'}{dw}\right)^{-\frac{1}{2}} \psi^{(i),\alpha A}(w') dw'. \tag{5.2}$$

Thus when mapping to the z plane via $z = e^w$, we have

$$\psi^{\alpha A}(w) dw \rightarrow z^{-1/2} \psi^{\alpha A}(z) dz, \tag{5.3}$$

and the fermionic modes become:

$$\begin{aligned} d_r^{(1)f,\alpha A} &\rightarrow \frac{1}{2\pi i} \int_{\sigma=0}^{2\pi} \psi^{(1)f,\alpha A}(z) z^{r-1/2} dz \\ d_r^{(2)f,\alpha A} &\rightarrow \frac{1}{2\pi i} \int_{\sigma=0}^{2\pi} \psi^{(2)f,\alpha A}(z) z^{r-1/2} dz. \end{aligned} \tag{5.4}$$

We now move to the covering t plane in the same manner, using the map (3.7). We then find:

$$\begin{aligned} d_r^{(1)f,\alpha A} &\rightarrow \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,\alpha A}(t) t^{-r-1/2} (t + \sqrt{ab})^{1/2} (t - \sqrt{ab})^{1/2} (t + a)^{r-1/2} (t + b)^{r-1/2} dt \\ d_r^{(2)f,\alpha A} &\rightarrow -\frac{1}{2\pi i} \int_{t=0} \psi^{f,\alpha A}(t) t^{-r-1/2} (t + \sqrt{ab})^{1/2} (t - \sqrt{ab})^{1/2} (t + a)^{r-1/2} (t + b)^{r-1/2} dt, \end{aligned} \tag{5.5}$$

where again the copy 2 mode gains a minus sign from the reversal of the direction of the contour as $z \sim t^{-1}$.

5.2 Applying the spectral flows

We now perform the spectral flows required to eliminate the spin fields in the t plane (3.33). Under a spectral flow by α units at a point t_0 , the fermion field ψ behaves as:

$$\psi^{\pm A}(t) \rightarrow (t - t_0)^{\mp \frac{1}{2}\alpha} \psi^{\pm A}(t). \quad (5.6)$$

We perform each of the four spectral flows in turn.

$\alpha = 1$ at $t = -b$

$$\begin{aligned} d_r^{(1)f,+A} &\rightarrow \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,+A}(t) t^{-r-1/2} (t + \sqrt{ab})^{1/2} (t - \sqrt{ab})^{1/2} (t + a)^{r-1/2} (t + b)^{r-1} dt \\ d_r^{(1)f,-A} &\rightarrow \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,-A}(t) t^{-r-1/2} (t + \sqrt{ab})^{1/2} (t - \sqrt{ab})^{1/2} (t + a)^{r-1/2} (t + b)^r dt \\ d_r^{(2)f,+A} &\rightarrow -\frac{1}{2\pi i} \int_{t=0} \psi^{f,+A}(t) t^{-r-1/2} (t + \sqrt{ab})^{1/2} (t - \sqrt{ab})^{1/2} (t + a)^{r-1/2} (t + b)^{r-1} dt \\ d_r^{(2)f,-A} &\rightarrow -\frac{1}{2\pi i} \int_{t=0} \psi^{f,-A}(t) t^{-r-1/2} (t + \sqrt{ab})^{1/2} (t - \sqrt{ab})^{1/2} (t + a)^{r-1/2} (t + b)^r dt. \end{aligned} \quad (5.7)$$

$\alpha = 1$ at $t = -a$

$$\begin{aligned} d_r^{(1)f,+A} &\rightarrow \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,+A}(t) t^{-r-1/2} (t + \sqrt{ab})^{1/2} (t - \sqrt{ab})^{1/2} (t + a)^{r-1} (t + b)^{r-1} dt \\ d_r^{(1)f,-A} &\rightarrow \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,-A}(t) t^{-r-1/2} (t + \sqrt{ab})^{1/2} (t - \sqrt{ab})^{1/2} (t + a)^r (t + b)^r dt \\ d_r^{(2)f,+A} &\rightarrow -\frac{1}{2\pi i} \int_{t=0} \psi^{f,+A}(t) t^{-r-1/2} (t + \sqrt{ab})^{1/2} (t - \sqrt{ab})^{1/2} (t + a)^{r-1} (t + b)^{r-1} dt \\ d_r^{(2)f,-A} &\rightarrow -\frac{1}{2\pi i} \int_{t=0} \psi^{f,-A}(t) t^{-r-1/2} (t + \sqrt{ab})^{1/2} (t - \sqrt{ab})^{1/2} (t + a)^r (t + b)^r dt. \end{aligned} \quad (5.8)$$

$\alpha = -1$ at $t = -\sqrt{ab}$

$$\begin{aligned} d_r^{(1)f,+A} &\rightarrow \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,+A}(t) t^{-r-1/2} (t + \sqrt{ab}) (t - \sqrt{ab})^{1/2} (t + a)^{r-1} (t + b)^{r-1} dt \\ d_r^{(1)f,-A} &\rightarrow \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,-A}(t) t^{-r-1/2} (t - \sqrt{ab})^{1/2} (t + a)^r (t + b)^r dt \\ d_r^{(2)f,+A} &\rightarrow -\frac{1}{2\pi i} \int_{t=0} \psi^{f,+A}(t) t^{-r-1/2} (t + \sqrt{ab}) (t - \sqrt{ab})^{1/2} (t + a)^{r-1} (t + b)^{r-1} dt \\ d_r^{(2)f,-A} &\rightarrow -\frac{1}{2\pi i} \int_{t=0} \psi^{f,-A}(t) t^{-r-1/2} (t - \sqrt{ab})^{1/2} (t + a)^r (t + b)^r dt. \end{aligned} \quad (5.9)$$

$\alpha = -1$ at $t = \sqrt{ab}$

$$\begin{aligned}
 d_r^{(1)f,+A} &\rightarrow d_r^{\prime(1)f,+A} = \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,+A}(t) t^{-r-1/2} (t^2 - ab) (t+a)^{r-1} (t+b)^{r-1} dt \\
 d_r^{(1)f,-A} &\rightarrow d_r^{\prime(1)f,-A} = \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,-A}(t) t^{-r-1/2} (t+a)^r (t+b)^r dt \\
 d_r^{(2)f,+A} &\rightarrow d_r^{\prime(2)f,+A} = -\frac{1}{2\pi i} \int_{t=0} \psi^{f,+A}(t) t^{-r-1/2} (t^2 - ab) (t+a)^{r-1} (t+b)^{r-1} dt \\
 d_r^{(2)f,-A} &\rightarrow d_r^{\prime(2)f,-A} = -\frac{1}{2\pi i} \int_{t=0} \psi^{f,-A}(t) t^{-r-1/2} (t+a)^r (t+b)^r dt.
 \end{aligned} \tag{5.10}$$

We have now removed all spin field insertions in the t -plane and can close all punctures with the state $|0_{NS}\rangle_t$. This allows us to smoothly deform our contours.

5.3 Expanding the fermion modes

The operators $d_r^{\prime(1)f,\pm A}$ are given by contour integrals at large t while the operators $d_r^{\prime(2)f,\pm A}$ are given by contour integrals around $t = 0$. We need to express these operators in terms of the fermion modes natural to the t plane, which are contour integrals of powers of t . We thus expand the integrand of each mode in powers of t around its appropriate region in the t -plane.

$d_r^{\prime(1)f,+A}$

$$\begin{aligned}
 t^{-r-1/2} (t^2 - ab) (t+a)^{r-1} (t+b)^{r-1} &= t^{r-5/2} (t^2 - ab) \sum_{q,q' \geq 0}^{r-1} C_q^{r-1} C_{q'}^{r-1} a^q b^{q'} t^{-q-q'} \\
 &= \sum_{q,q' \geq 0}^{r-1} C_q^{r-1} C_{q'}^{r-1} a^q b^{q'} t^{r-q-q'-1/2} \\
 &\quad - \sum_{q,q' \geq 0}^{r-1} C_q^{r-1} C_{q'}^{r-1} a^{q+1} b^{q'+1} t^{r-q-q'-5/2}.
 \end{aligned} \tag{5.11}$$

$d_r^{\prime(1)f,-A}$

$$\begin{aligned}
 t^{-r-1/2} (t+a)^r (t+b)^r &= t^{r-1/2} \sum_{q,q' \geq 0}^r C_q^r C_{q'}^r a^q b^{q'} t^{-q-q'} \\
 &= \sum_{q,q' \geq 0}^r C_q^r C_{q'}^r a^q b^{q'} t^{r-q-q'-1/2}.
 \end{aligned} \tag{5.12}$$

$d_r^{\prime(2)f,+A}$

$$\begin{aligned}
 t^{-r-1/2} (t^2 - ab) (t+a)^{r-1} (t+b)^{r-1} &= t^{-r-1/2} (t^2 - ab) a^{r-1} b^{r-1} \sum_{q,q' \geq 0}^{r-1} C_q^{r-1} C_{q'}^{r-1} a^{-q} b^{-q'} t^{q+q'} \\
 &= \sum_{q,q' \geq 0}^{r-1} C_q^{r-1} C_{q'}^{r-1} a^{r-q-1} b^{r-q'-1} t^{q+q'-r+3/2} \\
 &\quad - \sum_{q,q' \geq 0}^{r-1} C_q^{r-1} C_{q'}^{r-1} C_q^{r-1} C_{q'}^{r-1} a^{r-q} b^{r-q'} t^{q+q'-r-1/2}.
 \end{aligned} \tag{5.13}$$

$d_r^{\prime(2)f,-A}$

$$\begin{aligned}
 t^{-r-1/2}(t+a)^r(t+b)^r &= t^{-r-1/2}a^r b^r \sum_{q,q' \geq 0} {}^r C_q^r {}^r C_{q'}^r a^{-q} b^{-q'} t^{q+q'} \\
 &= \sum_{q,q' \geq 0} {}^r C_q^r {}^r C_{q'}^r a^{r-q} b^{r-q'} t^{q+q'-r-1/2}.
 \end{aligned} \tag{5.14}$$

With these expansions, (5.10) becomes:

$$\begin{aligned}
 d_r^{\prime(1)f,+A} &= \sum_{q,q' \geq 0} {}^{r-1} C_q^{r-1} {}^{r-1} C_{q'}^{r-1} a^q b^{q'} \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,+A}(t) t^{r-q-q'-1/2} dt \\
 &\quad - \sum_{q,q' \geq 0} {}^{r-1} C_q^{r-1} {}^{r-1} C_{q'}^{r-1} a^{q+1} b^{q'+1} \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,+A}(t) t^{r-q-q'-5/2} dt \\
 d_r^{\prime(1)f,-A} &\rightarrow \sum_{q,q' \geq 0} {}^r C_q^r {}^r C_{q'}^r a^q b^{q'} \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,-A}(t) t^{r-q-q'-1/2} dt \\
 d_r^{\prime(2)f,+A} &\rightarrow - \sum_{q,q' \geq 0} {}^{r-1} C_q^{r-1} {}^{r-1} C_{q'}^{r-1} a^{r-q-1} b^{r-q'-1} \frac{1}{2\pi i} \int_{t=0} \psi^{f,+A}(t) t^{q+q'-r+3/2} dt \\
 &\quad + \sum_{q,q' \geq 0} {}^{r-1} C_q^{r-1} {}^{r-1} C_{q'}^{r-1} a^{r-q} b^{r-q'} \frac{1}{2\pi i} \int_{t=0} \psi^{f,+A}(t) t^{q+q'-r-1/2} dt \\
 d_r^{\prime(2)f,-A} &\rightarrow - \sum_{q,q' \geq 0} {}^r C_q^r {}^r C_{q'}^r a^{r-q} b^{r-q'} \frac{1}{2\pi i} \int_{t=0} \psi^{f,-A}(t) t^{q+q'-r-1/2} dt.
 \end{aligned} \tag{5.15}$$

In terms of modes natural to the t -plane, we have:

$$\begin{aligned}
 d_r^{\prime(1)f,+A} &= \sum_{q,q' \geq 0} {}^{r-1} C_q^{r-1} {}^{r-1} C_{q'}^{r-1} a^q b^{q'} \tilde{d}_{r-q-q'}^{+A} - \sum_{q,q' \geq 0} {}^{r-1} C_q^{r-1} {}^{r-1} C_{q'}^{r-1} a^{q+1} b^{q'+1} \tilde{d}_{r-q-q'-2}^{+A} \\
 d_r^{\prime(1)f,-A} &= \sum_{q,q' \geq 0} {}^r C_q^r {}^r C_{q'}^r a^q b^{q'} \tilde{d}_{r-q-q'}^{-A} \\
 d_r^{\prime(2)f,+A} &= - \sum_{q,q' \geq 0} {}^{r-1} C_q^{r-1} {}^{r-1} C_{q'}^{r-1} a^{r-q-1} b^{r-q'-1} \tilde{d}_{q+q'-r+2}^{+A} + \sum_{q,q' \geq 0} {}^{r-1} C_q^{r-1} {}^{r-1} C_{q'}^{r-1} a^{r-q} b^{r-q'} \tilde{d}_{q+q'-r}^{+A} \\
 d_r^{\prime(2)f,-A} &= - \sum_{q,q' \geq 0} {}^r C_q^r {}^r C_{q'}^r a^{r-q} b^{r-q'} \tilde{d}_{q+q'-r}^{-A}.
 \end{aligned} \tag{5.16}$$

5.4 Computing $\gamma_{NS,rs}^{F(i)(j)}$

Here we shall compute the $\gamma_{NS}^{F(i)(j)}$ coefficients. We present the computation for $\gamma_{NS}^{F(1)(1)}$ explicitly. The other computations are performed in the same way, and we give only the results. Using (3.34) and (5.16), we find:

$$\begin{aligned}
 \gamma_{NS,rs}^{F(1)(1)} &= \sum_{p,p',q,q' \geq 0} s^{-1} C_q^{s-1} C_{q'}^{s-1} C_p^r C_{p'}^r C_p^r C_{p'}^r a^{q+p} b^{q'+p'} \frac{t \langle 0_{NS} | \tilde{d}_{s-q-q'}^{++} \tilde{d}_{r-p-p'}^{--} | 0_{NS} \rangle_t}{t \langle 0_{NS} | 0_{NS} \rangle_t} \\
 &\quad - \sum_{p,p',q,q' \geq 0} s^{-1} C_q^{s-1} C_{q'}^{s-1} C_p^r C_{p'}^r C_p^r C_{p'}^r a^{q+p+1} b^{q'+p'+1} \frac{t \langle 0_{NS} | \tilde{d}_{s-q-q'-2}^{++} \tilde{d}_{r-p-p'}^{--} | 0_{NS} \rangle_t}{t \langle 0_{NS} | 0_{NS} \rangle_t}.
 \end{aligned} \tag{5.17}$$

For the first term, it is clear that the summand vanishes except when:

$$\begin{aligned} r + s - p - p' - q - q' = 0 &\implies p' = r + s - q - q' - p \\ s - q - q' > 0 &\implies s - q > q'. \end{aligned} \quad (5.18)$$

Furthermore,

$$\begin{aligned} q' \geq 0 &\implies s > q \\ p' \geq 0 &\implies r + s - q - q' \geq p. \end{aligned} \quad (5.19)$$

We can thus rewrite (5.17) as:

$$\begin{aligned} \gamma_{NS,rs}^{F(1)(1)} &= - \sum_{q=0}^{[s]} \sum_{q'=0}^{[s]-q} \sum_{p=0}^{r+s-q-p} s^{-1} C_q^{s-1} C_{q'}^r C_p^r C_{r+s-q-q'-p} a^{q+p} b^{r+s-q-q'} \\ &\quad - \sum_{p,p',q,q' \geq 0} s^{-1} C_q^{s-1} C_{q'}^r C_p^r C_{p'} a^{q+p+1} b^{q'+p'+1} \frac{\langle 0_{NS} | \tilde{d}_{s-q-q'-2}^{++} \tilde{d}_{r-p-p'}^{--} | 0_{NS} \rangle_t}{t \langle 0_{NS} | 0_{NS} \rangle}. \end{aligned} \quad (5.20)$$

Applying the same line of reasoning for the second term, we find:

$$\begin{aligned} r + s - p - p' - q - q' - 2 = 0 &\implies p' = r + s - q - q' - p - 2 \\ s - q - q' - 2 > 0 &\implies s - q - 2 > q' \\ p' \geq 0 &\implies r + s - q - q' - 2 \geq p \\ q' \geq 0 &\implies s - 2 > q. \end{aligned} \quad (5.21)$$

So (5.20) becomes

$$\begin{aligned} \gamma_{NS,rs}^{F(1)(1)} &= - \sum_{q=0}^{[s]} \sum_{q'=0}^{[s]-q} \sum_{p=0}^{r+s-q-p} s^{-1} C_q^{s-1} C_{q'}^r C_p^r C_{r+s-q-q'-p} a^{q+p} b^{r+s-q-q'} \\ &\quad + \sum_{q=0}^{[s]-2} \sum_{q'=0}^{[s]-q-2} \sum_{p=0}^{r+s-q-q'-2} s^{-1} C_q^{s-1} C_{q'}^r C_p^r C_{r+s-q-q'-p-2} a^{q+p+1} b^{r+s-p-q-1}, \end{aligned} \quad (5.22)$$

where the second term is instead zero when the sums' ranges are invalid ($s \leq \frac{3}{2}$). We shall adopt this convention generally. Henceforth, all sums are to be treated as zero when the given range is invalid.

The other cases proceed along the same lines.

$$\begin{aligned} \gamma_{NS,rs}^{F(1)(2)} &= - \sum_{q=0}^{[s]-2} \sum_{q'=0}^{[s]-q-2} \sum_{p=0}^{r-s+q+q'+2} s^{-1} C_q^{s-1} C_{q'}^r C_p^r C_{r-s+q+q'-p+2} a^{s-q+p-1} b^{r+q-p+1} \\ &\quad + \sum_{q=0}^{[s]} \sum_{q'=0}^{[s]-q} \sum_{p=0}^{r-s+q+q'} s^{-1} C_q^{s-1} C_{q'}^r C_p^r C_{r-s+q+q'-p} a^{s-q+p} b^{r+q-p} \end{aligned} \quad (5.23)$$

$$\begin{aligned}
 \gamma_{NS,rs}^{F(2)(1)} &= - \sum_{q=0}^{[s]-2} \sum_{q'=0}^{[s]-q-2} \sum_{p=0}^{r-s+q+q'+2} s^{-1} C_q^{s-1} C_{q'}^r C_p^r C_{r-s+q+q'-p+2} a^{r+q-p+1} b^{s-q+p-1} \\
 &+ \sum_{q=0}^{[s]} \sum_{q'=0}^{[s]-q} \sum_{p=0}^{r-s+q+q'} s^{-1} C_q^{s-1} C_{q'}^r C_p^r C_{r-s+q+q'-p} a^{r+q-p} b^{s-q+p}
 \end{aligned} \tag{5.24}$$

$$\begin{aligned}
 \gamma_{NS,rs}^{F(2)(2)} &= - \sum_{q=0}^{[s]} \sum_{q'=0}^{[s]-q} \sum_{p=0}^{r+s-q-q'} s^{-1} C_q^{s-1} C_{q'}^r C_p^r C_{r+s-q-q'-p} a^{r+s-q-p} b^{q+p} \\
 &+ \sum_{q=0}^{[s]-2} \sum_{q'=0}^{[s]-q-2} \sum_{p=0}^{r+s-q-q'-2} s^{-1} C_q^{s-1} C_{q'}^r C_p^r C_{r+s-q-q'-p-2} a^{r+s-p-q-1} b^{q+p+1}.
 \end{aligned} \tag{5.25}$$

Using the interchange symmetry $a \leftrightarrow b$, we find:

$$\gamma_{NS,rs}^{F(2)(2)} = \gamma_{NS,rs}^{F(1)(1)} \tag{5.26}$$

$$\gamma_{NS,rs}^{F(1)(2)} = \gamma_{NS,rs}^{F(2)(1)}. \tag{5.27}$$

Just as in the bosonic case, we shall now bring the $\gamma_{NS}^{F(i)(j)}$ coefficients into a form that involves only one summation over a pair of hypergeometric functions, binomial coefficients, and a and b terms raised to the appropriate powers. We present the computation for $i = j = 1$ and simply give the results for the remaining cases. We start with:

$$\begin{aligned}
 \gamma_{NS,rs}^{F(1)(1)} &= - \sum_{q=0}^{[s]} \sum_{q'=0}^{[s]-q} \sum_{p=0}^{r+s-q-q'} s^{-1} C_q^{s-1} C_{q'}^r C_p^r C_{r+s-q-q'-p} a^{q+p} b^{r+s-q-q'} \\
 &+ \sum_{q=0}^{[s]-2} \sum_{q'=0}^{[s]-q-2} \sum_{p=0}^{r+s-q-q'-2} s^{-1} C_q^{s-1} C_{q'}^r C_p^r C_{r+s-q-q'-p-2} a^{q+p+1} b^{r+s-p-q-1}.
 \end{aligned} \tag{5.28}$$

We now redefine our summation indices:

$$\begin{aligned}
 \text{Term 1: } j &\equiv s - q - q' \implies q' = s - q - j \\
 \text{Term 2: } j &\equiv s - q - q' - 2 \implies q' = s - q - j - 2,
 \end{aligned} \tag{5.29}$$

where in both cases j is a positive half-integer. Since q and q' are both non-negative, we find:

$$\begin{aligned}
 \text{Term 1: } q' \geq 0 &\implies q \leq s - j, & q \geq 0 &\implies j \leq s \\
 \text{Term 2: } q' \geq 0 &\implies q \leq s - j - 2, & q \geq 0 &\implies j \leq s - 2.
 \end{aligned} \tag{5.30}$$

This gives:

$$\begin{aligned}
 \gamma_{NS,rs}^{F(1)(1)} &= - \sum_{j=1/2}^s b^{r+s} \left(\sum_{q=0}^{s-j} s^{-1} C_q^{s-1} C_{s-q-j} a^q b^{-q} \right) \left(\sum_{p=0}^{r+j} r C_p^r C_{r+j-p} a^p b^{-p} \right) \\
 &+ \sum_{j=1/2}^{s-2} a b^{r+s-1} \left(\sum_{q=0}^{s-j-2} s^{-1} C_q^{s-1} C_{s-q-j-2} a^q b^{-q} \right) \left(\sum_{p=0}^{r+j} r C_p^r C_{r+j-p} a^p b^{-p} \right).
 \end{aligned} \tag{5.31}$$

Evaluating the q and p sums in *Mathematica*, we find:

$$\begin{aligned} \gamma_{NS,rs}^{F(1)(1)} &= - \sum_{j=\frac{1}{2}}^s s^{-1} C_{s-j}^r C_{r+j} b^{r+s} \\ &\quad \times {}_2F_1\left(1-s, j-s; j; \frac{a}{b}\right) {}_2F_1\left(-r, -j-r; 1-j; \frac{a}{b}\right) \\ &\quad + \sum_{j=\frac{1}{2}}^{s-2} s^{-1/2} C_{s-j-2}^r C_{r+j} a b^{r+s-1} \\ &\quad \times {}_2F_1\left(1-s, j-s+2; j+2; \frac{a}{b}\right) {}_2F_1\left(-r, -j-r; 1-j; \frac{a}{b}\right). \end{aligned} \quad (5.32)$$

The other $\gamma_{NS,rs}^{F(i)(j)}$ coefficients are handled in the same manner. We present the final result along with the appropriate index redefinition that was made in each case. For $\gamma_{NS,rs}^{F(1)(2)}$, we find:

$$\begin{aligned} \gamma_{NS,rs}^{F(1)(2)} &= - \sum_{j=\frac{1}{2}}^{s-2} s^{-1} C_{s-j-2}^r C_{r-j} a^{s-1} b^{r+1} \\ &\quad \times {}_2F_1\left(-r, j-r; j; \frac{a}{b}\right) {}_2F_1\left(1-s, j-s+2; j+2; \frac{b}{a}\right) \\ &\quad + \sum_{j=\frac{1}{2}}^s s^{-1} C_{s-j}^r C_{r-j} a^s b^r \\ &\quad \times {}_2F_1\left(-r, j-r; j+1; \frac{a}{b}\right) {}_2F_1\left(1-s, j-s; j; \frac{b}{a}\right), \end{aligned} \quad (5.33)$$

where in (5.23), we substituted:

$$\begin{aligned} \text{Term 1: } q' &= s + j - q - 2 \\ \text{Term 2: } q' &= s + j - q. \end{aligned} \quad (5.34)$$

For $\gamma_{NS,rs}^{F(2)(1)}$, we find:

$$\begin{aligned} \gamma_{NS,rs}^{F(2)(1)} &= \sum_{j=\frac{1}{2}}^s s^{-1} C_{s-j}^r C_{r-j} a^r b^s \\ &\quad \times {}_2F_1\left(-r, j-r; j+1; \frac{b}{a}\right) {}_2F_1\left(1-s, j-s; j; \frac{a}{b}\right) \\ &\quad - \sum_{j=\frac{1}{2}}^{s-2} s^{-1} C_{s-j-2}^r C_{r-j} a^{r+1} b^{s-1} \\ &\quad \times {}_2F_1\left(-r, j-r; j+1; \frac{b}{a}\right) {}_2F_1\left(1-s, j-s+2; j+2; \frac{a}{b}\right), \end{aligned} \quad (5.35)$$

where in (5.24), we substituted:

$$\begin{aligned} \text{Term 1: } q' &= s - q - j \\ \text{Term 2: } q' &= s - q - j - 2. \end{aligned} \quad (5.36)$$

For $\gamma_{NS,rs}^{F(2)(2)}$, we find:

$$\begin{aligned} \gamma_{NS,rs}^{F(2)(2)} &= \sum_{j=\frac{1}{2}}^{s-2} s^{-1} C_{s-j-2}^r C_{r+j} a^{r+s-1} b \\ &\quad \times {}_2F_1\left(-r, -j-r; 1-j; \frac{b}{a}\right) {}_2F_1\left(1-s, j-s+2; j+2; \frac{b}{a}\right) \\ &\quad - \sum_{j=\frac{1}{2}}^s s^{-1} C_{s-j}^r C_{r+j} a^{r+s} \\ &\quad \times {}_2F_1\left(-r, -j-r; 1-j; \frac{b}{a}\right) {}_2F_1\left(1-s, j-s; j; \frac{b}{a}\right), \end{aligned} \tag{5.37}$$

where in (5.25), we substituted:

$$\begin{aligned} \text{Term 1: } q' &= s + j - q - 2 \\ \text{Term 2: } q' &= s + j - q. \end{aligned} \tag{5.38}$$

6 Computation of $\gamma_{R+-}^{F(i)(j)}$

Here we perform the computation of the $\gamma_{R+-,mn}^{F(i)(j)}$ coefficients, where the $R+-$ indicates that we are capping the squeezed state with the positive Ramond vacuum on copy 1 and the negative Ramond vacuum on copy 2. In order to obtain an empty t -plane from this capping state we must apply one additional spectral flow beyond those performed in the NS case. We then expand the resulting modes in terms of modes natural to the t plane in order to compute $\gamma_{R+-,mn}^{F(i)(j)}$.

6.1 Modes in t -plane from additional spectral flow by $\alpha = -1$

Here we obtain the t -plane modes after one additional spectral of $\alpha = -1$ around the point $t = 0$. We remind the reader that for the NS sector computation, we performed four spectral flows:

$$\begin{aligned} \alpha &= 1 \text{ around } t = -b \\ \alpha &= 1 \text{ around } t = -a \\ \alpha &= -1 \text{ around } t = -\sqrt{ab} \\ \alpha &= -1 \text{ around } t = \sqrt{ab}. \end{aligned} \tag{6.1}$$

The resulting modes were given by

$$\begin{aligned} d_r^{(1)f,+A} &= \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,+A}(t) t^{-r-1/2} (t^2 - ab) (t+a)^{r-1} (t+b)^{r-1} dt \\ d_r^{(1)f,-A} &= \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,-A}(t) t^{-r-1/2} (t+a)^r (t+b)^r dt \\ d_r^{(2)f,+A} &= -\frac{1}{2\pi i} \int_{t=0} \psi^{f,+A}(t) t^{-r-1/2} (t^2 - ab) (t+a)^{r-1} (t+b)^{r-1} dt \\ d_r^{(2)f,-A} &= -\frac{1}{2\pi i} \int_{t=0} \psi^{f,-A}(t) t^{-r-1/2} (t+a)^r (t+b)^r dt. \end{aligned} \tag{6.2}$$

We now apply one additional spectral flow of $\alpha = -1$ around the point, $t = 0$. We also note that the Ramond sector has integer-indexed fermions, so we will switch from the half-integer mode index r to the integer index n .

Under this last spectral flow, the fermion field changes as follows:

$$\psi^{(i)f,\pm A}(t) \rightarrow t^{\pm 1/2} \psi^{(i)f,\pm}(t). \quad (6.3)$$

The modes then become

$$\begin{aligned} d_n^{\prime(1)f,+A} &\rightarrow \hat{d}_n^{(1)f,+A} = \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,+A}(t) t^{-n} (t^2 - ab)(t+a)^{n-1} (t+b)^{n-1} dt \\ d_n^{\prime(1)f,-A} &\rightarrow \hat{d}_n^{(1)f,-A} = \frac{1}{2\pi i} \int_{t=\infty} \psi^{f,-A}(t) t^{-n-1} (t+a)^n (t+b)^n dt \\ d_n^{\prime(2)f,+A} &\rightarrow \hat{d}_n^{(2)f,+A} = -\frac{1}{2\pi i} \int_{t=0} \psi^{f,+A}(t) t^{-n} (t^2 - ab)(t+a)^{n-1} (t+b)^{n-1} dt \\ d_n^{\prime(2)f,-A} &\rightarrow \hat{d}_n^{(2)f,-A} = -\frac{1}{2\pi i} \int_{t=0} \psi^{f,-A}(t) t^{-n-1} (t+a)^n (t+b)^n dt. \end{aligned} \quad (6.4)$$

We now expand the modes (6.4) in terms of modes natural to the t plane. Since the additional spectral flow produced a factor of $t^{\pm 1/2}$, the binomials we wish to expand are identical to the ones found in the NS case. We thus present only the results:

$$\begin{aligned} \hat{d}_n^{(1)f,+A} &= \sum_{j,j' \geq 0}^{n-1} C_j^{n-1} C_{j'} a^j b^{j'} \tilde{d}_{n-j-j'+1/2}^{+A} \\ &\quad - \sum_{j,j' \geq 0}^{n-1} C_j^{n-1} C_{j'} a^{j+1} b^{j'+1} \tilde{d}_{n-j-j'-3/2}^{+A} \\ \hat{d}_n^{(1)f,-A} &= \sum_{j,j' \geq 0}^n C_j^n C_{j'} a^j b^{j'} \tilde{d}_{n-j-j'-1/2}^{-A} \\ \hat{d}_n^{(2)f,+A} &= - \sum_{j,j' \geq 0}^{n-1} C_j^{n-1} C_{j'} a^{n-j-1} b^{n-j'-1} \tilde{d}_{j+j'-n+5/2}^{+A} \\ &\quad + \sum_{j,j' \geq 0}^{n-1} C_j^{n-1} C_{j'} a^{n-j} b^{n-j'} \tilde{d}_{j+j'-n+1/2}^{+A} \\ \hat{d}_n^{(2)f,-A} &= - \sum_{j,j' \geq 0}^n C_j^n C_{j'} a^{n-j} b^{n-j'} \tilde{d}_{j+j'-n-1/2}^{-A}. \end{aligned} \quad (6.5)$$

6.2 Computing $\gamma_{R+-,mn}^{F(i)(j)}$

Here we show the computation of the $\gamma_{R+-}^{F(i)(j)}$ coefficients. We shall explicitly show the computation for the case $i = j = 1$. The other cases are computed in the same way and we shall simply state their results. Using (3.37) and (6.5), we find:

$$\begin{aligned} \gamma_{R+-,mn}^{F(1)(1)} &= \sum_{j,j',k,k' \geq 0}^{n-1} C_k^{n-1} C_{k'}^{n-1} C_j^m C_{j'}^m C_{j'} a^{k+j} b^{k'+j'} \frac{t \langle 0_{NS} | \tilde{d}_{n-k-k'+1/2}^{++} \tilde{d}_{m-j-j'-1/2}^{--} | 0_{NS} \rangle_t}{t \langle 0_{NS} | 0_{NS} \rangle_t} \\ &\quad - \sum_{j,j',k,k' \geq 0}^{n-1} C_k^{n-1} C_{k'}^{n-1} C_j^m C_{j'}^m C_{j'} a^{k+j+1} b^{k'+j'+1} \\ &\quad \times \frac{t \langle 0_{NS} | \tilde{d}_{n-k-k'-3/2}^{++} \tilde{d}_{m-j-j'-1/2}^{--} | 0_{NS} \rangle_t}{t \langle 0_{NS} | 0_{NS} \rangle_t}. \end{aligned} \quad (6.6)$$

Looking at the first term, it is clear that the summand vanishes except when:

$$\begin{aligned} m+n-j-j'-k-k' &= 0 \implies j' = m+n-k-k'-j \\ n-k-k'+\frac{1}{2} &> 0 \implies n-k \geq k'. \end{aligned} \quad (6.7)$$

Furthermore,

$$\begin{aligned} k' \geq 0 &\implies n \geq k \\ j' \geq 0 &\implies m+n-k-k' \geq j. \end{aligned} \quad (6.8)$$

We can thus re-write (6.6) as:

$$\begin{aligned} \gamma_{R+-,mn}^{F(1)(1)} &= \sum_{k=0}^n \sum_{k'=0}^{n-k} \sum_{j=0}^{m+n-k-k'} n^{-1} C_k^{n-1} C_{k'}^m C_j^m C_{m+n-k-k'-j} a^{k+j} b^{m+n-j-k} \\ &\quad - \sum_{j,j',k,k' \geq 0} n^{-1} C_k^{n-1} C_{k'}^m C_j^m C_{j'} a^{k+j+1} b^{k'+j'+1} \\ &\quad \times \frac{t \langle 0_{NS} | \tilde{d}_{n-k-k'-3/2}^{++} \tilde{d}_{m-j-j'-1/2}^{--} | 0_{NS} \rangle t}{t \langle 0_{NS} | 0_{NS} \rangle}. \end{aligned} \quad (6.9)$$

Applying the same line of reasoning for the second term, we see:

$$\begin{aligned} m+n-j-j'-k-k'-2 &= 0 \implies j' = m+n-k-k'-j-2 \\ n-k-k'-\frac{3}{2} &> 0 \implies n+2-k \geq k' \\ j' \geq 0 &\implies m+n-k-k'-2 \geq j. \end{aligned} \quad (6.10)$$

So (6.9) becomes

$$\begin{aligned} \gamma_{R+-,mn}^{F(1)(1)} &= - \sum_{k=0}^n \sum_{k'=0}^{n-k} \sum_{j=0}^{m+n-k-k'} n^{-1} C_k^{n-1} C_{k'}^m C_j^m C_{m+n-k-k'-j} a^{j+k} b^{m+n-j-k} \\ &\quad + \sum_{k=0}^{n-2} \sum_{k'=0}^{n-k-2} \sum_{j=0}^{m+n-k-k'-2} n^{-1} C_k^{n-1} C_{k'}^m C_j^m C_{m+n-k-k'-j-2} a^{j+k+1} b^{m+n-j-k-1}. \end{aligned} \quad (6.11)$$

Following these same steps for the other cases, we obtain:

$$\begin{aligned} \gamma_{R+-,mn}^{F(1)(2)} &= - \sum_{k=0}^{n-3} \sum_{k'=0}^{n-k-3} \sum_{j=0}^{m-n+k+k'+2} n^{-1} C_k^{n-1} C_{k'}^m C_j^m C_{m-n+k+k'-j+2} a^{n+j-k-1} b^{m-j+k+1} \\ &\quad + \sum_{k=0}^{n-1} \sum_{k'=0}^{n-k-1} \sum_{j=0}^{m-n+k+k'} n^{-1} C_k^{n-1} C_{k'}^m C_j^m C_{m-n+k+k'-j} a^{n+j-k} b^{m-j+k} \end{aligned} \quad (6.12)$$

$$\begin{aligned} \gamma_{R+-,mn}^{F(2)(1)} &= \sum_{k=0}^n \sum_{k'=0}^{n-k} \sum_{j=0}^{m-n+k+k'} n^{-1} C_k^{n-1} C_{k'}^m C_j^m C_{m-n+k+k'-j} a^{m-j+k} b^{n+j-k} \\ &\quad - \sum_{k=0}^{n-2} \sum_{k'=0}^{n-k-2} \sum_{j=0}^{m-n+k+k'+2} n^{-1} C_k^{n-1} C_{k'}^m C_j^m C_{m-n+k+k'-j+2} a^{m-j+k+1} b^{n+j-k-1} \end{aligned} \quad (6.13)$$

$$\begin{aligned}
 \gamma_{R+-;mn}^{F(2)(2)} &= \sum_{k=0}^{n-3} \sum_{k'=0}^{n-k-3} \sum_{j=0}^{m+n-k-k'-2} n^{-1} C_k^{n-1} C_{k'}^m C_j^m C_{m+n-k-k'-j-2} a^{m+n-j-k-1} b^{j+k+1} \\
 &\quad - \sum_{k=0}^{n-1} \sum_{k'=0}^{n-k-1} \sum_{j=0}^{m+n-k-k'} n^{-1} C_k^{n-1} C_{k'}^m C_j^m C_{m+n-k-k'-j} a^{m+n-j-k} b^{j+k}. \quad (6.14)
 \end{aligned}$$

The physics is symmetric under Copy 1 \leftrightarrow Copy 2. This symmetry manifests in the following relations:

$$\begin{aligned}
 \gamma_{R+-;m,0}^{F(1)(1)} &= -\gamma_{R+-;m,0}^{(2)(1)}, \quad m > 1 \\
 \gamma_{R+-;mn}^{F(1)(1)} &= \gamma_{R+-;mn}^{F(2)(2)}, \quad m, n > 1 \\
 \gamma_{R+-;mn}^{F(1)(2)} &= \gamma_{R+-;mn}^{F(2)(1)}, \quad m, n > 1 \\
 \gamma_{R+-;0,n}^{(2)(1)} &= \delta_{n,0} \\
 \gamma_{R+-;0,n}^{(2)(2)} &= 0. \quad (6.15)
 \end{aligned}$$

The manner in which these relations enforce the copy symmetry is examined in detail in appendix D. The fact that these relations are satisfied will become evident when we re-express the coefficients in terms of hypergeometric functions. As usual we present the computation for $i = j = 1$ and simply give the results for the remaining cases.

The symmetry requirements of (6.15) suggests that $\gamma_{R+-;mn}^{F(1)(1)}$ may behave differently for $n = 0$. Indeed, it is convenient to treat the $n = 0$ case separately. Setting $n = 0$ in (6.11), we find:

$$\gamma_{R+-;m,0}^{F(1)(1)} = - \sum_{j=0}^m a^j b^{m-j} m C_j^m C_{m-j} = -b^m {}_2F_1 \left(-m, -m, 1, \frac{a}{b} \right). \quad (6.16)$$

When $n > 0$, we begin:

$$\begin{aligned}
 &\left[\gamma_{R+-;mn}^{F(1)(1)} \right]_{n>0} \quad (6.17) \\
 &= - \sum_{k=0}^n \sum_{k'=0}^{n-k} \sum_{j=0}^{m+n-k-k'} n^{-1} C_k^{n-1} C_{k'}^m C_j^m C_{m+n-k-k'-j} a^{j+k} b^{m+n-j-k} \\
 &\quad + \sum_{k=0}^{n-2} \sum_{k'=0}^{n-k-2} \sum_{j=0}^{m+n-k-k'-2} n^{-1} C_k^{n-1} C_{k'}^m C_j^m C_{m+n-k-k'-j-2} a^{j+k+1} b^{m+n-j-k-1}.
 \end{aligned}$$

We now make the following index redefinitions:

$$\begin{aligned}
 \text{Term 1: } p &\equiv n - k - k' + 1/2 \implies k' = n - k - p + 1/2 \\
 \text{Term 2: } p &\equiv n - k - k' - 3/2 \implies k' = n - k - p - 3/2, \quad (6.18)
 \end{aligned}$$

where in both cases p is a positive half-integer. We then find:

$$\begin{aligned}
 \text{Term 1: } k' \geq 0 &\implies k \leq n - p + 1/2 \quad \text{and} \quad k, k' \geq 0 \implies p \leq n + 1/2 \\
 \text{Term 2: } k' \geq 0 &\implies k \leq n - p - 3/2 \quad \text{and} \quad k, k' \geq 0 \implies p \leq n - 3/2. \quad (6.19)
 \end{aligned}$$

This gives:

$$\begin{aligned}
 \left[\gamma_{R^{+-}, mn}^{F(1)(1)} \right]_{n>0} &= - \sum_{p=1/2}^{n+1/2} b^{m+n} \left(\sum_{k=0}^{n-p+1/2} {}^{n-1}C_k {}^{n-1}C_{n-k-p+1/2} a^k b^{-k} \right) \\
 &\quad \times \left(\sum_{j=0}^{m+p-1/2} {}^mC_j {}^mC_{m+p-j-1/2} a^j b^{-j} \right) \\
 &\quad + \sum_{p=1/2}^{n-3/2} ab^{m+n-1} \left(\sum_{k=0}^{n-p-3/2} {}^{n-1}C_k {}^{n-1}C_{n-k-p-3/2} a^k b^{-k} \right) \\
 &\quad \times \left(\sum_{j=0}^{m+p-1/2} {}^mC_j {}^mC_{m+p-j-1/2} a^j b^{-j} \right). \tag{6.20}
 \end{aligned}$$

Careful examination of (6.20) reveals that the $p = \frac{1}{2}$ terms either independently vanish or precisely cancel for all values of n . We thus shift the lower bound of the sums to $p = \frac{3}{2}$. Performing the k and j sums in *Mathematica*, we are left with:

$$\begin{aligned}
 \left[\gamma_{R^{+-}, mn}^{F(1)(1)} \right]_{n>0} &= - \sum_{p=\frac{3}{2}}^{n+\frac{1}{2}} {}^{n-1}C_{n-p+\frac{1}{2}} {}^mC_{p-\frac{1}{2}} a^{p-\frac{1}{2}} b^{m+n-p+\frac{1}{2}} \tag{6.21} \\
 &\quad \times {}_2F_1 \left(1-n, p-n-\frac{1}{2}; p-\frac{1}{2}; \frac{a}{b} \right) {}_2F_1 \left(p-m-\frac{1}{2}, -m; p+\frac{1}{2}; \frac{a}{b} \right) \\
 &\quad + \sum_{p=\frac{3}{2}}^{n-\frac{3}{2}} {}^{n-1}C_{n-p-\frac{3}{2}} {}^mC_{p-\frac{1}{2}} a^{p+\frac{1}{2}} b^{m+n-p-\frac{1}{2}} \\
 &\quad \times {}_2F_1 \left(1-n, p-n+\frac{3}{2}; p+\frac{3}{2}; \frac{a}{b} \right) {}_2F_1 \left(p-m-\frac{1}{2}, -m; p+\frac{1}{2}; \frac{a}{b} \right).
 \end{aligned}$$

The other cases are performed similarly, and we present only the results along with the index redefinitions used.

$$\begin{aligned}
 \gamma_{R^{+-}, mn}^{F(1)(2)} &= - \sum_{p=\frac{1}{2}}^{\min(m-\frac{1}{2}, n-\frac{5}{2})} {}^{n-1}C_{n-p-\frac{5}{2}} {}^mC_{m-p-\frac{1}{2}} a^{n-1} b^{m+1} \\
 &\quad {}_2F_1 \left(p-m+\frac{1}{2}, -m; p+\frac{3}{2}; \frac{a}{b} \right) {}_2F_1 \left(1-n, p-n+\frac{5}{2}; p+\frac{5}{2}; \frac{b}{a} \right) \\
 &\quad + \sum_{p=\frac{1}{2}}^{\min(m-\frac{1}{2}, n-\frac{1}{2})} {}^{n-1}C_{n-p-\frac{1}{2}} {}^mC_{m-p-\frac{1}{2}} a^n b^m \\
 &\quad \times {}_2F_1 \left(p-m+\frac{1}{2}, -m; p+\frac{3}{2}; \frac{a}{b} \right) {}_2F_1 \left(1-n, p-n+\frac{1}{2}; p+\frac{1}{2}; \frac{b}{a} \right), \tag{6.22}
 \end{aligned}$$

where in (6.12), we substituted:

$$\begin{aligned} \text{Term 1: } k' &\equiv n + p - k - \frac{5}{2} \\ \text{Term 2: } k' &\equiv n + p - k - \frac{1}{2}. \end{aligned} \tag{6.23}$$

$$\begin{aligned} \left[\gamma_{R^{+-}, mn}^{F(2)(1)} \right]_{n=0} &= \sum_{j=0}^m {}^m C_j {}^m C_{m-j} a^{m-j} b^j = a^m {}_2F_1 \left(-m, -m, 1, \frac{b}{a} \right) \\ \left[\gamma_{R^{+-}, mn}^{F(2)(1)} \right]_{n>0} &= \sum_{p=\frac{1}{2}}^{\min(m-\frac{1}{2}, n-\frac{1}{2})} n^{-1} C_{n-p-\frac{1}{2}} {}^m C_{m-p-\frac{1}{2}} a^m b^n \\ &\quad \times {}_2F_1 \left(1-n, p-n+\frac{1}{2}; p+\frac{1}{2}; \frac{a}{b} \right) {}_2F_1 \left(p-m+\frac{1}{2}, -m; p+\frac{3}{2}; \frac{b}{a} \right) \\ &\quad - \sum_{p=\frac{1}{2}}^{\min(m-\frac{1}{2}, n-\frac{5}{2})} n^{-1} C_{n-p-\frac{5}{2}} {}^m C_{m-p-\frac{1}{2}} a^{m+1} b^{n-1} \\ &\quad \times {}_2F_1 \left(1-n, p-n+\frac{5}{2}; p+\frac{5}{2}; \frac{a}{b} \right) {}_2F_1 \left(p-m+\frac{1}{2}, -m; p+\frac{3}{2}; \frac{b}{a} \right), \end{aligned} \tag{6.24}$$

where in (6.13), we substituted:

$$\begin{aligned} \text{Term 1: } k' &\equiv n - p - k - \frac{1}{2} \\ \text{Term 2: } k' &\equiv n - p - k - \frac{5}{2}. \end{aligned} \tag{6.25}$$

$$\begin{aligned} \gamma_{R^{+-}, mn}^{F(2)(2)} &= \sum_{p=\frac{3}{2}}^{n+\frac{1}{2}} n^{-1} C_{p-\frac{3}{2}} {}^m C_{m-p+\frac{1}{2}} a^{m+n-p+\frac{1}{2}} b^{p-\frac{1}{2}} \\ &\quad \times {}_2F_1 \left(1-n, p-n-\frac{1}{2}; p-\frac{1}{2}; \frac{b}{a} \right) {}_2F_1 \left(p-m-\frac{1}{2}, -m; p+\frac{1}{2}; \frac{b}{a} \right) \\ &\quad - \sum_{p=\frac{3}{2}}^{n-\frac{3}{2}} n^{-1} C_{p+\frac{1}{2}} {}^m C_{m-p+\frac{1}{2}} a^{m+n-p-\frac{1}{2}} b^{p+\frac{1}{2}} \\ &\quad \times {}_2F_1 \left(1-n, p-n+\frac{3}{2}; p+\frac{3}{2}; \frac{b}{a} \right) {}_2F_1 \left(p-m-\frac{1}{2}, -m; p+\frac{1}{2}; \frac{b}{a} \right), \end{aligned} \tag{6.26}$$

where in (6.14), we substituted:

$$\begin{aligned} \text{Term 1: } k' &\equiv n + p - k - 5/2 \\ \text{Term 2: } k' &\equiv n + p - k - 1/2. \end{aligned} \tag{6.27}$$

Using the $a \leftrightarrow b$ symmetry, it is clear that the relations (6.15) are satisfied. We also show in appendix E the additional relationship:

$$\left[\gamma_{R^{+-}, mn}^{F(1)(1)} \right]_{n>0} = - \left[\gamma_{R^{+-}, mn}^{F(2)(1)} \right]_{m, n>0}. \tag{6.28}$$

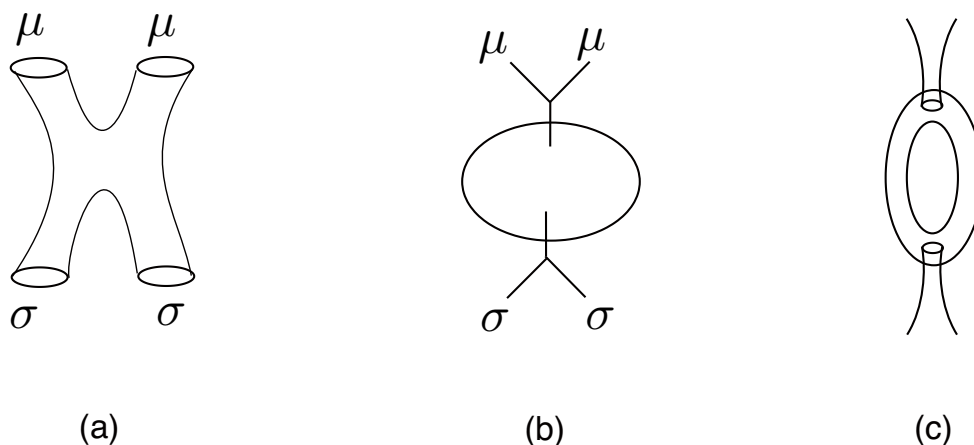


Figure 4. (a) An amplitude in the Ising model where we have put σ states on two legs and μ states on the other two. (b) The same amplitude drawn differently, using conformal invariance. (c) A 1-loop amplitude; only a single set of complete fields should run in the loop..

We thus have only a single linearly independent fermion coefficient for the non-zero modes (along with coefficients for zero modes).

7 NS vs R sectors in the final state

We have started with two singly wound component strings, both in the R sector. This sector was chosen because the physical D1D5 black hole has R periodicity for its fermions. We applied two twists to this initial state, and then tried to determine the final state by taking an inner product with all possible states.

Here we found the following. We can get a nonzero inner product by taking a final state which has two singly wound strings, each in the R sector. But we also find a nonzero inner product if we take a final state which has two singly wound strings, each in the NS sector. Does this mean that the initial R sector state can evolve to a state in the NS sector?

Such an evolution would be surprising, since it is known that all states of the D1D5 system can be accounted for by the different choices of R sector states. So why are we finding a nonzero inner product with NS sector states? To understand this, we consider an analogy: the example of the Ising model.

The Ising model has 3 primary operators: I, σ, ϵ . However it has a dual representation as well, where the operator order parameter σ is replaced by the disorder parameter μ .

Let us use σ as our basic variable; not the dual variable μ . Now consider the CFT amplitude shown in figure 4(a). The initial state has two copies of the CFT on a circle, each created by the application of a σ operator to the identity. The final state again has two copies of the CFT on a circle, but each state is generated by the application of μ to the identity state. Is this transition amplitude nonzero?

We can redraw this amplitude in the form shown in figure 4(b). Now it looks like the amplitude should be nonzero” the two σ operators fuse to the identity, and the two

μ operators do as well. The insertion of two identity operators into the sphere gives a nonzero amplitude.

So should we conclude that an initial state made of two σ operators can evolve to a state containing two μ operators? If this were true, then we would have to allow μ operators to run in loops like the one shown in figure 4(c), along with the σ operator that can run in the loop.

But at this stage we note that this looks like double counting: a complete set of operators is given by I, σ, ϵ , without the inclusion of μ . The operator μ does not need to be included in loops. This shows up in the fact that μ is not local with respect to σ .

In a similar way we note that the NS and R sectors are different. The R sector state is created by applying a spin field to the NS sector state, and this spin field is not local with respect to the fermions in the theory. We can start with the NS sector, and stay in this sector all along, or we can spectral flow to the R sector, and stay in the R sector. But we should not include both sectors. The formalism we have used started with the states in the R sector, thus we should keep only the inner products we computed for transitions to the R sector, and not the ones for the NS sector.

8 Analysis

In the single twist case, we were able to write down explicit closed form expressions for $\gamma_{mn}^B, \gamma_{mn}^F$. In the 2-twist case, we do not have such closed form expressions. In the single twist case the dependence on the position of the twist was trivial; it was just an overall phase. In the 2-twist case we expect a nontrivial dependence on the separation between the two twists Δw . We would also like to see how the $\gamma_{mn}^B, \gamma_{mn}^F$ fall off for large values of m, n . While it is not possible to see these features from the form of these functions as finite sums, numerical analysis turns out to reveal the information we seek. In this section we will explore the functions $\gamma_{mn}^B, \gamma_{mn}^F$ by computing them using Mathematica.

8.1 Wick rotation and origin choice

Note that the physical problem of the black hole needs us to do all computations for Minkowski signature on the cylinder. Thus we first rotate Euclidean time to Minkowski time.

$$\tau \rightarrow it \implies w = it + i\sigma. \tag{8.1}$$

Here w is purely imaginary.

Now that we have returned to Minkowski time, we make the convenient choice of setting our origin midway between w_1 and w_2 . We thus have:

$$t_2 = -t_1 = \frac{\Delta t}{2}, \quad \sigma_2 = -\sigma_1 = \frac{\Delta \sigma}{2}. \tag{8.2}$$

With the choice of coordinates, we make several observations that will simplify our algebra:

$$\begin{aligned} z_2 &= e^{w_1} = e^{i\frac{\Delta t + \Delta \sigma}{2}} = a + b + 2\sqrt{ab} \\ z_1 &= e^{w_1} = e^{-i\frac{\Delta t + \Delta \sigma}{2}} = a + b - 2\sqrt{ab}, \end{aligned} \tag{8.3}$$

from which we find:

$$\begin{aligned} a + b &= \frac{z_2 + z_1}{2} \\ \sqrt{ab} &= \frac{z_2 - z_1}{4}, \end{aligned} \tag{8.4}$$

and thus:⁴

$$\begin{aligned} \sqrt{a} &= \frac{1}{2} (\sqrt{z_2} + \sqrt{z_1}) = \cos\left(\frac{\Delta t + \Delta\sigma}{4}\right) \\ \sqrt{b} &= \frac{1}{2} (\sqrt{z_2} - \sqrt{z_1}) = i \sin\left(\frac{\Delta t + \Delta\sigma}{4}\right). \end{aligned} \tag{8.5}$$

This gives

$$\begin{aligned} a &= \cos^2\left(\frac{\Delta w}{4i}\right) \\ b &= -\sin^2\left(\frac{\Delta w}{4i}\right) = 1 - a, \end{aligned} \tag{8.6}$$

eliminating the parameter b . This also makes it clear that our γ parameters will be periodic in Δw , with a period of $4\pi i$.

8.2 Numerical analysis

Now that we have reduced our γ coefficients to a single spacetime parameter, we can numerically calculate their values for various choices of m , n , and Δw . We present here several figures plotting the behavior of the bosonic and fermionic γ coefficients as we vary these parameters.

Figures 5 and 6 show the behavior of the bosonic and fermionic $\gamma_{mn}^{(1)(1)}$ as m or n are varied. The value of Δw is held fixed. By considering a set of such graphs, ranging over different values of Δw , we can conjecture a simple approximate dependence of the γ on m, n :

$$\begin{aligned} \gamma_{mn}^{B,(1)(1)} &\sim \frac{1}{(m+n)} \frac{\Gamma(m + \frac{1}{2})\Gamma(n + \frac{1}{2})}{\Gamma(m+1)\Gamma(n+1)} \\ \gamma_{mn}^{F,(1)(1)} &\sim \frac{1}{(m+n)} \frac{\Gamma(m + \frac{1}{2})\Gamma(n + \frac{1}{2})}{\Gamma(m)\Gamma(n+1)}. \end{aligned} \tag{8.7}$$

This leads us to guess a simple relationship for the magnitudes of the bosonic and fermionic γ coefficients. Fixing the sign, we have:

$$\gamma_{mn}^{F,(1)(1)} = -m\gamma_{mn}^{B,(1)(1)}. \tag{8.8}$$

We prove this relationship analytically in appendix F. The relationship also exists for the first-order γ coefficients, though this was not noted in [25].

⁴We choose to have $b \rightarrow 0$ as $\Delta w \rightarrow 0$ so that the initial and final Copy 2 strands have the same image in this limit.

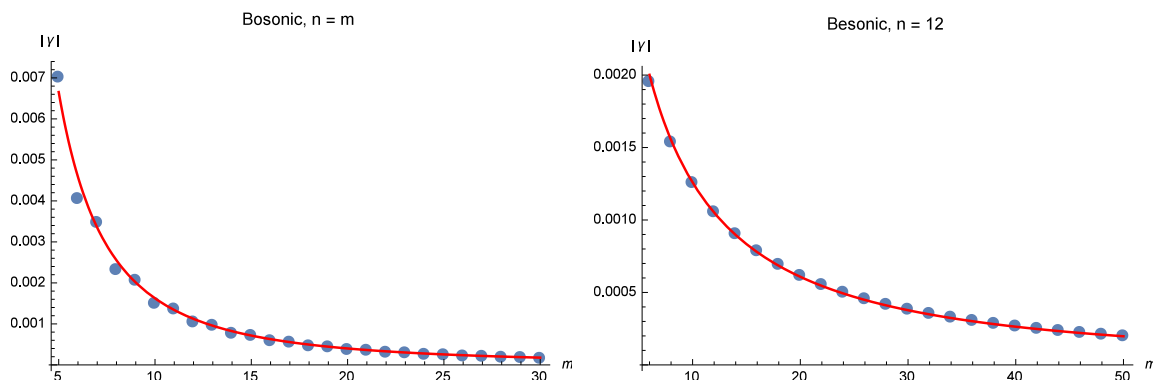


Figure 5. Numerical values of $|\gamma_{mn}^{B,(1)(1)}|$ for $\Delta w = i\pi/2$. For $n = m$ (left), a fit of the form $\Gamma^2(m + \frac{1}{2})/[m\Gamma^2(m + 1)]$ is shown. For $n = 12$ (right), a fit of the form $\Gamma(m + \frac{1}{2})/[\Gamma(m + 1)(m + 12)]$ is shown. The coefficient clearly behaves as $\Gamma(m + \frac{1}{2})\Gamma(n + \frac{1}{2})/[\Gamma(m + 1)\Gamma(n + 1)(m + n)]$. Both plots show only $m > 4$, as these simple fits do not work well for very low values of m and n . The plot on the right shows only even m , as the coefficient vanishes at $\Delta w = i\pi/2$ when $m + n$ is odd.

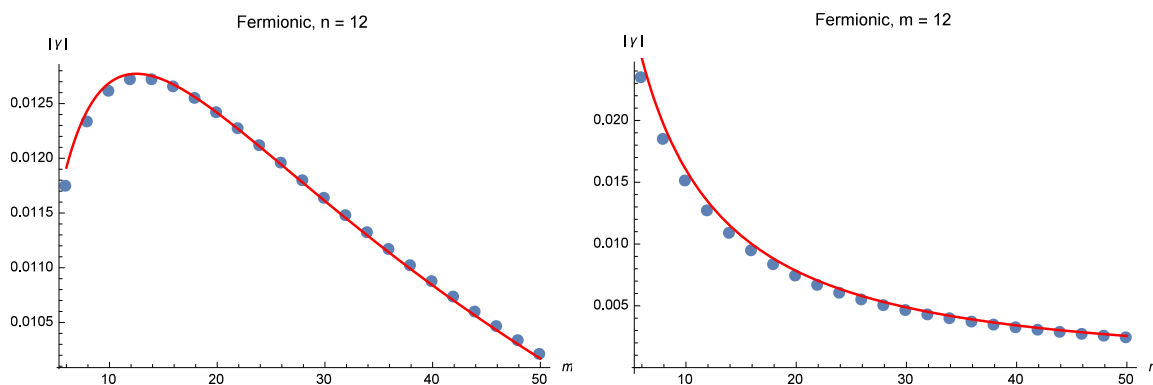


Figure 6. Numerical values of $|\gamma_{mn}^{F,(1)(1)}|$ for $\Delta w = i\pi/2$. For $n = 12$ (left), a fit of the form $\Gamma(m + \frac{1}{2})/[\Gamma(m)(m + 12)]$ is shown. For $m = 12$ (right), a fit of the form $\Gamma(n + \frac{1}{2})/[\Gamma(n + 1)(n + 12)]$ is shown. The coefficient clearly behaves as $\Gamma(m + \frac{1}{2})\Gamma(n + \frac{1}{2})/[\Gamma(m)\Gamma(n + 1)(m + n)]$. Both plots show only even $m, n > 4$, as these simple fits do not work well for very low values of m and n and this coefficient also vanishes at $\Delta w = i\pi/2$ when $m + n$ is odd.

For large m and n , (8.8) simplifies to:

$$\begin{aligned} \gamma_{mn}^{B,(1)(1)} &\sim \frac{1}{\sqrt{mn(m+n)}} \\ \gamma_{mn}^{F,(1)(1)} &\sim \sqrt{\frac{m}{n}} \frac{1}{(m+n)}, \end{aligned} \tag{8.9}$$

which is exactly the same behavior as the single-twist γ . Thus for any specific separation, each two-twist γ looks just like a re-scaled single-twist γ .

Next we fix values for m and n while varying Δw . As seen in figure 7, this analysis reveals that the Δw dependance is an oscillation with a frequency set by $m + n$. The amplitude of the oscillations, however does not remain constant over the full range of Δw .

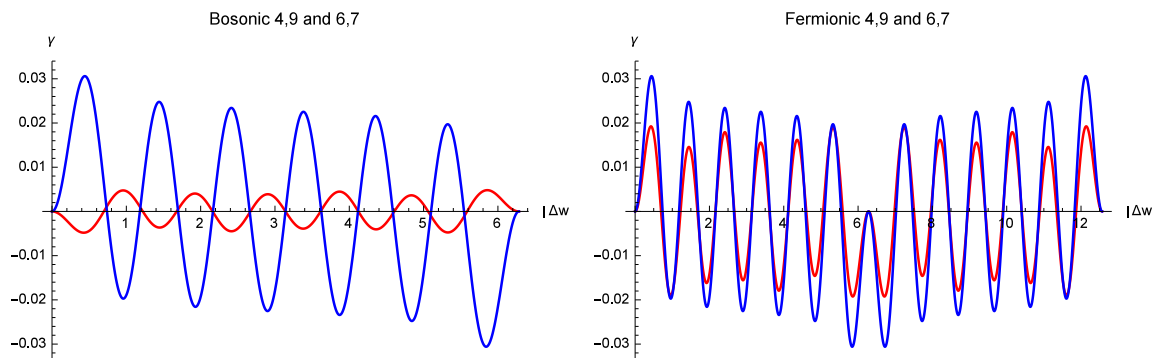


Figure 7. Numerical values of $|\gamma_{mn}^{F,(1)(1)}|$ for $m = 4, n = 9$ (red) and $m = 6, n = 7$ (blue). The Δw dependence is clearly an oscillatory function. The number of peaks is determined by $m + n$.

Combined with the previous result we reach a remarkable conclusion: the two-twist γ coefficients have the exact same large m, n behavior as their one-twist counterparts, except for the addition of an oscillation in Δw .

9 Discussion

Our goal is to see thermalization in the D1D5 CFT. This thermalization is expected to be dual to the process of black hole formation in the gravity theory. Thus if can understand thermalization in the CFT, we get a window on the some of the deepest questions associated to black holes.

The free D1D5 CFT itself has been very useful; it gives correctly the extremal and near-extremal entropies of the black hole, as well as the greybody factors of radiation from the near-extremal hole. But the free theory cannot show thermalization, since the excitations on the D1D5 brane system are free fields at the orbifold point. To reach the supergravity point, we have to consider the deformation operator O that takes us away from the orbifold point.

In our earlier studies of this problem, we had worked out the effect of one insertion of the deformation operator. Such an insertion takes two singly wound component strings, for example, to one doubly wound component string. Though this is a nontrivial interaction, creating excitations on the doubly wound string, it is hard to see evidence of thermalization at this level. One reason is that the vacuum state of the doubly wound string is different from the vacuum state of the two initial singly wound strings, so it is hard to disentangle the effect of this vacuum change from any changes due to interactions among the excitations on the string. In the present paper we looked at a 1-loop effect where we have two insertions of O . The initial state has two singly wound strings. The first twist converts this to a doubly wound string, and the second twist returns us to two singly wound strings. Thus the vacuum structure of the initial and final states is the same, and any changes to the state can be seen as actual excitations that have been generated on the strings.

In the single twist case we learned that the effect of a twist operator generated a squeezed state, described by Bogoliubov coefficients γ^B, γ^F . We have a similar structure

for the case of two twists. We have in this paper focused on the effect of the twists σ_2^+ , and left out the action of the supercharge contours that surround these twist insertions. Though these supercharge insertions are essential to the form of the final state after the deformation, they can be split off from the twist insertions and written as supercharge actions on the initial and final states. This computation is straightforward but messy, so we hope to return to it elsewhere and in this paper have focused only on the more essential physics: the computation of the Bogoliubov coefficients arising from the twists.

In the single twist case one could find closed form expressions for the Bogoliubov coefficients. In the two twist case the best we could do was write the Bogoliubov coefficients as a finite sum of hypergeometric functions. This form does not make the properties of the coefficients obvious. In particular, we expect the coefficients $\gamma_{mn}^B, \gamma_{mn}^F$ to fall at large values of m, n . This is not obvious from any term in the sums. But numerical work showed that the coefficients do fall off as desired. Further, there was a very simple relation between the coefficients with one twist and the coefficients with two twists that can be obtained as a numerical approximation in the limit of large m, n . The two twist case also brings in a new feature that was not present in the 1-twist case: the coefficients depend on the separation Δw between the twists. We found that there was a simple oscillatory behavior of the coefficients with Δw .

To obtain a complete analysis of the deformed theory at second order, we must use the Bogoliubov coefficients obtained here, but we must also allow excitations above the vacuum in the initial state, and apply the supercharge contours arising from the deformation operators. We expect to return to this in a future work.

A CFT notation and conventions

We follow the notation of [25, 26], which we record here for convenience. We have 4 real left moving fermions $\psi_1, \psi_2, \psi_3, \psi_4$ which we group into doublets $\psi^{\alpha A}$ as follows:

$$\begin{pmatrix} \psi^{++} \\ \psi^{-+} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1 + i\psi_2 \\ \psi_3 + i\psi_4 \end{pmatrix} \tag{A.1}$$

$$\begin{pmatrix} \psi^{+-} \\ \psi^{--} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_3 - i\psi_4 \\ -(\psi_1 - i\psi_2) \end{pmatrix}. \tag{A.2}$$

Here $\alpha = (+, -)$ is an index of the subgroup $SU(2)_L$ of rotations on S^3 and $A = (+, -)$ is an index of the subgroup $SU(2)_1$ from rotations in T^4 . The reality conditions on the individual fermions are

$$(\psi_i)^\dagger = \psi_i \quad \Rightarrow \quad (\psi^{\alpha A})^\dagger = -\epsilon_{\alpha\beta} \epsilon_{AB} \psi^{\beta B}. \tag{A.3}$$

One can introduce doublets ψ^\dagger , whose components are given by

$$(\psi^\dagger)_{\alpha A} = (\psi^{\alpha A})^\dagger, \tag{A.4}$$

from which the reality condition is given by

$$(\psi^\dagger)_{\alpha A} = -\epsilon_{\alpha\beta} \epsilon_{AB} \psi^{\beta B}. \tag{A.5}$$

The 2-point functions are

$$\langle \psi^{\alpha A}(z)(\psi^\dagger)_{\beta B}(w) \rangle = \delta_\beta^\alpha \delta_B^A \frac{1}{z-w}, \quad \langle \psi^{\alpha A}(z)\psi^{\beta B}(w) \rangle = -\epsilon^{\alpha\beta} \epsilon^{AB} \frac{1}{z-w}, \quad (\text{A.6})$$

where we have:

$$\epsilon_{12} = 1, \quad \epsilon^{12} = -1, \quad \psi_A = \epsilon_{AB} \psi^B, \quad \psi^A = \epsilon^{AB} \psi_B. \quad (\text{A.7})$$

There are 4 real left moving bosons X_1, X_2, X_3, X_4 , which can be grouped into a matrix:

$$X_{A\dot{A}} = \frac{1}{\sqrt{2}} X_i \sigma_i = \frac{1}{\sqrt{2}} \begin{pmatrix} X_3 + iX_4 & X_1 - iX_2 \\ X_1 + iX_2 & -X_3 + iX_4 \end{pmatrix}, \quad (\text{A.8})$$

where $\sigma_i = (\sigma_a, iI)$. The reality condition on the individual bosons is given by

$$(X_i)^\dagger = X_i \quad \Rightarrow \quad (X_{A\dot{A}})^\dagger = -\epsilon^{AB} \epsilon^{\dot{A}\dot{B}} X_{B\dot{B}}. \quad (\text{A.9})$$

One can introduce a matrix, X^\dagger , with components

$$(X^\dagger)^{A\dot{A}} = (X_{A\dot{A}})^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} X_3 - iX_4 & X_1 + iX_2 \\ X_1 - iX_2 & -X_3 - iX_4 \end{pmatrix}, \quad (\text{A.10})$$

from which the reality condition is given by

$$(X^\dagger)^{A\dot{A}} = -\epsilon^{AB} \epsilon^{\dot{A}\dot{B}} X_{B\dot{B}}. \quad (\text{A.11})$$

The 2-point functions are

$$\begin{aligned} \langle \psi X_{A\dot{A}}(z)(\psi X^\dagger)^{B\dot{B}}(w) \rangle &= -\frac{1}{(z-w)^2} \delta_A^B \delta_{\dot{A}}^{\dot{B}}, \\ \langle \psi X_{A\dot{A}}(z)\psi X_{B\dot{B}}(w) \rangle &= \frac{1}{(z-w)^2} \epsilon_{AB} \epsilon_{\dot{A}\dot{B}}. \end{aligned} \quad (\text{A.12})$$

The chiral algebra is generated by the operators

$$J^a = -\frac{1}{4} (\psi^\dagger)_{\alpha A} (\sigma^{Ta})^\alpha{}_\beta \psi^{\beta A} \quad (\text{A.13})$$

$$G_A^\alpha = \psi^{\alpha A} \psi X_{A\dot{A}}, \quad (G^\dagger)_\alpha^{\dot{A}} = (\psi^\dagger)_{\alpha A} \psi (X^\dagger)^{A\dot{A}} \quad (\text{A.14})$$

$$T = -\frac{1}{2} (\psi X^\dagger)^{A\dot{A}} \psi X_{A\dot{A}} - \frac{1}{2} (\psi^\dagger)_{\alpha A} \psi \psi^{\alpha A} \quad (\text{A.15})$$

$$(G^\dagger)_\alpha^{\dot{A}} = -\epsilon_{\alpha\beta} \epsilon^{\dot{A}\dot{B}} G_{\dot{B}}^\beta, \quad G_A^\alpha = -\epsilon^{\alpha\beta} \epsilon_{\dot{A}\dot{B}} (G^\dagger)_{\dot{B}}^{\dot{A}}. \quad (\text{A.16})$$

These operators generate the OPE algebra

$$J^a(z)J^b(z') \sim \delta^{ab} \frac{1}{(z-z')^2} + i\epsilon^{abc} \frac{J^c}{z-z'} \quad (\text{A.17})$$

$$J^a(z)G_{\dot{A}}^\alpha(z') \sim \frac{1}{(z-z')^2} \frac{1}{2} (\sigma^{aT})^\alpha{}_\beta G_{\dot{A}}^\beta \quad (\text{A.18})$$

$$G_{\dot{A}}^\alpha(z)(G^\dagger)_{\dot{\beta}}^\beta(z') \sim -\frac{2}{(z-z')^3} \delta_\beta^\alpha \delta_{\dot{A}}^{\dot{\beta}} - \delta_{\dot{A}}^{\dot{\beta}} (\sigma^{Ta})^\alpha{}_\beta \left[\frac{2J^a}{(z-z')^2} + \frac{\psi J^a}{(z-z')} \right] - \frac{1}{(z-z')} \delta_\beta^\alpha \delta_{\dot{A}}^{\dot{\beta}} T \quad (\text{A.19})$$

$$T(z)T(z') \sim \frac{3}{(z-z')^4} + \frac{2T}{(z-z')^2} + \frac{\psi T}{(z-z')} \quad (\text{A.20})$$

$$T(z)J^a(z') \sim \frac{J^a}{(z-z')^2} + \frac{\psi J^a}{(z-z')} \quad (\text{A.21})$$

$$T(z)G_{\dot{A}}^\alpha(z') \sim \frac{\frac{3}{2}G_{\dot{A}}^\alpha}{(z-z')^2} + \frac{\psi G_{\dot{A}}^\alpha}{(z-z')} \quad (\text{A.22})$$

Note that

$$J^a(z)\psi^{\gamma C}(z') \sim \frac{1}{2} \frac{1}{z-z'} (\sigma^{aT})^\gamma{}_\beta \psi^{\beta C} \quad (\text{A.23})$$

The above OPE algebra gives the commutation relations

$$[J_m^a, J_n^b] = \frac{m}{2} \delta^{ab} \delta_{m+n,0} + i\epsilon^{abc} J_{m+n}^c \quad (\text{A.24})$$

$$[J_m^a, G_{\dot{A},n}^\alpha] = \frac{1}{2} (\sigma^{aT})^\alpha{}_\beta G_{\dot{A},m+n}^\beta \quad (\text{A.25})$$

$$\{G_{\dot{A},m}^\alpha, G_{\dot{B},n}^\beta\} = \epsilon_{\dot{A}\dot{B}} \left[\left(m^2 - \frac{1}{4} \right) \epsilon^{\alpha\beta} \delta_{m+n,0} + (m-n) (\sigma^{aT})^\alpha{}_\gamma \epsilon^{\gamma\beta} J_{m+n}^a + \epsilon^{\alpha\beta} L_{m+n} \right] \quad (\text{A.26})$$

$$[L_m, L_n] = \frac{m(m^2 - \frac{1}{4})}{2} \delta_{m+n,0} + (m-n) L_{m+n} \quad (\text{A.27})$$

$$[L_m, J_n^a] = -n J_{m+n}^a \quad (\text{A.28})$$

$$[L_m, G_{\dot{A},n}^\alpha] = \left(\frac{m}{2} - n \right) G_{\dot{A},m+n}^\alpha \quad (\text{A.29})$$

B Ramond vacua notation

Here we define our notation for the various Ramond vacua in the untwisted sector. There are two copies, which are not technically separate Hilbert spaces. We start with the vacuum

$$|0_R^-\rangle^{(1)} \otimes |0_R^-\rangle^{(2)} \equiv |v\rangle \quad (\text{B.1})$$

and act on it with various fermion zero modes to construct the other Ramond vacua. In order to be consistent with ACM, we also require something along the lines of

$$|0_R^+\rangle^{(i)} = d_0^{(i)++} d_0^{(i)-} |0_R^-\rangle^{(i)}, \quad (\text{B.2})$$

though we do not actually have states containing only one of the two copies.

We now present a table defining our notation for the various vacua.

$$\begin{aligned}
 |v\rangle &= |0_R^-\rangle^{(1)} \otimes |0_R^-\rangle^{(2)} \\
 d_0^{(1)+-} |v\rangle &= |0_R\rangle^{(1)} \otimes |0_R^-\rangle^{(2)} \\
 d_0^{(1)++} |v\rangle &= |\tilde{0}_R\rangle^{(1)} \otimes |0_R^-\rangle^{(2)} \\
 d_0^{(1)++} d_0^{(1)+-} |v\rangle &= |0_R^+\rangle^{(1)} \otimes |0_R^-\rangle^{(2)}
 \end{aligned} \tag{B.3}$$

$$\begin{aligned}
 d_0^{(2)+-} |v\rangle &= |0_R^-\rangle^{(1)} \otimes |0_R\rangle^{(2)} \\
 d_0^{(1)+-} d_0^{(2)+-} |v\rangle &= |0_R\rangle^{(1)} \otimes |0_R\rangle^{(2)} \\
 d_0^{(1)++} d_0^{(2)+-} |v\rangle &= |\tilde{0}_R\rangle^{(1)} \otimes |0_R\rangle^{(2)} \\
 d_0^{(1)++} d_0^{(1)+-} d_0^{(2)+-} |v\rangle &= |0_R^+\rangle^{(1)} \otimes |0_R\rangle^{(2)}
 \end{aligned} \tag{B.4}$$

$$\begin{aligned}
 d_0^{(2)++} |v\rangle &= |0_R^-\rangle^{(1)} \otimes |\tilde{0}_R\rangle^{(2)} \\
 d_0^{(1)+-} d_0^{(2)++} |v\rangle &= |0_R\rangle^{(1)} \otimes |\tilde{0}_R\rangle^{(2)} \\
 d_0^{(1)++} d_0^{(2)++} |v\rangle &= |\tilde{0}_R\rangle^{(1)} \otimes |\tilde{0}_R\rangle^{(2)} \\
 d_0^{(1)++} d_0^{(1)+-} d_0^{(2)++} |v\rangle &= |0_R^+\rangle^{(1)} \otimes |\tilde{0}_R\rangle^{(2)}
 \end{aligned} \tag{B.5}$$

$$\begin{aligned}
 d_0^{(2)++} d_0^{(2)+-} |v\rangle &= |0_R^-\rangle^{(1)} \otimes |0_R^+\rangle^{(2)} \\
 d_0^{(1)+-} d_0^{(2)++} d_0^{(2)+-} |v\rangle &= |0_R\rangle^{(1)} \otimes |0_R^+\rangle^{(2)} \\
 d_0^{(1)++} d_0^{(2)++} d_0^{(2)+-} |v\rangle &= |\tilde{0}_R\rangle^{(1)} \otimes |0_R^+\rangle^{(2)} \\
 d_0^{(1)++} d_0^{(1)+-} d_0^{(2)++} d_0^{(2)+-} |v\rangle &= |0_R^+\rangle^{(1)} \otimes |0_R^+\rangle^{(2)}.
 \end{aligned} \tag{B.6}$$

These relations hold for both the initial (pre-twists) and final (post-twists) sectors.

C Proof of the exponential form

In this section we prove the exponential form of $|\chi(w_1, w_2)\rangle$ for the case of fermions in the NS sector. The other cases may be treated similarly.

The method of the proof is straightforward. We look at capping the state $|\chi(w_1, w_2)\rangle$ with a general fermionic NS state $\hat{Q}^\dagger |0_{NS}\rangle^{(1)} |0_{NS}\rangle^{(2)}$. This provides the relationship

$$\mathcal{A}(\hat{Q}) \equiv \frac{{}^{(1)}\langle 0_{NS} | {}^{(2)}\langle 0_{NS} | \hat{Q} | \chi(w_1, w_2) \rangle}{{}^{(1)}\langle 0_{NS} | {}^{(2)}\langle 0_{NS} | \chi(w_1, w_2) \rangle} = \frac{{}_t\langle 0_{NS} | \hat{Q}' | 0_{NS} \rangle_t}{{}_t\langle 0_{NS} | 0_{NS} \rangle}. \tag{C.1}$$

The rightmost expression can be calculated from the behavior of the fermion modes under coordinate transformations and spectral flows, while the middle expression can be calculated from our guess for the form of $|\chi(w_1, w_2)\rangle$. Showing that these two methods are consistent demonstrates that we have the correct form for $|\chi(w_1, w_2)\rangle$.

C.1 Middle expression

Let us begin by taking a close look at the form presented in (3.24). Because we are working with pairs of fermion creation operators in the exponent, all of the terms in the exponent

commute. We can thus rewrite the fermion portion in the NS sector as

$$\begin{aligned}
 |\chi(w_1, w_2)\rangle_{NS}^F &= e^{\sum_{(i),(j)} \sum_{r,s>0} \gamma_{NS,rs}^{F(i)(j)} \left(d_{-r}^{(i)f,++} d_{-s}^{(j)f,--} - d_{-r}^{(i)f,+} d_{-s}^{(j)f,-} \right)} |0_{NS}\rangle^{(1)} |0_{NS}\rangle^{(2)} \\
 &= \prod_{(i),(j)} \prod_{r,s>0} e^{\gamma_{NS,rs}^{F(i)(j)} \left(d_{-r}^{(i)f,++} d_{-s}^{(j)f,--} - d_{-r}^{(i)f,+} d_{-s}^{(j)f,-} \right)} |0_{NS}\rangle^{(1)} |0_{NS}\rangle^{(2)} \\
 &= \prod_{A,B} \prod_{(i),(j)} \prod_{r,s>0} \left[1 + \epsilon_{AB} \gamma_{NS,rs}^{F(i)(j)} d_{-r}^{(i)f,+A} d_{-s}^{(j)f,-B} \right] |0_{NS}\rangle^{(1)} |0_{NS}\rangle^{(2)}. \quad (C.2)
 \end{aligned}$$

From this form, we can see clearly that any state with an odd number of fermion excitations will have no overlap with $|\chi(w_1, w_2)\rangle$. So let us start with a two-excitation state. This case is quite simple:

$$\begin{aligned}
 \mathcal{A} \left(d_r^{(i)f,+A} d_s^{(j)f,-B} \right) &= \frac{{}^{(1)}\langle 0_{NS} | {}^{(2)}\langle 0_{NS} | d_r^{(i)f,+A} d_s^{(j)f,-B} | \chi(w_1, w_2) \rangle}{({}^{(1)}\langle 0_{NS} | {}^{(2)}\langle 0_{NS} | \chi(w_1, w_2) \rangle)} \\
 &= \epsilon^{AC} \epsilon^{BD} \epsilon_{CD} \gamma_{NS,rs}^{F(i)(j)} \\
 &= \epsilon^{AB} \gamma_{NS,rs}^{F(i)(j)}, \quad (C.3)
 \end{aligned}$$

where the fact that the coefficient is proportional to ϵ^{AB} simply means that the combinations $d^{++}d^{--}$ and $d^{+-}d^{-+}$ yield coefficients which differ only by an overall sign.

Now what happens when we cap with a state that contains more fermion excitations? We know that we must have an even number of excitations, so they come in pairs. We can thus calculate \mathcal{A} in two steps. First, we write out all possible ways to group the fermion excitations into pairs, accounting for the overall sign required to anticommute the operators into the appropriate pairings. Then each pairing combination provides a contribution equal to the product of the amplitude for each individual pair within that combination. Adding the contributions for each combination then gives us the total amplitude. We can write this schematically as:

$$\mathcal{A}(d_1 d_2 d_3 \dots) = \sum_{\text{pairing combinations}} (-1)^p \left(\prod_{\{d_i d_j\}} \mathcal{A}(d_i d_j) \right), \quad (C.4)$$

where p is the number of anticommutations we need to perform to achieve the pairing configuration.

C.2 Right expression

Let us compare the previous result to the calculation in the t plane. In section 3 we outlined a series of coordinate maps and spectral flows which bring $|\chi(w_1, w_2)\rangle$ to an empty NS vacuum in the t plane. Under any combination of such transformations, the modes $d_r^{(i)f,\alpha A}$ behave in general as:

$$d_r^{(1)f,\alpha A} \rightarrow d_r^{\prime(1)f,\alpha A} = \frac{1}{2\pi i} \oint_{t=\infty} \psi^{\alpha A}(t) h_r^\alpha(t) dt \quad (C.5)$$

$$d_r^{(2)f,\alpha A} \rightarrow d_r^{\prime(2)f,\alpha A} = \frac{1}{2\pi i} \oint_{t=0} \psi^{\alpha A}(t) h_r^\alpha(t) dt, \quad (C.6)$$

where the function h_r^α is identical for each copy but will in general depend on the original mode number r and the relevant spectral flow charge α .

Now we expand the function h_r^α as a polynomial in t , which can in general include negative powers of t . It turns out that the function h_r^α consists of a product of powers of t and binomials in t , which makes such an expansion straightforward. We perform the expansion with large t for Copy 1 and small t for copy 2. We then find:

$$d_r^{(1)f,\alpha A} = \sum_{p=-\infty}^{\infty} C_{rp}^{(1)\alpha} \frac{1}{2\pi i} \oint_{t=\infty} \psi^{\alpha A}(t) t^p dt \tag{C.7}$$

$$d_r^{(2)f,\alpha A} = \sum_{p=-\infty}^{\infty} C_{rp}^{(2)\alpha} \frac{1}{2\pi i} \oint_{t=0} \psi^{\alpha A}(t) t^p dt, \tag{C.8}$$

where the coefficients are not in general equal because we are expanding the function in two different regions.

Recall that after these coordinate maps and spectral flows, we have removed all insertions in the t plane. We can thus close all punctures with the NS vacuum and smoothly deform our contours. This allows us to deform the contour around $t = \infty$ for the copy 1 mode into a contour around $t = 0$. We then find that in general:

$$d_r^{(i)f,\alpha A} = \sum_{p=-\infty}^{\infty} C_{rp}^{(i)\alpha} \frac{1}{2\pi i} \oint_{t=0} \psi^{\alpha A}(t) t^p dt \tag{C.9}$$

$$= \sum_{p=-\infty}^{\infty} C_{rp}^{(i)\alpha} \tilde{d}_p^{\alpha A}. \tag{C.10}$$

From this expression we see that when capping with a two-excitation state, the right side of (C.1) becomes:

$$\mathcal{A} \left(d_r^{(i)f,+A} d_s^{(j)f,-B} \right) = \sum_{p,q=-\infty}^{\infty} C_{rp}^{(i)+} C_{sq}^{(j)-} \frac{\langle 0_{NS} | : \tilde{d}_p^{+A} \tilde{d}_q^{-B} : | 0_{NS} \rangle_t}{t \langle 0_{NS} | 0_{NS} \rangle}, \tag{C.11}$$

where the colons indicate radial ordering of the contours in the modes, which may not be the same as the initial τ ordering on the cylinder, as copy 1 modes always map to contours at larger radii than copy 2 modes.

The amplitude in the numerator on the right of (C.11) is only nonzero when the two fermion modes can contract together, with the outer mode annihilating the inner mode. However, we know the anticommutator is:

$$\left\{ \tilde{d}_p^{+A}, \tilde{d}_q^{-B} \right\} = -\epsilon^{+-} \epsilon^{AB} \delta_{p+q,0}, \tag{C.12}$$

which is proportional to ϵ^{AB} . We thus find that for any fermion pair,

$$\mathcal{A} \left(d_r^{(i)f,+A} d_s^{(j)f,-B} \right) = \epsilon^{AB} D_{rs}^{(i)(j)}, \tag{C.13}$$

which matches the result from (C.3).

Now what happens when we cap with a state that contains more fermion excitations? In general, we have:

$$\mathcal{A}(d_1 d_2 d_3 \dots) = \sum_{p_1, p_2, p_3, \dots} (C_{p_1} C_{p_2} C_{p_3} \dots) \frac{{}_t\langle 0_{NS} | : \tilde{d}_{p_1} \tilde{d}_{p_2} \tilde{d}_{p_3} \dots : | 0_{NS} \rangle_t}{{}_t\langle 0_{NS} | 0_{NS} \rangle}. \quad (\text{C.14})$$

It is clear that the right side vanishes when there are an odd number of fermion modes, as such a case will always leave at least one uncontracted mode to annihilate the vacuum bra or ket. With an even number of modes, the amplitude can be calculated by performing all possible combinations of contractions in the expression on the right side and adding each contribution together. However, we also have the relation

$$\frac{{}_t\langle 0_{NS} | \overbrace{\tilde{d}_{p_1} \tilde{d}_{p_2}} \overbrace{\tilde{d}_{p_3} \tilde{d}_{p_4}} | 0_{NS} \rangle_t}{{}_t\langle 0_{NS} | 0_{NS} \rangle} = \frac{{}_t\langle 0_{NS} | : \tilde{d}_{p_1} \tilde{d}_{p_2} : | 0_{NS} \rangle_t}{{}_t\langle 0_{NS} | 0_{NS} \rangle} \frac{{}_t\langle 0_{NS} | : \tilde{d}_{p_3} \tilde{d}_{p_4} : | 0_{NS} \rangle_t}{{}_t\langle 0_{NS} | 0_{NS} \rangle}. \quad (\text{C.15})$$

We thus find:

$$\begin{aligned} \mathcal{A}(d_1 d_2 d_3 \dots) &= \sum_{\text{pairing combinations}} (-1)^p \left[\prod_{\{p_i, p_j\}} \left(\sum_{p_i, p_j} C_{p_i} C_{p_j} \frac{{}_t\langle 0_{NS} | : \tilde{d}_{p_i} \tilde{d}_{p_j} : | 0_{NS} \rangle_t}{{}_t\langle 0_{NS} | 0_{NS} \rangle} \right) \right] \\ &= \sum_{\text{pairing combinations}} (-1)^p \left(\prod_{\{d_i, d_j\}} \mathcal{A}(d_i d_j) \right), \end{aligned} \quad (\text{C.16})$$

where p is again the number of anticommutations we must perform to reach the particular pair combination. Since this overall sign depends only on how we group the operators in \hat{Q} and not on any of the specifics of the coordinate maps or spectral flows, each p that appears here takes the same value as in in (C.4). Thus the two relations are identical, and we see that the claimed form for $|\chi(w_1, w_2)\rangle$ is correct.

D Proof of copy 1-copy 2 interchange symmetry

Using the vacua listed in appendix (B), we can look at the effects of $\gamma_{R+-;mn}^{F(i)(j)}$ and check the desired (1) \leftrightarrow (2) symmetry. We first note that $\gamma_{R+-;0,0}^{F(2)(1)} = 1$. We will also introduce the shorthand

$$\gamma_{R+-;m,0}^{F(2)(1)} \equiv A_m \implies \gamma_{R+-;m,0}^{F(1)(1)} = -A_m. \quad (\text{D.1})$$

With this notation, we find:

$$\begin{aligned} |\chi(w_1, w_1)\rangle &= \left(1 + d_0^{(2)f,++} d_0^{(1)f,--} \right) \left(1 - d_0^{(2)f,+ -} d_0^{(1)f,-+} \right) \left(1 + A_m d_{-m}^{(2)f,++} d_0^{(1)f,--} \right) \\ &\quad \times \left(1 - A_{m'} d_{-m'}^{(2)f,+ -} d_0^{(1)f,-+} \right) \left(1 - A_n d_{-n}^{(1)f,++} d_0^{(1)f,--} \right) \\ &\quad \times \left(1 + A_{n'} d_{-n'}^{(1)f,+ -} d_0^{(1)f,-+} \right) d_0^{(1)f,++} d_0^{(1)f,+ -} e^{\hat{Q}} |0_R^-\rangle^{(1)} |0_R^-\rangle^{(2)}, \end{aligned} \quad (\text{D.2})$$

where \hat{Q} now contains no zero modes and the repeated mode indices are summed over positive integers

Noting the relations (4.22), (4.23) and (6.15), we see that the operator \hat{Q} is symmetric under (1) \leftrightarrow (2). We must then inquire as to the symmetry of the rest of (D.2). In doing so, let us write P as the operation which swaps copies (1) and (2). We then note that the notation outlined in (B) implies:

$$\begin{aligned}
 P\left(|0_R\rangle^{(1)}|\tilde{0}_R\rangle^{(2)}\right) &= P\left(d_0^{(1)f,+}d_0^{(2)f,++}|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)}\right) \\
 &= d_0^{(2)f,+}d_0^{(1)f,++}|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} \\
 &= -d_0^{(1)f,++}d_0^{(2)f,+}|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} \\
 &= -|\tilde{0}_R\rangle^{(1)}|0_R\rangle^{(2)}.
 \end{aligned} \tag{D.3}$$

And since $P^2 = 1$, we also have:

$$P\left(|\tilde{0}_R\rangle^{(1)}|0_R\rangle^{(2)}\right) = -|0_R\rangle^{(1)}|\tilde{0}_R\rangle^{(2)}. \tag{D.4}$$

On the other hand,

$$P\left(|0_R^+\rangle^{(1)}|0_R^-\rangle^{(2)}\right) = |0_R^-\rangle^{(1)}|0_R^+\rangle^{(2)} \tag{D.5}$$

$$P\left(|0_R^-\rangle^{(1)}|0_R^+\rangle^{(2)}\right) = |0_R^+\rangle^{(1)}|0_R^-\rangle^{(2)}. \tag{D.6}$$

Using these relations, we find:

$$\begin{aligned}
 |v\rangle &\equiv \left(1 + d_0^{(2)f,++}d_0^{(1)f,--}\right)\left(1 - d_0^{(2)f,+}d_0^{(1)f,++}\right)d_0^{(1)f,++}d_0^{(1)f,+}|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} \\
 &= |0_R^+\rangle^{(1)}|0_R^-\rangle^{(2)} + |0_R\rangle^{(1)}|\tilde{0}_R\rangle^{(2)} - |\tilde{0}_R\rangle^{(1)}|0_R\rangle^{(2)} + |0_R^-\rangle^{(1)}|0_R^+\rangle^{(2)},
 \end{aligned} \tag{D.7}$$

which is symmetric under P . We now write:

$$\begin{aligned}
 |\chi(w_1, w_1)\rangle &= \left(1 + A_m d_{-m}^{(2)f,++}d_0^{(1)f,--}\right)\left(1 - A_{m'} d_{-m'}^{(2)f,+}d_0^{(1)f,++}\right) \\
 &\quad \times \left(1 - A_n d_{-n}^{(1)f,++}d_0^{(1)f,--}\right)\left(1 + A_{n'} d_{-n'}^{(1)f,+}d_0^{(1)f,++}\right) e^{\hat{Q}}|v\rangle.
 \end{aligned} \tag{D.8}$$

At this point it is necessary to make use of an interesting property of the state $|v\rangle$. When applying zero modes to this state, we notice:

$$\begin{aligned}
 d_0^{(1)f,--}|v\rangle &= -|0_R\rangle^{(1)}|0_R^-\rangle^{(2)} + |0_R^-\rangle^{(1)}|0_R\rangle^{(2)} \\
 d_0^{(2)f,--}|v\rangle &= |0_R\rangle^{(1)}|0_R^-\rangle^{(2)} - |0_R^-\rangle^{(1)}|0_R\rangle^{(2)} \\
 d_0^{(1)f,-+}|v\rangle &= -|\tilde{0}_R\rangle^{(1)}|0_R^-\rangle^{(2)} + |0_R^-\rangle^{(1)}|\tilde{0}_R\rangle^{(2)} \\
 d_0^{(2)f,-+}|v\rangle &= |\tilde{0}_R\rangle^{(1)}|0_R^-\rangle^{(2)} - |0_R^-\rangle^{(1)}|\tilde{0}_R\rangle^{(2)}.
 \end{aligned} \tag{D.9}$$

Thus whenever we have a copy (1) zero mode with negative charge acting on $|v\rangle$ we can substitute a copy (2) zero mode in its place:

$$d_0^{(1)f,-\alpha}|v\rangle = -d_0^{(2)f,-\alpha}|v\rangle. \tag{D.10}$$

Applying this substitution to the second line of (D.8), we have:

$$\begin{aligned}
 |\chi(w_1, w_1)\rangle &= \left(1 + A_m d_{-m}^{(2)f,++}d_0^{(1)f,--}\right)\left(1 - A_{m'} d_{-m'}^{(2)f,+}d_0^{(1)f,++}\right) \\
 &\quad \times \left(1 + A_n d_{-n}^{(1)f,++}d_0^{(2)f,--}\right)\left(1 - A_{n'} d_{-n'}^{(1)f,+}d_0^{(2)f,++}\right) e^{\hat{Q}}|v\rangle,
 \end{aligned} \tag{D.11}$$

which is manifestly symmetric.

E Proof of γ^F and γ^B relations

In this section we analytically prove the relations:

$$\begin{aligned} \gamma_{mn}^{B(1)(1)} &= -\gamma_{mn}^{B(2)(1)} \\ \left[\gamma_{R^{+-}, mn}^{F(1)(1)} \right]_{n>0} &= - \left[\gamma_{R^{+-}, mn}^{F(2)(1)} \right]_{m, n>0}. \end{aligned} \quad (\text{E.1})$$

E.1 Proof of $\gamma_{mn}^{B(1)(1)} = -\gamma_{mn}^{B(2)(1)}$

Beginning with $\gamma_{mn}^{B(i)(j)}$, we have the relation

$$\gamma_{mn}^{B(1)(1)} = -\gamma_{mn}^{B(2)(1)} \quad (\text{E.2})$$

for $a \neq 0$. Let us write the expressions for both functions:

$$\gamma_{mn}^{B(1)(1)} = \frac{1}{mn} \sum_{l=1}^n l^n C_l^m C_l^a l^m b^{m+n-l} {}_2F_1 \left(-n, l-n; l+1; \frac{a}{b} \right) {}_2F_1 \left(-m, l-m; l+1; \frac{a}{b} \right), \quad (\text{E.3})$$

and

$$\gamma_{mn}^{B(2)(1)} = -\frac{1}{mn} \sum_{l=1}^n l^m C_l^n C_l^a m^m b^n {}_2F_1 \left(-m, l-m; l+1; \frac{b}{a} \right) {}_2F_1 \left(-n, l-n; l+1; \frac{a}{b} \right). \quad (\text{E.4})$$

Let us investigate $\gamma^{B(2)(1)}$. Considering the *first* hypergeometric function with the identity:

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \frac{\Gamma(1-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a-b+1)} \left(\frac{-1}{z} \right)^a \left(1 - \frac{1}{z} \right)^{c-a-b} \\ &\quad \times {}_2F_1 \left(1-b, c-b; c-a-b+1; 1 - \frac{1}{z} \right) \\ &\quad + \frac{\Gamma(1-a)\Gamma(c)}{\Gamma(b-a+1)\Gamma(c-b)} \left(\frac{1}{z} \right)^b {}_2F_1 \left(b, b-c+1; b-a+1; \frac{1}{z} \right), \end{aligned} \quad (\text{E.5})$$

we find:

$$\begin{aligned} {}_2F_1 \left(-m, l-m; l+1; \frac{b}{a} \right) &= \frac{\Gamma(m+1)\Gamma(l+\frac{1}{2})}{\Gamma(l-m)\Gamma(2m+2)} \left(\frac{-a}{b} \right)^{-m} \left(1 - \frac{a}{b} \right)^{2m+1} \\ &\quad \times {}_2F_1 \left(m-l+1, m+1; 2m+2; 1 - \frac{a}{b} \right) \\ &\quad + \frac{\Gamma(m+1)\Gamma(l+1)}{\Gamma(l+1)\Gamma(m+1)} \left(\frac{a}{b} \right)^{l-m} {}_2F_1 \left(l-m, -m; l+1; \frac{a}{b} \right). \end{aligned} \quad (\text{E.6})$$

The first term vanishes because

$$\frac{1}{\Gamma(l-m)} = 0 \quad \text{when } l-m \leq 0, \quad (\text{E.7})$$

which is always the case. We also note that the ratio of gamma functions for the second term is one. Using the symmetry relation

$${}_2F_1(a, b, c, z) = {}_2F_1(b, a, c, z), \quad (\text{E.8})$$

we find that

$${}_2F_1\left(l-m, -m; l+1; \frac{a}{b}\right) = {}_2F_1\left(-m, l-m; l+1; \frac{a}{b}\right). \quad (\text{E.9})$$

Inserting this into (E.4), we obtain:

$$\begin{aligned} \gamma_{mn}^{B(2)(1)} &= -\frac{1}{mn} \sum_{l=1}^n l^m C_l^n C_l a^l b^{m+n-l} {}_2F_1\left(-m, l-m; l+1; \frac{a}{b}\right) {}_2F_1\left(-n, l-n; l+1; \frac{a}{b}\right) \\ &= -\gamma_{mn}^{B(1)(1)}. \end{aligned} \quad (\text{E.10})$$

E.2 Proof of $\left[\gamma_{R+-,mn}^{F(1)(1)}\right]_{n>0} = -\left[\gamma_{R+-,mn}^{F(2)(1)}\right]_{m,n>0}$

Here we prove the $\gamma_{R+-,mn}^{F(i)(j)}$ relation

$$\left[\gamma_{R+-,mn}^{F(1)(1)}\right]_{n>0} = -\left[\gamma_{R+-,mn}^{F(2)(1)}\right]_{m,n>0} \quad (\text{E.11})$$

for $a \neq 0$. Let us first write expressions for both functions. We have:

$$\begin{aligned} \left[\gamma_{R+-,mn}^{F(1)(1)}\right]_{n>0} &= -\sum_{p=\frac{3}{2}}^{n+\frac{1}{2}} n^{-1} C_{n-p+\frac{1}{2}}^m C_{p-\frac{1}{2}} a^{p-\frac{1}{2}} b^{m+n-p+\frac{1}{2}} \\ &\quad \times {}_2F_1\left(1-n, p-n-\frac{1}{2}; p-\frac{1}{2}; \frac{a}{b}\right) {}_2F_1\left(p-m-\frac{1}{2}, -m; p+\frac{1}{2}; \frac{a}{b}\right) \\ &\quad + \sum_{p=\frac{3}{2}}^{n-\frac{3}{2}} n^{-1} C_{n-p-\frac{3}{2}}^m C_{p-\frac{1}{2}} a^{p+\frac{1}{2}} b^{m+n-p-\frac{1}{2}} \\ &\quad \times {}_2F_1\left(1-n, p-n+\frac{3}{2}; p+\frac{3}{2}; \frac{a}{b}\right) {}_2F_1\left(p-m-\frac{1}{2}, -m; p+\frac{1}{2}; \frac{a}{b}\right), \end{aligned} \quad (\text{E.12})$$

and

$$\begin{aligned} \left[\gamma_{R+-,mn}^{F(2)(1)}\right]_{m,n>0} &= \sum_{p=\frac{1}{2}}^{\min(m-\frac{1}{2}, n-\frac{1}{2})} n^{-1} C_{n-p-\frac{1}{2}}^m C_{m-p-\frac{1}{2}} a^m b^n \\ &\quad \times {}_2F_1\left(1-n, p-n+\frac{1}{2}; p+\frac{1}{2}; \frac{a}{b}\right) {}_2F_1\left(p-m+\frac{1}{2}, -m; p+\frac{3}{2}; \frac{b}{a}\right) \\ &\quad - \sum_{p=\frac{1}{2}}^{\min(m-\frac{1}{2}, n-\frac{5}{2})} n^{-1} C_{n-p-\frac{5}{2}}^m C_{m-p-\frac{1}{2}} a^{m+1} b^{n-1} \\ &\quad \times {}_2F_1\left(1-n, p-n+\frac{5}{2}; p+\frac{5}{2}; \frac{a}{b}\right) {}_2F_1\left(p-m+\frac{1}{2}, -m; p+\frac{3}{2}; \frac{b}{a}\right). \end{aligned} \quad (\text{E.13})$$

We shall now investigate $\gamma^{F(2)(1)}$. First let us make the index shift

$$p \rightarrow p-1. \quad (\text{E.14})$$

For $\gamma^{F(2)(1)}$ this gives

$$\begin{aligned}
 \left[\gamma_{R^{+-}, mn}^{F(2)(1)} \right]_{m, n > 0} &= \sum_{p=\frac{3}{2}}^{\min(m+\frac{1}{2}, n+\frac{1}{2})} n^{-1} C_{n-p+\frac{1}{2}}^m C_{m-p+\frac{1}{2}} a^m b^n \\
 &\times {}_2F_1 \left(1-n, p-n-\frac{1}{2}; p-\frac{1}{2}; \frac{a}{b} \right) {}_2F_1 \left(p-m-\frac{1}{2}, -m; p+\frac{1}{2}; \frac{b}{a} \right) \\
 &- \sum_{p=\frac{3}{2}}^{\min(m+\frac{1}{2}, n-\frac{3}{2})} n^{-1} C_{n-p-\frac{3}{2}}^m C_{m-p+\frac{1}{2}} a^{m+1} b^{n-1} \\
 &\times {}_2F_1 \left(1-n, p-n+\frac{3}{2}; p+\frac{3}{2}; \frac{a}{b} \right) {}_2F_1 \left(p-m-\frac{1}{2}, -m; p+\frac{1}{2}; \frac{b}{a} \right).
 \end{aligned} \tag{E.15}$$

Now using the identity given in (E.5) to transform the second hypergeometric function of each term as well as the interchange symmetry given in (E.8), we obtain:

$$\begin{aligned}
 {}_2F_1 \left(p-m-\frac{1}{2}, -m, p-\frac{1}{2}; \frac{b}{a} \right) &= {}_2F_1 \left(-m, p-m-\frac{1}{2}; p-\frac{1}{2}; \frac{b}{a} \right) \\
 &= \frac{\Gamma(m+1)\Gamma(p+1/2)}{\Gamma(p-m-1/2)\Gamma(2m+2)} \left(\frac{-a}{b} \right)^{-m} \left(1-\frac{a}{b} \right)^{2m+1} \\
 &\times {}_2F_1 \left(m-p+3/2, m; 2m+2; 1-\frac{a}{b} \right) \\
 &+ \frac{\Gamma(m+1)\Gamma(p+1/2)}{\Gamma(m+1)\Gamma(p+1/2)} \\
 &\left(\frac{a}{b} \right)^{p-m-1/2} {}_2F_1 \left(p-m-\frac{1}{2}, -m; p+\frac{1}{2}; \frac{a}{b} \right).
 \end{aligned} \tag{E.16}$$

We see that the first term vanishes because

$$\frac{1}{\Gamma(p-m-1/2)} = 0 \quad \text{when } p \leq m+1/2, \tag{E.17}$$

which is always the case. Additionally, the ratio of gamma functions in the second term is one. Therefore we have

$${}_2F_1 \left(p-m-\frac{1}{2}, -m, p-\frac{1}{2}; \frac{b}{a} \right) = \left(\frac{a}{b} \right)^{p-m-1/2} {}_2F_1 \left(p-m-\frac{1}{2}, -m; p+\frac{1}{2}; \frac{a}{b} \right). \tag{E.18}$$

Inserting this term into (E.15), we obtain:

$$\begin{aligned}
 \left[\gamma_{R^{+-}, mn}^{F(2)(1)} \right]_{m, n > 0} &= \sum_{p=\frac{3}{2}}^{\min(m+\frac{1}{2}, n+\frac{1}{2})} n^{-1} C_{n-p+\frac{1}{2}}^m C_{m-p+\frac{1}{2}} a^{p-\frac{1}{2}} b^{m+n-p+\frac{1}{2}} \\
 &\times {}_2F_1 \left(1-n, p-n-\frac{1}{2}; p-\frac{1}{2}; \frac{a}{b} \right) {}_2F_1 \left(p-m-\frac{1}{2}, -m; p+\frac{1}{2}; \frac{a}{b} \right) \\
 &- \sum_{p=\frac{3}{2}}^{\min(m+\frac{1}{2}, n-\frac{3}{2})} n^{-1} C_{n-p-\frac{3}{2}}^m C_{m-p+\frac{1}{2}} a^{p+\frac{1}{2}} b^{m+n-p-\frac{1}{2}} \\
 &\times {}_2F_1 \left(1-n, p-n+\frac{3}{2}; p+\frac{3}{2}; \frac{a}{b} \right) {}_2F_1 \left(p-m-\frac{1}{2}, -m; p+\frac{1}{2}; \frac{a}{b} \right) \\
 &= - \left[\gamma_{R^{+-}, mn}^{F(1)(1)} \right]_{n > 0}.
 \end{aligned} \tag{E.19}$$

F $\gamma_{mn}^{B(i)(j)}$ and $\gamma_{R+-,mn}^{F(i)(j)}$ relation

Here we prove the relationship

$$\gamma_{R+-,mn}^{F(i)(j)} = -m\gamma_{mn}^{B(i)(j)}. \quad (\text{F.1})$$

We start by applying $G_{0,+}^{+(p)}$ to $|\chi(w_1, w_2)\rangle$.

$$\begin{aligned} \sum_{(p)} G_{0,+}^{+(p)} |\chi(w_1, w_2)\rangle &= \sum_{(p)} G_{0,+}^{+(p)} \sigma_2^+(w_2) \sigma_2^+(w_1) |0_R^-\rangle^{(1)} |0_R^-\rangle^{(2)} \\ &= \left(\frac{1}{2\pi i} \int_{w_2} dw G_+^+(w) \sigma_2^+(w_2) \right) \sigma_2^+(w_1) |0_R^-\rangle^{(1)} |0_R^-\rangle^{(2)} \\ &\quad + \sigma_2^+(w_2) \left(\frac{1}{2\pi i} \int_{w_1} dw G_+^+(w) \sigma_2^+(w_1) \right) |0_R^-\rangle^{(1)} |0_R^-\rangle^{(2)} \\ &\quad + \sigma_2(w_1) \sigma_2^+(w_1) \sum_{(j)} G_{0,+}^{+(j)} |0_R^-\rangle^{(1)} |0_R^-\rangle^{(2)} \\ &= 0. \end{aligned} \quad (\text{F.2})$$

where $j, p \in \{1, 2\}$. We have used the fact that $G_{0,A}^+$ acting on a single $\sigma_2^+(w_i)$ is zero,⁵ as well as the fact that $G_{0,+}^+$ annihilates the negative Ramond vacuum. More explicitly:

$$\begin{aligned} \frac{1}{2\pi i} \int_{w_i} dw G_+^+(w) \sigma_2^+(w_i) &= 0 \\ G_{0,+}^{+(j)} |0_R^-\rangle^{(1)} |0_R^-\rangle^{(2)} &= 0. \end{aligned} \quad (\text{F.3})$$

Using the form of $|\chi(w_1, w_2)\rangle$ given in (3.24), where we have excluded the *NS* sector states, we find:

$$\begin{aligned} 0 &= \sum_{(p)} G_{0,+}^{+(p)} |\chi(w_1, w_2)\rangle \quad (\text{F.4}) \\ &= \sum_{(p)} G_{0,+}^{+(p)} C_{R+-} \exp \left[\sum_{(i),(j)} \sum_{k,l>0} \gamma_{kl}^{B(i)(j)} \left(-\alpha_{++,-k}^{(i)f} \alpha_{--,-l}^{(j)f} + \alpha_{+,-,-k}^{(i)f} \alpha_{-+,-l}^{(j)f} \right) \right] \\ &\quad \times \exp \left[\sum_{(i),(j)} \sum_{k,l \geq 0} \gamma_{R+-,kl}^{F(i)(j)} \left(d_{-k}^{(i)f,++} d_{-l}^{(j)f,--} - d_{-k}^{(i)f,+-} d_{-l}^{(j)f,-+} \right) \right] |0_R^+\rangle^{(1)} |0_R^-\rangle^{(2)} \\ &= C_{R+-} \sum_{(i),(j),(p)} \left[\sum_{k,l>0} \gamma_{kl}^{B(i)(j)} \left[G_{0,+}^{+(p)}, -\alpha_{++,-k}^{(i)f} \alpha_{--,-l}^{(j)f} + \alpha_{+,-,-k}^{(i)f} \alpha_{-+,-l}^{(j)f} \right] \right. \\ &\quad \left. + \sum_{k,l \geq 0} \gamma_{R+-,kl}^{F(i)(j)} \left\{ G_{0,+}^{+(p)}, d_{-k}^{(i)f,++} d_{-l}^{(j)f,--} - d_{-k}^{(i)f,+-} d_{-l}^{(j)f,-+} \right\} \right] |\chi(w_1, w_2)\rangle \end{aligned}$$

⁵This can be seen clearly from mapping to the t plane and performing the appropriate spectral flow, giving an annihilation operator on the t plane vacuum rather than the creation operator that a $G_{0,A}^-$ contour produces.

$$\begin{aligned}
 &= C_{R+-} \sum_{(i),(j),(p)} \left[\sum_{k,l>0} \gamma_{kl}^{B(i)(j)} \left(-\alpha_{++,-k}^{(i)f} \left[G_{0,+}^{(p)+}, \alpha_{--,-l}^{(j)f} \right] + \left[G_{0,+}^{(p)+}, \alpha_{+,-,-k}^{(i)f} \right] \alpha_{-+,-l}^{(j)f} \right) \right. \\
 &\quad \left. + \sum_{k,l \geq 0} \gamma_{kl}^{B(i)(j)} \left(-d_{-k}^{(i)f,++} \{ G_{0,+}^{(p)+}, d_{-l}^{(j)f,--} \} + \{ G_{0,+}^{(p)+}, d_{-l}^{(j)f,-+} \} d_{-k}^{(i)f,+} \right) \right] |\chi(w_1, w_2)\rangle.
 \end{aligned}$$

We now apply the (anti)commutation relations:

$$\begin{aligned}
 \left[G_{\dot{A},0}^{(i)\alpha}, \alpha_{\dot{B}\dot{B},m}^{(j)} \right] &= -im\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\delta^{(i)(j)}d_m^{(i)\alpha A} \\
 \{ G_{\dot{A},0}^{(i)\alpha}, d_m^{(j),\beta B} \} &= i\epsilon^{\alpha\beta}\epsilon^{AB}\delta^{(i)(j)}\alpha_{\dot{A}\dot{A},m}^{(i)}.
 \end{aligned} \tag{F.5}$$

This gives:

$$\begin{aligned}
 0 &= C_{R+-} \sum_{(i),(j),(p)} \left[\sum_{k,l>0} \gamma_{kl}^{B(i)(j)} \left(-\alpha_{++,-k}^{(i)f} (-il)\delta^{(p)(j)}d_{-l}^{(p)f,++} + (-ik)\delta^{(p)(i)}d_{-k}^{(p)f,+} \alpha_{-+,-l}^{(j)f} \right) \right. \\
 &\quad \left. + \sum_{k,l \geq 0} \gamma_{kl}^{F(i)(j)} \left(-d_{-k}^{(i)f,++} (i)\delta^{(p)(j)}\alpha_{++,-l}^{(p)f} + (-i)\delta^{(p)(j)}\alpha_{-+,-l}^{(p)f} d_{-k}^{(i)f,+} \right) \right] |\chi(w_1, w_2)\rangle \\
 &= -iC_{R+-} \sum_{(i),(p)} \sum_{k,l>0} \left(l\gamma_{kl}^{B(i)(p)}\alpha_{++,-k}^{(i)f}d_{-l}^{(p)f,++} + k\gamma_{kl}^{B(p)(i)}\alpha_{-+,-l}^{(i)f}d_{-k}^{(p)f,+} \right. \\
 &\quad \left. + \gamma_{kl}^{F(i)(p)}\alpha_{++,-l}^{(p)f}d_{-k}^{(i)f,++} + \gamma_{kl}^{F(i)(p)}\alpha_{-+,-l}^{(p)f}d_{-k}^{(i)f,+} \right) |\chi(w_1, w_2)\rangle \\
 &\quad - iC_{R+-} \sum_{(i),(p)} \sum_{l>0} \left(\gamma_{0,l}^{F(i)(p)}\alpha_{++,-l}^{(p)f}d_0^{(i)f,++} + \gamma_{0,l}^{F(i)(p)}\alpha_{-+,-l}^{(p)f}d_0^{(i)f,+} \right) |\chi(w_1, w_2)\rangle, \tag{F.6}
 \end{aligned}$$

where we have used the fact that all bosonic zero modes annihilate $|\chi(w_1, w_2)\rangle$. The fermionic zero modes do not annihilate this state, but from (6.15) we know that

$$\left[\gamma_{R+-;0,l}^{F(p)(i)} \right]_{l>0} = 0, \quad i, j \in 1, 2. \tag{F.7}$$

Thus the last line of (F.6) vanishes, and we are left with

$$\begin{aligned}
 0 &= -iC_{R+-} \sum_{(i),(p)} \sum_{k,l>0} \left(l\gamma_{kl}^{B(i)(p)}\alpha_{++,-k}^{(i)f}d_{-l}^{(p)f,++} + k\gamma_{kl}^{B(p)(i)}\alpha_{-+,-l}^{(i)f}d_{-k}^{(p)f,+} \right. \\
 &\quad \left. + \gamma_{R+-,kl}^{F(i)(p)}\alpha_{++,-l}^{(p)f}d_{-k}^{(i)f,++} + \gamma_{R+-,kl}^{F(i)(p)}\alpha_{-+,-l}^{(p)f}d_{-k}^{(i)f,+} \right) |\chi(w_1, w_2)\rangle.
 \end{aligned} \tag{F.8}$$

We now make the following convenient modifications to (F.8):

$$\begin{aligned}
 \text{First Term : } & \quad k \leftrightarrow l \\
 & \quad \gamma_{kl}^B = \gamma_{lk}^B \\
 & \quad (i) \leftrightarrow (p) \\
 \text{Fourth Term : } & \quad (i) \leftrightarrow (p).
 \end{aligned} \tag{F.9}$$

This gives

$$\begin{aligned}
 0 = & -iC_{R+-} \sum_{(i),(p)} \sum_{k,l>0} \alpha_{++,-l}^{(p)f} d_{-k}^{(i)f,++} \left(k\gamma_{kl}^{B(p)(i)} + \gamma_{R+-,kl}^{F(i)(p)} \right) \\
 & - iC_{R+-} \sum_{(i),(p)} \sum_{k,l>0} \alpha_{-+,-l}^{(i)f} d_{-k}^{(p)f,+} \left(k\gamma_{kl}^{B(p)(i)} + \gamma_{R+-,kl}^{F(i)(p)} \right), \tag{F.10}
 \end{aligned}$$

which implies

$$\begin{aligned}
 \gamma_{R+-,kl}^{F(p)(i)} &= -k\gamma_{kl}^{B(p)(i)} \\
 \gamma_{R+-,kl}^{F(p)(i)} &= \gamma_{R+-,kl}^{F(i)(p)}, \tag{F.11}
 \end{aligned}$$

for all $k, l > 0$. The second was seen in (6.15). The first is a manifestation of the supersymmetry in the theory.

We will note here that we were not required to map to a cover space beyond verifying that $G_{A,0}^+ \sigma_2^+$ vanishes. Since this vanishing should be a local property of the operators, independent of our winding configurations, we expect the relation proved here to extend to arbitrary twist configurations.

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