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Discrete symmetries in the Kaluza-Klein theories

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Abstract: In theories of the Kaluza-Klein kind there are spins or total angular moments in higher dimensions which manifest as charges in the observable $d = (3 + 1)$. The charge conjugation requirement, if following the prescription in $(3 + 1)$, would transform any particle state out of the Dirac sea into the hole in the Dirac sea, which manifests as an anti-particle having all the spin degrees of freedom in d, except S^{03}, the same as the corresponding particle state. This is in contradiction with what we observe for the anti-particle. In this paper we redefine the discrete symmetries so that we stay within the subgroups of the starting group of symmetries, while we require that the angular moments in higher dimensions manifest as charges in $d = (3 + 1)$. We pay attention on spaces with even d.

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1 Introduction

Since the theorem of CPT is general under the assumption of the Lorentz invariance and causality[1] it will be true in a world with a higher number of dimensions than the empirical $(3+1)$, independent of the details of the way the extra dimensional space is realized in such a "Kaluza Klein theory" as long as the assumption of the Lorentz invariance and causality is valid. Under these conditions the CPT symmetry is the symmetry of the system whatever are the extra dimensional space details.

The concept of what the other symmetries C , P and T separately mean is in effective theories somewhat a matter of definition partly arranged so as to make them conserved if possible. A theory, which would in the low energy regime explain all the observed phenomena, are expected, however, to have the concept of the discrete symmetries well understood.

The main questions to be discussed in this article are:

- The definition of the discrete symmetries to be discrete symmetries in the higher dimensional space-time of the Kaluza Klein type, we shall denote these symmetries by $\mathcal{C}_{\mathcal{H}}$, $\mathcal{P}_{\mathcal{H}}$ and $\mathcal{T}_{\mathcal{H}}$, which means that we require extension of the so far defined discrete symmetries.

[1]The conservation of the product of all three symmetries CPT is discussed in the refs. [1–5].

- The definition of the discrete symmetries in the $(3+1)$ dimensions after letting a series or rather a group of Killing transformations to manifest the corresponding Noether's charges in $(3+1)$, we shall denote these symmetries by $\mathcal{C}_\mathcal{N}$, $\mathcal{P}_\mathcal{N}$ and $\mathcal{T}_\mathcal{N}$, which means that we analyse the type of symmetries in the extra dimensional space leading to observed symmetries in $(3+1)$.

There are two special examples of spaces with extra dimensions to the observed $(3+1)$ on which we discuss the here proposed discrete symmetries:

I. The space of M^{5+1} which breaks into $M^{3+1} \times M^2$ with M^2 which due to the zweibein compactifies in an almost S^2.[2] Both, spin connections and vielbeins, have the rotational invariance around the axes perpendicular to the M^2 surface, manifesting correspondingly the U(1) charge in $d = (3+1)$.

II. The space of M^{13+1} which breaks into $M^{3+1}\times$ the rest,[3] manifesting again rotational symmetries responsible for the charges in $d = (3+1)$, required by the *standard model*. There are vielbein and spin connection fields in $d > 4$ which manifest in $d = (3+1)$ as the corresponding gauge vector (and scalar [15–30]) fields after the compactification.

In the Kaluza-Klein kind[4] of theories [31–39] total angular moments in higher dimensions ($d > (3+1)$) manifest as charges in $(3+1)$ and the corresponding spin connections and vielbeins as the gauge fields [35–37, 40]. In the low energy regime there are indeed the spin degrees of freedom[5] which manifest as the conserved Kaluza-Klein charges [6–12, 27–29].

There are several papers [41–53] discussing discrete symmetries in higher dimensional spaces in several contexts. Authors discuss mostly only the parity symmetry, some of them the charge conjugation and very rare all the three symmetries. All discussions on discrete symmetries concern particular models. We are proposing the definition of the

[2]We showed in the refs. [6–14] that in such an almost compactified space the appropriately chosen spin connections guarantees that ($2^{\frac{d-2}{2}-1} = 2$ families of) only (either) left (or right) handed spinors keep masslessness while being coupled with the Kaluza-Klein U(1) charge to the corresponding gauge fields.

[3]First the manifold M^{13+1} breaks into $M^{7+1}\times$ M^6, M^6 manifesting the Kaluza-Klein charges of $SU(3) \times U(1)$, with (eight families of) massless spinors, and then further to $M^{3+1}\times$ $M^4\times$ M^6, manifesting the symmetry of $SO(3,1)\times$ $SU(2)\times$ $SU(2) \times U(1)$ $\times SU(3)$. Further breaks bring masses to eight families [15–30] of spinors. These further breaks could go similarly as it does in some theories with the sigma model action [31, 32]. These studies are in progress.

[4]With the Kaluza-Klein type of theories we mean the theories in which fermions carry only spins, and (may be) family quantum numbers, as internal degrees of freedom and interact correspondingly only through spin connections and vielbeins [12, 15].

[5]The lowest energy state in any bound system is (almost always) the state with orbital excitation equal to zero. Like it is the $1s$ state of the hydrogen atom. A state which has no orbital or radial excitation when $M^{(d-1)+1}$ breaks into $M^{3+1}\times$ the rest could manifest the subgroups of the spin degrees of freedom in higher dimension as charges in $(3+1)$. As an example let us cite a toy model [6–14], where the rest is the infinite disc curled into an almost S^2. The lowest energy state, which appears to be massless, has the orbital angular momentum equal to zero, so that it is the spin in $d = (5,6)$, which manifests as the charge in the Kaluza-Klein sense. For all the other states, which are massive, there are subgroups of the total angular moments in higher dimensional space which determine the Kaluza-Klein charges in $d = (3+1)$.

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discrete symmetries for the Kaluza-Klein kind of theories in even dimensional spaces.[6] This definition leads after compactification of space-time into the (so far) observed $(3+1)$-dimensional space to the measured properties of particles and anti-particles.

Extending the prescription of the discrete symmetries from $d = (3+1)$ *to any* d (eq. (2.1), (2.6), section 2), *the anti-particle to a chosen particle would have in the second quantized theory all the components of spin, or total angular momentum* (except the S^{03} component which is involved in the boost and does not contribute to the spin component; in quantum mechanics time is a parameter), *the same as the starting particle*, which means that *it would have all the charges the same as the corresponding particle.* This would be in contradiction with what we observe, namely that the anti-particle to a chosen particle has opposite charges.

In this paper, section 3, we modify the d-dimensional discrete symmetries, for example the charge conjugation operator $\mathcal{C}_{\mathcal{H}}$ (eq. (2.1)) as it would follow from the $(3+1)$ case by analogy, so that they work effectively in the $(3+1)$ dimensional theory. As we shall see below, the connection between the effective three dimensional ones (eqs. (3.1), (3.4)), $\underline{\mathbb{C}}_{\mathcal{N}}$, $\mathcal{T}_{\mathcal{N}}$ and $\mathcal{P}^{(d-1)}_{\mathcal{N}}$, and the d-dimensional ones (eq. (2.3), (2.6)), $\underline{\mathbb{C}}_{\mathcal{H}}$, $\mathcal{T}_{\mathcal{H}}$ and $\mathcal{P}^{(d-1)}_{\mathcal{H}}$, is a multiplication with products of representatives of the Lorentz group corresponding to reflections and a parity operator in higher dimensions. Our notation is that we put index $\mathcal{H}$ on discrete symmetries $\mathcal{P}^{(d-1)}_{\mathcal{H}}$, $\mathcal{T}_{\mathcal{H}}$, and $\mathcal{C}_{\mathcal{H}}$ for the whole space, i.e. d dimensions, while we use $\mathcal{N}$ for the effective discrete symmetries in only our $(3+1)$ dimensions. We define three kinds of the charge conjugation operator: $\mathcal{C}_{(\mathcal{H},\mathcal{N})}$, $\mathbb{C}_{(\mathcal{H},\mathcal{N})}$, and $\underline{\mathbb{C}}_{(\mathcal{H},\mathcal{N})}$. The first one operates on the single particle state, put on the top of the Dirac sea, transforming the positive energy state into the corresponding negative energy state (eqs. (2.1), (3.1)). The second one does the job of the first one emptying [15] (eqs. (2.5), (2.8), (3.1), (3.4)) in addition the negative energy state, creating correspondingly a hole, which manifests as a positive energy anti-particle state, put on the top of the Dirac sea. (The corresponding single anti-particle state must also solve the equations of motion as the starting particle state does, although we must understand it as a hole in the Dirac sea in the context of the Fock space). The third one eqs. (2.5), (3.1) is the operator, operating on the second quantized state (eq. (2.3)).

Discrete symmetries presented in this paper commute with the family quantum numbers — the family groups defining the equivalent representations with respect to the spin and correspondingly to all the charge groups have no influence on the here presented discrete symmetries.[7]

Although we illustrate our proposed discrete symmetries in two special cases (sec-

[6] We demonstrate in the refs. [6–14] that the masslessness of fermions can be guaranteed only in even dimensional spaces.

[7] This paper is initiated by the theory, proposed by one of us (S.N.M.B) [15–30], and named the *spin-charge-family* theory. This theory, which is offering the mechanism for generating families, predicts consequently the number of observable families at low energies. It also predicts several scalar fields which at low energies manifest as the Higgs and Yukawa couplings of the *standard model*. The spin in higher dimensions manifests as the observed charges in $d = (3+1)$, as in all the Kaluza-Klein kind of theories. The generators of the groups, determining families in this theory, commute with the total angular momentum in all dimensions. Both authors have published together several papers, proving that in non-compact spaces the break of the starting symmetry in $d > 4$ might allow massless fermions after the break [11–14] for all the Kaluza-Klein theories.

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tions 2.1, 3.1, 3.2, 4), in which fermions, section 3.2, interact in the Kaluza-Klein way with the vielbein and spin connection fields, the proposed redefinition of the discrete symmetries, marked by index $\mathcal{N}$, is expected to be quite general, offering experimentally observed properties of anti-particles in $d = (3 + 1)$ for the Kaluza-Klein kind of theories, helping also to define the discrete symmetries in $d = (3+1)$ in other cases with higher dimensional spaces.

We allow, in general, curling up extra dimensions by various bosonic background fields (metric tensors, magnetic fields, ... in extra dimensions) as far as the equations of motion determining properties of fermions in extra dimensions keep the proposed discrete symmetries conserved.

We assumed that there are a few fixed points symmetries and particular rotational symmetries around these fixed points in higher than $d = (3 + 1)$ and that there are a series of Cartan subalgebra symmetries around fixed points. Various subgroups of the rotations around (a) fixed point(s) are the "Killing forms" manifesting charges in the $(3+1)$ effective theory. (The example of compactified two extra dimensions on an almost S^2-sphere with a Killing form transformation being a rotation of the sphere illustrates that typically there shall be two fixed points. But if we had for instance an infinite extra dimensional space, only one fixed point is also possible.)

Having such one or more fixed points attached to the "Killing forms" of the charges makes it very attractive and natural to assume parity symmetry under point inversions in the fixed point(s) (the parity operation should at the same time be inversion in both fixed points, if, say, there are two). Combining such a suggestively imposed parity inversion in the extra dimensions with the parity operation in $(3 + 1)$ would lead to parity operation $\mathcal{P}^{(d-1)}_{\mathcal{H}}$ (eq. (2.6)) in all the $(d - 1)$ spatial dimensions.

Our effective parity, $\mathcal{P}^{(d-1)}_{\mathcal{N}}$, eqs. (3.1), (3.4), *proposal* does, however, *not contain any transformation of the extra dimensional coordinates* and just *got the contribution* of the γ^a matrices *adjusted so that the extra dimensional gamma matrices* γ^5, γ^6, ..., γ^{d-1},γ^d *commute with* $\mathcal{P}^{(d-1)}_{\mathcal{N}}$. *This means that this operation is quite insensitive to the extra dimensions* in such a way that it is not important if the extra dimensional space obeys any parity like symmetry.

We pay attention on spaces with even d.[8]

We do not discuss the way how does an (almost) compactification happen in our here discussed two particular cases. In the ref. [11–14] we propose vielbein and spin connection fields which are responsible for the compactification of an infinite surface into an almost S^2, but do not tell what (fermion condensates) causes the appearance of these gauge fields. These studies are for the two cases, presented in this paper, under consideration. There are, however, several proposals in the literature which suggest the compactification scheme and discuss it [54, 55]. We are not yet able to comment them from the point of view of our two discussed cases.

Our new discrete symmetries are demonstrated in section 3, in which spins or total angular moments in higher dimensions manifest charges of massless and massive spinors in $d = (3 + 1)$, by showing how do the example wave functions and quite general La-

[8]We do not pay attention on renormalizability of the theory in this paper.

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grange density transform under the $\mathcal{C}_{(\mathcal{H},\mathcal{N})}$, $\mathcal{P}^{(d-1)}_{(\mathcal{H},\mathcal{N})}$, and $\mathcal{T}_{(\mathcal{H},\mathcal{N})}$ discrete symmetries. These two particular cases concern fermions, the charges of which originate in $d > 4$, in: i.) $d = (5+1)$, when SO(5,1) breaks into SO(3,1) × U(1) [11, 12], with U(1) manifesting as the Kaluza-Klein charge in $d = (3+1)$. ii.) $d = (13+1)$, the symmetry of which breaks into SO(3,1) × SU(3) × SU(2) ×U(1), while the subgroups determine charges of fermions, manifesting before the electroweak break left handed weak charged and right handed weak chargeless massless quarks and leptons [15–30] of the *standard model*. In these two demonstrations the technique [57–59] is used to treat spinor degrees of freedom, which is very convenient for this purpose, since it is transparent and simple.

We discuss the generality of our effective proposal for discrete symmetries in section 4, in subsection 4.1 of which we discuss our two special cases, commenting also possible way of compactifying the higher dimensional space.

We shall use the concept of the Dirac sea second quantized picture, which is equivalent to the formal ordinary second quantization, because it offers, in our opinion, a nice physical understanding.

We do not study in this paper the break of the CP and correspondingly of the T symmetry.

2 Discrete symmetries in d-dimensions following the definitions in $d = (3+1)$

We start with the definition of the discrete symmetries as they follow from the prescription in $d = (3+1)$. We treat particles which carry in d dimensions only spin, no charges. They also carry the family quantum numbers, which, however, commute with the discrete family operators.

We first treat free spinors. We define the $\mathcal{C}_\mathcal{H}$ operator to be distinguished from the $\mathbb{C}_\mathcal{H}$ operator. The first transforms any single particle state $\Psi^{\rm pos}_p$, index $_p$ denotes the fermion state, which solves the Weyl equation for a free massless spinor with a positive energy and it is in the second quantized theory understood as the state above the Dirac sea, into the charge conjugate one with the negative energy $\Psi^{\rm neg}_p$ and correspondingly belonging to a state in the Dirac sea

$$\mathcal{C}_\mathcal{H} = \prod_{\gamma^a \in \Im} \gamma^a \; K\,. \tag{2.1}$$

The product of the imaginary γ^a operators is meant in the ascending order. We make a choice of γ^0, γ^1 real, γ^2 imaginary, γ^3 real, γ^5 imaginary, γ^6 real, and alternating real and imaginary ones we end up in even dimensional spaces with real γ^d. K makes complex conjugation, transforming i into $-i$.

We define $\mathbb{C}_\mathcal{H}$ as the operator, which *emptyies the negative energy state in the Dirac sea following from the starting positive energy state, and creates an anti-particle with the positive energy and all the properties of the starting single particle state above the Dirac sea — that is with the same d-momentum and all the spin degrees of freedom the same, except the S^{03} value*, as the starting single particle state. The operator S^{03} is involved in the boost (contributing in $d = (3+1)$, together with the spin, to handedness) and does not determine the (ordinary) spin. Accordingly we do not have to keep the S^{03} value a priori unchanged under the charge conjugation. Had we instead considered CP we would also have kept S^{03}.

Let $\underline{\mathbf{\Psi}}^{\dagger}_{p}[\Psi^{\rm pos}_{p}]$ be the creation operator creating a fermion in the state $\Psi^{\rm pos}_{p}$ (which is a function of $\vec{x}$) and let $\mathbf{\Psi}_{p}(\vec{x})$ be the second quantized field creating a fermion at position $\vec{x}$. Then

$$\underline{\mathbf{\Psi}}^{\dagger}_{p}[\Psi^{\rm pos}_{p}] = \int \mathbf{\Psi}^{\dagger}_{p}(\vec{x})\, \Psi^{\rm pos}_{p}(\vec{x}) d^{(d-1)}x \tag{2.2}$$

or on a vacuum where it describes a single particle in the state $\Psi^{\rm pos}$

$$\{\underline{\mathbf{\Psi}}^{\dagger}_{p}[\Psi^{\rm pos}_{p}] = \int \mathbf{\Psi}^{\dagger}_{p}(\vec{x})\, \Psi^{\rm pos}_{p}(\vec{x}) d^{(d-1)}x\, \}\, |{\rm vac}\rangle$$

so that the anti-particle state becomes

$$\{\underline{\mathbb{C}}_{\mathcal{H}}\, \underline{\mathbf{\Psi}}^{\dagger}_{p}[\Psi^{\rm pos}_{p}] = \int \mathbf{\Psi}_{p}(\vec{x})\, (\mathcal{C}_{\mathcal{H}}\, \Psi^{\rm pos}_{p}(\vec{x})) d^{(d-1)}x\}\, |{\rm vac}\rangle\,.$$

We also can derive the relation

$$\underline{\mathbb{C}}_{\mathcal{H}}\, \mathbf{\Psi}(\vec{x})\, (\underline{\mathbb{C}}_{\mathcal{H}})^{-1} = \mathcal{C}_{\mathcal{H} formal}\, \mathbf{\Psi}(\vec{x}) = (\mathcal{C}_{\mathcal{H}}\, K)_{formal}\, \mathbf{\Psi}^{\dagger}(\vec{x})\,. \tag{2.3}$$

This formal operation $\mathcal{C}_{\mathcal{H} formal}$ means the action on the second quantized field $\mathbf{\Psi}$ as if it were a function of $\vec{x}$ and a column in gamma matrix space, and that the complex conjugation is replaced by the Hermitian conjugation (†) on the second quantized operator.[9]

Let us define the operator "emptying" [15–18] (arXiv:1312.1541) the Dirac sea, so that operation of "emptying" after the charge conjugation $\mathcal{C}_{\mathcal{H}}$ (which transforms the state put on the top of the Dirac sea into the corresponding negative energy state) creates the anti-particle state to the starting particle state, both put on the top of the Dirac sea and both solving the Weyl equation for a free massless fermions

$$\text{“emptying”} = \prod_{\Re\gamma^{a}} \gamma^{a}\, K = (-)^{\frac{d}{2}+1} \prod_{\Im\gamma^{a}} \gamma^{a}\, \Gamma^{(d)} K\,, \tag{2.4}$$

although we must keep in mind that indeed the anti-particle state is a hole in the Dirac sea from the Fock space point of view. The operator "emptying" is bringing the single particle operator $\mathcal{C}_{\mathcal{H}}$ into the operator on the Fock space. Then the anti-particle state creation operator - $\underline{\mathbf{\Psi}}^{\dagger}_{a}[\Psi^{\rm pos}_{p}]$ — to the corresponding particle state creation operator — can be obtained also as follows

$$\begin{aligned} \underline{\mathbf{\Psi}}^{\dagger}_{a}[\Psi^{\rm pos}_{p}]\, |{\rm vac}\rangle\, &= \underline{\mathbb{C}}_{\mathcal{H}}\, \underline{\mathbf{\Psi}}^{\dagger}_{p}[\Psi^{\rm pos}_{p}]\, |{\rm vac}\rangle\, = \int \mathbf{\Psi}^{\dagger}_{a}(\vec{x})\, (\mathbb{C}_{\mathcal{H}}\, \Psi^{\rm pos}_{p}(\vec{x}))\, d^{(d-1)}x\, |{\rm vac}\rangle\,, \\ \mathbb{C}_{\mathcal{H}}\, &= \text{“emptying”}\, \cdot\, \mathcal{C}_{\mathcal{H}}\,. \end{aligned} \tag{2.5}$$

The operator $\mathbb{C}_{\mathcal{H}} = \text{“emptying”} \cdot \mathcal{C}_{\mathcal{H}}$ operating on $\Psi^{\rm pos}_{p}(\vec{x})$ transforms the positive energy spinor state (which solves the Weyl equation for a massless free spinor) put on the top of the Dirac sea into the positive energy anti-spinor state, which again solves the Weyl

[9] This simply means that, for example, we can use Hermitian conjugate equations of motion for $(\mathcal{C}_{\mathcal{H}}\, K)_{formal}\, \mathbf{\Psi}(\vec{x})$ and then check the $\mathcal{C}_{\mathcal{H}}$ without the complex conjugation: $(\mathcal{C}_{\mathcal{H}}\, K)_{formal}$.

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equation for a massless free anti-spinor put on the top of the Dirac sea. Let us point out that the operator "emptying" transforms the single particle operator $\mathcal{C}_{\mathcal{H}}$ into the operator operating in the Fock space.

The operator "emptying" operates meaningfully in all known cases when the higher dimensions manifest charges or masses or both in $d = (3+1)$ space.

We define the time reversal operator $\mathcal{T}_{\mathcal{H}}$ and the parity operator $\mathcal{P}^{(d-1)}_{\mathcal{H}}$ as follows

$$\begin{aligned}
\mathcal{T}_{\mathcal{H}} &= \gamma^0 \prod_{\gamma^a \in \Re} \gamma^a \; K \, I_{x^0} \,, \\
\mathcal{P}^{(d-1)}_{\mathcal{H}} &= \gamma^0 \, I_{\vec{x}} \,, \\
I_x x^a &= -x^a \,, \quad I_{x^0} x^a = (-x^0, \vec{x}) \,, \quad I_{\vec{x}} \vec{x} = -\vec{x} \,, \\
I_{\vec{x}_3} x^a &= (x^0, -x^1, -x^2, -x^3, x^5, x^6, \ldots, x^d) \,.
\end{aligned} \tag{2.6}$$

Again the product $\prod \gamma^a$ is meant in the ascending order in γ^a.

Let us calculate now the product of $\mathcal{C}_{\mathcal{H}} \, \mathcal{P}^{(d-1)}_{\mathcal{H}} \, \mathcal{T}_{\mathcal{H}}$ and $\mathbb{C}_{\mathcal{H}} \, \mathcal{P}^{(d-1)}_{\mathcal{H}} \, \mathcal{T}_{\mathcal{H}}$

$$\begin{aligned}
\mathcal{C}_{\mathcal{H}} \mathcal{P}^{(d-1)}_{\mathcal{H}} \mathcal{T}_{\mathcal{H}} &\propto \Gamma^{(d)} \; I_x \,, \\
\mathbb{C}_{\mathcal{H}} \mathcal{P}^{(d-1)}_{\mathcal{H}} \mathcal{T}_{\mathcal{H}} &= \text{“emptying”} \cdot \mathcal{C}_{\mathcal{H}} \mathcal{P}_{\mathcal{H}} \mathcal{T}_{\mathcal{H}} \propto \prod_{\gamma^a \in \Im} \gamma^a \; I_x \, K \,,
\end{aligned} \tag{2.7}$$

with

$$\mathbb{C}_{\mathcal{H}} = \prod_{\Re \gamma^a} \gamma^a \, K \, \mathcal{C}_{\mathcal{H}} \; \propto \Gamma^{(d)} \,. \tag{2.8}$$

$\propto$ stays for up to a phase. It follows

$$\underline{\mathbb{C}}_{\mathcal{H}} \mathcal{P}^{(d-1)}_{\mathcal{H}} \mathcal{T}_{\mathcal{H}} \, \underline{\mathbf{\Psi}}^{\dagger}_{p} [\Psi^{\rm pos}_{p}] \, (\underline{\mathbb{C}}_{\mathcal{H}} \mathcal{P}^{(d-1)}_{\mathcal{H}} \mathcal{T}_{\mathcal{H}})^{-1} = \underline{\mathbf{\Psi}}^{\dagger}_{a} [\mathbb{C}_{\mathcal{H}} \mathcal{P}^{(d-1)}_{\mathcal{H}} \mathcal{T}_{\mathcal{H}} \, \Psi^{\rm pos}_{p}] \,. \tag{2.9}$$

$\Gamma^{(d)}$ is defined in eq. (2.11)

2.1 Free spinors case

To demonstrate what do the discrete symmetry operators of eqs. (2.1), (2.6), (2.8) do on the spinor states let us first look for the solutions of the Weyl equation for a free spinor in $d = (d-1) + 1$ for d even,

$$\gamma^a p_a \, \psi = 0 \,, \tag{2.10}$$

and show the application of the above defined discrete symmetries on the solutions for two particular cases: **i.** $d = (5+1)$, the properties of which we study in several papers [6–14], and **ii.** $d = (13+1)$, which one of the authors of this paper uses in her *spin-charge-family* theory [15–30], since it manifests in $d = (3+1)$ in the low energy regime the family members (explaining correspondingly the appearance of families) with the family members assumed by the *standard model* (extended with the right handed neutrino). Let us recognize that the operator of handedness, expressed in terms of the Cartan subalgebra members, is as follows

$$\Gamma^{((d-1)+1)} = (-2i)^{\frac{d}{2}} \, S^{03} S^{12} S^{56} \ldots S^{(d-1)d} \,. \tag{2.11}$$

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For the choice of the coordinate system so that d-momentum manifests $p^a = (p^0, 0, 0, p^3, 0\ldots 0)$ the Weyl equation simplifies to

$$(-2iS^{03}p^0 = p^3)\psi\,. \tag{2.12}$$

We shall make use of this choice. Solutions in the coordinate representation are plane waves: $e^{-ip^a x_a}$. In this part $\mathcal{T}_{\mathcal{H}}$ and $\mathcal{P}_{\mathcal{H}}$ manifest as follows

$$\mathcal{T}_{\mathcal{H}}(\cdots)e^{-ip^0x^0+ip^3x^3} = (\cdots)e^{-ip^0x^0-ip^3x^3}\,, \qquad \mathcal{P}_{\mathcal{H}}(\cdots)e^{-ip^0x^0+ip^3x^3} = (\cdots)e^{-ip^0x^0-ip^3x^3}\,, \tag{2.13}$$

since in the momentum representation only p^a is a vector, while x^a is just a parameter (and opposite in the coordinate representation). (With $\mathcal{T}_{\mathcal{H}}$ transformed wave function develops the usual Schroedinger way for x^0 is replaced by $-x^0$.)

$d = (5 + 1)$ case. Let us now demonstrate the application of the discrete operators $\mathbb{C}_{\mathcal{H}}$, $\mathcal{T}_{\mathcal{H}}$ and $\mathcal{P}_{\mathcal{H}}$ on one Weyl representation from table 1, which represents the positive and negative energy solutions of the Weyl equation (2.12) in $d = (5 + 1)$. Here and in what follows we do not pay attention on the normalization factor of the single particle states. Let us make a choice of the positive energy state $\psi_1^{\rm pos} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\; e^{-ip^0x^0+ip^3x^3}$, for example. We use the technique of the refs. [57–59]. A short overview can be found in the appendix. The reader is kindly asked to look for more detailed explanation in [15, 59]. It follows for $p^0 = |p^0|$ and $p^3 = |p^3|$

$$\mathcal{C}_{\mathcal{H}}\psi_1^{\rm pos} \rightarrow \stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{[-]}\; e^{ip^0x^0-ip^3x^3} = \psi_2^{\rm neg}\,. \tag{2.14}$$

This state is the solution of the Weyl equation for the negative energy state. But the hole of this state in the Dirac sea makes a positive energy state (above the Dirac sea) with the *properties* of the starting state, but *it is an anti-particle* state: $\Psi^{pos}_{a1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\; e^{-ip^0x^0+ip^3x^3}$, defined[10] on the Dirac sea with the hole belonging to the negative energy single-particle state $\psi_2^{\rm neg}$. Namely, $\mathbb{C}_{\mathcal{H}}\, \underline{\mathbf{\Psi}}[\Psi_p^{\rm pos}]\, \mathbb{C}_{\mathcal{H}}^{-1}$, when applied on the vacuum state, represents an anti-particle.

This anti-particle state is correspondingly the solution of the same Weyl equation, and it belongs to the same representation as the starting state (and $\mathbb{C}_{\mathcal{H}}$ is obviously a good symmetry in this $d = 2 \pmod 4$ space). The operator $\mathbb{C}_{\mathcal{H}}$ from eq. (2.8), applied on the state $\psi_{p1}^{\rm pos}$, gives the same result: $\psi_{a1}^{\rm pos}$, which belong to the same representation of the Weyl equation as the starting state. But this state has the S^{56} spin, which should represent in $d = (3 + 1)$ the charge of the anti-particle, the same as the starting state. This is not in agreement with what we observe.

Since both $\mathcal{T}_{\mathcal{H}}$ ($\mathcal{T}_{\mathcal{H}}\psi_1^{\rm pos} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\; e^{-ip^0x^0-ip^3x^3}$) and $\mathcal{P}_{\mathcal{H}}$ ($\mathcal{P}_{\mathcal{H}}\psi_1^{\rm pos} = \stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{(+)}\; e^{-ip^0x^0-ip^3x^3}$) are defined with an odd number of γ^a operators, none of them are the symmetry (the conserved operators) within one Weyl representation, since both transform

[10] If one would like a more detailed meaning of $\Psi_a^{\rm pos}$ one can imagine the second quantization of the whole theory using anti-particles instead of particles in the theory and so obtaining the original particles as holes. In such a theory an anti-particle state corresponding to $\Psi_a^{\rm pos}$ would be $\underline{\mathbf{\Psi}}^\dagger[\Psi^{\rm pos}]|{\rm antivac}\rangle$, therefore $\underline{\mathbf{\Psi}}^\dagger[\Psi^{\rm neg}]|{\rm antivac}\rangle \rightarrow \underline{\mathbf{\Psi}}^\dagger[\Psi^{\rm pos}]|{\rm antivac}\rangle$.

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correspondingly the starting state into a state of another Weyl representation. (This is true for all the spaces with $d = 2\,(\mod 4)$, while in the spaces with $d = 0\,(\mod 4)$ the operator $\mathcal{T}_{\mathcal{H}}$ has an even product of γ^a, while $\mathcal{C}_{\mathcal{H}}$ contains an odd number of γ^a.)

The product of $\mathcal{T}_{\mathcal{H}}$ and $\mathcal{P}^{(d-1)}_{\mathcal{H}}$ is again a good symmetry, transforming the starting state, say $\psi^{\rm pos}_1$, into a positive energy state of the same Weyl representation, $\mathcal{T}_{\mathcal{H}}\,\mathcal{P}^{(d-1)}_{\mathcal{H}}\psi^{\rm pos}_1 = \overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]}\, e^{-ip^0x^0+ip^3x^3} = \psi^{\rm pos}_2$, and solving the Weyl equation.

Also the product of all three discrete symmetries is correspondingly a good symmetry as well, transforming the starting state (put on the top of the Dirac sea) into the positive energy anti-particle state, $\underline{\mathbb{C}}_{\mathcal{H}}\ \mathcal{T}_{\mathcal{H}}\,\mathcal{P}^{(d-1)}_{\mathcal{H}}\ \underline{\mathbf{\Psi}}^{\dagger}[\Psi^{\rm pos}_1](\underline{\mathbb{C}}_{\mathcal{H}}\ \mathcal{T}_{\mathcal{H}}\,\mathcal{P}^{(d-1)}_{\mathcal{H}})^{-1}$ $= \underline{\mathbf{\Psi}}^{\dagger}_a[\mathbb{C}_{\mathcal{H}}\mathcal{T}_{\mathcal{H}}\,\mathcal{P}^{(d-1)}_{\mathcal{H}}\Psi^{\rm pos}_1] \rightarrow \underline{\mathbf{\Psi}}^{\dagger}_a[\Psi^{\rm pos}_2]$, which is the hole in the state $\psi^{\rm neg}_1$ in the Dirac sea.

$d = (13 + 1)$ case. Let us now look at $d = (13 + 1)$ case, the positive energy states of which are presented in table 2. Following the procedure used in the previous case of $d = (5 + 1)$, the operator $\mathcal{C}_{\mathcal{H}}$ transforms, let say the first state in table 2, which represents due to its quantum numbers the right handed (with respect to $d = (3 + 1)$) u-quark with spin up, weak chargeless, carrying the colour charge $(\frac{1}{2}, \frac{1}{(2\sqrt{3})})$, the third component of the second $\mathrm{SU}(2)_{II}$ charge $\frac{1}{2}$, the hyper charge $\frac{2}{3}$ and the electromagnetic charge $\frac{2}{3}$, while it carries the momentum $p^a = (p^0, 0, 0, p^3, 0, \ldots, 0)$, as follows

$$\mathcal{C}_{\mathcal{H}} u_{1R} \rightarrow \overset{03}{(+i)}\overset{12}{[-]}\,|\,\overset{56}{[-]}\overset{78}{[-]}\,||\,\overset{9\,10}{[-]}\overset{11\,12}{[+]}\ \overset{13\,14}{[+]}\ \ e^{ip^0x^0-ip^3x^3}\,. \tag{2.15}$$

This state solves the Weyl equation for the negative energy and inverse momentum, carrying all the eigenvalues of the Cartan subalgebra operators (S^{12}, S^{56}, S^{78}, $S^{9\,10}$, $S^{11\,12}$, $S^{13\,14}$), except S^{03}, of the opposite values than the starting state (this negative energy state is a part of the starting Weyl representation, not presented in table 2, but the reader can find this state in the ref. [29, 30]). The second quantized charge conjugation operator $\underline{\mathbb{C}}_{\mathcal{H}}$ empties $\mathcal{C}_{\mathcal{H}} u_{1R}$ in the Dirac sea, creating the anti-particle state to the starting state with all the quantum numbers of the starting state, obviously in contradiction with the observations, that the anti-particle state has the same momentum in $d = (3 + 1)$ but opposite charges than the starting state.

We conclude that *the second quantized anti-particle state* (the hole in the Dirac sea) *manifests correspondingly all the quantum numbers of the starting state, but it is the anti-particle*. Requiring that the eigenvalues of the Cartan subalgebra members in $d \geq 5$ represent charges in $d = (3 + 1)$, the *charges should have opposite values*, which the definition of the discrete symmetries operators in eqs. (2.1), (2.6) does not offer. The charge conjugation operation is a good symmetry in any $d = 2\,(\mod 4)$ from the point of view that in any of spaces with $d = 2(\mod 4)$ $\mathcal{C}_{\mathcal{H}}\,\psi^{\rm pos}_i$ defines the state within the same Weyl representation due to the fact that it is defined as the product of an even number of imaginary operators γ^a. The product of the time reversal and the parity operation is in the space with $d = 2\,(\mod 4)$ again a good symmetry, which means that it transforms the starting state of a chosen Weyl representation into the state belonging to the same Weyl representation, with the same d-momentum as the starting state.

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$\psi_i^{\rm pos}$	positive energy state	$\frac{p^0}{\vert p^0\vert}$	$\frac{p^3}{\vert p^3\vert}$	$(-2iS^{03})$	$\Gamma^{(3+1)}$	S^{56}
$\psi_1^{\rm pos}$	$\overset{03}{(+i)}\,\overset{12}{(+)}\mid\overset{56}{(+)}\;e^{-i\vert p^0\vert x^0+i\vert p^3\vert x^3}$	$+1$	$+1$	$+1$	$+1$	$\frac{1}{2}$
$\psi_2^{\rm pos}$	$\overset{03}{(+i)}\,\overset{12}{[-]}\mid\overset{56}{[-]}\;e^{-i\vert p^0\vert x^0+i\vert p^3\vert x^3}$	$+1$	$+1$	$+1$	-1	$-\frac{1}{2}$
$\psi_3^{\rm pos}$	$\overset{03}{[-i]}\,\overset{12}{[-]}\mid\overset{56}{(+)}\;e^{-i\vert p^0\vert x^0-i\vert p^3\vert x^3}$	$+1$	-1	-1	$+1$	$\frac{1}{2}$
$\psi_4^{\rm pos}$	$\overset{03}{[-i]}\,\overset{12}{(+)}\mid\overset{56}{[-]}\;e^{-i\vert p^0\vert x^0-i\vert p^3\vert x^3}$	$+1$	-1	-1	-1	$-\frac{1}{2}$
$\psi_i^{\rm neg}$	negative energy state	$\frac{p^0}{\vert p^0\vert}$	$\frac{p^3}{\vert p^3\vert}$	$(-2iS^{03})$	$\Gamma^{(3+1)}$	S^{56}
$\psi_1^{\rm neg}$	$\overset{03}{(+i)}\,\overset{12}{(+)}\mid\overset{56}{(+)}\;e^{i\vert p^0\vert x^0-i\vert p^3\vert x^3}$	-1	-1	$+1$	$+1$	$\frac{1}{2}$
$\psi_2^{\rm neg}$	$\overset{03}{(+i)}\,\overset{12}{[-]}\mid\overset{56}{[-]}\;e^{i\vert p^0\vert x^0-i\vert p^3\vert x^3}$	-1	-1	$+1$	-1	$-\frac{1}{2}$
$\psi_3^{\rm neg}$	$\overset{03}{[-i]}\,\overset{12}{[-]}\mid\overset{56}{(+)}\;e^{i\vert p^0\vert x^0+i\vert p^3\vert x^3}$	-1	$+1$	-1	$+1$	$\frac{1}{2}$
$\psi_4^{\rm neg}$	$\overset{03}{[-i]}\,\overset{12}{(+)}\mid\overset{56}{[-]}\;e^{i\vert p^0\vert x^0+i\vert p^3\vert x^3}$	-1	$+1$	-1	-1	$-\frac{1}{2}$

Table 1. Four positive energy states and four negative energy states, the solutions of eq. (2.12), half have $\frac{p^3}{|p^3|}$ positive and half negative. $p^a = (p^0, 0, 0, p^3, 0, 0)$, $\Gamma^{(5+1)} = -1$, S^{56} defines charges in $d = (3+1)$. Nilpotents $\overset{ab}{(k)}$ and projectors $\overset{ab}{[k]}$ operate on the vacuum state $|\text{vac}\rangle_{\text{fam}}$ not written in the table.

2.1.1 Solutions of the Weyl equations in $d = (5 + 1)$

There are $2^{\frac{d}{2}-1} = 4$ basic spinor states of one family representation in $d = (5+1)$.[11] Since the operators of eqs. (2.1), (2.6) do not distinguish among the families, all the families behave equivalently with respect to these discrete symmetry operators. One of the family representation, with four basic spinor states, is in the technique [59], described in terms of nilpotents $\overset{ab}{(k)}$ and projectors $\overset{ab}{[k]}$ (see appendix A), as follows

$$\begin{aligned}
\Psi_1 &= \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}\ |\text{vac}\rangle_{\text{fam}}, \\
\Psi_2 &= \overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]}\ |\text{vac}\rangle_{\text{fam}}, \\
\Psi_3 &= \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)}\ |\text{vac}\rangle_{\text{fam}}, \\
\Psi_4 &= \overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[-]}\ |\text{vac}\rangle_{\text{fam}}\,,
\end{aligned} \tag{2.16}$$

where $|\text{vac}\rangle_{\text{fam}}$ is defined so that there are $2^{\frac{d}{2}-1}$ family members (this is, however, not a second quantized vacuum). All the basic states are eigenstates of the Cartan subalgebra (of the Lorentz transformation Lie algebra), for which we take: S^{03}, S^{12}, S^{56}, with the eigenvalues, which can be read from eq. (2.16) if taking $\frac{1}{2}$ of the numbers $\pm i$ or ± 1 in the parentheses ( ) (nilpotents) and [ ] (projectors). We look for the solutions of eq. (2.12) for a particular choice of the d-momentum $p^a = (p^0, 0, 0, p^3, 0, 0)$, and find what is presented in table 1.

[11] There are for $d = 6$ in the *spin-charge-family* proposal $2^{\frac{d}{2}-1} = 4$ families of spinors.

$\psi_i^{\rm pos}$	positive energy state	$\frac{p^0}{\vert p^0\vert}$	$\frac{p^3}{\vert p^3\vert}$	$(-2iS^{03})$	$\Gamma^{(3+1)}$	τ^{13}	τ^{23}	τ^4	Y	Q
$u_{1\,R}$	$\overset{03}{(+i)}\,\overset{12}{(+)} \vert \overset{56}{(+)}\,\overset{78}{(+)} \Vert \overset{9\,10}{(+)}\;\;\overset{11\,12}{(-)}\;\;\overset{13\,14}{(-)}\;\; e^{-i\vert p^0\vert x^0+i\vert p^3\vert x^3}$	$+1$	$+1$	$+1$	$+1$	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
$u_{2\,R}$	$\overset{03}{[-i]}\,\overset{12}{[-]} \vert \overset{56}{(+)}\,\overset{78}{(+)} \Vert \overset{9\,10}{(+)}\;\;\overset{11\,12}{(-)}\;\;\overset{13\,14}{(-)}\;\; e^{-i\vert p^0\vert x^0-i\vert p^3\vert x^3}$	$+1$	-1	-1	$+1$	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
$d_{1\,R}$	$\overset{03}{(+i)}\,\overset{12}{(+)} \vert \overset{56}{[-]}\,\overset{78}{[-]} \Vert \overset{9\,10}{(+)}\;\;\overset{11\,12}{(-)}\;\;\overset{13\,14}{(-)}\;\; e^{-i\vert p^0\vert x^0+i\vert p^3\vert x^3}$	$+1$	$+1$	$+1$	$+1$	0	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
$d_{2\,R}$	$\overset{03}{[-i]}\,\overset{12}{[-]} \vert \overset{56}{[-]}\,\overset{78}{[-]} \Vert \overset{9\,10}{(+)}\;\;\overset{11\,12}{(-)}\;\;\overset{13\,14}{(-)}\;\; e^{-i\vert p^0\vert x^0-i\vert p^3\vert x^3}$	$+1$	-1	-1	$+1$	0	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
$d_{1\,L}$	$\overset{03}{[-i]}\,\overset{12}{(+)} \vert \overset{56}{[-]}\,\overset{78}{(+)} \Vert \overset{9\,10}{(+)}\;\;\overset{11\,12}{(-)}\;\;\overset{13\,14}{(-)}\;\; e^{-i\vert p^0\vert x^0-i\vert p^3\vert x^3}$	$+1$	-1	-1	-1	$-\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
$d_{2\,L}$	$\overset{03}{(+i)}\,\overset{12}{[-]} \vert \overset{56}{[-]}\,\overset{78}{(+)} \Vert \overset{9\,10}{(+)}\;\;\overset{11\,12}{(-)}\;\;\overset{13\,14}{(-)}\;\; e^{-i\vert p^0\vert x^0+i\vert p^3\vert x^3}$	$+1$	$+1$	$+1$	-1	$-\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
$u_{1\,L}$	$\overset{03}{[-i]}\,\overset{12}{(+)} \vert \overset{56}{(+)}\,\overset{78}{[-]} \Vert \overset{9\,10}{(+)}\;\;\overset{11\,12}{(-)}\;\;\overset{13\,14}{(-)}\;\; e^{-i\vert p^0\vert x^0-i\vert p^3\vert x^3}$	$+1$	-1	-1	-1	$\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
$u_{2\,L}$	$\overset{03}{(+i)}\,\overset{12}{[-]} \vert \overset{56}{(+)}\,\overset{78}{[-]} \Vert \overset{9\,10}{(+)}\;\;\overset{11\,12}{(-)}\;\;\overset{13\,14}{(-)}\;\; e^{-i\vert p^0\vert x^0+i\vert p^3\vert x^3}$	$+1$	$+1$	$+1$	-1	$\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$

Table 2. One SO$(7,1)$ sub representation of the representation of SO$(13,1)$, the one representing quarks, which carry the colour charge ($\tau^{33} = 1/2$, $\tau^{38} = 1/(2\sqrt{3})$). All members have $\Gamma^{(13+1)} = -1$. All states are the eigenstates of the Cartan subalgebra $(S^{03}, S^{12}, S^{56}, S^{78}, S^{9\,10}, S^{11\,12}, S^{13\,14})$ with the eigenvalues defined in eq. (A.2) and solve the Weyl equation (2.12) for the choice of the coordinate system $p^a = (p^0, 0, 0, p^3, 0, \ldots, 0)$. The infinitesimal generators of the weak charge SU(2) group are defined as $(\vec{\tau}^1 = \frac{1}{2}(S^{58} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78}))$, of another SU(2) as $(\vec{\tau}^2 = \frac{1}{2}(S^{58} + S^{67}, S^{57} - S^{68}, S^{56} + S^{78}))$, of the τ^4 charge as $(-\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14}))$ and of the colour charge group as $(\vec{\tau}^3 = (\frac{1}{2}(S^{9\,12} - S^{10\,11}, S^{9\,11} + S^{10\,12}, S^{9\,10} - S^{11\,12}, S^{9\,14} - S^{10\,13}, S^{9\,13} + S^{10\,14}, S^{11\,14} - S^{12\,13}, S^{11\,13} + S^{12\,14}, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14}))$, $Y = \tau^{23} + \tau^4$, $Q = \tau^{13} + Y$. Nilpotents $\overset{ab}{(k)}$ and projectors $\overset{ab}{[k]}$ operate on the vacuum state $|{\rm vac}\rangle_{\rm fam}$ not written in the table.

2.1.2 Solutions of the Weyl equations in $d = (13 + 1)$

There are $2^{\frac{d}{2}-1} = 64$ basic spinor states of one family representation in $d = (13 + 1)$. (We again do not pay attention on the families, since all behave equivalently with respect to the discrete symmetries presented in eqs. (2.1), (2.6).) We present in this subsection positive energy states for quarks of a particular charge ($\tau^{33} = 1/2$, $\tau^{38} = 1/(2\sqrt{3})$). The solution for, say, the right handed u-quark with spin up, $u_{1\,R}$, with the colour charge ($\tau^{33} = -1/2$ and $\tau^{38} = 1/(2\sqrt{3})$), weak chargeless and with a positive momentum p^3 is proportional to $(\overset{03}{(+i)}\,\overset{12}{(+)} \mid \overset{56}{(+)}\,\overset{78}{(+)} \parallel \overset{9\,10}{[-]}\;\;\overset{11\,12}{[+]}\;\;\overset{13\,14}{(-)}\;\; e^{-ip^0x^0+ip^3x^3})$. All the other colour states follow from this one by the application of the generators τ^{3i} of the colour group SU(3), the definition of which, expressed as the superposition of S^{ab}, can be found in the caption of table 2. One can as well define the generators of the total angular momentum $J^{ab} = L^{ab} + S^{ab}$. The definition of the generators of the charge groups, presented in the caption of table 2, then changes correspondingly by replacing S^{ab} by J^{ab}.

3 Discrete symmetries in d even with the desired properties in $d = (3+1)$

In section 2 we define the discrete symmetries in spaces with $d > (3 + 1)$ as they follow from the definition in $d = (3 + 1)$. This definition, however, does not allow to interpret the angular momentum (the spin, indeed, at the low energy regime) in higher than four dimensions as charges in $(3 + 1)$. The proposed charge conjugated states have, namely, the same charges as the starting states.

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We look for new discrete symmetries, which would lead to the desired properties of the anti-particle state to any second quantized state:

i. The anti-particle state has the same momentum in $d = (3+1)$ as the starting state.

ii. The anti-particle state has the opposite values of the Cartan subalgebra of the total angular momentum $J^{st} = L^{st} + S^{st}, (s,t) \in (5,6,\ldots,d)$ (or at low energies rather the opposite values of the Cartan subalgebra of $S^{st}, (s,t) \in (5,6,\ldots,d)$) as the starting state.

The manifestation of the total angular momentum (in the low energy regime rather the spin degrees of freedom) in $d > 4$ as charges in $d \leq 4$ depends on the symmetries that (non-)compact spaces manifest [11–13]. (For the toy model [11–13] in $d = (5+1)$ the spin on the infinite surface, curled into an almost sphere, manifests for a massless spinor as a charge in $d = (3+1)$. Only to the massive states the total angular momentum in $d = (5,6)$ contributes.) In the case of the *spin-charge-family* theory in $d = (13+1)$, which manifests at low energies properties of the *standard model*, the operators $\vec{\tau}^1, \vec{\tau}^2, \vec{\tau}^3, Y, \tau^4, Q$, or rather their superposition (which all are superposition of $S^{ab}, a,b \in \{5,6,\ldots,14\}$) define the conserved charges in $d = (3+1)$ before and after the electroweak break.

We define new discrete symmetries by transforming the above defined discrete symmetries ($\underline{\mathbb{C}}_{\mathcal{H}}$, $\mathbb{C}_{\mathcal{H}}$, $\mathcal{C}_{\mathcal{H}}$, $\mathcal{T}_{\mathcal{H}}$, $\mathcal{P}_{\mathcal{H}}$) so that, while remaining within the same groups of symmetries, the redefined discrete symmetries manifest the experimentally acceptable properties in $d = (3+1)$, which is of the essential importance for all the Kaluza-Klein theories [33–39] without any degrees of freedom of fermions besides the spin and family quantum numbers [12–30]. We define new discrete symmetries as follows

$$\begin{aligned}
\mathcal{C}_{\mathcal{N}} &= \mathcal{C}_{\mathcal{H}}\, \mathcal{P}^{(d-1)}_{\mathcal{H}}\, e^{i\pi J_{12}}\, e^{i\pi J_{35}}\, e^{i\pi J_{79}}\, e^{i\pi J_{11\,13}}, \ldots, e^{i\pi J_{(d-3)(d-1)}}\,,\\
\mathcal{T}_{\mathcal{N}} &= \mathcal{T}_{\mathcal{H}}\, \mathcal{P}^{(d-1)}_{\mathcal{H}}\, e^{i\pi J_{12}}\, e^{i\pi J_{36}}\, e^{i\pi J_{8\,10}}\, e^{i\pi J_{12\,14}}, \ldots, e^{i\pi J_{(d-2)d}}\,,\\
\mathcal{P}^{(d-1)}_{\mathcal{N}} &= \mathcal{P}^{(d-1)}_{\mathcal{H}}\, e^{i\pi J_{56}}\, e^{i\pi J_{78}}\, e^{i\pi J_{9\,10}}\, e^{i\pi J_{11\,12}}\, e^{i\pi J_{13\,14}}, \ldots, e^{i\pi J_{(d-1)d}}\,,\\
\mathbb{C}_{\mathcal{N}} &= \mathbb{C}_{\mathcal{H}}\, \mathcal{P}^{(d-1)}_{\mathcal{H}}\, e^{i\pi J_{12}}\, e^{i\pi J_{35}}\, e^{i\pi J_{79}}\, e^{i\pi J_{11\,13}}, \ldots, e^{i\pi J_{(d-3)(d-1)}}\,,\\
\underline{\mathbb{C}}_{\mathcal{N}} &= \underline{\mathbb{C}}_{\mathcal{H}}\, \mathcal{P}^{(d-1)}_{\mathcal{H}}\, e^{i\pi J_{12}}\, e^{i\pi J_{35}}\, e^{i\pi J_{79}}\, e^{i\pi J_{11\,13}}, \ldots, e^{i\pi J_{(d-3)(d-1)}}\,.
\end{aligned} \tag{3.1}$$

The operator for "emptying" is defined in eq. (2.4) as "emptying" $= \prod_{\Re\gamma^a} \gamma^a\, K$, the operator $\mathbb{C}_{\mathcal{N}} = \prod_{\Re\gamma^a} \gamma^a\, K\, \mathcal{C}_{\mathcal{N}}$, while the operator $\underline{\mathbb{C}}_{\mathcal{N}}$ is defined according to eq. (2.5) as

$$\underline{\mathbf{\Psi}}^{\dagger}_{aN}[\Psi^{\rm pos}_{p}]\,|{\rm vac}\rangle = \underline{\mathbb{C}}_{\mathcal{N}}\, \underline{\mathbf{\Psi}}^{\dagger}_{p}[\Psi^{\rm pos}_{p}]\,|{\rm vac}\rangle = \int \mathbf{\Psi}^{\dagger}_{aN}(\vec{x})\,(\mathbb{C}_{\mathcal{N}}\, \Psi^{\rm pos}_{p}(\vec{x}))\, d^{(d-1)}x\; |{\rm vac}\rangle\,. \tag{3.2}$$

The rotations ($e^{i\pi J_{12}}\, e^{i\pi J_{35}}\, e^{i\pi J_{79}}\, \ldots, e^{i\pi J_{(d-3)(d-1)}}$) together with (multiplied by) $\mathcal{P}^{(d-1)}_{\mathcal{H}}$, which are included in $\underline{\mathbb{C}}_{\mathcal{N}}$ (and in $\mathbb{C}_{\mathcal{N}}$ and $\mathcal{C}_{\mathcal{N}}$), keep p^i for $i = (1,2,3)$ unchanged, while they transform a state so that all the eigenvalues of the Cartan subalgebra except S^{03} and J^{12} (or at the low energy regime S^{12}) change sign.[12] Correspondingly this redefined

[12]Since in our extra dimension picture J_{35} is no longer a symmetry (for the metric taken as a background field) in coordinate space, the operation $e^{i\pi J_{35}}$ looks suspicious as being not a symmetry, but it is. Indeed, the operation $e^{i\pi J_{35}}$ is in the coordinate part composed just of a mirror reflection around the $x^3 = 0$ plane in usual space and reflection in the extra dimension space around the surface $x^5 = 0$.

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$\mathbb{C}_{\mathcal{H}}$ transforms a second quantized state into the anti-particle state with the same four momentum as the starting state but with the opposite values of the total angular momentum (or at the low energy regime rather the spin) determined by the Cartan subalgebra eigenvalues, except for S^{03} and J^{12} (or at the low energy regime S^{12}). The parity operator $\mathcal{P}^{(d-1)}_{\mathcal{N}}$ changes p^i into $-p^i$ only for $i = (1, 2, 3)$, while the time reversal operator corrects all the properties of the new $\mathbb{C}_{\mathcal{N}}$ (and $\mathbb{C}_{\mathcal{N}}$) and $\mathcal{P}^{(d-1)}_{\mathcal{N}}$ so that

$$\begin{aligned}
\mathcal{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}} &= \mathcal{C}_{\mathcal{H}}\mathcal{P}^{(d-1)}_{\mathcal{H}}\mathcal{T}_{\mathcal{H}} \to \Gamma^{(d)}\, I_x\,, \\
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}} &= \mathbb{C}_{\mathcal{H}}\mathcal{P}^{(d-1)}_{\mathcal{H}}\mathcal{T}_{\mathcal{H}}\,.
\end{aligned} \tag{3.3}$$

All three new operators commute among themselves as also the old ones do. The shorter expressions for the same discrete operators of eq. (3.1) are up to a phase

$$\begin{aligned}
\mathcal{C}_{\mathcal{N}} &= \prod_{\Im\gamma^m, m=0}^{3} \gamma^m\ \Gamma^{(3+1)}\, K\, I_{x^6,x^8,\ldots,x^d}\,, \\
\mathcal{T}_{\mathcal{N}} &= \prod_{\Re\gamma^m, m=1}^{3} \gamma^m\ \Gamma^{(3+1)}\, K\, I_{x^0}\, I_{x^5,x^7,\ldots,x^{d-1}}\,, \\
\mathcal{P}^{(d-1)}_{\mathcal{N}} &= \gamma^0\, \Gamma^{(3+1)}\, \Gamma^{(d)}\, I_{\vec{x}_3}\,, \\
\mathbb{C}_{\mathcal{N}} &= \prod_{\Re\gamma^a, a=0}^{d} \gamma^a\ K \prod_{\Im\gamma^m, m=0}^{3} \gamma^m\ \Gamma^{(3+1)}\, K\, I_{x^6,x^8,\ldots,x^d} \\
&= \prod_{\Re\gamma^s, s=5}^{d} \gamma^s\ I_{x^6,x^8,\ldots,x^d}\,.
\end{aligned} \tag{3.4}$$

Operators I operates as follows: $I_{x^5,x^7,\ldots,x^{d-1}}\ (x^0, x^1, x^2, x^3, x^5, x^6, x^7, x^8, \ldots, x^{d-1}, x^d) = (x^0, x^1, x^2, x^3, -x^5, x^6, -x^7, \ldots, -x^{d-1}, x^d)$; $I_{x^6,x^8,\ldots,x^d}\ (x^0, x^1, x^2, x^3, x^5, x^6, x^7, x^8, \ldots, x^{d-1}, x^d) = (x^0, x^1, x^2, x^3, x^5, -x^6, x^7, -x^8, \ldots, x^{d-1}, -x^d)$, $d = 2n$.

The above defined operators $\mathbb{C}_{\mathcal{H}}, \mathcal{P}^{(d-1)}_{\mathcal{H}}$ and $\mathcal{T}_{\mathcal{H}}$ (eqs. (2.1), (2.6), (2.8)), indexed by $\mathcal{H}$, are good symmetries only when also boson fields, in the Kaluza-Klein theories the gravitational fields, in higher than $(3 + 1)$ dimensions are correspondingly transformed and not considered as *background* fields. However, the operators $\mathbb{C}_{\mathcal{N}}, \mathcal{P}^{(d-1)}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{N}}$ with index $\mathcal{N}$ will be good symmetries even if we take it that there is a *background* field depending only on the extra dimension coordinates — independent of whether the extra dimension space is compactified or not — so that they are not transformed.

One can namely easily see that the transformations of the coordinates of the extra dimensions in eqs. (3.1), (3.4) are *cancelled* between the π-rotations and the actions of e.g. $P_{\mathcal{H}}$ on the extra dimensional coordinates. Thus it can be easily seen that even if we consider a background gravitational field for the extra dimensions — but the $(3+1)$ dimensional space is either flat or their gravitational field is considered dynamical so as to be also transformed — these operators with index $\mathcal{N}$, $\mathbb{C}_{\mathcal{N}}, \mathcal{P}^{(d-1)}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{N}}$, are good symmetries with respect to the space-time transformations. They are indeed good symmetries according to their action

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on the Weyl field. The crucial point really is that the $\mathcal{N}$-indexed operators $\mathbb{C}_\mathcal{N}, \mathcal{P}^{(d-1)}_\mathcal{N}$ and $\mathcal{T}_\mathcal{N}$ with their associated x-transformations do *not transform the extra* $(d-4)$ *coordinates* so that background fields depending on these extra dimension coordinates do not matter.

3.1 Free spinors

Let us now see on two cases, for $d=(5+1)$ and for $d=(13+1)$, how do the new proposals for the discrete symmetries, $\underline{\mathbb{C}}_\mathcal{N}$, $\mathcal{P}^{(d-1)}_\mathcal{N}$, $\mathcal{T}_\mathcal{N}$, manifest for non interacting spinors.

Charge conjugation symmetry $\underline{\mathbb{C}}_\mathcal{N}$. Let us start with $\psi_1^{\rm pos}$ from table 1. In $d=(5+1)$ the charge conjugation operator $\underline{\mathbb{C}}_\mathcal{N}$ equals to $\mathbb{C}_\mathcal{H}\ \mathcal{P}^{(d-1)}_\mathcal{H}\ e^{i\pi J_{12}}\, e^{i\pi J_{35}}$. To test this symmetry on the second quantized state $\underline{\boldsymbol{\Psi}}^\dagger[\Psi_1^{\rm pos}]$ one can start with eq. (2.14) and the recognition below this equation that $\mathbb{C}_\mathcal{H}$ transforms a second quantized state $\underline{\boldsymbol{\Psi}}^\dagger[\Psi_1^{\rm pos}]$ into the anti-particle second quantized state with the properties as the starting state: The same d-momentum and the same eigenvalues of the Cartan subalgebra operators (S^{03}, J^{12}, J^{56}, or rather S^{12}, S^{56}). One can easily check that the operation of $\mathcal{P}^{(d-1)}_\mathcal{H}\ e^{i\pi J_{12}}\, e^{i\pi J_{35}}$ on this anti-particle state (the hole in the Dirac sea) with the properties $S^{03}=\frac{i}{2}$, $S^{12}=\frac{1}{2}$, $S^{56}=\frac{1}{2}$ and the momentum $(|p^0|,0,0,|p^3|,0,0)$ (manifesting in $e^{-ip^0x^0+ip^3x^3}$) transforms this anti-particle state into the anti-particle state $\stackrel{03}{(+i)}\stackrel{12}{(+)}|\stackrel{56}{[-]}\ e^{-ip^0x^0+ip^3x^3}$ put on the top of the Dirac sea, with the same spin and the same handedness in $d=(3+1)$ and the opposite "charge": $S^{56}=-\frac{1}{2}$ — if we recognize the spin in $d=(5,6)$ as the charge in $d=(3+1)$ — as the starting second quantized state. But $\mathcal{C}_\mathcal{N}\psi_1^{\rm pos} = \stackrel{03}{(+i)}\stackrel{12}{[-]}|\stackrel{56}{(+)}\ e^{ip^0x^0-ip^3x^3}$ (solving the Weyl equation (2.12)) does not belong to the same Weyl representation as the starting state $\Psi_1^{\rm pos}$ and also $\stackrel{03}{(+i)}\stackrel{12}{(+)}|\stackrel{56}{[-]}\ e^{-ip^0x^0+ip^3x^3}$ does not. We can conclude that the charge conjugation operator $\underline{\mathbb{C}}_\mathcal{N}$,

$$\underline{\mathbb{C}}_\mathcal{N}\underline{\boldsymbol{\Psi}}^\dagger_p[\Psi_1^{\rm pos}](\underline{\mathbb{C}}_\mathcal{N})^{-1} = \underline{\boldsymbol{\Psi}}^\dagger_{aN}[\mathbb{C}_\mathcal{N}\,\Psi_1^{\rm pos}]\,, \tag{3.5}$$

is not a good symmetry.

Let us make the charge conjugation operation $\underline{\mathbb{C}}_\mathcal{N}$ on the second quantized state $\underline{\boldsymbol{\Psi}}^\dagger[u_{1R}]$, the corresponding single-particle state of which, put on the top of the Dirac sea, is presented in the first line of table 2. We find in eq. (2.15) that $\mathcal{C}_\mathcal{H}u_{1R} = \stackrel{03}{(+i)}\stackrel{12}{[-]}|\stackrel{56}{[-]}\stackrel{78}{[-]}||\stackrel{9\,10}{[-]}\stackrel{11\,12}{[+]}\stackrel{13\,14}{[+]}\ e^{ip^0x^0-ip^3x^3}$. To apply $\mathcal{C}_\mathcal{N}$ on u_{1R} we must, according to the definition in the first line of eq. (3.1), multiply $\mathcal{C}_\mathcal{H}u_{1R}$ by $\mathcal{P}^{(d-1)}_\mathcal{H}\ e^{i\pi J_{1\,2}}\, e^{i\pi J_{3\,5}}\, e^{i\pi J_{7\,9}}\, e^{i\pi J_{11\,13}}$. We end up with

$$\mathcal{C}_\mathcal{N}u_{1R} = \stackrel{03}{(+i)}\stackrel{12}{[-]}|\stackrel{56}{(+)}\stackrel{78}{(+)}||\stackrel{9\,10}{(+)}\stackrel{11\,12}{(-)}\stackrel{13\,14}{(-)}\ e^{ip^0x^0-ip^3x^3}\,. \tag{3.6}$$

The corresponding second quantized state is the hole in this single particle negative energy state in the Dirac sea (Fock space), which solves the Weyl equation for the negative energy state. It is the state

$$\mathbb{C}_\mathcal{N}\,u_{1R} = \stackrel{03}{(+i)}\stackrel{12}{(+)}|\stackrel{56}{[-]}\stackrel{78}{[-]}||\stackrel{9\,10}{[-]}\stackrel{11\,12}{[+]}\stackrel{13\,14}{[+]}\ e^{-ip^0x^0+ip^3x^3}\,. \tag{3.7}$$

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This state, put on the top of the Dirac sea, is the anti-particle state. But neither the state of eq. (3.6) nor the state of eq. (3.7) does belong to the same Weyl representation, similarly as it was in the case with $d = (5+1)$. Although the corresponding second quantized state, that is the hole of the state of eq. (3.6) in the Dirac sea, which is the same as the state of eq. (3.7) put on the top of the Dirac sea, $\mathbb{C}_{\mathcal{N}}\, \mathbf{\underline{u}}_{1R}\, (\mathbb{C}_{\mathcal{N}})^{-1}$ $(\rightarrow \stackrel{03}{(+i)}\stackrel{12}{(+)} \mid \stackrel{56}{[-]}\stackrel{78}{[-]} \mid\mid \stackrel{9\,10}{[-]}\stackrel{11\,12}{[+]}\stackrel{13\,14}{[+]}\; e^{-ip^0x^0+ip^3x^3}\, |\mathrm{vac}\rangle_{\mathrm{fam}})$ has the right charges, that is the opposite ones to those of the corresponding particle state, it is not a good symmetry. Again this is not within the same Weyl representation and correspondingly $\mathbb{C}_{\mathcal{N}}$ is not a good symmetry in $d = (13+1)$.

In all the spaces with $d = 2 (\mod 4)$ the charge conjugation operator $\mathbb{C}_{\mathcal{N}}$ is not a good symmetry within one Weyl representation: with a product of an odd number of γ^a it jumps out of the starting Weyl representation.

Parity symmetry $\boldsymbol{\mathcal{P}^{(d-1)}_{\mathcal{N}}}$. $\mathcal{P}^{(d-1)}_{\mathcal{N}}$ (the third lines in eqs. (3.1), (3.4)) reflects only in the $d = (3+1)$ and multiplies spinors with γ^0. It does not keep the transformed state within the same Weyl representation, either in the case $d = (5+1)$ or in the case $d = (13+1)$. In $d = (5+1)$ it transforms the single particle state Ψ^{pos}_1 into $\stackrel{03}{[-i]}\stackrel{12}{(+)} \mid \stackrel{56}{(+)}$ $e^{-ip^0x^0-ip^3x^3}\, |\mathrm{vac}\rangle_{\mathrm{fam}}$, which is not within the same Weyl representation. In $d = (13+1)$ $\mathcal{P}^{(d-1)}_{\mathcal{N}}$ transforms u_{1R} into $\stackrel{03}{[-i]}\stackrel{12}{(+)} \mid \stackrel{56}{(+)}\stackrel{78}{(+)} \mid\mid \stackrel{9\,10}{(+)}\stackrel{11\,12}{(-)}\stackrel{13\,14}{(-)}\;\; e^{-ip^0x^0-ip^3x^3}\, |\mathrm{vac}\rangle_{\mathrm{fam}}$, manifesting that $\mathcal{P}^{(d-1)}_{\mathcal{N}}$ is not a good symmetry in spaces with $d = 2 (\mod 4)$.

$\boldsymbol{\mathbb{C}_{\mathcal{N}}\times\, \mathcal{P}^{(d-1)}_{\mathcal{N}}}$ symmetry. Let us now check the $\mathbb{C}_{\mathcal{N}}\, \mathcal{P}^{(d-1)}_{\mathcal{N}}$ symmetry. According to the third and the fourth line of eq. (3.1), (3.4)) and to eqs. (2.1), (2.6) it contains an even number of γ^a operators. Correspondingly the application of $\mathcal{C}_{\mathcal{N}}\, \mathcal{P}^{(d-1)}_{\mathcal{N}}$ on any state transforms the state again into the state within the same Weyl representation.

In $d = (5+1)$ we apply $\mathbb{C}_{\mathcal{N}}\, \mathcal{P}^{(d-1)}_{\mathcal{N}}$ on $\mathbf{\underline{\Psi}}^{\dagger}_{p}[\Psi^{\mathrm{pos}}_1]$ by applying $\mathbb{C}_{\mathcal{N}}\, \mathcal{P}^{(d-1)}_{\mathcal{N}}$ on Ψ^{pos}_1 as follows: $\mathbb{C}_{\mathcal{N}}\, \mathcal{P}^{(d-1)}_{\mathcal{N}}\; \mathbf{\underline{\Psi}}^{\dagger}_{p}[\Psi^{\mathrm{pos}}_1]\; (\mathbb{C}_{\mathcal{N}}\, \mathcal{P}^{(d-1)}_{\mathcal{N}})^{-1} = \mathbf{\underline{\Psi}}^{\dagger}_{aN}[\mathbb{C}_{\mathcal{N}}\; \mathcal{P}^{(d-1)}_{\mathcal{N}}\, \Psi^{\mathrm{pos}}_1]$. One recognizes that it is $\mathbb{C}_{\mathcal{N}}\, \mathcal{P}^{(d-1)}_{\mathcal{N}}\;\; \Psi^{\mathrm{pos}}_1 \;=\; \Psi^{\mathrm{pos}}_4$ (table 1), which must be put on the top of the Dirac sea, representing the hole in the state ψ^{neg}_3 in the Dirac sea. The state is within the same Weyl and solves the Weyl equation. The $\mathbb{C}_{\mathcal{N}}\, \mathcal{P}^{(d-1)}_{\mathcal{N}}$ manifests as a good symmetry.

Let in $d = (13+1)$ the operator $\mathbb{C}_{\mathcal{N}}\; \mathcal{P}^{(d-1)}_{\mathcal{N}}$ apply on $\mathbf{\underline{\Psi}}^{\dagger}_{p}[u_{1R}]$. One applies correspondingly $\mathbb{C}_{\mathcal{N}}\; \mathcal{P}^{(d-1)}_{\mathcal{N}}$ on u_{1R}, which gives the state $\stackrel{03}{[-i]}\stackrel{12}{(+)} \mid \stackrel{56}{[-]}\stackrel{78}{[-]} \mid\mid \stackrel{9\,10}{[-]}\stackrel{11\,12}{[+]}\stackrel{13\,14}{[+]}\;\; e^{-ip^0x^0-ip^3x^3}$. This state (which solves the Weyl equation $\gamma^a p_a \Psi = 0$) gives, put on the top of the Dirac sea, the corresponding anti-particle, belonging to the same Weyl representation, and it is left handed with respect $d = (3+1)$. This anti-particle is recognized as a left handed weak chargeless anti u-quark, of the anti-colour charge, belonging to the same Weyl representation (see the ref. [30], table 4., line 35).

$\mathbb{C}_{\mathcal{N}}\, \mathcal{P}^{(d-1)}_{\mathcal{N}}$ *is a good symmetry in* $d = 2(2n+1) (= 2 (\mod 4))$ *spaces.*

Following eq. (2.9), the creation operator for an anti-particle state, which is $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}$ transformed creation operator for the particle state is therefore

$$\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\; \mathbf{\underline{\Psi}}^{\dagger}_{p}[\Psi^{\mathrm{pos}}_1]\, (\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}})^{-1} = \mathbf{\underline{\Psi}}^{\dagger}_{aN}[\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\; \Psi^{\mathrm{pos}}_1]\,. \tag{3.8}$$

$I_{\vec{x}_3}$ reflects (x^1, x^2, x^3) and $I_{x^6, x^8, \ldots x^d}$ reflects even coordinates in $d > 3$.

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Time reversal $\mathcal{T}_{\mathcal{N}}$. The application of the time reversal operator $\mathcal{T}_{\mathcal{N}}$ (the second equation in eqs. (3.1), (3.4), constructed in spaces with even d out of an even number of γ^a operators, does keep the transformed state within the same Weyl representation.

Let us test on $d = (5+1)$ case first, applying $\mathcal{T}_{\mathcal{N}}$ on $\Psi^{\rm pos}_1$. The transformed state is $\Psi^{\rm pos}_3$ from table 1: the state has the same handedness in $d = (3+1)$ as the starting state, the same S^{56} eigenvalue and opposite p^3 and S^{12}. Obviously $\mathcal{T}_{\mathcal{N}}$ is a good symmetry.

In the case of $d = (13+1)$ operator $\mathcal{T}_{\mathcal{N}}$ transforms u_{1R} with spin up from table 2 into the state with spin down ($u_{2R} =\stackrel{03}{[-i]}\stackrel{12}{[-]} \mid \stackrel{56}{(+)}\stackrel{78}{(+)} \mid\mid \stackrel{9\,10}{(+)}\stackrel{11\,12}{(-)}\stackrel{13\,14}{(-)}\; e^{-ip^0x^0 - ip^3x^3}$), keeping all the quantum numbers except eigenvalue of S^{03} and S^{12} the same and p^3 changes the sign. The state solves the Weyl equation.

$\mathcal{T}_{\mathcal{N}}$ *is a good symmetry* $d = 2 \,(\mod 4)$. It keeps states within the same Weyl representation and commutes with the operator $\gamma^a p_a$.

$\underline{\mathbb{C}}_{\mathcal{N}} \times \mathcal{P}^{(d-1)}_{\mathcal{N}} \times \mathcal{T}_{\mathcal{N}}$ symmetry. In $d = (5+1)$ the operator $\underline{\mathbb{C}}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\; \mathcal{T}_{\mathcal{N}}$ transforms $\underline{\mathbf{\Psi}}^{\dagger}_{p}[\Psi^{\rm pos}_1]$, with $\Psi^{\rm pos}_1$ from table 1 and creating the particle state, into the creation operator for the positive energy anti-particle state $\underline{\mathbf{\Psi}}^{\dagger}_{aN}[\Psi^{\rm pos}_2]$, since $\mathbb{C}_{\mathcal{N}}\; \mathcal{P}^{(d-1)}_{\mathcal{N}}\; \mathcal{T}_{\mathcal{N}}\; \Psi^{\rm pos}_1 = \Psi^{\rm pos}_2$. This state has an opposite handedness in $d = (3+1)$ and also the opposite spin and the opposite "charge".

In $d = (13+1)$ the operator $\mathbb{C}_{\mathcal{N}}\; \mathcal{P}^{(d-1)}_{\mathcal{N}}\; \mathcal{T}_{\mathcal{N}}$ transforms the right handed weakless $u_{1\,R}$ quark with spin up and colour $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$ from table 1, put on the top of the Dirac sea, into the positive energy anti-particle state with the properties of $\bar{u}_{1\,L}$ from the ref. [30], table 4., line 36) (put on the top of the Dirac sea): weak chargeless, with the spin down and of the anti-colour charge $(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$.

$\underline{\mathbb{C}}_{\mathcal{N}}\; \mathcal{P}^{(d-1)}_{\mathcal{N}}\; \mathcal{T}_{\mathcal{N}}$ *is a good symmetry*, as it is expected to be.

3.2 Interacting spinors

Let us assume quite a general Lagrange density for a spinor in $d = ((d-1)+1)$ dimensional space, which carries, like in the Kaluza-Klein theories, the spins and no charges

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\, E\, \Psi^{\dagger}\, \gamma^0\, \gamma^a\, p_{0a}\, \Psi + \, h.c.\,, \\ p_{0a} &= f^{\alpha}_{a} p_{\alpha} + \frac{1}{2E}\{p_{\alpha}, f^{\alpha}_{a} E\}_{-} - \frac{1}{2}\, S^{cd}\, f^{\alpha}_{a}\, \omega_{cd\alpha}\,. \end{aligned} \tag{3.9}$$

f^{α}_{a} are vielbein and $\omega_{cd\alpha}$ spin connection fields, the gauge fields of p^a and S^{ab}, respectively. In this paper we do not discuss the families quantum numbers, which commute with here defined discrete symmetries operators. Let the vielbeins and spin connections be responsible for the break of symmetry of $M^{(d-1)+1}$ into $M^{3+1}\times\, M^{d-4}$ so that the manifold M^{d-4} is (almost) compactified and let the spinor manifest in $d = (3+1)$ the ordinary spin and the charges.[13] Looking for the subgroups (denoted by B, C) of the SO$((d-1)+1)$ group and assuming no gravity in $d = (3+1)$, the Lagrange density of eq. (3.9) can be

[13] In the references [8, 11–14] it is demonstrated on the toy model how such an almost compactification could occur.

rewritten in a more familiar shape

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\, E\, \Psi^{\dagger}\, \gamma^{0}\, (\gamma^{m}\, p_{0m} + \gamma^{s}\, p_{0s})\; \Psi\; + h.c.\,, \\
p_{0m} &= p_{m} - \sum_{B} \vec{\tau}^{B}\, \vec{A}^{B}_{m}\,, \\
p_{0s} &= f^{\sigma}_{s}\, p_{\sigma} + \frac{1}{2E}\{p_{\sigma}, f^{\sigma}_{s} E\}_{-} - \sum_{C} \vec{\tau}^{C}\, \vec{A}^{C}_{s}\,,
\end{aligned}
\tag{3.10}
$$

with $m = (0,1,2,3)$, $s = (5,6,\ldots,d)$. We have $\tau^{Bi} = \sum_{st} b^{Bi}{}_{st}\, S^{st}$, $\tau^{Ci} = \sum_{st} c^{Ci}{}_{st}\, S^{st}$, $\sum_{B} \vec{\tau}^{B}\, \vec{A}^{B}_{m} = \frac{1}{2}\sum_{st} S^{st}\, \omega_{stm}$, $\sum_{C} \vec{\tau}^{C}\, \vec{A}^{C}_{s} = \frac{1}{2}\sum_{st} S^{st}\, f^{\sigma}_{s}\omega_{st\sigma}$.

One finds that

$$
\begin{aligned}
\mathbb{C}_{\mathcal{N}}\; \tau^{Ai}\; \mathbb{C}^{-1}_{\mathcal{N}} &= -\tau^{Ai}\,, \\
\mathbb{C}_{\mathcal{N}}\; A^{Ai}_{m}(x^{0},\vec{x}_{3})\; \mathbb{C}^{-1}_{\mathcal{N}} &= -A^{Ai}_{m}(x^{0},\vec{x}_{3})\,, \\
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\; \tau^{Ai}\; (\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}})^{-1} &= -\tau^{Ai}\,, \\
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\; A^{Ai}_{m}(x^{0},\vec{x}_{3})\; (\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}})^{-1} &= -A^{Ai\,m}(x^{0},-\vec{x}_{3})\,, \\
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}\; \tau^{Bi}\; A^{Bi}_{m}(x)\; (\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}})^{-1} &= (-\tau^{Bi})\,(-A^{Bi*}_{m}(-x))\,,
\end{aligned}
\tag{3.11}
$$

for τ^{Ai} from the Cartan subalgebra for each A, but it is always true that $\tau^{Ai}A^{Ai}_{m}$ transforms either to $(-\tau^{Ai})$ $(-A^{Ai}_{m})$ or to $\tau^{Ai}A^{Ai}_{m}$, for each Ai, all in agreement with the standard knowledge for the gauge vector fields and charges in $d = (3+1)$ [60].

One can check also that $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}\;\; \gamma^{a}\;\; (\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}})^{-1} = \gamma^{a}$; $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}\;\; S^{ab}\;\; (\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}})^{-1} = -S^{ab}$; $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}\;\; f^{\alpha}_{a}(x)\, p_{\alpha}\;\; (\mathbb{C}_{\mathcal{N}}\mathcal{P}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}})^{-1} = f^{\alpha\,*}_{a}(-x)\, p_{\alpha}$; $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}\;\; \omega_{abc}(x)\;\; (\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}})^{-1} = -\omega^{*}_{abc}(-x)$. We also have $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}\, \tau^{Ci}\, A^{Ci}_{s}(x) (\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}})^{-1} = (-\tau^{Ci})\,(-A^{Ci*}_{s})(-x))$, concerning in $d = (3+1)$ the gauge scalar fields. The later determine massless and massive solutions for spinors and, if gaining nonzero vacuum expectation values, contribute not only to masses of spinors but also to those gauge fields, to which they couple.

There exist in (almost) compactified spaces $\mathcal{M}^{d-4}$, for particular choices of vielbeins and spin connection fields in eq. (3.9), massless and massive solutions [6–14]. In subsection 4.1 we discuss such a case for $d = (5+1)$. One finds that the operator $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}$ transforms either the massless or massive solutions of the Weyl equation, which represent particle states on the top of the Dirac sea, into their anti-particle states, which are holes in the Dirac sea. It follows also for the case that the infinite surface in the fifth and the sixth dimensions compactifies into an almost S^{2} with the radius ρ_{0} that the massive state $\psi^{(6)(m\rho_{0})}_{(n+\frac{1}{2})}$ $= (\mathcal{A}_{n}\; \stackrel{56}{(+)}\; \psi^{(4)}_{(+)}(\vec{p}) + \mathcal{B}_{n+1}\, e^{i\phi}\; \stackrel{56}{[-]}\; \psi^{(4)}_{(-)}(\vec{p}))\, e^{in\phi}$ ($\psi^{(4)}_{(\pm)}(\vec{p})$ are the corresponding plane wave solutions in $d = (3+1)$ with the three momentum $\vec{p}$) with the charge $(n+\frac{1}{2})$ ($M^{56}\, \psi^{(6)(m\rho_{0})}_{(n+\frac{1}{2})}$ $= (n+\frac{1}{2})\, \psi^{(6)(m\rho_{0})}_{(n+\frac{1}{2})}$, M^{56} is the total angular momentum) transforms under $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}$ into $\psi^{(6)(m\rho_{0})}_{-(n+\frac{1}{2})} = (\mathcal{B}_{n+1}\; \stackrel{56}{(+)}\; \psi^{(4)}_{(+)}(-\vec{p}) + \mathcal{A}_{n}\; e^{i\phi}\; \stackrel{56}{[-]}\; \psi^{(4)}_{(-)}(-\vec{p}))\, e^{-i(n+1)\phi}$, which is the hole in the

Dirac sea. This state $\psi^{(6)(m\rho_0)}_{-(n+\frac{1}{2})}$ solves the by $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}$ transformed Weyl equation (4.7) with $F \to -F$ and $(\mathcal{B}_{n+1} = \mathcal{A}_{-n-1}, \mathcal{A}_n = \mathcal{B}_{-n})$, as one can check in eq. (4.10).

The Hermiticity requirement for the Lagrange density (eq. (3.10)), $\mathcal{L}^{\dagger} = \mathcal{L}$, leads to

$$\omega^{*}_{abc}(x) = (\mp)\,\omega_{abc}(x)\,;(-)\,\text{if}\; a = c\,\text{or}\, b = c\,, (+)\,\text{otherwise}\,, \tag{3.12}$$

which is to be taken into account together with the $\mathbb{C}_{\mathcal{N}}\mathcal{P}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}$ invariance.

4 Discussions on generality of our proposal for discrete symmetries

Searching for the appropriate definition of the discrete symmetries, when starting at higher dimensions than 4, for any even d, we indeed limited ourselves to a simple example with two different choices of dimension. We assumed that there is a central point symmetry (there might be several) and particular rotational symmetries around the central point. We do not really study how could an almost compactification occur. There are several papers in the literature [31, 32, 54, 55] studying the way of compactification. It is not easy to say whether the experiences from this literature can usefully be used in our cases. In one of our cases we so far just use the appropriate gauge fields, zweibeins and spin connections, without paying attention where do they originate and then study the properties of the scalar and vector gauge fields and spinors. We shortly present their properties in the subsections of this section.

Let us ask first how general is our proposal for Kaluza-Klein type of theories. Although for examples like the one when dimensions are compactified into a (compact[14]) torus with momenta as the conserved charges would not be of our type, still our proposal might be of a help to find the definition of appropriate symmetries also for such cases.

We got the proposals for the discrete symmetries for the effective $(3 + 1)$ theory (eqs. (3.1), (3.4)) from analysing our special case, for which one immediately sees that the proposal for $\mathcal{P}^{(d-1)}_{\mathcal{N}}$ does not contain any transformation of the extra dimension coordinates, while getting the contribution of the γ^a matrices adjusted so that the extra dimensional gamma matrices $\gamma^5, \gamma^6, \ldots, \gamma^{d-1}, \gamma^d$ commute with $\mathcal{P}^{(d-1)}_{\mathcal{N}}$. This means that this operation is quite insensitive to the extra dimensions in such a way that it is not important if the extra dimensional space obeys any parity like symmetry. Correspondingly there should be no transformation of the extra dimension boson fields in the sense that the extra dimensional components should not be changed except for their transformation due to the coordinate flipping in the first three dimensions. The components with vector or tensor indices among the first three spatial components bring correspondingly the signs shifted, but otherwise the boson fields are not to be transformed under $\mathcal{P}^{(d-1)}_{\mathcal{N}}$. This means successively that the general type of background fields describing the extra dimensional curling up in some way will not be modified under this operation and thus one takes such fields as background fields in the sense of this $\mathcal{N}$-marked parity operation $\mathcal{P}^{(d-1)}_{\mathcal{N}}$, which means that one leaves such background fields untouched.

[14] An almost from the infinite surface compactified torus has no conserved charges.

Looking at eq. (3.4) one sees that the background fields have to obey some reflection symmetry in order that $\mathcal{T}_{\mathcal{N}}$ and $\mathcal{C}_{\mathcal{N}}$ be well defined symmetries. (One needs well defined discrete symmetries even if in particular cases each of them is not a good symmetry, when the handedness of spinors prevent them to be a good symmetries, while the product of the two is then a good symmetry.)

So, unless the extra dimensional back ground fields obey in even d the reflection symmetry for $\mathcal{C}_{\mathcal{N}}$

$$(x^5, x^6, \cdots, x^d) \to (x^5, -x^6, x^7, -x^8, \cdots, x^{d-1}, -x^d)\,, \tag{4.1}$$

while for $\mathcal{T}_{\mathcal{N}}$ they obey

$$(x^5, x^6, \cdots, x^d) \to (-x^5, x^6, -x^7, x^8, \cdots, -x^{d-1}, x^d)\,, \tag{4.2}$$

the equations of motion for spinors do not have these symmetries of $\mathcal{T}_{\mathcal{N}}$ and $\mathcal{C}_{\mathcal{N}}$. One easily checks that the toy model [11–14] has the above (eqs. (3.4), (4.1), (4.2)) symmetry.

These requirements for the extra dimensional reflection for background and fermion fields of eqs. (4.1), (4.2) are due to our request that anti-particles should manifest in $(3+1)$ dimensions opposite charges as particles (the charges of which correspond to appropriate "Killing forms"). (So that $\mathbb{C}_{\mathcal{N}}$ inverts the charges.) One can understand the alternating reflection properties of x^s, $s \geq 5$, eq. (4.1), example of the toy model [11–14], by the requirement that the "Killing forms", which are circles around the fixed point, must change the orientation.

Concerning the alternating reflection (in coordinate space) of $\mathcal{T}_{\mathcal{N}}$ in eq. (4.2) one can understand this alternation by again looking at our example of the toy model [11–14]. Since $\mathcal{T}_{\mathcal{H}}$ (eq. (2.6)) reflects the momentum $\vec{p}$ in $(d-1)$ dimensions, the "Killing forms" acquire a change in the direction. To compensate the change of the sign of the "Killing forms" we need the alternative reflection offered by $\mathcal{T}_{\mathcal{N}}$. In this way one namely obtains the usually wanted property of the $(3+1)$-dimensional time reversal operator $\mathcal{T}_{\mathcal{N}}$ that it leaves the charges untouched.

While $\mathcal{T}_{\mathcal{H}}$ does change the signs of "Killing forms", $\mathbb{C}_{\mathcal{H}}$ does not. So, both, $\mathbb{C}_{\mathcal{H}}$ and $\mathcal{T}_{\mathcal{H}}$ are cured by the same reflection of "Killing forms": in an example, when compactification is made by a torus (let us say again that almost compactified torus has no rotational symmetry), where the generators of translations around the torus are declared as charges in $(3+1)$, we must replace the reflection symmetry of eqs. (4.1), (4.2) by the reflection which again inverts the corresponding "Killing forms". This means that x^s goes to $-x^s$, $s = 5, 6, \ldots$ around any point.

In the torus case we need the true parity $\mathcal{P}_{\mathcal{H}} \times \mathcal{P}_{\mathcal{N}}$ in extra dimensions to change the signs of "Killing forms".

In complicated cases we can a priori imagine that constructing appropriate reflections inverting the signs of all the to be charges "Killing forms" could be complicated.

If the background fields are mainly just the metric tensor fields with extra dimensional components and the charges commute, it would not be difficult to find for each separate charge a reflection symmetry, reflecting just that symmetry, just that charge. Combining these reflections for the separate charges to a combined reflection reflecting all the charges would then be a proposal for the replacement for (4.2) and (4.1).

Let us mention the ref. [56] with one of the authors of this paper (H.B.N.) as a coauthor. The book stresses that symmetries can often be derived from small assumptions which we put into a theory. For the discrete symmetries for the strong and electromagnetic interactions one ought to assume: i. Anomaly cancellations, ii. Small group representations and iii. Charge quantization rule. This author understands their derivation as a competitive way of deriving the discrete symmetries operators without knowing the theory behind.

Let us add that the Calabi-Yau kind of spaces [31, 32] seems to have the symmetry so that our proposed discrete symmetries work.

4.1 Comments on two special cases

In the subsection 3.2 we discuss how do our proposed discrete symmetries, eq. (3.1), (3.4), behave in cases when there are the vielbein and spin connection fields (eq. (3.9)) of the Kaluza-Klein kinds, which determine the spinor interactions. We demonstrate there how do spinors manifest in $d = (3+1)$ the Kaluza-Klein charges, interact with the Kaluza-Klein vector gauge fields and with the scalar gauge fields (these last ones determine masses of spinors in $(3 + 1)$ and, after gaining nonzero vacuum expectation values, besides the masses of spinors also the masses of those vector gauge fields which they interact with) and how do spinors, vector gauge fields and scalar gauge fields transform under our proposed discrete symmetries.

In this subsection we shortly present the fields, zweibeins and spin connections, which in our toy model [11–14] in $d = (5+1)$ cause an almost compactification. We also comment briefly our "realistic case" in $d = (13+1)$ which is offering the explanation for all the charges and gauge fields of the *standard model*, with the families and scalar fields included, although we do not discuss in this paper the appearance of families and correspondingly a possible explanation for the Yukawa couplings [15–30].

A toy model in $d = (5 + 1)$. In the ref. [11–13] we present the zweibeins and the spin connection fields, assumed to be caused by a kind of spinor condensates, which allow after the compactification of the manifold M^{5+1} into $M^{3+1}\times$ an almost S^2 one massless and mass protected solution and the chain of massive solutions of the Weyl equation following from the Lagrange density in eq. (3.9). We assume a flat $(3 + 1)$ space and the zweibein in $d = (5, 6)$

$$e^{s}{}_{\sigma} = f^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f^{\sigma}{}_{s} = f \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.3}$$

with

$$\begin{aligned} f &= 1 + \left(\frac{\rho}{2\rho_0}\right)^2 = \frac{2}{1+\cos\vartheta}, \\ x^{(5)} &= \rho\,\cos\phi, \quad x^{(6)} = \rho\,\sin\phi, \\ E &= \det(e^{s}{}_{\sigma}) = f^{-2}, e^{s}{}_{\sigma}\, f^{\sigma}{}_{t} = \delta^{s}_{t}, \end{aligned} \tag{4.4}$$

and the spin connection field

$$f^{\sigma}{}_{s'}\,\omega_{st\sigma} = iF\,f\,\varepsilon_{st}\,\frac{e_{s'\sigma}x^{\sigma}}{(\rho_0)^2}\,, \quad 0 < 2F \leq 1\,, \quad s = 5,6, \quad \sigma = (5),(6)\,, \tag{4.5}$$

where ρ_0 is the radius of S^2. It follows that this choice of the spin connection field on an almost S^2 allows for $0 < 2F \leq 1$ only one normalizable (square integrable) massless solution - the left handed spinor with the Kaluza-Klein charge in $d = (3+1)$ equal to $\frac{1}{2}$. The massless and massive solutions preserve the rotational symmetry around the axis perpendicular to the surface in the fifth and the sixth dimension and are correspondingly the eigenfunctions of the total angular momentum $M^{56} = x^5p^6 - x^6p^5 + S^{56} = -i\frac{\partial}{\partial\phi} + S^{56}$, $M^{56}\psi^{(6)} = (n+\frac{1}{2})\,\psi^{(6)}$. For the choice of the coordinate system $p^a = (p^0,0,0,p^3,p^5,p^6)$ the massive solution with the Kaluza-Klein charge $n + 1/2$

$$\psi^{(6)(\rho_0 m)}_{n+1/2} = (\mathcal{A}_n \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)} + \mathcal{B}_{n+1}\,e^{i\phi}\stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{[-]})\cdot e^{in\phi}e^{-i(p^0x^0-p^3x^3)}\,, \tag{4.6}$$

solves the equation of motion, derived from the Lagrange function eq. (3.9), with $\mathcal{A}_n$ and $\mathcal{B}_{n+1}$ determined by the equations

$$\begin{aligned}
-if\left\{\left(\frac{\partial}{\partial\rho} + \frac{n+1}{\rho}\right) - \frac{1}{2f}\frac{\partial f}{\partial\rho}\,(1+2F)\right\}\mathcal{B}_{n+1} + m\mathcal{A}_n &= 0\,,\\
-if\left\{\left(\frac{\partial}{\partial\rho} - \frac{n}{\rho}\right) - \frac{1}{2f}\frac{\partial f}{\partial\rho}\,(1-2F)\right\}\mathcal{A}_n + m\mathcal{B}_{n+1} &= 0\,.
\end{aligned} \tag{4.7}$$

There exists the massless left handed spinor with the Kaluza-Klein charge in $d = (3+1)$ equal to $\frac{1}{2}$

$$\psi^{(6)(m=0)}_{\frac{1}{2}} = \mathcal{N}_0\,f^{-F+1/2}\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\;e^{-i(p^0x^0-p^3x^3)}\,. \tag{4.8}$$

For $F = \frac{1}{2}$ and $p^1 = 0 = p^2$ this solution corresponds to the particle described by $\psi^{\rm pos}_1$ and put on the top of the Dirac sea. The corresponding $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}$ transformed state, put on the top of the Dirac sea, that is the anti-particle state, the hole indeed in the Dirac sea, is the state $\psi^{\rm pos}_4$ corresponding to the empty $\psi^{\rm neg}_3$ in the Dirac sea, in accordance with what we have discussed in section 3. With the operator $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}$ transformed state $\psi^{(6)(\rho_0 m)}_{n+1/2}$ is the state

$$\psi^{(6)(\rho_0 m)}_{-(n+1/2)} = (\mathcal{A}_{-(n+1)} \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)} + \mathcal{B}_{-n}\,e^{i\phi}\stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{[-]})\cdot e^{-i(n+1)\phi}e^{-i(p^0x^0+p^3x^3)}\,, \tag{4.9}$$

with the two functions $\mathcal{A}_{-(n+1)}$ and $\mathcal{B}_{-n}$, which solve the equations

$$\begin{aligned}
-if\left\{\left(\frac{\partial}{\partial\rho} - \frac{n}{\rho}\right) - \frac{1}{2f}\frac{\partial f}{\partial\rho}\,(1-2F)\right\}\mathcal{B}_{-n} + m\mathcal{A}_{-(n+1)} &= 0,\\
-if\left\{\left(\frac{\partial}{\partial\rho} + \frac{n+1}{\rho}\right) - \frac{1}{2f}\frac{\partial f}{\partial\rho}\,(1+2F)\right\}\mathcal{A}_{-(n+1)} + m\mathcal{B}_{-n} &= 0\,,
\end{aligned} \tag{4.10}$$

where F goes to $-F$, in accordance with the $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}} = \gamma^0\gamma^5 I_{\vec{x}_3} I_{x_6}$ transformation requirement for the fields.

One easily sees that $\psi^{(6)(\rho_0 m)}_{-(n+1/2)} = -\,\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\;\psi^{(6)(\rho_0 m)}_{(n+1/2)}$.

Would the scalar (with respect to $(d = (3+1))$) $f^{\sigma}_{s}\omega_{56\sigma}$ achieve nonzero vacuum expectation values breaking the rotational symmetry on the $(5,6)$ surface, the charge S^{56} would no longer be conserved and the scalar fields would behave similar as the Higgs of the *standard model*, carrying in this case the "hypercharge" S^{56}.

The case with $d = (13+1)$. In the case of $d = (13+1)$ the compactification is again assumed to be triggered by spinor condensates which then cause the appearance of vielbeins and spin connection fields. The compactification from the symmetry SO$(13,1)$ (first to SO$(7,1) \times$ U$(1)_{II} \times$ SU(3) and then) to SO$(3,1) \times$ SU$(2)_I \times$ SU$(2)_{II}$ $\times$U$(1)_{II}$ $\times$SU(3), leaving all the family members massless (in the toy model case we found the solution for the compactification of the (x_5, x_6) surface into an almost S^2 for particular spin connections and vielbeins) ensure that the spins in $d > 4$ (in the low energy limit, otherwise the total angular momenta) manifest in $d = (3+1)$ all the observed charges. (There are in the theory [15–30] two kinds of spin connection fields. The second one, not discussed in this paper, takes care of families. Correspondingly there are before the electroweak break four, rather than three so far observed, massless families of quarks and leptons.)

We don't yet have the solution for the compactification procedure not even comparable with the one for the toy model in $d = (5+1)$. This study is under consideration.

However, analysing a massless left handed representation in $d = (13+1)$ — similarly as in the case of the toy model but in this case taking into account the charge groups of quarks and leptons assumed by the *standard model*, they are subgroups of SO$(13,1)$ — one easily sees that one (each) family representation in $d = (13+1)$ contains [15–30] the left handed (with respect to $d = (3+1)$) weak charged coloured quarks and colourless leptons with particular spinor quantum number ($\frac{1}{6}$ for quarks and $-\frac{1}{2}$ for leptons) and zero hyper charge and the right handed weak chargeless quarks and leptons, with the spinor charge of the left handed ones but with the hyper charges as required by the *standard model*. In table 2 are u and d quarks of a particular colour presented, left and right handed ones. Leptons distinguish from the quarks in the colour and in the spinor quantum numbers. One can find the whole one family representation in the ref. [30] and in table 3 of appendix A.

When the scalar spin connection fields of the two kinds (bringing appropriate weak and hyper charges to the right handed members of one family) gain nonzero vacuum expectation values, the electroweak break occurs, causing that the fermions and the weak bosons become massive, while the U(1) electromagnetic field stay massless.

The effective Lagrange density is presented in eqs. (3.9), (3.10).

The term $\bar{\psi}\gamma^s p_{0s}\,\psi$ is responsible for masses of spinors in $d = (3+1)$, with $\gamma^0\gamma^s\,, s = (7,8)$ transforming the right handed quarks and leptons, weak chargeless and of particular hypercharge into the left handed weak charged partners.

Similarly as in the case of the toy model the discrete symmetries of eq. (3.4) keep their meaning also in this case.

5 Conclusions

We define in this paper the discrete symmetries, $\mathbb{C}_{\mathcal{N}}$, $\mathcal{P}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{N}}$ (eqs. (3.1), (3.4)) in even dimensional spaces leading in $d = (3+1)$ to the experimentally observed symmetries, if the Kaluza-Klein kind of a theory [33–39] with $d > (3+1)$ determining charges in $d = (3+1)$

(among them also the *spin-charge-family* proposal of one of us (N.S.M.B. [15–30, 57–59] offering also the mechanism for generating families) is the right way to understand the assumptions of the *standard model*. We indeed define three kinds of the charge conjugation operators: besides $\underline{\mathbb{C}}_{\mathcal{N}}$, which operates on the creation operator for a particle, also $\mathcal{C}_{\mathcal{N}}$ transforming the positive energy state representing a particle when put on the top of the Dirac sea into its negative energy partner, and $\mathbb{C}_{\mathcal{N}}$ which empties this negative energy state in the Dirac sea representing on the top of the Dirac sea the anti-particle state (3.2).

Although we designed this discrete symmetry operators for cases with a central point symmetry (see section 4) (there might be several) and particular rotational symmetries around the central point in higher dimensions, yet our proposal might help to define these discrete symmetries also in more complicated cases, as discussed in section 4.

Our ($\underline{\mathbb{C}}_{\mathcal{N}}$, $\mathcal{P}_{\mathcal{N}}$, $\mathcal{T}_{\mathcal{N}}$) discrete symmetries are rotated and reflected with respect to the symmetries as they would follow if extending the definition of the discrete symmetries from $d = (3+1)$ to any even d: ($\underline{\mathbb{C}}_{\mathcal{H}}$, $\mathcal{P}_{\mathcal{H}}$ and $\mathcal{T}_{\mathcal{H}}$), presented in eqs. (2.1), (2.6). The discrete symmetries ($\underline{\mathbb{C}}_{\mathcal{H}}$, $\mathcal{P}_{\mathcal{H}}$ and $\mathcal{T}_{\mathcal{H}}$) do not lead, namely, to the experimentally observed definitions, since if using $\underline{\mathbb{C}}_{\mathcal{H}}$ on a second quantized state $\mathbf{\Psi}^{\dagger}$, the charge conjugated state has the same charge as the starting state. The proposed new discrete symmetries ($\underline{\mathbb{C}}_{\mathcal{N}}$, $\mathcal{P}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{N}}$) behave as they should — in agreement with the observed properties of fermions and anti-fermions.

We do not study in this paper the break of $\underline{\mathbb{C}}_{\mathcal{N}}$, $\mathcal{P}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{N}}$ symmetries.

We analyse properties of the proposed symmetries from the point of view of the observables in $d = (3+1)$. Our definition of discrete symmetries is, as discussed in this paper and in particular in section 4, more general and valid for spaces with the central points and rotational symmetries around these points and might be helpful also for finding appropriate discrete symmetries operators in examples, when compactification is made by a torus, where the generators of translations around the torus are declared as charges in $(3+1)$.

These discrete symmetries do not distinguish among families of fermions as long as the family groups form equivalent representations with respect to the charge groups.

We illustrate our definition of the discrete symmetries on two cases: i. $d = (5+1)$ and ii. $d = (13+1)$. The first case is a toy model which we show [6–14] that the Kaluza-Klein kind of theories can lead in non-compact spaces to observable (almost massless) properties of fermions. We present in table 1 one family of fermions of positive and negative energy states. We also presented a way for a possible compactification in this toy model to demonstrate that our definition of the discrete symmetries is meaningful 4.1.

For the second illustration of the proposed discrete symmetries the one family spinor representation of the *spin-charge-family* theory, which explains the assumptions of the *standard model*, is taken. We present in table 2 the representation of quarks of particular colour charge, in table 3 we present all the members of one representation. It contains quarks and leptons and the corresponding charge conjugated states.

The discrete symmetries proceed similarly to the case of $d = (5+1)$. In this second illustration fermions carry the experimentally recognized properties: $\mathbb{C}_{\mathcal{N}}$ $\mathcal{P}_{\mathcal{N}}$ transforms the right handed u-quark with the spin up, weak chargeless and of the colour charge $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$ and the hyper charge equal to $\frac{2}{3}$ into the left handed weak chargeless anti-quark with spin up and with the anti-colour charge $(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ and anti-hyper charge $-\frac{2}{3}$ (see appendix A lines 1 and 35 and also the ref. [30], table 2. line 1 and table 4. line 35). $\mathbb{C}_{\mathcal{N}}\, \mathcal{P}_{\mathcal{N}}$ transforms the weak charged $(\frac{1}{2})$ left handed neutrino, with spin up and colour chargeless into the right handed weak anti-charged $(-\frac{1}{2})$ anti-neutrino with the spin up, anti-colour chargeless (see appendix A table 3, line 31 and 61 and also the ref. [30], table 3, line 31 and table 5, line 61).

We also discuss about an acceptable compactification procedure, which leads in this case to the *standard model* as a low energy effective theory of the *spin-charge-family* theory. This study is in progress.

We concentrated on discrete symmetries of fermions, but discussed also the properties of bosonic fields in higher dimensions, which are assumed to be treated as background fields, discussing in section 4 their behaving with respect to both kinds of the discrete symmetries: $\underline{\mathbb{C}}_{\mathcal{N}}$, $\mathcal{P}^{(d-1)}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{N}}$ and $\underline{\mathbb{C}}_{\mathcal{H}}$, $\mathcal{P}^{(d-1)}_{\mathcal{H}}$ and $\mathcal{T}_{\mathcal{H}}$.

The proposed discrete symmetries $\underline{\mathbb{C}}_{\mathcal{N}}$, $\mathcal{P}^{(d-1)}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{N}}$, defined for spaces with dimensions d even have obviously the desired properties in the observable part of space in cases with central point symmetries and the rotational symmetries around such central points [11, 12, 15–30], in which the way of curling up the higher dimensional space into (almost) compact spaces or non compact spaces do not break a parity.

To discuss discrete symmetries of Kaluza-Klein kind of theories proposed in the literature [31, 32, 41–53] from the point of view of our proposal would require our complete understanding of these models and in addition discussions with the authors.

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A The technique for representing spinors [6, 20, 57–59], a shortened version of the one presented in [15–18]

The technique [6, 20, 57–59] can be used to construct a spinor basis for any dimension d and any signature in an easy and transparent way. Equipped with the graphic presentation of basic states, the technique offers an elegant way to see all the quantum numbers of states with respect to the Lorentz groups, as well as transformation properties of the states under any Clifford algebra object.

The objects γ^a have properties $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}\, I$, for any d, even or odd. I is the unit element in the Clifford algebra.

The Clifford algebra objects S^{ab} close the algebra of the Lorentz group $S^{ab} := (i/4)(\gamma^a \gamma^b - \gamma^b \gamma^a)\,,\ \{S^{ab}, S^{cd}\}_- = i(\eta^{ad} S^{bc} + \eta^{bc} S^{ad} - \eta^{ac} S^{bd} - \eta^{bd} S^{ac})\,.$ The “Hermiticity” property for γ^a's: $\gamma^{a\dagger} = \eta^{aa} \gamma^a$ is assumed in order that γ^a are formally unitary, i.e. $\gamma^{a\,\dagger}\, \gamma^a = I$.

The Cartan subalgebra of the algebra is chosen in even dimensional spaces as follows: $S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\, d}, \quad \text{if} \quad d = 2n \geq 4.$

The choice for the Cartan subalgebra in $d > 4$ is straightforward. It is useful to define one of the Casimir operators of the Lorentz group - the handedness Γ ($\{\Gamma, S^{ab}\}_- = 0$) in any d, for even dimensional spaces it follows: $\Gamma^{(d)} := (i)^{d/2} \quad \prod_a \quad (\sqrt{\eta^{aa}}\gamma^a)$, if $\quad d = 2n$. The product of γ^a's in the ascending order with respect to the index a: $\gamma^0\gamma^1 \cdots \gamma^d$ is understood. It follows for any choice of the signature η^{aa} that $\Gamma^\dagger = \Gamma$, $\Gamma^2 = I$. For d even the handedness anticommutes with the Clifford algebra objects γ^a ($\{\gamma^a, \Gamma\}_+ = 0$).

To make the technique simple the graphic presentation is introduced

$$\stackrel{ab}{(k)}:= \frac{1}{2}\left(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b\right), \qquad \stackrel{ab}{[k]}:= \frac{1}{2}\left(1 + \frac{i}{k}\gamma^a\gamma^b\right), \tag{A.1}$$

where $k^2 = \eta^{aa}\eta^{bb}$. One can easily check by taking into account the Clifford algebra relation and the definition of S^{ab} that if one multiplies from the left hand side by S^{ab} the Clifford algebra objects $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$, it follows that

$$S^{ab} \stackrel{ab}{(k)}= \frac{1}{2} k \stackrel{ab}{(k)}, \qquad S^{ab} \stackrel{ab}{[k]}= \frac{1}{2} k \stackrel{ab}{[k]}, \tag{A.2}$$

which means that we get the same objects back multiplied by the constant $\frac{1}{2}k$. This also means that when $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$ act from the left hand side on a vacuum state $|\psi_0\rangle$ the obtained states are the eigenvectors of S^{ab}. One can further recognize that γ^a transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[-k]}$, never to $\stackrel{ab}{[k]}$:

$$\gamma^a \stackrel{ab}{(k)}= \eta^{aa} \stackrel{ab}{[-k]},\ \gamma^b \stackrel{ab}{(k)}= -ik \stackrel{ab}{[-k]},\ \gamma^a \stackrel{ab}{[k]}=\stackrel{ab}{(-k)},\ \gamma^b \stackrel{ab}{[k]}= -ik\eta^{aa} \stackrel{ab}{(-k)}\ . \tag{A.3}$$

Let us deduce some useful relations

$$\begin{aligned}
&\stackrel{ab}{(k)}\stackrel{ab}{(k)} = 0\,, \qquad &&\stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa} \stackrel{ab}{[k]}\,, \qquad &&\stackrel{ab}{[k]}\stackrel{ab}{[k]} =\stackrel{ab}{[k]}\,, \qquad &&\stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0\,,\\
&\stackrel{ab}{(k)}\stackrel{ab}{[k]} = 0\,, &&\stackrel{ab}{[k]}\stackrel{ab}{(k)} =\stackrel{ab}{(k)}\,, &&\stackrel{ab}{(k)}\stackrel{ab}{[-k]} =\stackrel{ab}{(k)}\,, &&\stackrel{ab}{[k]}\stackrel{ab}{(-k)} = 0\,.
\end{aligned} \tag{A.4}$$

Taking into account the above equations it is easy to find a Weyl spinor irreducible representation for d-dimensional space.

For d even we simply make a starting state as a product of $d/2$, let us say, only nilpotents $\stackrel{ab}{(k)}$, one for each S^{ab} of the Cartan subalgebra elements, applying it on an (unimportant) vacuum state. Then the generators S^{ab}, which do not belong to the Cartan subalgebra, being applied on the starting state from the left, generate all the members of one Weyl spinor.

$$\begin{array}{c}
\stackrel{0d}{(k_{0d})}\stackrel{12}{(k_{12})}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\,d-2}{(k_{d-1\,d-2})}\ \psi_0\\
\stackrel{0d}{[-k_{0d}]}\stackrel{12}{[-k_{12}]}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\,d-2}{(k_{d-1\,d-2})}\ \psi_0\\
\vdots
\end{array}$$

$$\overset{od}{(k_{0d})}\overset{12}{[-k_{12}]}\overset{35}{[-k_{35}]}\cdots\overset{d-1\,d-2}{(k_{d-1\,d-2})}\;\psi_0$$

$$\vdots \tag{A.5}$$

All the states have the handedness Γ, since $\{\Gamma, S^{ab}\}_- = 0$. States, belonging to one multiplet with respect to the group $\mathrm{SO}(q, d-q)$, that is to one irreducible representation of spinors (one Weyl spinor), can have any phase. We made a choice of the simplest one, taking all phases equal to one.

There are two kinds of the Clifford algebra objects [13–16, 57, 58]: besides the Dirac γ^a ones also $\tilde{\gamma}^a$, with the properties [57, 58]

$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab}\,, \qquad \{\gamma^a, \tilde{\gamma}^b\}_+ = 0\,, \tag{A.6}$$

for any d, even or odd. $\tilde{\gamma}^a$ form the equivalent representations with respect to γ^a. If γ^a multiply any Clifford algebra object $\mathbf{B} = \sum_{i=0,d} a_{a_1\cdots a_i}\gamma^{a_1}\cdots\gamma^{a_i}$ from the left hand side ($\gamma^a\mathbf{B}|\mathrm{vac}\rangle_{\mathrm{fam}}$, $|\mathrm{vac}\rangle_{\mathrm{fam}}$ is the vacuum state), multiply $\tilde{\gamma}^a$ the same $\mathbf{B}$ from the right hand side ($\tilde{\gamma}^a\mathbf{B}|\mathrm{vac}\rangle_{\mathrm{fam}} = i(-)^{n_B}\,\mathbf{B}\gamma^a|\mathrm{vac}\rangle_{\mathrm{fam}}$. $(-)^{n_B} = +1, -1$, when the object $\mathbf{B}$ has an even or odd Clifford character, respectively.

Correspondingly transforms γ^a the object $\overset{ab}{(k)}$ into $\overset{ab}{[-k]}$, never to $\overset{ab}{[k]}$, and $\overset{ab}{[k]}$ to $\overset{ab}{(-k)}$, never to $\overset{ab}{(k)}$, while $\tilde{\gamma}^a$ transforms $\overset{ab}{(k)}$ into $\overset{ab}{[k]}$, never to $\overset{ab}{[-k]}$, and $\overset{ab}{[k]}$ to $\overset{ab}{(k)}$, never to $\overset{ab}{(-k)}$.

The generators S^{ab} (or superposition of S^{ab}) take care of spins and charges of the family members, while $\tilde{S}^{ab}$ (or superposition of $\tilde{S}^{ab}$) take care of families.

We present below the multiplet of states and charge conjugated states of quarks and leptons belonging to one SO(13, 1) multiplet of quarks and leptons. To find families of the subgroup SO(7, 1) the generators of $\tilde{S}^{ab}, (a, b) \in (0, \ldots, 7, 8)$, with the property $\{\tilde{S}^{ab}, S^{ab}\}_- = 0$ must be taken into account.

i		$\vert^a\psi_i >$	$\Gamma^{(3,1)}$	S^{12}	$\Gamma^{(4)}$	τ^{13}	τ^{23}	τ^{31}	τ^{38}	τ^4	Y	Q
		Octet, $\Gamma^{(1,7)} = 1,\ \Gamma^{(6)} = -1$, of quarks and leptons										
1	u_R^{c1}	$\overset{03}{(+i)}\overset{12}{(+)} \vert \overset{56}{(+)}\overset{78}{(+)} \Vert \overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
2	u_R^{c1}	$\overset{03}{[-i]}\overset{12}{[-]} \vert \overset{56}{(+)}\overset{78}{(+)} \Vert \overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	1	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
3	d_R^{c1}	$\overset{03}{(+i)}\overset{12}{(+)} \vert \overset{56}{[-]}\overset{78}{[-]} \Vert \overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	1	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	d_R^{c1}	$\overset{03}{[-i]}\overset{12}{[-]} \vert \overset{56}{[-]}\overset{78}{[-]} \Vert \overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	1	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
5	d_L^{c1}	$\overset{03}{[-i]}\overset{12}{(+)} \vert \overset{56}{[-]}\overset{78}{(+)} \Vert \overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	-1	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
6	d_L^{c1}	$\overset{03}{(+i)}\overset{12}{[-]} \vert \overset{56}{[-]}\overset{78}{(+)} \Vert \overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	-1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
7	u_L^{c1}	$\overset{03}{[-i]}\overset{12}{(+)} \vert \overset{56}{(+)}\overset{78}{[-]} \Vert \overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	-1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
8	u_L^{c1}	$\overset{03}{(+i)}\overset{12}{[-]} \vert \overset{56}{(+)}\overset{78}{[-]} \Vert \overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	-1	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
9	u_R^{c2}	$\overset{03}{(+i)}\overset{12}{(+)} \vert \overset{56}{(+)}\overset{78}{(+)} \Vert \overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{[-]}$	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
10	u_R^{c2}	$\overset{03}{[-i]}\overset{12}{[-]} \vert \overset{56}{(+)}\overset{78}{(+)} \Vert \overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{[-]}$	1	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
⋯												
17	u_R^{c3}	$\overset{03}{(+i)}\overset{12}{(+)} \vert \overset{56}{(+)}\overset{78}{(+)} \Vert \overset{9\,10}{[-]}\;\overset{11\,12}{[-]}\;\overset{13\,14}{(+)}$	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
18	u_R^{c3}	$\overset{03}{[-i]}\overset{12}{[-]} \vert \overset{56}{(+)}\overset{78}{(+)} \Vert \overset{9\,10}{[-]}\;\overset{11\,12}{[-]}\;\overset{13\,14}{(+)}$	1	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
⋯												

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25	ν_R	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
26	ν_R	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\mid\stackrel{56}{(+)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
27	e_R	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\mid\stackrel{56}{[-]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
28	e_R	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\mid\stackrel{56}{[-]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
29	e_L	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\mid\stackrel{56}{[-]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
30	e_L	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\mid\stackrel{56}{[-]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
31	ν_L	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
32	ν_L	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\mid\stackrel{56}{(+)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
33	$\bar{d}_L^{\bar{c}1}$	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
34	$\bar{d}_L^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\mid\stackrel{56}{(+)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
35	$\bar{u}_L^{\bar{c}1}$	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\mid\stackrel{56}{[-]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ -	1	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
36	$\bar{u}_L^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\mid\stackrel{56}{[-]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
37	$\bar{d}_R^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
38	$\bar{d}_R^{\bar{c}1}$	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\mid\stackrel{56}{(+)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
39	$\bar{u}_R^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\mid\stackrel{56}{[-]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
40	$\bar{u}_R^{\bar{c}1}$	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\mid\stackrel{56}{[-]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
41	$\bar{d}_L^{\bar{c}2}$	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
…												
49	$\bar{d}_L^{\bar{c}3}$	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
…												
57	$\bar{e}_L$	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
58	$\bar{e}_L$	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\mid\stackrel{56}{(+)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
59	$\bar{\nu}_L$	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\mid\stackrel{56}{[-]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
60	$\bar{\nu}_L$	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\mid\stackrel{56}{[-]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
61	$\bar{\nu}_R$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\mid\stackrel{56}{[-]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
62	$\bar{\nu}_R$	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\mid\stackrel{56}{[-]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
63	$\bar{e}_R$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
64	$\bar{e}_R$	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\mid\stackrel{56}{(+)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Table 3. The left handed ($\Gamma^{(13,1)} = -1 = \Gamma^{(7,1)} \times \Gamma^{(6)}$) multiplet of spinors — the members of the SO(13, 1) group, manifesting the subgroup SO(7, 1) — of the colour charged quarks and anti-quarks and the colourless leptons and anti-leptons is presented in the massless basis using the technique of this appendix. It contains the left handed ($\Gamma^{(3,1)}$) weak charged (τ^{13}) and $\mathrm{SU}(2)_{II}$ chargeless (τ^{23}) quarks and the right handed weak chargeless and $\mathrm{SU}(2)_{II}$ charged quarks of three colours ($c^i = (\tau^{33}, \tau^{38})$), with the spinor charge (τ^4) and the colourless left handed weak charged leptons and the right handed weak chargeless leptons. S^{12} defines the ordinary spin $\pm\frac{1}{2}$. Additional notations are presented in table 2. The vacuum state $|\mathrm{vac}\rangle_{\mathrm{fam}}$, on which the nilpotents and projectors operate, is not shown. The reader can find this Weyl representation also in the refs. [15–17, 29, 30]. Two anti-octets of anti-quarks of the rest two anti-triplet colours follow from the presented one by substituting the part $\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ of the basis by $\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ for $\bar{q}_{L,R}^{\bar{c}2}$ and by $\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ for $\bar{q}_{L,R}^{\bar{c}3}$. Correspondingly the charges are $(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ and $(0, \frac{1}{\sqrt{3}})$, respectively.[15]

[15]The family in table 3 differs from the family in table 2 in the part concerning coordinates $x^9, x^{10}, x^{11}, x^{12}, x^{13}, x^{14}$, but both are equivalent with respect to the discrete symmetries, charges and spins.

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