Conservation of asymptotic charges from past to future null infinity: supermomentum in general relativity

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ABSTRACT: We show that the BMS-supertranslations and their associated supermomenta on past null infinity can be related to those on future null infinity, proving the conjecture of Strominger for a class of spacetimes which are asymptotically-flat in the sense of Ashtekar and Hansen. Using a cylindrical 3-manifold of both null and spatial directions of approach towards spatial infinity, we impose appropriate regularity conditions on the Weyl tensor near spatial infinity along null directions. The asymptotic Einstein equations on this 3-manifold and the regularity conditions imply that the relevant Weyl tensor components on past null infinity are antipodally matched to those on future null infinity. The subalgebra of totally fluxless supertranslations near spatial infinity provides a natural isomorphism between the BMS-supertranslations on past and future null infinity. This proves that the flux of the supermomenta is conserved from past to future null infinity in a classical gravitational scattering process provided additional suitable conditions are satisfied at the timelike infinities.

Contents

1 Introduction

For asymptotically-flat spacetimes describing isolated systems in general relativity, it is well-known that at both future and past null infinities one obtains an infinite-dimensional asymptotic symmetry group — the Bondi-Metzner-Sachs (BMS) group — along with the corresponding charges and fluxes due to gravitational radiation [1–7]. Similarly, at spatial infinity one again obtains an infinite-dimensional asymptotic symmetry group — the Spi-group — and the Arnowitt-Deser-Misner (ADM) energy and angular momentum as conserved charges corresponding to a Poincaré subgroup [8–17]. For a detailed review of asymptotic structures in general relativity see [18].

Recently it has been conjectured by Strominger [19] that the (a priori independent) BMS groups on past and future null infinities can be related through an antipodal reflection near spatial infinity. Such a matching gives a *global* "diagonal" asymptotic symmetry group for general relativity. If similar matching conditions (which were assumed as "boundary

conditions" in [19]) relate the gravitational fields, it would imply infinitely many conservation laws in classical gravitational scattering in the sense that the incoming fluxes associated to the BMS group at past null infinity would equal the outgoing fluxes of the corresponding BMS group at future null infinity. It has been further conjectured that this diagonal group is also a symmetry of the scattering matrix in quantum gravity [19] and the corresponding conservation laws (in linearised gravity around Minkowski spacetime) have been related to various soft theorems [20, 21]. These conservation laws are also speculated to play a role in the resolution of the black hole information loss problem [22, 23] (see however [24] for a contrarian view).

However, the validity of such matching conditions for the asymptotic symmetries and charges has not been proven even in classical general relativity except in certain special cases discussed below. The main difficulty in resolving the matching problem is the limited structure available at spatial infinity in general spacetimes. The asymptotic behaviour of the gravitational field for any asymptotically-flat spacetime can be conveniently described in a conformally-related unphysical spacetime, the Penrose conformal-completion. In the unphysical spacetime, null infinities $\mathscr{I}^{\pm}$ are smooth null boundaries while spatial infinity is a boundary point i^0 acting as the vertex of "the light cone at infinity". For Minkowski spacetime the unphysical spacetime is smooth (in fact, analytic) at i^0, and so a natural identification exists between the null generators (and fields) on $\mathscr{I}^-$ with those on $\mathscr{I}^+$ by "passing through" i^0. However, in more general spacetimes, the unphysical metric is not even once-differentiable at spatial infinity unless the ADM mass of the spacetime vanishes [12], and the unphysical spacetime manifold *does not* have a smooth differential structure at i^0. Thus the identification between the null generators of $\mathscr{I}^-$ and $\mathscr{I}^+$, and the corresponding symmetries and fields, becomes much more difficult.

Nevertheless, Ashtekar and Magnon-Ashtekar [25] showed that the limit of Bondi energy-momentum to i^0 along both $\mathscr{I}^{\pm}$ equals the ADM energy-momentum. Similar result for the Bondi and ADM angular momentum was obtained by Ashtekar and Streubel [26] for spacetimes which are stationary near i^0. For general supertranslations, the antipodal matching of all the infinite number of symmetries and charges has been shown in linearised gravity around a Minkowski background spacetime by Troessaert [27].[1] In full nonlinear general relativity, it was argued by Strominger [19] that a similar result should hold for all supertranslations in the Christodoulou-Klainerman class of spacetimes (CK-spacetimes) [29]. However, in such spacetimes only the Bondi energy-momentum associated to translations is non-vanishing in the limit to i^0 along $\mathscr{I}$ and the supermomenta associated to the rest of the supertranslations vanish (see [30]). Thus, although CK-spacetimes form an open ball in some suitable topology around Minkowski spacetime, they are not general enough to address the non-trivial aspects of the matching problem for supertranslations. A non-trivial result in the nonlinear theory was obtained by Herberthson and Ludvigsen [28] who proved that the Weyl tensor component entering the Bondi mass formula (denoted by ψ_2 in appendix A) on $\mathscr{I}^-$ matches antipodally with the corresponding quantity on

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[1]The result of [27] can be viewed as the linearisation around a Minkowski background of the more general analysis in [28].

$\mathscr{I}^+$ on spacetimes which are sufficiently regular in the limit to i^0 along $\mathscr{I}$. The matching of the supertranslation symmetries was not addressed in [28], but assuming an antipodal matching of supertranslations (as proposed in [19]), the earlier result of [28] resolves the matching problem for a class of spacetimes (more general than the ones in [19]) where the News tensor and certain connection components falloff fast enough in the limit to i^0. We note that a key improvement in [27, 28] over [19] is that the antipodal matching of the relevant fields is not imposed a priori as a boundary condition, but follows from the regularity of solutions to the Einstein equation on $\mathscr{I}$ and at i^0.

In this paper we prove the matching conditions for *all* asymptotic supertranslations in general relativity on asymptotically-flat spacetimes satisfying suitably regularity conditions near spatial infinity (see Definition 4.1). We will use the methods of [31] where an analogous result was shown for Maxwell fields on any asymptotically-flat spacetime. our result can be viewed as a generalisation of [25] to include all supertranslations, or of [27] to full nonlinear general relativity, or of [28] to show the antipodal matching of not only the Weyl tensor but also the relevant supertranslation symmetries.

Both null and spatial infinities can be treated in a unified spacetime-covariant manner using the definition of asymptotic-flatness given by Ashtekar and Hansen [12, 14] (Definition 2.1). In the Ashtekar-Hansen formalism, instead of working directly at the point i^0 where sufficiently smooth structure is unavailable, one works on the space of spatial directions at i^0 given by a timelike-unit-hyperboloid $\mathscr{H}$ in the tangent space at i^0 (figure 1). The Weyl tensor of the unphysical spacetime (suitably conformally rescaled) admits limits to i^0 which depend on the direction of approach and thus induces *smooth* fields on $\mathscr{H}$. The asymptotic Spi-supertranslations at spatial infinity then give us infinitely many charges on $\mathscr{H}$ in terms of these smooth limiting fields.

However, for the matching problem we are interested in the behaviour of fields at i^0 along $\mathscr{I}^\pm$ i.e., along null directions at i^0. Such null directions can be incorporated into the Ashtekar-Hansen formalism using a space of both null and spatial directions of approach to i^0 constructed in [31]. This space is a cylinder $\mathscr{C}$ in the tangent space at i^0, which is diffeomorphic to a conformal-completion of $\mathscr{H}$ (see figure 2). The two boundaries $\mathscr{N}^\pm$ of $\mathscr{C}$ correspond to the directions of approach to i^0 in null directions along $\mathscr{I}^\pm$. Using this diffeomorphism, we can study the asymptotic gravitational fields and supertranslations on $\mathscr{C}$, instead of on $\mathscr{H}$. There is a reflection map, which acts as a conformal isometry on $\mathscr{C}$ eq. (2.39), which allows us to identify the null generators of $\mathscr{I}^\pm$ represented by the spaces $\mathscr{N}^\pm$ of null directions at i^0.

With this geometric setup we can ask about two different limits of the gravitational fields and supertranslations: (i) first take the limit to $\mathscr{I}^\pm$ and then towards i^0, or, (i) first take the limit to i^0 along spatial directions (now represented by $\mathscr{C} \setminus \mathscr{N}^\pm$) and then take the limit where the direction of approach becomes null i.e., a limit to $\mathscr{N}^\pm$. In general, neither of these limits might exist given the conditions by Ashtekar and Hansen. Thus, we impose additional *null-regularity* conditions on some of the gravitational fields (Definition 4.1). These conditions imply that both limits, taken as described above, exist and the induced limiting fields on the boundaries $\mathscr{N}^\pm$ obtained by both limiting procedures match. Thus, the null-regularity conditions act as "continuity" conditions on the grav-

itational fields at i^0 and further ensure that the flux of charges on $\mathscr{I}^\pm$ is finite. This will lead us to a partial matching of the supertranslation symmetries, whereby any BMS-supertranslation on $\mathscr{I}^\pm$ gives some (not unique) Spi-supertranslation on $\mathscr{C}$ such that they match continuously at $\mathscr{N}^\pm$.

Using the asymptotic Einstein equations on $\mathscr{C}$ we show that, with our null-regularity conditions, the fields entering the expression for the charges from the past null directions $\mathscr{N}^-$ match antipodally to those from the future null directions $\mathscr{N}^+$ (under the reflection map on $\mathscr{C}$). Finally, we isolate a subalgebra of Spi-supertranslations for which the total flux of the corresponding supermomenta across *all* of $\mathscr{C}$ vanishes. This corresponds to the physical requirement that in a scattering process one only is concerned with the fields on null infinity, and any flux through spatial infinity is "non-dynamical". We emphasise that this is not a restriction on the kinds of spacetimes we consider (unlike the previously discussed null-regularity conditions), but a choice of supertranslations relevant to a scattering process. Such totally fluxless Spi-supertranslations on $\mathscr{C}$ then give us the desired isomorphism between the BMS-supertranslations and a conservation law for the fluxes of BMS-supermomenta between $\mathscr{I}^-$ and $\mathscr{I}^+$, proving the conjecture in [19].

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The rest of the paper is organised as follows. In section 2 we review the Ashtekar-Hansen structure of spacetimes that are asymptotically-flat at both null and spatial infinity. In section 2.1, we summarise the construction of the space $\mathscr{C}$ at spatial infinity that includes both null and spatial directions and its relation to the unit-hyperboloid $\mathscr{H}$ in the Ashtekar-Hansen framework. In section 3 we review the asymptotic supertranslation symmetries and the associated supermomenta at $\mathscr{I}$ and in the Ashtekar-Hansen formalism at i^0. In section 4, we impose suitable regularity conditions on the gravitational fields on null infinity which ensure that the BMS-supermomenta defined on null infinity remain finite as we approach spatial infinity. We introduce the subalgebra of Spi-supertranslations for which the total flux of Spi-supermomenta across spatial infinity vanishes, which then gives us the antipodal matching conditions, the global diagonal symmetry algebra and the flux conservation between past and future null infinity. We end with section 5 summarising and discussing our analysis and results.

We collect the computations generalising the relevant field equations on null infinity to arbitrary conformal choices in appendix A. In appendix B we collect the definitions of direction-dependent differential structures and tensors and summarise an explicit construction of the space of directions at spatial infinity. We analyse the solutions of the electric Einstein equations at spatial infinity in appendix C. In appendix D we compare our formula for the Spi-supermomenta with the expression derived by Compère and Dehouck in [32]. In appendix E, we relate our covariant geometric construction to the approaches based on Bondi-Sachs and Beig-Schmidt coordinates.

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We use an abstract index notation with indices $a, b, c, \ldots$ for tensor fields. Quantities defined on the physical spacetime will be denoted by a "hat", while the ones on the

Normal vector field	Orientation
n^a	null and future-pointing at $\mathscr{I}^+$, past-pointing at $\mathscr{I}^-$
l^a	null and future-pointing at $\mathscr{I}^+$, past-pointing at $\mathscr{I}^-$
$\boldsymbol{\eta}^a$	spatial and inward-pointing at i^0
$\boldsymbol{u}^a$	timelike and future-pointing at some cross-section S of $\mathscr{H}$
$\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^a$	timelike and future-pointing at $\mathscr{N}^+$, past-pointing at $\mathscr{N}^-$ on $\mathscr{C}$

Table 1. Conventions for orientation of normals.

conformally-completed unphysical spacetime are without the "hat" e.g. $\hat{g}_{ab}$ is the physical metric while g_{ab} is the unphysical metric on the conformal-completion. We will raise and lower indices on tensors with g_{ab} and explicitly write out $\hat{g}_{ab}$ when used to do so. We denote directions at i^0 by an overhead arrow e.g. $\vec{N}$ denotes directions which are either null or spatial while $\vec{\eta}$ denotes spatial directions. Regular direction-dependent limits of tensor fields will be denoted by a boldface symbol e.g. $\boldsymbol{C}_{abcd}(\vec{\eta})$ is the limit of the (rescaled) unphysical Weyl tensor along spatial directions at i^0. We collect our conventions on the orientations of the normals defined on various manifolds in table 1.

2 Asymptotic-flatness at null and spatial infinity: Ashtekar-Hansen structure

We define spacetimes which are asymptotically-flat at null and spatial infinity using an Ashtekar-Hansen structure [12, 14] as follows.

Definition 2.1 (Ashtekar-Hansen structure [14]). A *physical* spacetime $(\hat{M}, \hat{g}_{ab})$ has an *Ashtekar-Hansen structure* if there exists another *unphysical* spacetime (M, g_{ab}), such that

(1) M is C^∞ everywhere except at a point i^0 where it is $C^{>1}$,

(2) the metric g_{ab} is C^∞ on $M - i^0$, and C^0 at i^0 and $C^{>0}$ along spatial directions at i^0

(3) there is an embedding of $\hat{M}$ into M such that $\overline{J}(i^0) = M - \hat{M}$,

(4) there exists a function Ω on M, which is C^∞ on $M - i^0$ and C^2 at i^0 so that $g_{ab} = \Omega^2 \hat{g}_{ab}$ on $\hat{M}$ and

  (a) $\Omega = 0$ on $\dot{J}(i^0)$
  (b) $\nabla_a \Omega \neq 0$ on $\mathscr{I}$
  (c) at i^0, $\nabla_a \Omega = 0$, $\nabla_a \nabla_b \Omega = 2g_{ab}$

(5) There exists a neighbourhood N of $\dot{J}(i^0)$ such that (N, g_{ab}) is strongly causal and time orientable, and in $N \cap \hat{M}$ the physical metric $\hat{g}_{ab}$ satisfies the vacuum Einstein equation $\hat{R}_{ab} = 0$,

(6) The space of integral curves of $n^a = g^{ab}\nabla_b \Omega$ on $\dot{J}(i^0)$ is diffeomorphic to the space of null directions at i^0.

(7) The vector field $\varpi^{-1}n^a$ is complete on $\mathscr{I}$ for any smooth function ϖ on $M - i^0$ such that $\varpi > 0$ on $\hat{M} \cup \mathscr{I}$ and $\nabla_a(\varpi^4 n^a) = 0$ on $\mathscr{I}$.

In the above we have used the following the notation for causal structures from [33]: $J(i^0)$ is the causal future of a point i^0 in M, $\overline{J}(i^0)$ is its closure, $\dot{J}(i^0)$ is its boundary and $\mathscr{I} := \dot{J}(i^0) - i^0$. We also use the definition and notation for direction-dependent tensors from appendix B. One can also use much weaker differentiability requirements on the unphysical metric g_{ab} (as discussed on section 5), but we choose not to do so for simplicity.

The physical role of the conditions in Definition 2.1 are explained in [14]. In particular, these conditions imply that (i) The point i^0 is spacelike related to all points in the physical spacetime $\hat{M}$, and represents *spatial infinity*. (ii) $\mathscr{I} := \dot{J}(i^0) - i^0$ consists of two disconnected pieces — the future piece $\mathscr{I}^+$ and the past piece $\mathscr{I}^-$ — which are both smooth null submanifolds of M, representing future and past *null infinities*, respectively. Note that the metric g_{ab} is only $C^{>0}$ at i^0 along spatial directions, that is, the metric is continuous but the metric connection (or Christoffel symbols in some $C^{>1}$ coordinate chart, see appendix B) is allowed to have limits which depend on the direction of approach to i^0. As mentioned in the Introduction this low differentiability structure is necessary to accomodate spacetimes with non-vanishing ADM mass. Note that the unphysical metric is only required to be C^0 approaching i^0 along null directions; later we will impose additional regularity conditions (Definition 4.1) which ensure that the flux of all supermomenta through $\mathscr{I}$ is finite.

For a given physical spacetime $(\hat{M}, \hat{g}_{ab})$, the choice of an Ashtekar-Hansen structure is not unique. There is an ambiguity in the choice of the $C^{>1}$ differential structure at i^0 given by a 4-parameter family of *logarithmic translations* which simultaneously change the $C^{>1}$-structure and the conformal factor at i^0 [34–36] (see Remark B.1). Given a choice of the $C^{>1}$-structure the additional ambiguity in the choice of the conformal factor Ω is as follows.

Remark 2.1 (Freedom in the conformal factor [12, 14]). The freedom in the choice of the conformal factor in Definition 2.1 is given by $\Omega \mapsto \omega\Omega$ where the function ω satisfies

(1) $\omega > 0$ on M

(2) ω is smooth on $M - i^0$

(3) ω is $C^{>0}$ in spatial directions at i^0 and $\omega|_{i^0} = 1$

We say that a tensor field $T^{a\cdots}{}_{b\ldots}$ has a *conformal weight* w if under the above change of conformal factor it transforms as

$$T^{a\cdots}{}_{b\ldots} \mapsto \omega^w \ T^{a\cdots}{}_{b\ldots} \tag{2.1}$$

For instance the unphysical metric g_{ab} has conformal weight $w = 2$.

In the following we will work with a fixed the unphysical spacetime given by some choice of the $C^{>1}$-structure at i^0 and some choice of conformal factor Ω. In the end, we argue that our results are independent of these choices (see Remarks 4.3–4.5). Note that, all spacetimes satisfying Definition 2.1 have the same metric at i^0, that is, the unphysical metric g_{ab} at i^0 is *universal* (isometric to the Minkowski metric at i^0) and cannot even be further conformally-rescaled (since $\omega|_{i^0} = 1$) [12].

Using the conformal transformation relating the unphysical Ricci tensor R_{ab} to the physical Ricci tensor $\hat{R}_{ab}$ (see appendix D [37]), the vacuum Einstein equation $\hat{R}_{ab} = 0$ can be written as

$$S_{ab} = -2\Omega^{-1}\nabla_{(a}n_{b)} + \Omega^{-2}n^c n_c g_{ab} \tag{2.2}$$

where S_{ab} is given by

$$S_{ab} := R_{ab} - \tfrac{1}{6}R g_{ab} \tag{2.3}$$

Further, the Bianchi identity $\nabla_{[a}R_{bc]de} = 0$ on the uphysical Riemann tensor along with eq. (2.2) gives the following equations for the unphysical Weyl tensor C_{abcd} (see [18] for details)

$$\nabla_{[a}(\Omega^{-1}C_{bc]de}) = 0 \tag{2.4a}$$

$$\nabla^d C_{abcd} = -\nabla_{[a}S_{b]c} \tag{2.4b}$$

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On $\mathscr{I}$, let us introduce the function

$$\Phi := \tfrac{1}{4}\nabla_a n^a|_{\mathscr{I}}, \quad \Phi|_{i^0} = 2 \tag{2.5}$$

where the second condition follows from Definition 2.1. Since S_{ab} is smooth at $\mathscr{I}$, by the assumptions in Definition 2.1, eq. (2.2) implies

$$\lim_{\to \mathscr{I}} \Omega^{-1}n^a n_a = 2\Phi, \quad \nabla_a n_b|_{\mathscr{I}} = \Phi g_{ab} \tag{2.6}$$

that is, the vector field n^a is a null geodesic generator of $\mathscr{I}^\pm \cong \mathbb{R} \times \mathbb{S}^2$, which is future/past pointing on $\mathscr{I}^\pm$ respectively.

Denote by q_{ab} the pullback of g_{ab} to $\mathscr{I}$. This defines a degenerate metric on $\mathscr{I}$ with $q_{ab}n^b = 0$. It is convenient to introduce a foliation of $\mathscr{I}$ by a family of cross-sections diffeomorphic to $\mathbb{S}^2$. The pullback of q_{ab} to any cross-section S defines a Riemannian metric on S. Then, for any choice of foliation, there is a unique *auxilliary normal* vector field l^a at $\mathscr{I}$ such that

$$l^a l_a = 0, \quad l^a n_a = -1, \quad q_{ab}l^b = 0 \tag{2.7}$$

In our conventions, l^a is future/past pointing at $\mathscr{I}^\pm$, respectively (table 1). We further have

$$q_{ab} = g_{ab} + 2n_{(a}l_{b)}, \quad \varepsilon_{abc} = l^d\varepsilon_{dabc}, \quad \varepsilon_{ab} = n^c\varepsilon_{cab} \tag{2.8}$$

where ε_{abc} defines a volume element on $\mathscr{I}$ and ε_{ab} is the area element on any cross-section S of the foliation. Evaluating the pullback of $\pounds_n g_{ab}$ and using eq. (2.6) we have on $\mathscr{I}$

$$\pounds_n q_{ab} = 2\Phi q_{ab} \tag{2.9}$$

that is, Φ measures the expansion of the chosen cross-sections of $\mathscr{I}$ along the null generator n^a while, their shear and twist identically vanishes.

Let

$$\tau_a := q_a{}^c n^b \nabla_b l_c \tag{2.10}$$

so we have $n^b \nabla_b l_a = \tau_a - \Phi l_a$ i.e., τ_a represents change in the direction of l_a along the null generators of n^a. The shear of the auxilliary normal l^a on the cross-sections S of the foliation is defined by

$$\sigma_{ab} := \left(q_a{}^c q_b{}^d - \tfrac{1}{2} q_{ab} q^{cd} \right) \nabla_c l_d \tag{2.11}$$

while the twist $\varepsilon^{ab} \nabla_a l_b$ vanishes on account of l_a being normal to the cross-sections. We will not require the expansion of l^a in our analysis.

For any smooth v_a satisfying $n^a v_a = l^a v_a = 0$ on $\mathscr{I}$ we define the derivative $\mathscr{D}_a$ on the cross-sections by

$$\mathscr{D}_a v_b := q_a{}^c q_b{}^d \nabla_c v_d \tag{2.12}$$

It is easily verified that $\mathscr{D}_a q_{bc} = 0$, i.e., $\mathscr{D}_a$ is the metric-compatible covariant derivative on cross-sections of $\mathscr{I}$. We also note the following identity for integration-by-parts on any cross-section S of $\mathscr{I}$

$$\int_S \varepsilon_2 \, f \mathscr{D}^a v_a = - \int_S \varepsilon_2 \, (\nabla_a f q^{ab} + f \nabla_a q^{ab}) v_b = - \int_S \varepsilon_2 \, (\mathscr{D}_a + \tau_a) f v^a \tag{2.13}$$

where we have discarded a boundary term on $S \cong \mathbb{S}^2$, used eqs. (2.8) and (2.10) and that v_a is orthogonal to both n^a and l^a. The generalisation to arbitrary tensors on $\mathscr{I}$ that are orthogonal to both n^a and l^a is immediate.

On $\mathscr{I}$, with the choice of foliation of $\mathscr{I}$ held fixed, we have the following conformal weights (eq. (2.1))[2]

$$(n_a, l_a) : w = 1 \,, \quad (n^a, l^a) : w = -1 \,, \quad q_{ab} : w = 2 \,, \quad \sigma_{ab} : w = 1 \tag{2.14}$$

while Φ and τ_a are not conformally-weighted but transform as

$$\Phi \mapsto \omega^{-1}(\Phi + \pounds_n \ln \omega) \,, \quad \tau_a \mapsto \tau_a + \mathscr{D}_a \ln \omega \tag{2.15}$$

Remark 2.2 (Choices of conformal factor). It has been conventional to choose the conformal factor Ω so that the *Bondi condition* holds

$$\nabla_a \nabla_b \Omega |_{\mathscr{I}} = 0 \tag{2.16}$$

This can be seen to be equivalent to the condition that $\Phi = 0$ at $\mathscr{I}$. However, such a choice of conformal factor violates Definition 2.1 i.e., $\Phi|_{i^0} = 2$, and is ill-behaved at i^0. It can be verified that the function ϖ in Definition 2.1 used to define a complete divergence-free normal $\varpi^{-1} n^a$ cannot be used as a conformal-rescaling at i^0 since ϖ will diverge at i^0 (see footnote 2, section 11.1 [37] and appendix E). In particular, the unphysical metric in the Bondi conformal frame and the corresponding Bondi-Sachs coordinates are ill-behaved near i^0. It is however always possible to choose the conformal factor so that $\Phi = 2$ on $\mathscr{I}$ in some neighbourhood around i^0. This choice was made, for instance, in [25, 28] and simplifies many of the subsequent computations. We prefer to keep the choice of conformal factor arbitrary, subject to Remark 2.1, so that the conformal invariance of our result can be easily verified.

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[2]Note that the normal n_a also transforms away from $\mathscr{I}$ as $n_a \mapsto \omega n_a + \Omega \nabla_a \omega$.
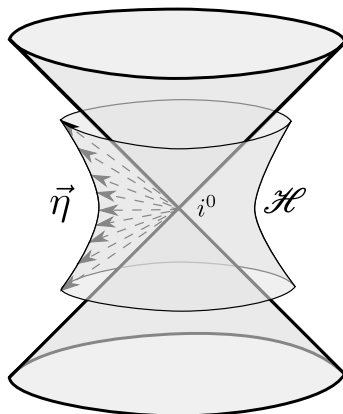
Figure 1. The (non-compact) unit-hyperboloid $\mathscr{H}$ in Ti^0 representing spatial directions $\vec{\eta}$ at i^0.

At spatial infinity represented by a single point i^0 in M, the gravitational fields of interest, in general, only admit *direction-dependent* limits, and hence it is rather awkward to study such fields directly at i^0. Instead of working at the point i^0 one works on the space of directions at i^0 i.e., a *blowup* of i^0 (see [38]). The fields which have regular direction-dependent limits to i^0 (as defined in appendix B) induce smooth tensor fields on the space of directions. The space of spatial directions at i^0 was constructed in [12], we review the aspects of this construction needed in our analysis below. A different, but related, blowup of i^0 which includes the null directions at i^0 was constructed in [31] (summarised in section 2.1) and is more useful for relating the gravitational fields and symmetries at spatial infinity to those on null infinity.

Along spatial directions $\nabla_a \Omega^{1/2}$ is $C^{>-1}$ at i^0 and

$$\boldsymbol{\eta}^a := \lim_{\to i^0} \nabla^a \Omega^{1/2} \tag{2.17}$$

determines a $C^{>-1}$ spatial unit vector field at i^0 representing the spatial directions $\vec{\eta}$ at i^0. The space of directions $\vec{\eta}$ in Ti^0 is a unit-hyperboloid $\mathscr{H}$ depicted in figure 1.

If $T^{a\cdots}{}_{b\cdots}$ is a $C^{>-1}$ tensor field at i^0 in spatial directions then, $\lim_{\to i^0} T^{a\cdots}{}_{b\cdots} = \boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta})$ is a smooth tensor field on $\mathscr{H}$. Further, the derivatives of $\boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta})$ to all orders with respect to the direction $\vec{\eta}$ satisfy[3]

$$\boldsymbol{\partial}_c \cdots \boldsymbol{\partial}_d \boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta}) = \lim_{\to i^0} \Omega^{1/2} \nabla_c \cdots \Omega^{1/2} \nabla_d T^{a\cdots}{}_{b\cdots} \tag{2.18}$$

where $\boldsymbol{\partial}_a$ is the derivative with respect to the directions $\vec{\eta}$ defined by

$$\begin{aligned}
\boldsymbol{v}^c \boldsymbol{\partial}_c \boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta}) &:= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta} + \epsilon \vec{v}) - \boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta}) \right] \quad \text{for all } \boldsymbol{v}^a \in T\mathscr{H} \\
\boldsymbol{\eta}^c \boldsymbol{\partial}_c \boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta}) &:= 0
\end{aligned} \tag{2.19}$$

It can be checked that, along spatial directions, eq. (2.18) is equivalent to the definition eq. (B.1) given in terms of a $C^{>1}$ coordinate chart.

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[3]The factors of $\Omega^{1/2}$ on the right-hand-side of eq. (2.18) convert between ∇_a and the derivatives with respect to the directions [14, 18].

The metric $\boldsymbol{h}_{ab}$ induced on $\mathscr{H}$ by the universal metric $\boldsymbol{g}_{ab}$ at i^0, satisfies

$$\boldsymbol{h}_{ab} := \boldsymbol{g}_{ab} - \boldsymbol{\eta}_a \boldsymbol{\eta}_b = \boldsymbol{\partial}_a \boldsymbol{\eta}_b \tag{2.20}$$

Further, if $\boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta})$ is orthogonal to $\boldsymbol{\eta}^a$ in all its indices then it defines a tensor field $\boldsymbol{T}^{a\cdots}{}_{b\cdots}$ intrinsic to $\mathscr{H}$. In this case, projecting *all* the indices in eq. (2.18) using $\boldsymbol{h}_{ab}$ to defines a derivative operator $\boldsymbol{D}_a$ intrinsic to $\mathscr{H}$ which is also the covariant derivative operator associated to $\boldsymbol{h}_{ab}$.[4] We also define

$$\boldsymbol{\varepsilon}_{abc} := -\boldsymbol{\eta}^d \boldsymbol{\varepsilon}_{dabc}, \quad \boldsymbol{\varepsilon}_{ab} := \boldsymbol{u}^c \boldsymbol{\varepsilon}_{cab} \tag{2.21}$$

where $\boldsymbol{\varepsilon}_{abcd}$ is volume element at i^0 corresponding to the metric $\boldsymbol{g}_{ab}$, $\boldsymbol{\varepsilon}_{abc}$ is the induced volume element on $\mathscr{H}$, and $\boldsymbol{\varepsilon}_{ab}$ is the induced area element on some cross-section S of $\mathscr{H}$ with a future-pointing timelike normal $\boldsymbol{u}^a$ such that $\boldsymbol{h}_{ab}\boldsymbol{u}^a\boldsymbol{u}^b = -1$.

We note that $\mathscr{H}$ admits a reflection isometry as follows. On the unit-hyperboloid we can introduce coordinates (τ, θ^A) — where $\tau \in (-\infty, \infty)$, and $\theta^A = (\theta, \phi)$ are the standard coordinates on $\mathbb{S}^2$ with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ — so that the metric on $\mathscr{H}$ is

$$\boldsymbol{h}_{ab} \equiv -d\tau^2 + \cosh^2\tau(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.22}$$

The metric $\boldsymbol{h}_{ab}$ has a reflection isometry Υ

$$\Upsilon : \mathscr{H} \to \mathscr{H} : (\tau, \theta^A) \mapsto (-\tau, -\theta^A)$$
$$\text{with } \Upsilon \circ \boldsymbol{h}_{ab} = \boldsymbol{h}_{ab} \tag{2.23}$$

where $\theta^A = (\theta, \phi) \mapsto -\theta^A = (\pi - \theta, \phi \pm \pi)$ is the antipodal reflection on $\mathbb{S}^2$ — the sign is chosen so that $\phi \pm \pi \in [0, 2\pi)$. We have used $\Upsilon \circ$ to denote the natural action of the reflection map Υ on tensor fields on $\mathscr{H}$.

2.1 The space $\mathscr{C}$ of null and spatial directions at i^0

For the matching problem we are interested in analysing the gravitational fields and super-translations in the null directions along $\mathscr{I}^\pm$ at i^0. The hyperboloid $\mathscr{H}$ is not well-suited for this as the null directions correspond to "points at infinity" on $\mathscr{H}$. Thus, we use a different blowup $\mathscr{C}$ of i^0, constructed in [31], which includes both null and spatial directions. The basic strategy of the construction is as follows. (1) We rescale n^a so that the rescaled vector field is non-vanishing and represents "good" null directions at i^0. (2) We also conformally-complete $\mathscr{H}$ to get a new manifold whose boundaries represent the points in the infinite future or past along $\mathscr{H}$. (3) Then, we can identify the null directions given by the rescaled n^a with the boundaries of the conformal-completion of $\mathscr{H}$ in a "sufficiently smooth" way, to get the new manifold $\mathscr{C}$. The final picture obtained is depicted in figure 2.

To implement the construction of $\mathscr{C}$ described above, we work in a neighbourhood of i^0 in M, and use M to mean such a neighbourhood unless otherwise specified. In M, we define a *rescaling function* Σ as follows:

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[4]This follows from eq. (2.20), and $\boldsymbol{\partial}_c \boldsymbol{g}_{ab} = 0$ since $\boldsymbol{g}_{ab}$ is direction-independent at i^0.
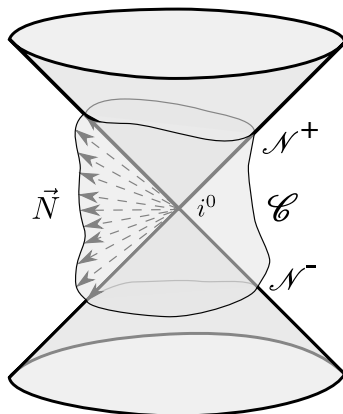
Figure 2. The space $\mathscr{C}$ of null and spatial directions $\vec{N}$ at i^0. The boundaries $\mathscr{N}^\pm \cong \mathbb{S}^2$, diffeomorphic to the space of generators of $\mathscr{I}^\pm$ respectively, represent the space of null directions. $\mathscr{C}\backslash\mathscr{N}^\pm$ is the space of *rescaled* spatial directions conformally diffeomorphic to the unit-hyperboloid $\mathscr{H}$. $\mathscr{C}$ depends on the choice of the rescaling function Σ (defined below) and *need not* be a cylinder of unit radius in Ti^0 — we have drawn a "wiggly" cylinder to emphasise this.

Definition 2.2 (Rescaling function Σ)**.** Let Σ be a function in M such that

(1) $\Sigma^{-1} > 0$ is smooth on $M - i^0$

(2) Σ^{-1} is $C^{>0}$ at i^0 in both null and spatial directions,

(3) $\Sigma^{-1}|_{i^0} = 0$, $\lim\limits_{\to i^0} \nabla_a \Sigma^{-1} \neq 0$ and

(4) $\Sigma \pounds_n \Sigma^{-1} = 2$ at i^0 and on $\mathscr{I}$

Remark 2.3 (Freedom in the rescaling function). If Σ is a choice of rescaling function for the conformal factor Ω then $\sigma\Sigma$ is a choice for the conformal factor $\omega\Omega$ (where ω satisfies the conditions in Remark 2.1) if

(1) $\sigma > 0$ in M

(2) σ is smooth on $M - i^0$

(3) σ is $C^{>-1}$ at i^0 in both null and spatial directions

(4) $\pounds_n \ln \sigma|_{\mathscr{I}} = 2(1-\omega)|_{\mathscr{I}}$

Similar, to the conformal weight (eq. (2.1)), we say that a tensor field $T^{a\cdots}{}_{b\cdots}$ has a *rescaling weight* r if under the above change of rescaling function (with fixed conformal factor, i.e. $\omega = 1$) it transforms as

$$T^{a\cdots}{}_{b\cdots} \mapsto \sigma^r \, T^{a\cdots}{}_{b\cdots} \tag{2.24}$$

The existence of such a rescaling function was shown in appendix B [31] (summarised in appendix B below). We emphasise that the rescaling by Σ is *not* an alternative choice of the conformal factor Ω, in particular the unphysical metric g_{ab} is not rescaled by Σ. Any choice of the rescaling function Σ allows us to construct certain regular fields near i^0 which will be useful in our analysis. We list below their essential properties which can be verified following in appendix B [31].

Since $\Sigma^{-1}|_{i^0} = 0$ and Σ^{-1} is $C^{>0}$, there exists a function $\boldsymbol{\Sigma}(\vec{\eta})$, which is $C^{>-1}$ along spatial directions, such that

$$\boldsymbol{\Sigma}^{-1}(\vec{\eta}) = \lim_{\to i^0}(\Omega^{1/2}\Sigma)^{-1} \tag{2.25}$$

Rescaled normal N^a and the space $\mathscr{C}$ of null and spatial directions. The rescaled vector field (the factor of half in definition of N^a is for later convenience)

$$N^a := \tfrac{1}{2}\Sigma n^a = \tfrac{1}{2}\Sigma\nabla^a\Omega \tag{2.26}$$

is $C^{>-1}$ at i^0 such that $\boldsymbol{N}^a = \lim_{\to i^0} N^a \neq 0$ in both null and spatial directions. Thus, along $\mathscr{I}$, we have $\boldsymbol{N}^a$ as a direction-dependent null vector representing the null directions at i^0 which are future/past directed along $\mathscr{I}^\pm$ respectively. Along spatial directions at i^0 we have $\boldsymbol{N}^a(\vec{\eta}) = \boldsymbol{\Sigma}(\vec{\eta})\boldsymbol{\eta}^a \neq 0$ which represents the rescaled spatial directions at i^0. The space of these directions $\vec{N}$ can be represented by a cylinder $\mathscr{C}$ with two boundaries $\mathscr{N}^\pm \cong \mathbb{S}^2$ (as in figure 2). The boundaries $\mathscr{N}^\pm$ represent the null directions along $\mathscr{I}^\pm$ respectively, while $\mathscr{C}\backslash\mathscr{N}^\pm$ is the space of the rescaled spatial directions at i^0.

Rescaled auxilliary normal L^a and foliation of $\mathscr{I}$. Define a vector field L^a in M by

$$L^a := -\nabla^a\Sigma^{-1} + \tfrac{1}{2}\nabla_b\Sigma^{-1}\nabla^b\Sigma^{-1}\ N^a \tag{2.27}$$

which is $C^{>-1}$ at i^0 and $\lim_{\to i^0} L^a \neq 0$ in both null and spatial directions. Further, using eq. (2.26) and Definition 2.2, we have

$$N^a L_a|_{\mathscr{I}} = -1\,, \quad L^a L_a|_{\mathscr{I}} = 0 \tag{2.28}$$

The pullback to $\mathscr{I}$ of L_a equals the pullback of $-\nabla_a\Sigma^{-1}$, thus L^a defines a rescaled auxilliary normal to the foliation of $\mathscr{I}$ by a family of cross-sections S_Σ with $\Sigma^{-1} = $ constant. From Definition 2.2 and Definition 2.1, the limiting cross-section S_Σ as $\Sigma^{-1} \to 0$, is diffeomorphic to the space of null directions $\mathscr{N}^\pm$. The auxilliary normal l^a to this foliation, satisfying eq. (2.7), is obtained by

$$l^a := \tfrac{1}{2}\Sigma L^a \tag{2.29}$$

We also extend l^a into M by the above formula and eq. (2.27). In such a foliation, we have (using eqs. (2.26) and (2.28))

$$N^a|_{\mathscr{I}} \equiv \partial_{\Sigma^{-1}}\,, \quad n^a|_{\mathscr{I}} \equiv 2\Sigma^{-1}\partial_{\Sigma^{-1}} \tag{2.30}$$

Further, we can show that the tensor τ_a defined in eq. (2.10) vanishes in this choice of foliation. From eqs. (2.27) and (2.29) we can compute

$$\tau_a = -\tfrac{1}{2}q_a{}^c n^b\nabla_b\nabla_c\ln\Sigma^{-1} = -\tfrac{1}{2}q_a{}^c n^b\nabla_c\nabla_b\ln\Sigma^{-1} = \tfrac{1}{2}\Phi\mathscr{D}_a\ln\Sigma^{-1} = 0 \tag{2.31}$$

where we have used eq. (2.6) and Definition 2.2 on $\mathscr{I}$. Finally, the vanishing of τ_a follows since each cross-section of the foliation has $\Sigma^{-1} = $ constant.[5]

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[5]Note that under a conformal transformation τ_a changes according to eq. (2.15). But the rescaling function Σ, and hence the chosen foliation by S_Σ, also changes according to Remark 2.3, so that the condition $\tau_a = 0$ also holds in the new (and thus any) choice of conformal factor and choice of foliation of $\mathscr{I}$ by S_Σ.

Conformal-completion of $\mathscr{H}$. Let $\boldsymbol{\Sigma}$ be the function induced on $\mathscr{H}$ by $\boldsymbol{\Sigma}(\vec{\eta})$ (eq. (2.25)). Let $(\tilde{\mathscr{H}}, \tilde{\boldsymbol{h}}_{ab})$ be a conformal-completion of $(\mathscr{H}, \boldsymbol{h}_{ab})$ with metric $\tilde{\boldsymbol{h}}_{ab} = \boldsymbol{\Sigma}^2 \boldsymbol{h}_{ab}$. There exists a diffeomorphism from $\tilde{\mathscr{H}}$ onto $\mathscr{C}$ such that $\mathscr{H}$ is mapped onto $\mathscr{C} \backslash \mathscr{N}^{\pm}$ and $\boldsymbol{\Sigma}$, as a function on $\mathscr{C} \backslash \mathscr{N}^{\pm}$, extends smoothly to the boundaries $\mathscr{N}^{\pm}$ where

$$\boldsymbol{\Sigma}|_{\mathscr{N}^{\pm}} = 0 \tag{2.32}$$

Using eqs. (2.17), (2.18), (2.20) and (2.27), the limit of L^a to i^0 along spatial directions gives the direction-dependent vector field

$$\boldsymbol{L}^a(\vec{\eta}) = -\boldsymbol{h}^{ab} \boldsymbol{D}_b \boldsymbol{\Sigma}^{-1} + (\boldsymbol{h}^{bc} \boldsymbol{D}_b \boldsymbol{\Sigma}^{-1} \boldsymbol{D}_c \boldsymbol{\Sigma}^{-1}) \boldsymbol{\eta}^a \tag{2.33}$$

The projection of $\boldsymbol{L}^a(\vec{\eta})$ onto $\mathscr{H}$ is the vector field

$$\boldsymbol{U}^a := \boldsymbol{h}^a{}_b \boldsymbol{L}^b(\vec{\eta}) = \tilde{\boldsymbol{h}}^{ab} \boldsymbol{D}_b \boldsymbol{\Sigma} \tag{2.34}$$

Viewed as a vector field on $\tilde{\mathscr{H}}$, and hence $\mathscr{C}$, we have

$$\lim_{\to \mathscr{N}^{\pm}} \tilde{\boldsymbol{h}}_{ab} \boldsymbol{U}^a \boldsymbol{U}^b = -1 \, , \quad \lim_{\to \mathscr{N}^{\pm}} \boldsymbol{\Sigma}^{-1} \boldsymbol{U}^a \neq 0 \tag{2.35}$$

Note that $\boldsymbol{\Sigma}^{-1} \boldsymbol{U}^a$ is future/past directed at $\mathscr{N}^{\pm}$ respectively (see table 1). Note, from eq. (2.35), the metric $\tilde{\boldsymbol{h}}_{ab}$ is *not* smooth at $\mathscr{N}^{\pm}$ on $\mathscr{C}$, but still provides a useful relation between $\mathscr{C}$ and the conformal-completion of $\mathscr{H}$.

Metric on $\mathscr{N}^{\pm}$. On $\mathscr{I}$ consider the rescaled metric

$$\tilde{q}_{ab} := \Sigma^2 q_{ab} \tag{2.36}$$

Along the foliation S_{Σ} as $\Sigma^{-1} \to 0$, $\lim_{\to i^0} \tilde{q}_{ab}(\vec{N})$ exists along null directions $\vec{N}$ and defines a direction-dependent Riemannian metric $\tilde{\boldsymbol{q}}_{ab}$ on the space of null directions $\mathscr{N}^{\pm}$. Further, this metric coincides with the metric induced on $\mathscr{N}^{\pm}$ by $\tilde{\boldsymbol{h}}_{ab}$ on $\mathscr{C}$, that is,

$$\tilde{\boldsymbol{q}}_{ab} = \lim_{\to \mathscr{N}^{\pm}} (\tilde{\boldsymbol{h}}_{ab} + \boldsymbol{D}_a \boldsymbol{\Sigma} \boldsymbol{D}_b \boldsymbol{\Sigma}) \tag{2.37}$$

Similarly, the rescaled area element $\tilde{\varepsilon}_{ab} := \Sigma^2 \varepsilon_{ab}$ on the foliation S_{Σ} induces an area element $\tilde{\boldsymbol{\varepsilon}}_{ab}$ on $\mathscr{N}^{\pm}$ such that

$$\tilde{\boldsymbol{\varepsilon}}_{ab} = \lim_{\to \mathscr{N}^{\pm}} \boldsymbol{U}^c \tilde{\boldsymbol{\varepsilon}}_{cab} \tag{2.38}$$

where $\tilde{\boldsymbol{\varepsilon}}_{abc} := \boldsymbol{\Sigma}^3 \boldsymbol{\varepsilon}_{abc}$ is the volume element on $\mathscr{C}$ defined by the metric $\tilde{\boldsymbol{h}}_{ab} = \boldsymbol{\Sigma}^2 \boldsymbol{h}_{ab}$.

Reflection conformal isometry of $\mathscr{C}$. The reflection isometry Υ of $\mathscr{H}$ (see eq. (2.23)) extends to a reflection conformal isometry of $\mathscr{C}$ i.e., there exists a reflection map $\Upsilon : \mathscr{C} \to \mathscr{C}$ and a smooth function $\boldsymbol{\varsigma} > 0$ on $\mathscr{C}$ such that

$$\Upsilon \circ \tilde{\boldsymbol{h}}_{ab} = \boldsymbol{\varsigma}^2 \tilde{\boldsymbol{h}}_{ab} \tag{2.39}$$

where $\boldsymbol{\varsigma} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Upsilon} \circ \boldsymbol{\Sigma})$ accounts for the fact that choice of rescaling function need not be invariant under the reflection $\boldsymbol{\Upsilon}$. Further, under this reflection map we also have $\boldsymbol{\Upsilon} : \mathscr{N}^- \to \mathscr{N}^+$, such that

$$\begin{aligned}
\boldsymbol{\Upsilon} \circ (\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^a)|_{\mathscr{N}^-} &= -\boldsymbol{\varsigma}^{-2}\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^a|_{\mathscr{N}^+} \\
\boldsymbol{\Upsilon} \circ \tilde{\boldsymbol{\varepsilon}}_{ab}|_{\mathscr{N}^-} &= -\boldsymbol{\varsigma}^2\tilde{\boldsymbol{\varepsilon}}_{ab}|_{\mathscr{N}^+}
\end{aligned} \tag{2.40}$$

where the negative signs on the right-hand-side are due to our orientation conventions (table 1). From Definition 2.1, $\mathscr{N}^\pm$ are diffeomorphic to the space of null generators of $\mathscr{I}^\pm$, and thus the reflection map $\boldsymbol{\Upsilon} : \mathscr{N}^- \to \mathscr{N}^+$ provides an antipodal map between the null generators of $\mathscr{I}^-$ and those of $\mathscr{I}^+$. This mapping between the generators of $\mathscr{I}^\pm$ is singled out by the fact that it is a symmetry of those solutions to the Einstein equation on $\mathscr{C}$ which smoothly extend to $\mathscr{N}^\pm$ (see appendix C).

The abstract manifold $\mathscr{C}$ with a conformal-class of metrics. If we choose a different rescaling function $\Sigma' = \sigma\Sigma$ with σ satisfying the conditions in Remark 2.3, we get a different space $\mathscr{C}'$ of directions $\vec{N}' = \boldsymbol{\sigma}(\vec{N})\vec{N}$ at i^0, where $\boldsymbol{\sigma}(\vec{N}) = \lim_{\to i^0} \sigma$ along the directions $\vec{N}$. This new space $\mathscr{C}'$ is naturally diffeomorphic to $\mathscr{C}$ under the above mapping of the directions with the metric $\tilde{\boldsymbol{h}}'_{ab} = \boldsymbol{\sigma}^2\tilde{\boldsymbol{h}}_{ab}$. Thus, we consider $\mathscr{C}$ as an abstract manifold with this conformal-class of metrics — note that here the conformal-class corresponds to a change of rescaling function and *not* the conformal factor Ω. This point of view will be useful to show that our results are Lorentz-invariant; Remark 4.4. The transformation of the various fields defined above can be computed directly from the defining equations.

$C^{>-1}$ functions on $\mathscr{C}$. Consider any function f which is smooth at $\mathscr{I}$ and $C^{>-1}$ at i^0 in *both* null and spatial directions. Then along null directions, $\lim_{\to i^0} f|_{\mathscr{I}^\pm}$ induces a smooth function $\boldsymbol{f}^\pm$ on $\mathscr{N}^\pm$. Similarly, along spatial directions, $\lim_{\to i^0} f$ induces a smooth function $\boldsymbol{f}$ on $\mathscr{H}$. Using the diffeomorphism between $\mathscr{H}$ and $\mathscr{C}\backslash\mathscr{N}^\pm$, we can consider $\boldsymbol{f}$ as a smooth function on $\mathscr{C}\backslash\mathscr{N}^\pm$. Since, f is $C^{>-1}$ along both null and spatial directions, the function $\boldsymbol{f}$ extends to $\mathscr{N}^\pm$ as a smooth function, and on $\mathscr{N}^\pm$ satisfies

$$\boldsymbol{f}^\pm = \boldsymbol{f}|_{\mathscr{N}^\pm} \tag{2.41}$$

That is, for functions which are $C^{>-1}$ in both null and spatial directions, the fields induced on $\mathscr{N}^\pm$ by, first taking the limit to $\mathscr{I}^\pm$ and then to i^0, or by, first taking the limit to i^0 in spatial directions and then to the space of null directions $\mathscr{N}^\pm$, coincide. Thus, such functions are continuous at the space of null directions $\mathscr{N}^\pm$ when going from $\mathscr{C}$ to $\mathscr{I}^\pm$.

3 Supertranslations and supermomentum in general relativity

The well-known expressions for the BMS-supertranslations and the corresponding supermomenta hold only in choices of the conformal factor which satisfy the Bondi condition on $\mathscr{I}$ which, as mentioned above (Remark 2.2), are incompatible with the Definition 2.1. Thus, we first need to extend these expressions to arbitrary choices of the conformal factor.

The computation of these expressions is most easily done in Geroch-Held-Penrose (GHP) formalism, and so we defer the details to appendix A. The relation to the expressions existing in the literature is given in Remark 3.1.

By the peeling theorem, we have $C_{abcd} = 0$ at $\mathscr{I}$, and thus $\Omega^{-1}C_{abcd}$ admits a limit to $\mathscr{I}$ (see Theorem 11 [18]).[6] In any choice of a foliation of $\mathscr{I}$ we define the fields

$$\mathcal{R}_{ab} := (\Omega^{-1}C_{cdef})q_a{}^c n^d q_b{}^e n^f\,, \qquad\qquad \mathcal{S}_a := (\Omega^{-1}C_{cdef})l^c n^d q_a{}^e n^f \qquad (3.1a)$$

$$\mathcal{P} := (\Omega^{-1}C_{cdef})l^c n^d l^e n^f\,, \qquad\qquad \mathcal{P}^* := \tfrac{1}{2}(\Omega^{-1}C_{cdef})l^c n^d \varepsilon^{ef} \qquad (3.1b)$$

$$\mathcal{J}_a := (\Omega^{-1}C_{cdef})n^c l^d q_a{}^e l^f\,, \qquad\qquad \mathcal{I}_{ab} := (\Omega^{-1}C_{cdef})q_a{}^c l^d q_b{}^e l^f \qquad (3.1c)$$

All the above tensors are orthogonal to both n^a and l^a in all indices and so can be considered as tensor fields on the cross-sections S of the chosen foliation of $\mathscr{I}$. Further, $\mathcal{R}_{ab}$ and $\mathcal{I}_{ab}$ are symmetric and traceless with respect to the metric q^{ab} on the cross-sections. These fields have the following conformal weights

$$(\mathcal{R}_{ab}, \mathcal{I}_{ab}) : w = -1\,, \quad (\mathcal{S}_a, \mathcal{J}_a) : w = -2\,, \quad (\mathcal{P}, \mathcal{P}^*) : w = -3 \qquad (3.2)$$

The relation of these tensors to the Weyl scalar components in the Newman-Penrose notation is given in eq. (A.11).

For the fields defined in eq. (3.1), eq. (2.4a) implies the following evolution equations along $\mathscr{I}$, which can be verified to be conformally-invariant (eq. (A.12) in the GHP formalism)

$$(\pounds_n + 2\Phi)\mathcal{S}_a = (\mathscr{D}^b + \tau^b)\mathcal{R}_{ab} \qquad (3.3a)$$

$$(\pounds_n + 3\Phi)\mathcal{P} = (\mathscr{D}^a + 2\tau^a)\mathcal{S}_a - \sigma^{ab}\mathcal{R}_{ab} \qquad (3.3b)$$

$$(\pounds_n + 3\Phi)\mathcal{P}^* = -\varepsilon^{ab}(\mathscr{D}_a + 2\tau_a)\mathcal{S}_b + \varepsilon_b{}^c \sigma^{ab}\mathcal{R}_{ac} \qquad (3.3c)$$

$$(\pounds_n + 2\Phi)\mathcal{J}_a = \tfrac{1}{2}(\mathscr{D}_b + 3\tau_b)(q_a{}^b\mathcal{P} - \varepsilon_a{}^b\mathcal{P}^*) - 2\sigma_a{}^b\mathcal{S}_b \qquad (3.3d)$$

$$(\pounds_n + \Phi)\mathcal{I}_{ab} = \left(q_a{}^c q_b{}^d - \tfrac{1}{2}q_{ab}q^{cd}\right)(\mathscr{D}_c + 4\tau_c)\mathcal{J}_d - \tfrac{3}{2}\sigma_{ac}(q_b{}^c\mathcal{P} - \varepsilon_b{}^c\mathcal{P}^*) \qquad (3.3e)$$

We define the *News tensor* by

$$N_{ab} := 2(\pounds_n - \Phi)\sigma_{ab} \qquad (3.4)$$

which satisfies $N_{ab}n^b = 0$, $N_{ab}q^{ab} = 0$ and is conformally-invariant on $\mathscr{I}$. The News tensor is related to the curvature tensor S_{ab} (eq. (2.3)) by (see eq. (A.13) in the GHP formalism)

$$N_{ab} = \left(q_a{}^c q_b{}^d - \tfrac{1}{2}q_{ab}q^{cd}\right)[S_{cd} - 2\Phi\sigma_{cd} + 2(\mathscr{D}_c\tau_d + \tau_c\tau_d)] \qquad (3.5)$$

and to the Weyl tensor on $\mathscr{I}$ by (from eq. (2.4b))

$$\mathcal{R}_{ab} = \tfrac{1}{2}\pounds_n N_{ab}\,, \quad \mathcal{S}_a = \tfrac{1}{2}\mathscr{D}^b N_{ab} \qquad (3.6)$$

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[6]The peeling theorem requires that the unphysical metric is atleast C^3 at $\mathscr{I}$. In our analysis we do not require the full strength of the peeling theorem — in particular the limits defining $\mathcal{J}_a$ and $\mathcal{I}_{ab}$ in eq. (3.1c) need not exist on $\mathscr{I}$.

The form of the asymptotic BMS-supertranslation in a general conformal frame can be derived either by transforming between the usual form in the Bondi frame (see appendix A) or directly from the universal structure at $\mathscr{I}$ (Remark A.1). The final result is as follows: the BMS-supertranslation algebra $\mathfrak{s}^{\pm}$ at $\mathscr{I}^{\pm}$, respectively, is generated by vector fields on the form $\xi^a = fn^a$ on $\mathscr{I}^{\pm}$, where the function f satisfies

$$\mathfrak{s}^{\pm} = \{f \in C^{\infty}_{w=1}(\mathscr{I}^{\pm}) \ : \ (\pounds_n - \Phi)f|_{\mathscr{I}^{\pm}} = 0, \ f|_{i^0} = 0\} \tag{3.7}$$

Note that f has conformal weight $w = 1$, indicated by the subscript in eq. (3.7). In any fixed choice of conformal factor, such functions f are completely determined by their value at some chosen cross-section $S \cong \mathbb{S}^2$ of $\mathscr{I}$. But, since we are allowed to change the conformal factor on $\mathscr{I}$ away from the chosen cross-section S, we cannot (yet) characterise $\mathfrak{s}^{\pm}$ by conformally-weighted functions on $\mathbb{S}^2$, as is usually done in the Bondi conformal frame. We will show later (see eq. (3.24)) how such a characterisation can be obtained on limiting cross-sections of $\mathscr{I}$ near i^0.

The BMS-supertranslations contain a (unique; see [3]) 4-dimensional subalgebra $\mathfrak{t}^{\pm}$ of BMS-translations given by

$$\mathfrak{t}^{\pm} = \left\{f \in \mathfrak{s}^{\pm} \ : \ \left(q_a{}^c q_b{}^d - \tfrac{1}{2}q_{ab}q^{cd}\right)(\mathscr{D}_c + \tau_c)(\mathscr{D}_d - \tau_d)f = 0\right\} \tag{3.8}$$

On some cross-section S of $\mathscr{I}$ the BMS-supermomentum corresponding to some $f \in \mathfrak{s}^{\pm}$ is given by

$$\mathcal{Q}[f; S] = -\int_S \varepsilon_2 \ f \left[\mathcal{P} + \tfrac{1}{2}\sigma^{ab}N_{ab}\right] \tag{3.9}$$

where $\varepsilon_2 \equiv \varepsilon_{ab}$ is the area element on S (eq. (2.8)). The flux of these charges in the region $\Delta\mathscr{I}$ of $\mathscr{I}$ bounded by two cross-sections S_2 and S_1 is given by the exterior derivative of the integrand in eq. (3.9) which, using eqs. (2.8) and (3.7), can be expressed as $\varepsilon_3 f(\pounds_n + 3\Phi)\left[\mathcal{P} + \tfrac{1}{2}q^{ac}q^{bd}\sigma_{ab}N_{cd}\right]$ where $\varepsilon_3 \equiv \varepsilon_{abc}$ is the volume element on $\mathscr{I}$ (eq. (2.8)). Then using eqs. (2.9), (3.3) and (3.6) we get

$$\mathcal{F}[f; \Delta\mathscr{I}] := \mathcal{Q}[f; S_2] - \mathcal{Q}[f; S_1]$$
$$= -\int_{\Delta\mathscr{I}} \varepsilon_3 \ f \left[\tfrac{1}{4}N^{ab}N_{ab} + (\mathscr{D}^a + 2\tau^a)\mathcal{S}_a\right] \tag{3.10a}$$

$$= -\int_{\Delta\mathscr{I}} \varepsilon_3 \ \left[\tfrac{1}{4}fN^{ab}N_{ab} - \mathcal{S}^a(\mathscr{D}_a - \tau_a)f\right] \tag{3.10b}$$

$$= -\int_{\Delta\mathscr{I}} \varepsilon_3 \ \left[\tfrac{1}{4}fN^{ab}N_{ab} - \tfrac{1}{2}\mathscr{D}_b N^{ab}(\mathscr{D}_a - \tau_a)f\right] \tag{3.10c}$$

$$= -\int_{\Delta\mathscr{I}} \varepsilon_3 \ \left[\tfrac{1}{4}fN^{ab}N_{ab} + \tfrac{1}{2}N^{ab}(\mathscr{D}_a + \tau_a)(\mathscr{D}_b - \tau_b)f\right] \tag{3.10d}$$

where eqs. (3.10b)–(3.10d) are obtained by integrating-by-parts on the cross-sections of $\mathscr{I}$ using eqs. (2.13) and (3.6). Note, due to our orientation conventions (table 1), if S_2 is in the future of S_1 the fluxes eq. (3.10) measure the *incoming flux* on both $\mathscr{I}^{\pm}$. Further, from eqs. (3.8) and (3.10d), we reproduce the Bondi energy flux for BMS-translations $f \in \mathfrak{t}^{\pm}$.

Remark 3.1 (Relation to previously obtained formulae). We note that in the Bondi-Sachs conformal frame and with the choice of foliation corresponding to constant Bondi time Φ and τ_a vanish on $\mathscr{I}$ (see section 9.8 [39]) and our expressions eqs. (3.4), (3.5), (3.7)–(3.9) and (3.10d) are equivalent to the usual formulae [6, 7, 40]. In general conformal choices, if we choose the foliation of $\mathscr{I}$ (and hence the auxilliarly normal l^a) such that $\tau_a = 0$, then eq. (3.5) reduces to the News tensor used in [25]. Further consider the BMS-momentum in eq. (3.9), with the News tensor given by eq. (3.5); using eq. (3.8) for BMS-translations we can replace the term $(\mathscr{D}_a\tau_b + \tau_a\tau_b)f$ by $\mathscr{D}_a\mathscr{D}_b f$, reproducing the expression for the BMS-momentum used in eq. 1 [25].

$$* * *$$

Since the metric g_{ab} is $C^{>0}$, $\Omega^{1/2}C_{abcd}$ is $C^{>-1}$, at i^0 along spatial directions, and let $\boldsymbol{C}_{abcd}(\vec{\eta}) := \lim_{\to i^0} \Omega^{1/2}C_{abcd}$ [12, 14]. The *electric* part of $\boldsymbol{C}_{abcd}(\vec{\eta})$ defined by

$$\boldsymbol{E}_{ab}(\vec{\eta}) := \boldsymbol{C}_{acbd}(\vec{\eta})\boldsymbol{\eta}^c\boldsymbol{\eta}^d \tag{3.11}$$

is orthogonal to $\boldsymbol{\eta}^a$ and thus induces an intrinsic field $\boldsymbol{E}_{ab}$ on $\mathscr{H}$. Since the Weyl tensor is trace-free and satisfies eq. (2.4a), $\boldsymbol{E}_{ab}$ on $\mathscr{H}$ satisfies (see [12] for details)

$$\boldsymbol{h}^{ab}\boldsymbol{E}_{ab} = 0 \,, \quad \boldsymbol{D}_{[a}\boldsymbol{E}_{b]c} = 0 \tag{3.12}$$

We defer the analysis of solutions to these equations to appendix C.

At spatial infinity Spi-supertranslations are generated by vector fields ξ^a in M such that $\Omega^{-1}\xi^a$ is $C^{>-1}$ at i^0 in spatial directions and satisfies (see eq. (E.4) for the corresponding transformations in the coordinates used by Beig and Schmidt [11])

$$\lim_{\to i^0} \Omega^{-1}\xi^a = \boldsymbol{f}\boldsymbol{\eta}^a - \boldsymbol{D}^a\boldsymbol{f} \tag{3.13}$$

for some smooth function $\boldsymbol{f}$ on $\mathscr{H}$. Thus, the Spi-supertranslation algebra $\mathfrak{s}^0$ is given by all smooth functions $\boldsymbol{f}$ on $\mathscr{H}$ i.e.

$$\mathfrak{s}^0 \cong C^\infty(\mathscr{H}) \tag{3.14}$$

The 4-dimensional Spi-translation subalgebra $\mathfrak{t}^0$ is given by

$$\mathfrak{t}^0 = \left\{ \boldsymbol{f} \in \mathfrak{s}^0 \ : \ \boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{f} + \boldsymbol{h}_{ab}\boldsymbol{f} = 0 \right\} \tag{3.15}$$

Remark 3.2 (Translation vectors at i^0). Given any element $\boldsymbol{f} \in \mathfrak{t}^0$, consider the vector at i^0 defined by (note the sign difference in the hyperboloidal component relative to eq. (3.13))

$$\boldsymbol{v}^a := \boldsymbol{f}\boldsymbol{\eta}^a + \boldsymbol{D}^a\boldsymbol{f} \tag{3.16}$$

Using eq. (2.20) we have

$$\boldsymbol{\partial}_a\boldsymbol{v}_b = \boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{f} + \boldsymbol{h}_{ab}\boldsymbol{f} = 0 \tag{3.17}$$

so, $\boldsymbol{v}^a$ is *direction-independent* i.e., $\boldsymbol{v}^a \in Ti^0$. Thus, the Spi-translations $\mathfrak{t}^0$ can be represented by vectors in Ti^0.

The conserved charges associated to Spi-translations $\boldsymbol{f} \in \mathfrak{t}^0$ were given by Ashtekar and Hansen [12]. We define the Spi-supermomenta for any Spi-supertranslation $\boldsymbol{f} \in \mathfrak{s}^0$ using the same expression as [12] — in appendix D we show that our final result is unchanged if we use, instead, the expression for the Spi-supermomenta given by Compère and Dehouck [32]. Thus, on a cross-section S of $\mathscr{H}$, with unit future-pointing timelike normal $\boldsymbol{u}^a$, we take the Spi-supermomenta to be given by

$$\mathcal{Q}[\boldsymbol{f}; S] = \int\limits_S \boldsymbol{\varepsilon}_2 \, \boldsymbol{E}_{ab} \boldsymbol{u}^a \boldsymbol{D}^b \boldsymbol{f} \tag{3.18}$$

where $\boldsymbol{\varepsilon}_2 \equiv \boldsymbol{\varepsilon}_{ab}$ is the area element on S (eq. (2.21)). From eq. (3.12) it follows that $\boldsymbol{D}^a \boldsymbol{E}_{ab} = 0$, and the flux of the Spi-supermomenta between any region $\Delta\mathscr{H}$ of $\mathscr{H}$ bounded by the cross-sections S_2 and S_1 (where S_2 is in the future of S_1) is

$$\mathcal{F}[\boldsymbol{f}; \Delta\mathscr{H}] := \mathcal{Q}[\boldsymbol{f}; S_2] - \mathcal{Q}[\boldsymbol{f}; S_1] = \int\limits_{\Delta\mathscr{H}} \boldsymbol{\varepsilon}_3 \, \boldsymbol{E}_{ab} \boldsymbol{D}^a \boldsymbol{D}^b \boldsymbol{f} \tag{3.19}$$

where $\boldsymbol{\varepsilon}_3 \equiv \boldsymbol{\varepsilon}_{abc}$ is the volume element on $\mathscr{H}$.

For Spi-translations $\boldsymbol{f} \in \mathfrak{t}^0$, using eqs. (3.12) and (3.15), the fluxes eq. (3.19) vanish across *any* region $\Delta\mathscr{H}$. Thus corresponding Spi-momenta eq. (3.18) are independent of the choice of cross-section of $\mathscr{H}$ and are well-defined at i^0; these are the usual ADM-momenta at i^0 [12, 41]. For any Spi-translation $\boldsymbol{f}$ let $\boldsymbol{v}^a$ be the direction-independent vector at i^0 determined by eq. (3.16). The charge eq. (3.18) is then a linear map from Ti^0 to the reals which determines the *direction-independent* ADM-momentum covector $\boldsymbol{P}_a$ at i^0 by

$$\boldsymbol{P}_a \boldsymbol{v}^a = \mathcal{Q}[\boldsymbol{f}; S] \tag{3.20}$$

where S is any cross-section of $\mathscr{H}$. The Spi-supermomenta defined by eq. (3.18) where $\boldsymbol{f}$ is a general Spi-supertranslation can only be associated to the blowup $\mathscr{H}$ and not to the asymptotic boundary i^0 itself; however, these will be useful when relating the BMS-supermomenta on $\mathscr{I}^-$ to those on $\mathscr{I}^+$.

Remark 3.3 (Supertranslation vector fields near i^0). Let f be any function in the unphysical spacetime M with conformal weight $w = 1$, which satisfies eq. (3.7) on $\mathscr{I}$, and is $C^{>0}$ in spatial directions at i^0. Consider the vector field in M given by

$$\xi^a := f n^a - \Omega \nabla^a f \tag{3.21}$$

Clearly $\xi^a|_{\mathscr{I}}$ generates a BMS-supertranslation on $\mathscr{I}$. Since, f is $C^{>0}$ along spatial directions at i^0 and $f|_{i^0} = 0$, $\Omega^{-1/2} f$ is $C^{>-1}$ at i^0. Then, defining $\boldsymbol{f}(\vec{\eta}) := \lim\limits_{\to i^0} \Omega^{-1/2} f$ along spatial directions, using eqs. (2.17) and (2.18), we can verify eq. (3.13), that is, ξ^a generates a Spi-supertranslation along spatial directions at i^0. Further, since under a change of conformal factor $\Omega \mapsto \omega\Omega$ we have $n^a \mapsto \omega^{-1}(n^a + \Omega\nabla^a \ln\omega)$ and f has conformal weight $w = 1$, ξ^a is independent of the choice of conformal factor used in the conformal-completion of the physical spacetime. Thus, asymptotically the vector field defined by eq. (3.21) generates a BMS-supertranslation on $\mathscr{I}$ and a Spi-supertranslation at i^0 along spatial directions.

Thus, in this picture the asymptotic supertranslation algebra including all the asymptotic regions is the direct sum

$$\mathfrak{s} = \mathfrak{s}^- \oplus \mathfrak{s}^0 \oplus \mathfrak{s}^+ \cong C^\infty_{w=1}(\mathscr{I}^-) \oplus C^\infty(\mathscr{H}) \oplus C^\infty_{w=1}(\mathscr{I}^+) \tag{3.22}$$

This structure of $\mathfrak{s}$ arises essentially because the hyperboloid $\mathscr{H}$ does not "attach" to null directions along $\mathscr{I}$ at i^0 and one cannot demand any "continuity" between the supertranslations as they approach i^0 along null directions and spatial directions. An additional issue not present in the case of Maxwell fields studied in [31], is the presence of the conformal weights for the BMS-supertranslations $\mathfrak{s}^\pm$ which is absent for the Spi-supertranslations $\mathfrak{s}^0$. Both of these issues can be resolved by using the space $\mathscr{C}$ (see section 2.1) which includes both null and spatial directions at i^0 as follows.

Consider first a BMS-supertranslation $f^\pm \in \mathfrak{s}^\pm$ on $\mathscr{I}^\pm$. From eq. (3.7), $f^\pm|_{i^0} = 0$ and so let $F^\pm = \Sigma f^\pm$. Using eq. (3.7) and Definition 2.2 we have on $\mathscr{I}$

$$\pounds_n F^\pm = (\Phi - 2)F^\pm \tag{3.23}$$

Integrating this along the null generators, since $\Phi|_{i^0} = 2$, we have $\boldsymbol{F}^\pm = \lim\limits_{\to i^0} F^\pm$ along $\mathscr{I}^\pm$ is a smooth function on $\mathscr{N}^\pm$. The limiting rescaled functions $\boldsymbol{F}^\pm$ are invariant under a change of conformal factor (since $\omega|_{i^0} = 1$) but transform as $\boldsymbol{F}^\pm \mapsto \boldsymbol{\sigma}^\pm \boldsymbol{F}^\pm$ under a change of rescaling function $\Sigma \mapsto \sigma\Sigma$, that is, $\boldsymbol{F}^\pm$ has rescaling weight $r = 1$ on $\mathscr{N}^\pm$. Thus, the BMS-supertranslations can be characterised near i^0 as

$$\mathfrak{s}^\pm = \left\{ \boldsymbol{F}^\pm := \lim\limits_{\to i^0} \Sigma f^\pm \right\} \cong C^\infty_{r=1}(\mathscr{N}^\pm) \tag{3.24}$$

Since the metric $\tilde{\boldsymbol{q}}_{ab}$ on $\mathscr{N}^\pm$ transforms according to $\tilde{\boldsymbol{q}}_{ab} \mapsto \boldsymbol{\sigma}^\pm \tilde{\boldsymbol{q}}_{ab}$ (see eq. (2.36)), in the limit to i^0 we recover the characterisation of BMS-supertranslations as conformally-weighted functions on $\mathscr{N}^\pm \cong \mathbb{S}^2$ where now the rescaling freedom $\boldsymbol{\sigma}^\pm$ is viewed as a conformal transformation on $\mathscr{N}^\pm$. We show in appendix E that the functions $\boldsymbol{F}^\pm$ are precisely the usual BMS-supertranslation functions used in the Bondi conformal choice.

Next consider a Spi-supertranslation $\boldsymbol{f} \in \mathfrak{s}^0$ which is a smooth function on $\mathscr{H}$ and hence on $\mathscr{C}\backslash\mathscr{N}^\pm$. However, $\boldsymbol{f}$ does not have a limit to $\mathscr{N}^\pm$ even for Spi-translations, though $\boldsymbol{\Sigma f}$ does extend smoothly to $\mathscr{N}^\pm$ (see eqs. (C.12) and (C.13)). Thus, we restrict to Spi-supertranslations $\boldsymbol{f}$ such that the rescaled function $\boldsymbol{F} := \boldsymbol{\Sigma f}$ extends smoothly to $\mathscr{N}^\pm$.[7] As in the case of a BMS-supertranslation the rescaled function $\boldsymbol{F}$ is invariant under changes of the conformal factor but has rescaling weight $r = 1$. Note that such Spi-supertranslation $\boldsymbol{F} \in \mathfrak{s}^0$ determines unique BMS-supertranslations $\boldsymbol{F}^\pm \in \mathfrak{s}^\pm$ by "continuity" at $\mathscr{N}^\pm$ that is, $\boldsymbol{F}|_{\mathscr{N}^\pm} = \boldsymbol{F}^\pm$. Thus, there is a natural subalgebra $\mathfrak{s}^{\mathrm{nr}}$ of $\mathfrak{s}$ (eq. (3.22)) given by the *null-regular* supertranslations of the form

$$\mathfrak{s}^{\mathrm{nr}} = \left\{ (\boldsymbol{F}^-, \boldsymbol{F}, \boldsymbol{F}^+) \in \mathfrak{s} \ : \ \boldsymbol{F}^\pm = \boldsymbol{F}|_{\mathscr{N}^\pm} \right\} \cong C^\infty_{r=1}(\mathscr{C}) \tag{3.25}$$

Note that we have replaced the conformal weight $w = 1$ of the BMS-supertranslations represented by $f^\pm$ with the rescaling weight $r = 1$ of the limiting functions $\boldsymbol{F}^\pm$, and

---

[7]This condition on the Spi-supertranslations at $\mathscr{N}^\pm$ is also suggested in the last footnote of [14].

similarly introduced the Spi-supertranslation function $\boldsymbol{F}$ of rescaling weight $r = 1$. Thus, the null-regular subalgebra in eq. (3.25) is both conformal and rescaling invariant.

As shown in [25, 26], there exists an isomorphism between Spi-translations $\mathfrak{t}^0$ and both the BMS-translations $\mathfrak{t}^\pm$. We illustrate this isomorphism explicitly using eq. (3.25) as follows. Since eq. (3.25) is invariant under a change of rescaling function, we choose Σ such that the metric $\tilde{\boldsymbol{q}}_{ab}$ on $\mathscr{N}^\pm$ is that of a unit 2-sphere as in appendix C. Then as shown in eq. (C.13), for any element of the Spi-translations $\mathfrak{t}^0$, the functions $\boldsymbol{F}|_{\mathscr{N}^\pm}$ are spanned by the $\ell = 0, 1$ spherical harmonics. For such functions the BMS-supertranslations $f^\pm$ such that $\lim_{\to i^0} \Sigma f^\pm = \boldsymbol{F}^\pm = \boldsymbol{F}|_{\mathscr{N}^\pm}$ are precisely the ones which satisfy eq. (3.8) (note that $\tau_a = 0$ in our choice of foliation; eq. (2.31)). Thus, any Spi-translation in $\mathfrak{t}^0$ determines *unique* elements of both $\mathfrak{t}^\pm$ satisfying eq. (3.25). Thus, $\mathfrak{s}^{\mathrm{nr}}$ contains a unique subalgebra of translations which is isomorphic to $\mathfrak{t}^0$ and $\mathfrak{t}^\pm$. Note from eq. (C.13) this isomorphism naturally provides an antipodal identification between $\mathfrak{t}^-$ and $\mathfrak{t}^+$.

For supertranslations, which are not translations, eq. (3.25) does not provide a unique isomorphism from $\mathfrak{s}^-$ to $\mathfrak{s}^+$. Given a BMS-supertranslation $\boldsymbol{F}^-$ on $\mathscr{I}^-$ we can get any BMS-supertranslation $\boldsymbol{F}^+$ on $\mathscr{I}^+$ simply by choosing $\boldsymbol{F}$ suitably on $\mathscr{C}$ since general Spi-supertranslations can be arbitrary functions on $\mathscr{C}$. Since the conformal reflection isometry of $\mathscr{C}$ provides a natural map $\Upsilon : \mathscr{N}^- \to \mathscr{N}^+$ between the space of null generators of $\mathscr{I}^-$ and $\mathscr{I}^+$, one can impose some matching condition relating $\Upsilon \circ \boldsymbol{F}^-$ to $\boldsymbol{F}^+$. However, our considerations so far do not provide any physical criteria for such a matching condition — except by analogy to translations. As we show below, using the Einstein equation on $\mathscr{C}$ and demanding that the total flux of Spi-supermomenta through $\mathscr{C}$ vanish in any scattering process provides a natural matching condition (eq. (4.19)) which is precisely the one conjectured in [19].

4 Null-regular spacetimes at i^0

Even with the "partial matching" of the supertranslations provided by eq. (3.25), we cannot immediately conclude that the corresponding supermomenta match in the same manner at i^0. In fact, since we have not specified the behaviour of the gravitational fields on $\mathscr{I}$ in the limit to i^0, the BMS-supermomenta eq. (3.9) can diverge as the cross-sections of $\mathscr{I}$ tend towards i^0. Such spacetimes will have an infinite flux of BMS-supermomenta through null infinity and are thus unphysical in a scattering process. We will now impose the following additional restrictions on the class of spacetimes we consider, from here on, to discard such "pathological" solutions to the Einstein equation. Even though we argue that these conditions are physically reasonable for a scattering process, the existence of (a suitably large class of) such spacetimes is very much an open question which we discuss in section 5.

Definition 4.1 (Null-regular spacetime at i^0). Let l^a be the vector field defined by eq. (2.29) in M. We call a spacetime with an Ashtekar-Hansen structure (Definition 2.1) *null-regular* at i^0 if

(1) the rescaled quantity

$$\Sigma^{-3}\Omega^{-1}C_{abcd}l^a n^b l^c n^d \text{ is } C^{>-1} \text{ in } both \text{ null and spatial directions at } i^0 \qquad (4.1)$$

(2) in the limit to i^0 along each null generator of $\mathscr{I}$, the components of $\Sigma^{1+\epsilon}N_{ab}$ and $\Sigma^{1+\epsilon}\mathcal{R}_{ab}$ in a $C^{>1}$-chart at i^0 remain bounded for some small $\epsilon > 0$, that is,[8]

$$N_{ab} = O(\Sigma^{-(1+\epsilon)}), \quad \mathcal{R}_{ab} = O(\Sigma^{-(1+\epsilon)}) \text{ as } \Sigma^{-1} \to 0 \text{ along } \mathscr{I} \qquad (4.2)$$

Our falloff condition on the News tensor is stronger than the one required for a finite flux of Bondi momentum (associated to translations) through $\mathscr{I}$ [25], but is weaker than the falloff assumed in [19] which was motivated by CK-spacetimes [29]. As we show below (Remark 4.1), this falloff is needed so that the flux of *all* BMS-supermomenta through $\mathscr{I}$ is finite. The falloffs in eq. (4.2) also ensure that the radiative phase space on $\mathscr{I}$ has a well-defined symplectic structure [6, 42]. The form of these conditions in terms of the Bondi-Sachs parameter on $\mathscr{I}$ is given in eq. (E.1).

Let us explore the consequences of the regularity conditions Definition 4.1 on the gravitational fields on $\mathscr{I}$ and at i^0. From eq. (3.1) we have on $\mathscr{I}$, $\Sigma^{-3}\Omega^{-1}C_{abcd}l^a n^b l^c n^d = \Sigma^{-3}\mathcal{P}$, and from eq. (4.1) $\lim_{\to i^0}\Sigma^{-3}\mathcal{P}$ along $\mathscr{I}$ exists and induces a smooth function on the space of null directions $\mathscr{N}^\pm$. Similarly, the limit of $\Sigma^{-3}\Omega^{-1}C_{abcd}l^a n^b l^c n^d$ to i^0 in spatial directions can be rewritten as (using eqs. (2.25), (2.26), (2.29), (2.34) and (3.11))

$$\lim_{\to i^0}\Sigma^{-3}\Omega^{-1}C_{abcd}l^a n^b l^c n^d = \lim_{\to i^0}\Sigma^{-3}\Omega^{-1}C_{abcd}L^a N^b L^c N^d = (\boldsymbol{\Sigma}^{-1}\boldsymbol{E}_{ab}\boldsymbol{U}^a\boldsymbol{U}^b)(\vec{\eta}) \qquad (4.3)$$

Thus, $\Sigma^{-3}\Omega^{-1}C_{abcd}l^a n^b l^c n^d$ induces the field $\boldsymbol{\Sigma}^{-1}\boldsymbol{E}_{ab}\boldsymbol{U}^a\boldsymbol{U}^b$ on $\mathscr{H}$ which can be viewed as a field on $\mathscr{C}\backslash\mathscr{N}^\pm$. Since $\Sigma^{-3}\Omega^{-1}C_{abcd}l^a n^b l^c n^d$ is $C^{>-1}$ in both null and spatial directions (eq. (4.1)), $\boldsymbol{\Sigma}^{-1}\boldsymbol{E}_{ab}\boldsymbol{U}^a\boldsymbol{U}^b$ has a limit as a smooth function on $\mathscr{N}^\pm$, which further coincides with the field induced by $\lim_{\to i^0}\Sigma^{-3}\Omega^{-1}C_{abcd}l^a n^b l^c n^d = \lim_{\to i^0}\Sigma^{-3}\mathcal{P}$ on $\mathscr{N}^\pm$ from $\mathscr{I}^\pm$ that is,

$$\lim_{\to i^0}(\Sigma^{-3}\mathcal{P}) \text{ along } \mathscr{I}^\pm = \lim_{\to \mathscr{N}^\pm}(\boldsymbol{\Sigma}^{-1}\boldsymbol{E}_{ab}\boldsymbol{U}^a\boldsymbol{U}^b) \text{ along } \mathscr{C} \qquad (4.4)$$

Now, consider the foliation of $\mathscr{I}$ by cross-sections S_Σ with $\Sigma^{-1} = $ constant as described section 2.1. From eq. (3.4) and Definition 2.2 it follows that the shear tensor σ_{ab} of such cross-sections satisfies

$$\pounds_n(\Sigma\sigma_{ab}) = \tfrac{1}{2}\Sigma N_{ab} + (\Phi - 2)\Sigma\sigma_{ab} \qquad (4.5)$$

We can integrate this equation along the null generators (using eqs. (2.30) and (4.2)), noting that $\Phi|_{i^0} = 2$, to conclude that

$$\Sigma\sigma_{ab} \text{ remains bounded as } \Sigma^{-1} \to 0 \qquad (4.6)$$

Using these conditions we can evaluate the supermomenta induced on the space of null directions $\mathscr{N}^\pm$ in the limit from $\mathscr{I}^\pm$ and from $\mathscr{C}$. First, consider the BMS-supermomentum associated to a BMS-supertranslation $f^\pm \in \mathfrak{s}^\pm$ on the cross-sections S_Σ of $\mathscr{I}^\pm$ tend towards

---

[8]Note that the falloffs of N_{ab} and $\mathcal{R}_{ab}$ are compatible with eq. (3.6). However, integrating eq. (3.6) to derive the falloffs on the News tensor N_{ab} from those on $\mathcal{R}_{ab}$, in general, leads to a "constant of integration" in N_{ab}. The faster falloff of the News tensor imposed in eq. (4.2) ensures that this constant of integration vanishes.

i^0, i.e., as $\Sigma^{-1} \to 0$. From eq. (3.9), the BMS-supermomentum induced on $\mathscr{N}^\pm$ from $\mathscr{I}^\pm$ is given by

$$
\begin{aligned}
\mathcal{Q}[f^\pm; \mathscr{N}^\pm] &= - \lim_{\Sigma^{-1} \to 0} \int_{S_\Sigma} \varepsilon_2 \, f^\pm \left[\mathcal{P} + \tfrac{1}{2} q^{ac} q^{bd} \sigma_{ab} N_{cd} \right] \\
&= - \lim_{\Sigma^{-1} \to 0} \int_{S_\Sigma} (\Sigma^2 \varepsilon_2) \, (\Sigma f^\pm)(\Sigma^{-3}\mathcal{P} + \tfrac{1}{2} \Sigma \tilde{q}^{ac} \tilde{q}^{bd} \sigma_{ab} N_{cd}) \\
&= - \int_{\mathscr{N}^\pm} \tilde{\boldsymbol{\varepsilon}}_2 \, \boldsymbol{F}^\pm (\Sigma^{-3}\mathcal{P})
\end{aligned}
\tag{4.7}
$$

where in the third line we have used the falloffs eqs. (4.2) and (4.6), the rescaled area element $\tilde{\varepsilon}_{ab} = \lim_{\to i^0} \Sigma^2 \varepsilon_{ab}$ and the rescaled supertranslation function $\boldsymbol{F}^\pm = \lim_{\to i^0} \Sigma f^\pm$ on $\mathscr{N}^\pm$. The final limit exists due to our null-regularity condition eq. (4.1) as discussed above.

Remark 4.1 (Finiteness of flux through $\mathscr{I}$). Let $\Delta \mathscr{I}^\pm$ be a region of $\mathscr{I}^\pm$ foliated by cross-sections S_Σ with $\Sigma_0^{-1} \leq \Sigma^{-1} \leq \Sigma_1^{-1}$. Since the limit to i^0 of the BMS-supermomenta evaluated on S_Σ exists, the flux must be finite as $\Sigma_0^{-1} \to 0$. However, the volume element ε_{abc} on $\mathscr{I}^\pm$ (appearing in eq. (3.10b)) is ill-defined in this limit since l^a diverges at i^0. But the rescaled volume element

$$
L^d \varepsilon_{dabc} = 2\Sigma^{-1}\varepsilon_{abc} = 3\nabla_{[a}\Sigma^{-1}\varepsilon_{bc]} \text{ is } C^{>-1} \text{ at } i^0
\tag{4.8}
$$

Thus, on $\Delta \mathscr{I}^\pm$ foliated by S_Σ we can rewrite the flux eq. (3.10b), using the rescaled quantities $F^\pm = \Sigma f^\pm$, $\tilde{q}^{ab} = \Sigma^{-2} q^{ab}$ and $\tilde{\varepsilon}_{ab} = \Sigma^2 \varepsilon_{ab}$, as

$$
\mathcal{F}[f^\pm; \Delta \mathscr{I}^\pm] = -\tfrac{1}{2} \int_{\Sigma_0^{-1}}^{\Sigma_1^{-1}} d\Sigma^{-1} \, \Sigma \int_{S_\Sigma} \tilde{\varepsilon}_2 \, \left[\tfrac{1}{4} F^\pm \Sigma \, \tilde{q}^{ac} \tilde{q}^{bd} N_{ab} N_{cd} - \Sigma^{-1} \tilde{q}^{ab} \mathcal{S}_a \nabla_b F^\pm \right]
\tag{4.9}
$$

Since the flux on the left-hand-side is finite, in the limit $\Sigma_0^{-1} \to 0$, and $F^\pm$, $\tilde{q}^{ab}$ and $\tilde{\varepsilon}_2$ on the right-hand-side induce smooth fields on $\mathscr{N}^\pm$ and from the falloff on N_{ab} eq. (4.2), we get

$$
\Sigma^{-1} \mathcal{S}_a = O(\Sigma^{-\epsilon}) \quad \text{along null directions}
\tag{4.10}
$$

for some small $\epsilon > 0$. Similarly, using the alternative forms for the flux from eq. (3.10) we can also conclude that in the limit $\Sigma^{-1} \to 0$ towards i^0 along null directions[9]

$$
\begin{aligned}
\Sigma^{-1}\tilde{q}^{ab} \mathscr{D}_a \mathcal{S}_b = O(\Sigma^{-\epsilon}) \quad &\Sigma \tilde{q}^{bc} \mathscr{D}_b N_{ac} = O(\Sigma^{-\epsilon}) \\
\left(\tilde{q}_a{}^c \tilde{q}_b{}^d - \tfrac{1}{2} \tilde{q}_{ab} \tilde{q}^{cd} \right) \mathscr{D}_c \mathscr{D}_d F^\pm \quad &\text{remains bounded}
\end{aligned}
\tag{4.11}
$$

Note that, from eqs. (3.8) and (3.10), the term linear in the News tensor in the flux of BMS-supermomenta vanishes identically for BMS-translations. Thus, if one is only interested in the BMS-translations, the slower falloff $N_{ab} = O(\Sigma^{-(1+\epsilon)/2})$ (equivalent to the falloff used in [25]) suffices to ensure that the BMS-momentum flux is finite.

---

[9]If the unphysical metric is also $C^{>0}$ in null directions then, eq. (4.11) would follow from eqs. (3.6), (4.2) and (4.10).

Remark 4.2 (Classical vacua on $\mathscr{I}$). From eq. (3.6), if the $N_{ab} = 0$ vanishes in some open region of $\mathscr{I}$ then $\mathcal{R}_{ab} = \mathcal{S}_a = 0$. Conversely if $\mathcal{R}_{ab} = \mathcal{S}_a = 0$ in a region of $\mathscr{I}$ which contains the entire space of generators (topologically $\mathbb{S}^2$), then the News tensor N_{ab} is a symmetric traceless tensor on $\mathbb{S}^2$ (as $\pounds_n N_{ab} = 0$) satisfying $\mathscr{D}^b N_{ab} = 0$. Since the News tensor is conformally-invariant, we can choose the unit-metric on $\mathbb{S}^2$ to show that any such smooth tensor must vanish; see for instance, appendix A.4 [40], appendix C [41], or Prop. 4.15.59 [43]. If an asymptotically-flat spacetime is such that its News tensor vanishes on *all* of $\mathscr{I}$, then it corresponds to a *classical vacuum* in the radiative phase space on $\mathscr{I}$ (see [6] or [42] for a recent review). Our falloff conditions eqs. (4.2), (4.6) and (4.10) (equivalently eq. (E.1)) then imply that all null-regular spacetimes approach some classical vacuum near i^0, and the tensor $\Sigma\sigma_{ab}$ (which remains bounded; eq. (4.6)) measures the deviation of the given spacetime relative to this vacuum.

Next we evaluate the Spi-supermomentum associated to a Spi-supertranslation $\boldsymbol{f} \in \mathfrak{s}^0$ on $\mathscr{C}$ as the cross-section $S \to \mathcal{N}^\pm$. Since, the normals $\boldsymbol{u}^a$ and $\boldsymbol{U}^a$ are timelike and unit with respect to the metrics $\boldsymbol{h}_{ab}$ and $\tilde{\boldsymbol{h}}_{ab} = \boldsymbol{\Sigma}^2 \boldsymbol{h}_{ab}$, respectively, we have

$$\lim_{\to \mathcal{N}^\pm} \boldsymbol{\Sigma}^{-1} \boldsymbol{U}^a = \pm \lim_{\to \mathcal{N}^\pm} \boldsymbol{\Sigma}^{-2} \boldsymbol{u}^a \tag{4.12}$$

where the signs on the right-hand-side are due to our orientation conventions from table 1. Similarly, we have for the area element on S in the limit to $\mathcal{N}^\pm$ (from eqs. (2.21) and (2.38))

$$\tilde{\boldsymbol{\varepsilon}}_{ab} = \pm \lim_{S \to \mathcal{N}^\pm} \boldsymbol{\Sigma}^2 \boldsymbol{\varepsilon}_{ab} \tag{4.13}$$

Thus, rewriting the Spi-supermomenta eq. (3.18) in terms of $\boldsymbol{F} = \boldsymbol{\Sigma} \boldsymbol{f}$, using eq. (2.34) we have on $\mathcal{N}^\pm$

$$\begin{aligned}
\mathcal{Q}[\boldsymbol{f}; \mathcal{N}^\pm] = -\int_{\mathcal{N}^\pm} \tilde{\boldsymbol{\varepsilon}}_2 \, &\Big[(\boldsymbol{\Sigma}^{-1} \boldsymbol{E}_{ab} \boldsymbol{U}^a \boldsymbol{U}^b) \boldsymbol{F} \\
&- \boldsymbol{\Sigma}^2 (\boldsymbol{\Sigma}^{-1} \boldsymbol{E}_{ab} \boldsymbol{U}^a \boldsymbol{U}^b)(\boldsymbol{\Sigma}^{-1} \boldsymbol{U}^c) \boldsymbol{D}_c \boldsymbol{F} - \boldsymbol{E}_{ab} \boldsymbol{U}^a \tilde{\boldsymbol{q}}^{bc} \boldsymbol{D}_c \boldsymbol{F} \Big] \\
= -\int_{\mathcal{N}^\pm} \tilde{\boldsymbol{\varepsilon}}_2 \, &\boldsymbol{F} (\boldsymbol{\Sigma}^{-1} \boldsymbol{E}_{ab} \boldsymbol{U}^a \boldsymbol{U}^b)
\end{aligned} \tag{4.14}$$

where in the last line we have restricted to the Spi-supertranslations for which $\boldsymbol{F}$ admits a smooth limit to $\mathcal{N}^\pm$ as discussed at the end of section 3. We have also used eq. (2.35) and eq. (C.8) for the electric field on $\mathscr{C}$ in null-regular spacetimes.

Comparing eqs. (4.7) and (4.14) using eq. (4.4), we see that the supermomenta induced on $\mathcal{N}^\pm$ by the BMS-supermomenta on $\mathscr{I}$ and the Spi-supermomenta on $\mathscr{C}$ are equal iff $\boldsymbol{F}^\pm = \boldsymbol{F}|_{\mathcal{N}^\pm}$, that is, for the null-regular supertranslations in eq. (3.25) we have

$$\mathcal{Q}[f^\pm; \mathcal{N}^\pm] \text{ along } \mathscr{I}^\pm = \mathcal{Q}[\boldsymbol{f}; \mathcal{N}^\pm] \text{ along } \mathscr{C} \quad \text{for } \boldsymbol{F} \in \mathfrak{s}^{\mathrm{nr}} \tag{4.15}$$

For a Spi-translation the flux of supermomenta on $\mathscr{C}$ vanishes and so the Spi-momenta on $\mathcal{N}^\pm$ are equal. Thus, from eq. (4.15), the BMS-momenta eq. (3.9) on $\mathscr{I}^\pm$ and the Spi-momenta eq. (3.18) on $\mathscr{C}$ determine the unique ADM-momentum covector $\boldsymbol{P}_a$ at i^0

through eq. (3.20) which reproduces the result of [25]. For a general Spi-supertranslation the Spi-supermomenta on $\mathscr{N}^\pm$ need not be equal. We argue next that the total flux of Spi-supermomenta on $\mathscr{C}$ is "non-dynamical" and that the subalgebra of Spi-supertranslations on $\mathscr{C}$ which are *totally fluxless* provides a natural isomorphism from $\mathfrak{s}^-$ to $\mathfrak{s}^+$.

4.1 Totally fluxless supertranslations on $\mathscr{C}$, antipodal matching and conservation laws

As mentioned above, for some arbitrary choice of Spi-supertranslation $\boldsymbol{F} \in \mathfrak{s}^{\mathrm{nr}}$, the flux of supermomenta through all of $\mathscr{C}$ need not vanish. The flux through all of $\mathscr{C}$ is given by the difference of the Spi-supermomenta integrals on $\mathscr{N}^\pm$

$$\mathcal{F}[\boldsymbol{f};\mathscr{C}] = -\int_{\mathscr{N}^+} \tilde{\varepsilon}_2 \; \boldsymbol{F}^+ (\boldsymbol{\Sigma}^{-1} \boldsymbol{E}_{ab} \boldsymbol{U}^a \boldsymbol{U}^b) + \int_{\mathscr{N}^-} \tilde{\varepsilon}_2 \; \boldsymbol{F}^- (\boldsymbol{\Sigma}^{-1} \boldsymbol{E}_{ab} \boldsymbol{U}^a \boldsymbol{U}^b) \tag{4.16}$$

where $\boldsymbol{F}^\pm = \boldsymbol{F}|_{\mathscr{N}^\pm}$. From the analysis in appendix C, the only solutions to Einstein equation on $\mathscr{C}$ for which $\lim_{\to \mathscr{N}^\pm} \boldsymbol{\Sigma}^{-1} \boldsymbol{E}_{ab} \boldsymbol{U}^a \boldsymbol{U}^b$ exists (and thus corresponds to null-regular spacetimes) are the ones which satisfy

$$\Upsilon \circ (\boldsymbol{\Sigma}^{-1} \boldsymbol{E}_{ab} \boldsymbol{U}^a \boldsymbol{U}^b)|_{\mathscr{N}^-} = (\boldsymbol{\varsigma}^{-3} \boldsymbol{\Sigma}^{-1} \boldsymbol{E}_{ab} \boldsymbol{U}^a \boldsymbol{U}^b)|_{\mathscr{N}^+} \tag{4.17}$$

where Υ is the reflection conformal isometry of $\mathscr{C}$ (eq. (2.39)). Then, using the transformation of $\tilde{\varepsilon}_{ab}$ under Υ, eq. (2.40), the total flux is given by

$$\mathcal{F}[\boldsymbol{f};\mathscr{C}] = -\int_{\mathscr{N}^+} \tilde{\varepsilon}_2 \; (\boldsymbol{F}^+ + \boldsymbol{\varsigma}^{-1} \Upsilon \circ \boldsymbol{F}^-)(\boldsymbol{\Sigma}^{-1} \boldsymbol{E}_{ab} \boldsymbol{U}^a \boldsymbol{U}^b) \tag{4.18}$$

Since eq. (4.18) is invariant under a change of rescaling function, we consider its value in the choice of Σ from appendix B, where the metric on $\mathscr{N}^\pm$ is that of a unit-sphere and $\boldsymbol{\varsigma} = 1$. In this choice, for any Schwarzschild spacetime which is "at rest" with respect to the chosen $C^{>1}$-coordinates at i^0, the scalar $\boldsymbol{\Sigma}^{-1} \boldsymbol{E}_{ab} \boldsymbol{U}^a \boldsymbol{U}^b$ is a $\ell = 0$ spherical harmonic on $\mathscr{N}^+$ (see appendix C). However, for general Spi-supertranslations $(\boldsymbol{F}^+ + \Upsilon \circ \boldsymbol{F}^-)$ can be any arbitrary function on $\mathscr{N}^+$, and the total flux of Spi-supermomenta through $\mathscr{C}$ can take any value even for Schwarzschild spacetime. This suggests that the total flux through $\mathscr{C}$ is "spurious" and is not associated to any dynamics of the gravitational scattering process. Thus, we further restrict the symmetry algebra to those elements which have vanishing total flux on $\mathscr{C}$. From eq. (4.18), the only Spi-supertranslations $\boldsymbol{f}$ which satisfy $\mathcal{F}[\boldsymbol{f};\mathscr{C}] = 0$ are the ones for which

$$\Upsilon \circ \boldsymbol{F}^- = -\boldsymbol{\varsigma} \boldsymbol{F}^+ \tag{4.19}$$

Note that this is *not* a restriction on the spacetimes we consider, unlike the null-regularity conditions in Definition 4.1. The behaviour of $\boldsymbol{F}$ in $\mathscr{C} \backslash \mathscr{N}^\pm$ can be arbitrary, and only the boundary values at $\mathscr{N}^\pm$ are required to satisfy eq. (4.19). Thus, we define the equivalence class $[\boldsymbol{F}]$ for any $\boldsymbol{F}$ satisfying eq. (4.19) by

$$\boldsymbol{F}' \in [\boldsymbol{F}] \iff \boldsymbol{F}'|_{\mathscr{N}^\pm} = \boldsymbol{F}|_{\mathscr{N}^\pm} \tag{4.20}$$

Each equivalence class $[\boldsymbol{F}]$ is uniquely determined by a smooth function on $\mathbb{S}^2$ (with rescaling weight $r = 1$), either considered as a function on $\mathscr{N}^-$ or $\mathscr{N}^+$ related by eq. (4.19). Thus, the condition that the total flux through $\mathscr{C}$ vanish gives us the "diagonal" subalgebra of the null-regular supertranslation algebra $\mathfrak{s}^{\mathrm{nr}}$

$$\mathfrak{s}^\times := \left\{ [\boldsymbol{F}] \ : \ \boldsymbol{F} \in \mathfrak{s}^{\mathrm{nr}} \text{ and } \Upsilon \circ \boldsymbol{F}^- = -\varsigma \boldsymbol{F}^+ \right\} \cong C^\infty_{r=1}(\mathbb{S}^2) \tag{4.21}$$

The supertranslations in $\mathfrak{s}^\times$ provide a natural isomorphism between the BMS-super- translations $\mathfrak{s}^-$ and $\mathfrak{s}^+$ on null infinity as follows. In any choice of rescaling function Σ, any BMS-supertranslation f^- on $\mathscr{I}^-$ determines a unique $[\boldsymbol{F}] \in \mathfrak{s}^\times$ on $\mathscr{C}$ so that along $\mathscr{I}^-$ we have $\lim_{\to i^0} \Sigma f^- = \boldsymbol{F}|_{\mathscr{N}^-} = \boldsymbol{F}^-$. From eq. (4.21) this determines a unique symmetry f^+ on $\mathscr{I}^+$ so that along $\mathscr{I}^+$ we have $\lim_{\to i^0} \Sigma f^+ = \boldsymbol{F}|_{\mathscr{N}^+} = \boldsymbol{F}^+$. Thus we have the isomorphism

$$\mathfrak{s}^- \to \mathfrak{s}^+ : (\lim_{\to i^0} \Sigma f^-)(\theta^A) \text{ along } \mathscr{I}^- \mapsto -(\lim_{\to i^0} \Sigma f^+)(-\theta^A) \text{ along } \mathscr{I}^+ \tag{4.22}$$

That is, for the subalgebra $\mathfrak{s}^\times$, the symmetries on $\mathscr{I}^-$ can be matched to those on $\mathscr{I}^+$ through an antipodal reflection on $\mathbb{S}^2$. Note that the negative sign in eq. (4.22) is due to our orientation conventions (see table 1) — if instead, n^a is taken to be future-directed on both $\mathscr{I}^\pm$ this reproduces the matching condition proposed by Strominger [19].

From eq. (4.15), we see that under the isomorphism eq. (4.22) we have

$$\mathcal{Q}[f^-; \mathscr{N}^-] \text{ along } \mathscr{I}^- = \mathcal{Q}[f^+; \mathscr{N}^+] \text{ along } \mathscr{I}^+ \tag{4.23}$$

that is, the BMS-supermomenta on $\mathscr{I}^-$ and $\mathscr{I}^+$ are equal at i^0 which is a direct consequence of the corresponding Spi-supertranslation $\boldsymbol{f}$ on $\mathscr{C}$ being totally fluxless.

Remark 4.3 (Change of conformal factor and rescaling function). Our analysis is conformally-invariant on $\mathscr{I}$, while at i^0, $\omega|_{i^0} = 1$ and hence our analysis is independent of the choice of conformal factor. Thus it suffices to consider the change of the rescaling function $\Sigma \mapsto \sigma\Sigma$ (Remark 2.3) where $\mathcal{L}_n \sigma = 0$ on $\mathscr{I}$. Under this change of rescaling function we have on $\mathscr{I}$

$$n^a \mapsto n^a, \quad l^a \mapsto l^a + \tfrac{1}{2} q^{ab} \nabla_b \ln \sigma + \tfrac{1}{8} q^{bc} \nabla_b \ln \sigma \nabla_c \ln \sigma \ n^a \tag{4.24}$$

where the transformation of l^a follows from eqs. (2.27) and (2.29). Using eq. (3.1) we have

$$\begin{aligned} \Sigma^{-3}\mathcal{P} &\mapsto \sigma^{-3}\Sigma^{-3}\big[\mathcal{P} + q^{ab}\mathcal{S}_a \nabla_b \ln \sigma + \tfrac{1}{4} q^{ac} q^{bd} \mathcal{R}_{ab} \nabla_c \ln \sigma \nabla_d \ln \sigma \big] \\ &= \sigma^{-3}\big[\Sigma^{-3}\mathcal{P} + \Sigma^{-1}\tilde{q}^{ab}\mathcal{S}_a \nabla_b \ln \sigma + \tfrac{1}{4}\Sigma \tilde{q}^{ac}\tilde{q}^{bd}\mathcal{R}_{ab}\nabla_c \ln \sigma \nabla_d \ln \sigma \big] \end{aligned} \tag{4.25a}$$

$$\begin{aligned} \Sigma^{-1}\mathcal{S}_a &\mapsto \sigma^{-1}\Sigma^{-1}\left[\mathcal{S}_a + \tfrac{1}{2} q^{bc}\mathcal{R}_{ab}\nabla_c \ln \sigma\right] \\ &= \sigma^{-1}\left[\Sigma^{-1}\mathcal{S}_a + \tfrac{1}{2}\Sigma \tilde{q}^{bc}\mathcal{R}_{ab}\nabla_c \ln \sigma\right] \end{aligned} \tag{4.25b}$$

$$\Sigma \mathcal{R}_{ab} \mapsto \sigma \Sigma \mathcal{R}_{ab} \tag{4.25c}$$

Since σ is $C^{>-1}$ in null directions it follows from the above transformations that the falloff conditions eqs. (4.1), (4.2) and (4.10) are preserved under a change of the rescaling function, and the field induced on $\mathcal{N}^{\pm}$ by $\lim_{\to i^0} \Sigma^{-3}\mathcal{P}$ transforms with rescaling weight $r = -3$ as

$$(\Sigma^{-3}\mathcal{P})|_{\mathcal{N}^{\pm}} \mapsto (\boldsymbol{\sigma}^{\pm})^{-3}(\Sigma^{-3}\mathcal{P})|_{\mathcal{N}^{\pm}} \tag{4.26}$$

where $\boldsymbol{\sigma}^{\pm}$ is the function induced on $\mathcal{N}^{\pm}$ by $\lim_{\to i^0} \sigma$ along $\mathscr{I}^{\pm}$. Similarly on $\mathscr{C}$, using eq. (2.34), we have

$$(\boldsymbol{\Sigma}^{-1}\boldsymbol{E}_{ab}\boldsymbol{U}^a\boldsymbol{U}^b)|_{\mathcal{N}^{\pm}} \mapsto (\boldsymbol{\sigma}^{\pm})^{-3}(\boldsymbol{\Sigma}^{-1}\boldsymbol{E}_{ab}\boldsymbol{U}^a\boldsymbol{U}^b)|_{\mathcal{N}^{\pm}} \tag{4.27}$$

which also has rescaling weight $r = -3$, where we have used the fact that σ is $C^{>-1}$ in both null and spatial directions so that on $\mathscr{C}$, $\lim_{\to \mathcal{N}^{\pm}} \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\pm}$. Further, the area element $\tilde{\varepsilon}_{ab}$ induced on $\mathcal{N}^{\pm}$ from both $\mathscr{I}$ and $\mathscr{C}$ has rescaling weight $r = 2$, and the function $\boldsymbol{F}^{\pm}$ representing the supertranslations has rescaling weight $r = 1$

$$\tilde{\varepsilon}_{ab} \mapsto (\boldsymbol{\sigma}^{\pm})^2 \tilde{\varepsilon}_{ab}\,, \quad \boldsymbol{F}^{\pm} \mapsto \boldsymbol{\sigma}^{\pm}\boldsymbol{F}^{\pm} \tag{4.28}$$

Thus, the supermomenta on $\mathcal{N}^{\pm}$ induced from both $\mathscr{I}$ and $\mathscr{C}$, and our matching result are independent of the choice of rescaling function and conformal factor.

Remark 4.4 (Lorentz invariance in Ti^0). Consider the action of a Lorentz transformation $\Lambda \in \mathrm{SO}(1,3)$ at i^0. Under the action of this transformation on Ti^0, a given unit-spatial direction $\vec{\eta} \in \mathscr{H}$ is mapped to another unit-spatial direction $\Lambda \circ \vec{\eta} \in \mathscr{H}$. Thus, Lorentz transformations act as isometries on the space of spatial directions $\mathscr{H}$.[10] Note, that the choice of rescaling function used to construct $\mathscr{C}$ need not be Lorentz-invariant, i.e., $\Lambda \circ \Sigma \neq \Sigma$. However, any null or (rescaled) spatial direction $\vec{N}$ is mapped to another direction $(\Lambda \circ \Sigma)(\Lambda \circ \vec{N})$ which is also null or spatial respectively. Thus, the space $\mathscr{C}$ constructed with some choice of rescaling function Σ is mapped to another space $\mathscr{C}'$ constructed using the rescaling function $\Sigma' = (\Lambda \circ \Sigma)$. Since we consider $\mathscr{C}$ as an abstract manifold with a conformal-class (under a change of rescaling function) of metrics $\tilde{\boldsymbol{h}}_{ab}$ as described in section 2.1, any Lorentz transformation Λ acts as a conformal isometry of $\mathscr{C}$ with the change in rescaling function given by $\sigma = \Sigma^{-1}(\Lambda \circ \Sigma)$. Further the action of any Lorentz transformation commutes with the reflection Υ. As we have already shown that our result is invariant under a change of the rescaling function, it follows that our matching result is also invariant under Lorentz transformations.

Remark 4.5 (Invariance under logarithmic translations). Consider the logarithmic translation ambiguity in the $C^{>1}$-structure at i^0. As described in Remark B.1 let x^i and $\tilde{x}^i$ be two *inequivalent* $C^{>1}$-charts at i^0, related by the coordinate transformation eq. (B.13) and the unphysical metrics by eq. (B.14). Since both $\partial x^i / \partial \tilde{x}^j$ and the relative conformal factor $\tilde{\omega}$ are C^0 at i^0, the space of spatial directions $\mathscr{H}$ is unaffected by logarithmic translations.

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[10]For any infinitesimal Lorentz transformation represented by a *direction-independent* antisymmetric tensor $\boldsymbol{\Lambda}_{ab}$ at i^0, the corresponding *direction-dependent* Killing field on $\mathscr{H}$ is given by $\boldsymbol{X}^a(\vec{\eta}) = \varepsilon^{abc}(\vec{\eta})\boldsymbol{\Lambda}_{bc}$ [12, 14].

Similarly, it can be verified that any choice of rescaling function Σ satisfies the properties Definition 2.2 in the $\tilde{x}^i$-chart if it does so in the x^i-chart. Thus, the space of null and spatial directions $\mathscr{C}$ is also unaffected by the logarithmic translations. Further, since $\boldsymbol{E}_{ab}$ is invariant under the logarithmic translations [35], it follows that the matching of super-translations and supermomenta is independent of the logarithmic translation ambiguity in the Ashtekar-Hansen structure. Note that since for null-regular spacetimes $\boldsymbol{E}_{ab}$ is even under the reflection map Υ (see appendix C), we can follow the analysis of [35] to eliminate the logarithmic translation ambiguity by demanding that the potential $\boldsymbol{E}$ (eq. (C.1)) is also reflection-even (for Minkowski spacetime this uniquely picks out the choice $\boldsymbol{E} = 0$). However this additional parity condition on the potential $\boldsymbol{E}$ is not required for our result.

$$* * *$$

With the diagonal symmetry algebra $\mathfrak{s}^\times$ we can now analyse the conservation of flux of BMS-supermomenta between $\mathscr{I}^-$ and $\mathscr{I}^+$. In any choice of the rescaling function Σ, consider any totally fluxless supertranslation $[\boldsymbol{F}] \in \mathfrak{s}^\times$, and let $f^\pm \in \mathfrak{s}^\pm$ be the unique BMS-supertranslations on $\mathscr{I}^\pm$ determined by the boundary values on $\mathscr{N}^\pm$ of any representative $\boldsymbol{F} \in [\boldsymbol{F}]$. Let $S^\pm$ be some (finite) cross-sections of $\mathscr{I}^\pm$, respectively, and let $\mathcal{F}^\pm$ denote the BMS-supermomentum flux between i^0 and $S^\pm$ corresponding to $f^\pm$. Note that in our convention both $\mathcal{F}^\pm$ are *incoming* fluxes into the physical spacetime.

From the preceding analysis we know that for any totally fluxless supertranslation $[\boldsymbol{F}] \in \mathfrak{s}^\times$ the corresponding BMS-supermomenta for $f^\pm$ at i^0 from both $\mathscr{I}^\pm$ match (eq. (4.23)), and the fluxes $\mathcal{F}^\pm$ are finite (Remark 4.1). This immediately gives us the conservation law for the BMS-supermomenta defined on the cross-sections $S^\pm$ of $\mathscr{I}^\pm$

$$\mathcal{Q}[f^+; S^+] - \mathcal{Q}[f^-; S^-] = \mathcal{F}^+[f^+] + \mathcal{F}^-[f^-] \tag{4.29}$$

Further, if $\mathscr{I}^\pm$ each have future/past boundaries at timelike infinities $i^\pm$, respectively, and the spacetime satisfies appropriate conditions at $i^\pm$ (see Remark 4.6) so that the BMS-supermomenta vanish as $S^\pm \to i^\pm$, then we have the *global* conservation law

$$\mathcal{F}^+[f^+; \mathscr{I}^+] + \mathcal{F}^-[f^-; \mathscr{I}^-] = 0 \tag{4.30}$$

This implies that the total *incoming* flux on $\mathscr{I}^-$ equals the total *outgoing* flux at $\mathscr{I}^+$ and thus the flux is conserved in the scattering from $\mathscr{I}^-$ to $\mathscr{I}^+$.

Remark 4.6 (Timelike infinities $i^\pm$). To obtain the global conservation law eq. (4.30), one needs suitable falloff conditions on the gravitational fields at the timelike infinities $i^\pm$. However, unlike at i^0 the behaviour of the fields at $i^\pm$ is completely determined by the Einstein equations along with suitable initial data. Any falloff conditions one imposes at $i^\pm$ should allow for, at the very least, solutions with initial data which is compactly-supported (in some suitable gauge), or more generally, initial data which is asymptotically-flat at i^0 in the sense of Definition 2.1 on a Cauchy surface. If one assumes that the spacetime becomes asymptotically-flat at $i^\pm$, analogous to the conditions at i^0 [44, 45], one can adapt our method above to derive global conservation laws as in eq. (4.30). In these cases, for

each $i^\pm$, the analogues of the blowups $\mathscr{H}$ and $\mathscr{C}$ are a spacelike-unit-hyperboloid and a 3-dimensional ball with a single boundary, respectively. However, in general relativity, even "low frequency" data on $\mathscr{I}^-$, in general, can lead to black hole formation [46]. In such spacetimes, the presence of an event horizon implies that the metric g_{ab} can only be assumed to be $C^{>-1}$ at $i^\pm$, and a more careful analysis is needed at the timelike infinities [47]. Further, one would also have to take into account the fluxes associated to symmetries on the event horizon [48].

5   Discussion

We showed that the matching conditions for asymptotic supertranslations and supermomentum charges conjectured in [19] are satisfied by suitably regular asymptotically-flat spacetimes. The essential ingredients in our analysis are (1) the space $\mathscr{C}$ of both null and spatial directions which allows us to simultaneously consider the limits of (suitably rescaled) Weyl tensor fields in both null and spatial directions at i^0. (2) the reflection conformal isometry Υ of $\mathscr{C}$ (eq. (2.39)) which provides an identification of the null generators of $\mathscr{I}^-$ to those on $\mathscr{I}^+$. (3) the null-regularity conditions on the gravitational fields along null directions in Definition 4.1 which ensure that the BMS-supermomenta defined on $\mathscr{I}$ admit limits as one approaches i^0. (4) the choice of the totally fluxless subalgebra of Spi-supertranslations on $\mathscr{C}$, ensuring the physical criteria that there is no flux across spatial infinity in a scattering process. The Einstein equations on $\mathscr{C}$ near spatial infinity, then imply that the relevant parts of the asymptotic Weyl tensor from $\mathscr{I}^-$ to $\mathscr{I}^+$ antipodally match at i^0 and reduce the supertranslation algebra to the diagonal subalgebra $\mathfrak{s}^\times$. As a consequence we showed that the BMS-supermomenta associated to $\mathfrak{s}^\times$ on $\mathscr{I}^\pm$ match at i^0 and the corresponding fluxes are conserved.

The smoothness conditions in Definition 2.1 can be weakened to allow for certain logarithmic terms in the asymptotic behaviour of the unphysical metric: at $\mathscr{I}$, g_{ab} can be polyhomogenous in the sense of [49] while at i^0, g_{ab} can be C^{0^+} in the sense of [50]. These weaker conditions still allow for a sufficiently regular Weyl tensor so that the fields $\mathcal{R}_{ab}, \mathcal{S}_a, \mathcal{P}$ at $\mathscr{I}$ (eq. (3.1)) and $\boldsymbol{E}_{ab}$ on $\mathscr{H}$ (eq. (3.11)) are well-defined. The rest of our analysis can then be carried out in the same manner. The null-regularity conditions (Definition 4.1) imposed on the gravitational fields are crucial in our analysis — they ensure that the flux of charges radiated through $\mathscr{I}^\pm$ in the scattering process are finite (Remark 4.1). At spatial infinity these conditions discard the solutions for which the electric part of the asymptotic Weyl tensor $\boldsymbol{E}_{ab}$ is odd under the reflection Υ (see appendix C). The supermomenta for such solutions on the space of null directions $\mathcal{N}^\pm$ diverges and the flux of supermomenta radiated through null infinity will not be finite; thus such solutions are unphysical from the point of view of a scattering problem. It can be verified that our null-regularity conditions are automatically satisfied when the unphysical metric g_{ab} is $C^{>0}$ in both null and spatial directions at i^0 — the Kerr family of spacetimes falls into this class as shown by Herberthson [51], and also under the weaker assumption that $X^a \nabla_a$ is $C^{>-1}$ for any vector field X^a which is tangent to $\mathscr{I}$ and $C^{>-1}$ at i^0 along null directions, that is, one only requires that the derivatives tangent to $\mathscr{I}$ admit regular direction-dependent

limits to i^0 [28]. They also hold when the unphysical metric is only C^{0^+} at i^0 in the sense of [50].

While we have argued that the conditions in Definitions 2.1 and 4.1 are physically reasonable, an important question we have left unanswered is the existence of a "large enough" class of solutions of the Einstein equation which satisfy these conditions. To resolve this, one would have to show that there exist initial data — with some appropriate topology and suitable falloffs near i^0 — either on a spatial Cauchy surface or on $\mathscr{I}^-$, which evolve through the Einstein equation to a spacetime satisfying Definitions 2.1 and 4.1. It seems this issue is best analysed in the formalism of Friedrich [16] which also involves a cylindrical blowup of spatial infinity, similar to the space $\mathscr{C}$ used in our analysis. The analysis of [27] suggests that there do exist many solutions of the linearised Einstein equation on Minkowski spacetime satisfying Definition 4.1. However, in the conformal-completion used by Friedrich the unphysical metric does not have a limit to i^0 along spatial directions, even for Minkowski spacetime (see [27]), and the relationship of this formalism to that of Ashtekar and Hansen is not clear, but certainly merits further investigation.

We can also compare the falloffs implied by Definition 4.1 to those in the class of spacetimes constructed in the global nonlinear stability analysis of Minkowski spacetime; we use the Bondi-Sachs conformal frame and the Bondi-Sachs parameter u described in appendix E. In the Christodoulou-Klainerman (CK) class of spacetimes [29], one has $N_{ab}^{(\mathrm{BS})} = O(1/u^{3/2})$ and $\mathcal{P}^{(\mathrm{BS})}$ becomes spherically-symmetric near i^0 as fast as $1/u^{1/2}$. Thus, CK-spacetimes have finite Bondi mass at i^0 but all other the BMS-supermomenta vanish [30]. The conditions used by Bieri [52–54] allow $N_{ab}^{(\mathrm{BS})} = O(1/u^{1/2})$, while the slightly stronger conditions used by Bieri and Chruściel [55] give $N_{ab}^{(\mathrm{BS})} = O(1/u^{1/2+\epsilon})$ (which is also the falloff used in [25]). In the latter case, the Bondi mass has a limit to i^0 and coincides with the ADM mass, but due to the slow falloff of the News tensor we expect that general BMS-supermomenta do not admit a limit to i^0 in such spacetimes. Our assumed falloff on the News tensor $N_{ab}^{(\mathrm{BS})} = O(1/u^{1+\epsilon})$ (eq. (E.1)) lies between the cases of Christodoulou-Klainerman and Bieri-Chruściel, and allow for a finite yet non-vanishing BMS-supermomenta at i^0. We expect our conditions to be compatible with another intermediate case considered by Bieri [56] (denoted by (B2) in [57]).

The analogous analysis for the charges associated to the full BMS group is trickier. Neither the BMS group nor the Spi group contain a unique Lorentz subgroup or a unique Poincaré subgroup. This results in the well-known problem of supertranslation ambiguities in defining the charges for a Lorentz subalgebra, such as angular momentum, on null infinity and at spatial infinity. For stationary spacetimes the angular momentum can be defined at the cost of reducing the BMS algebra to the Poincaré algebra [58]. Similarly at spatial infinity, the angular momentum can be defined by imposing, either the Regge-Teitelboim parity conditions [59] in a $(3+1)$-formalism, or, stronger falloff conditions on the magnetic part of the Weyl tensor at i^0 in the Ashtekar-Hansen framework [12], again reducing the asymptotic symmetry algebra at i^0 to the Poincaré algebra. In this case, the analysis of [26] proves the matching conditions for angular momentum (and other Lorentz charges). We defer the detailed investigation of this problem in the general case to future work.

Acknowledgments

A Null infinity in arbitrary conformal choices

In this section we obtain the evolution equations of the Weyl tensor, and expressions for the News tensor and supertranslations on $\mathscr{I}$ in an arbitrary choice of conformal factor Ω which is not restricted to satisfy the Bondi condition i.e. $\nabla_a \nabla_b \Omega \neq 0$ on $\mathscr{I}$. The evolution of the Weyl tensor on $\mathscr{I}$ can be directly obtained using the GHP-formalism. For the rest, it will be convenient to proceed in the following manner: we first write the relevant expressions in the GHP-formalism in the Bondi-Sachs conformal frame i.e., where the conformal factor $\Omega_{\text{(BS)}}$ is chosen such that

$$\nabla_a^{\text{(BS)}} \nabla_b^{\text{(BS)}} \Omega_{\text{(BS)}}|_{\mathscr{I}} = 0\,, \quad q_{ab}^{\text{(BS)}} = s_{ab} \tag{A.1}$$

where s_{ab} is the unit-metric on $\mathbb{S}^2$. Then we use the conformally-covariant version of the GHP-derivative operators to generalise these expressions to an arbitrary conformal choice. Finally, we convert these expressions from the GHP notation to the tensor notation used in the main body of the paper.

We begin with summarising the GHP-formalism (for details see [60, 61]). To conform to our conventions for the signature and Reimann curvature of the metric we will use the sign conventions of [62].[11] We pick a null tetrad $(n^a, l^a, m^a, \overline{m}^a)$ normalised so that $n_a l^a = -m_a \overline{m}^a = -1$ and all other inner products vanish. The metric can then be written as $g_{ab} = -2n_{(a}l_{b)} + 2m_{(a}\overline{m}_{b)}$. Given a metric the freedom in the choice of null tetrad is given by the *GHP-transformations*

$$l^a \mapsto \lambda\overline{\lambda}l^a\,, \quad n^a \mapsto (\lambda\overline{\lambda})^{-1}n^a\,, \quad m^a \mapsto \lambda(\overline{\lambda})^{-1}m^a \tag{A.2}$$

for any complex scalar λ. A scalar field ξ, associated to the choice of null tetrad, has a *GHP-weight* of (p, q) if under the GHP-transformations eq. (A.2) ξ transforms as

$$\xi \mapsto \lambda^p \overline{\lambda}^q \xi \tag{A.3}$$

We will denote the GHP-weight of such fields as $\xi \overset{\circ}{=} (p, q)$.

---

[11]In the notation of [62], their l^a corresponds to our n^a and their k^a corresponds to our l^a.

We define the GHP spin coefficients by (following the conventions of [62], these differ by a sign from [60, 61])

$$-\kappa := m^a l^b \nabla_b l_a \circeq (3,1), \qquad\qquad -\rho := m^a \overline{m}^b \nabla_b l_a \circeq (1,1) \qquad\text{(A.4a)}$$

$$-\sigma := m^a m^b \nabla_b l_a \circeq (3,-1), \qquad\qquad -\tau := m^a n^b \nabla_b l_a \circeq (1,-1) \qquad\text{(A.4b)}$$

$$-\kappa' := \overline{m}^a n^b \nabla_b n_a \circeq (-3,-1), \qquad\qquad -\rho' := \overline{m}^a m^b \nabla_b n_a \circeq (-1,-1) \qquad\text{(A.4c)}$$

$$-\sigma' := \overline{m}^a \overline{m}^b \nabla_b n_a \circeq (-3,1), \qquad\qquad -\tau' := \overline{m}^a l^b \nabla_b n_a \circeq (-1,1) \qquad\text{(A.4d)}$$

$$-\epsilon := \tfrac{1}{2}(n^a l^b \nabla_b n_a - \overline{m}^a l^b \nabla_b m_a), \qquad -\beta := \tfrac{1}{2}(n^a m^b \nabla_b l_a - \overline{m}^a m^b \nabla_b m_a) \qquad\text{(A.4e)}$$

$$-\epsilon' := \tfrac{1}{2}(l^a n^b \nabla_b n_a - m^a n^b \nabla_b \overline{m}_a), \qquad -\beta' := \tfrac{1}{2}(l^a \overline{m}^b \nabla_b n_a - m^a \overline{m}^b \nabla_b \overline{m}_a) \qquad\text{(A.4f)}$$

The spin coefficients $\epsilon, \beta, \epsilon', \beta'$ are not GHP-weighted quantities (their transformations contain derivatives of λ appearing in eq. (A.2)), but instead define the *GHP derivative operators* $(\text{þ}, \text{þ}', \eth, \eth')$ as follows

$$
\begin{aligned}
\text{þ}\,\xi &:= (l^a \nabla_a - p\epsilon - q\overline{\epsilon})\xi & &\circeq (p+1, q+1) \\
\text{þ}'\,\xi &:= (n^a \nabla_a + p\epsilon' + q\overline{\epsilon}')\xi & &\circeq (p-1, q-1) \\
\eth\,\xi &:= (m^a \nabla_a - p\beta + q\overline{\beta}')\xi & &\circeq (p+1, q-1) \\
\eth'\,\xi &:= (\overline{m}^a \nabla_a + p\beta' - q\overline{\beta})\xi & &\circeq (p-1, q+1)
\end{aligned}
\qquad\text{(A.5)}
$$

where $\xi \circeq (p,q)$ is any GHP-weighted scalar.

We also define the Weyl scalars in the chosen tetrad by

$$
\begin{aligned}
\Psi_4 &:= C_{abcd}\overline{m}^a n^b \overline{m}^c n^d & &\circeq (-4,0) \\
\Psi_3 &:= C_{abcd} l^a n^b \overline{m}^c n^d & &\circeq (-2,0) \\
\Psi_2 &:= \tfrac{1}{2}C_{abcd}(l^a n^b l^c n^d - l^a n^b m^c \overline{m}^d) & & \\
       &= C_{abcd} l^a m^b \overline{m}^c n^d & &\circeq (0,0) \\
\Psi_1 &:= C_{abcd} l^a m^b l^c n^d & &\circeq (2,0) \\
\Psi_0 &:= C_{abcd} l^a m^b l^c m^d & &\circeq (4,0)
\end{aligned}
\qquad\text{(A.6)}
$$

The relations between the spin coefficients, Weyl and Ricci tensors in the GHP-formalism can be found in [61].

$$* * *$$

Now we adapt the choice of the null tetrad in the unphysical spacetime to $\mathscr{I}$. Since the conformal factor Ω is independent of the choice of a null tetrad we have the GHP-weights $\nabla^a \Omega \circeq (0,0)$. To use the GHP formalism consistently with the transformations eq. (A.2) we define on $\mathscr{I}$

$$A n^a = \nabla^a \Omega \quad \text{where } A \circeq (1,1) \qquad\text{(A.7)}$$

and in the end we shall set $A = 1$. As in section 2, we choose l^a to be the auxilliary null normal to some choice of foliation of $\mathscr{I}$. Then the remaining tetrads $(m^a, \overline{m}^a)$ are a *complex* orthonormal basis for the cross-sections of $\mathscr{I}$ of the chosen foliation, and we have

$$q_{ab} = 2m_{(a}\overline{m}_{b)}, \quad \varepsilon_{ab} = 2im_{[a}\overline{m}_{b]} \qquad\text{(A.8)}$$

Using $\nabla_a \nabla_b \Omega|_{\mathscr{I}} = \Phi g_{ab}$ (from eq. (2.6)) and eq. (A.7) in eq. (A.4) we get on $\mathscr{I}$[12]

$$
\begin{aligned}
\kappa' = \sigma' = \tau' &= 0\,, & \rho' &= -A^{-1}\Phi \\
\epsilon &= \tfrac{1}{2}(A^{-1}\Phi + \overline{m}^a l^b \nabla_b m_a)\,, & \beta &= \tfrac{1}{2}(m^a \nabla_a \ln A + \overline{m}^a m^b \nabla_b m_a) \\
\epsilon' &= \tfrac{1}{2}(-\pounds_n \ln A + A^{-1}\Phi + m^a n^b \nabla_b \overline{m}_a)\,, & \beta' &= \tfrac{1}{2}(-\overline{m}^a \nabla_a \ln A + m^a \overline{m}^b \nabla_b \overline{m}_a)
\end{aligned}
\tag{A.9}
$$

Further since l_a is normal to the cross-sections of $\mathscr{I}$ we have $\rho = \overline{\rho}$ (i.e., the twist of l_a vanishes), while κ, σ, τ are arbitrary. The spin coefficients τ and σ are related to the tensors τ_a and σ_{ab} defined in eqs. (2.10) and (2.11) by

$$
\begin{aligned}
\tau_a &= -\tau \overline{m}_a - \overline{\tau} m_a\,, & \tau &= -\tau_a m^a \\
\sigma_{ab} &= -\sigma \overline{m}_a \overline{m}_b - \overline{\sigma} m_a m_b\,, & \sigma &= -\sigma_{ab} m^a m^b
\end{aligned}
\tag{A.10}
$$

By the peeling theorem all the Weyl scalars Ψ_i ($i = 0, \ldots, 4$) defined in eq. (A.6) vanish at $\mathscr{I}$, and $\psi_i := \Omega^{-1}\Psi_i$ has a limit to $\mathscr{I}$ (see Theorem 9.6.41 [39]). These are related to the tensor fields defined in (eq. (3.1)) by

$$
\psi_4 = \mathcal{R}_{ab}\overline{m}^a \overline{m}^b \quad \psi_3 = \mathcal{S}_a \overline{m}^a \quad \psi_2 = \tfrac{1}{2}(\mathcal{P} + i\mathcal{P}^*) \quad \psi_1 = \mathcal{J}_a m^a \quad \psi_0 = \mathcal{I}_{ab} m^a m^b
\tag{A.11}
$$

Note that each ψ_i has the same GHP-weight as the corresponding Ψ_i given in eq. (A.6).

We first note some relations on $\mathscr{I}$ which are independent of the choice of the conformal factor Ω and can be directly obtained from the GHP equations in [61]. Eq. (2.4a) implies that the Weyl scalars ψ_i on the unphysical spacetime satisfy the same Bianchi identity at $\mathscr{I}$ as the Weyl scalars $\hat{\Psi}_i$ defined with the physical Weyl tensor with all the physical Ricci tensor terms set to vanish by the vacuum Einstein equations (see eqs. 9.10.1 and 9.10.2 [39]). Taking the GHP-prime of the Bianchi identities eqs. 2.33–2.36 [61], with their Ψ_i replaced by ψ_i and the Ricci tensor terms set to vanish, we have

$$
(\text{\th}' - 4\rho')\psi_3 = (\text{ð} - \tau)\psi_4
\tag{A.12a}
$$

$$
(\text{\th}' - 3\rho')\psi_2 = (\text{ð} - 2\tau)\psi_3 + \sigma\psi_4
\tag{A.12b}
$$

$$
(\text{\th}' - 2\rho')\psi_1 = (\text{ð} - 3\tau)\psi_2 + 2\sigma\psi_3
\tag{A.12c}
$$

$$
(\text{\th}' - \rho')\psi_0 = (\text{ð} - 4\tau)\psi_1 + 3\sigma\psi_2
\tag{A.12d}
$$

If the conformal factor satisfies the Bondi condition i.e. $\rho' = 0$, these reduce to eqs. 9.10.4–9.10.7 [39]. Using eq. (A.11), along with eqs. (A.5) and (A.9) and setting $A = 1$, eq. (A.12) is equivalent to eq. (3.3). Further, from eq. 2.25 [61] and eq. (A.9) we have

$$
-\text{\th}'\sigma = \Phi_{02} - \rho'\sigma - (\text{ð} - \tau)\tau
\tag{A.13}
$$

where, in our conventions, $\Phi_{02} := \tfrac{1}{2}S_{ab}m^a m^b$ [62].

The expressions which lead to eq. (3.6) can also be derived directly from the GHP-formalism following the computations described in section 9.8 [39] but keeping a general

---

[12]Note that eq. (A.9) also implies that $(\text{\th}' + \rho')A = \text{ð}A = 0$ which is eq. 9.8.26 [39].

conformal factor. However, it will be easier to use the already available form of these expressions in a Bondi-Sachs conformal frame and then generalise them to arbitary conformal frames. In a Bondi-Sachs frame the *complex News function N* is defined by (eq. 9.8.75 [39])

$$\overline{N} := -\,\flat'\,\sigma \qquad \text{(Bondi-Sachs)} \tag{A.14}$$

which is related to the Weyl scalars through (eqs. 9.8.82 and 9.8.83 [39])

$$A\psi_4 = \flat'\,N\,, \quad A\psi_3 = \eth\,N \qquad \text{(Bondi-Sachs)} \tag{A.15}$$

Our goal is to generalise eqs. (A.14) and (A.15) to arbitrary conformal choices. Under a change of conformal factor $\Omega \mapsto \omega\Omega$, recall the transformations of the metric and null tetrad on $\mathscr{I}$

$$g_{ab} \mapsto \omega^2 g_{ab}\,, \quad (n^a, l^a, m^a, \overline{m}^a) \mapsto \omega^{-1}(n^a, l^a, m^a, \overline{m}^a) \tag{A.16}$$

with the GHP-weights $\omega \overset{\circ}{=} (0,0)$. Under this transformation a scalar field ξ associated to the choice of null frame we have the additional conformal weight defined in eq. (2.1). We will denote the combined *conformal-GHP-weight* of such scalars as $\xi \overset{\circ}{=} (p,q;w)$.

Under a conformal transformation eq. (A.16), the GHP-derivatives of a conformally-weighted GHP-scalar ξ (eq. (A.5)) will, in general, pick up derivatives of ω. However, the spin coefficients ρ' and τ are not conformally-weighted since (see also eq. (2.15))

$$\rho' \mapsto \omega^{-1}(\rho' - \flat'\ln\omega)\,, \quad \tau \mapsto \omega^{-1}(\tau - \eth\ln\omega) \tag{A.17}$$

Thus, we can "correct" the conformal behaviour of the GHP-derivatives by adding suitable combinations of ρ' and τ to define the *conformal-GHP-derivatives*. For our purposes we only need the conformal-GHP-derivatives tangential to $\mathscr{I}$ which are given by (see eq. 5.6.36 [43], where we use $w_0 = w_1 = -\tfrac{1}{2}$ as in see eq. 5.6.26 (iii) [43] to be compatible with the conformal transformations of the null tetrad eq. (A.16))[13]

$$\left[\flat' + (w + \tfrac{1}{2}(p+q))\rho'\right]\xi \overset{\circ}{=} (p-1, q-1; w-1)$$
$$\left[\eth + (w - \tfrac{1}{2}(p-q))\tau\right]\xi \overset{\circ}{=} (p+1, q-1; w-1) \tag{A.18}$$
$$\left[\eth' + (w + \tfrac{1}{2}(p-q))\overline{\tau}\right]\xi \overset{\circ}{=} (p-1, q+1; w-1)$$

where $\xi \overset{\circ}{=} (p,q;w)$ is any conformally-weighted GHP-scalar. In a Bondi-Sachs frame $\rho' = \tau = 0$ and so the conformal-GHP-derivatives eq. (A.18) are equivalent to the usual GHP-derivatives. Thus, to generalise any expression in the Bondi-Sachs frame to arbitrary conformal choices, we can replace the usual GHP-derivatives in that expression with the conformal-GHP-derivatives taking into account the appropriate weights. Eq. (A.12) is already in a conformal-invariant form where each ψ_i has conformal weight $w = -3$, and the conformal-invariance of eq. (A.13) can be verified though the computation is tedious.

---

[13]We note that the spin coefficient τ' is, in fact, conformally-invariant on $\mathscr{I}$ contrary to its transformation given in eq. 5.6.27 (iii) [43]. This difference arises since under $\Omega \mapsto \omega\Omega$ the normal n_a transforms away from $\mathscr{I}$ as $n_a \mapsto \omega n_a + \Omega A \nabla_a \omega$. For this reason we have defined the conformal-GHP-derivative corresponding to $\eth'$ with a $\overline{\tau}$ instead of τ'. This corresponds to the conformal operators $\tfrac{1}{2}(\eth_\mathscr{C} + \overline{\eth}'_\mathscr{C})$ and $\tfrac{1}{2}(\overline{\eth}_\mathscr{C} + \eth'_\mathscr{C})$ introduced in eq. 5.6.36 [43].

With this setup we return to eqs. (A.14) and (A.15). The spin coefficient σ is conformal-GHP-weighted according to $\sigma \overset{\circ}{=} (3, -1; -1)$. Replacing the $\text{þ}'$ in eq. (A.14) by the corresponding conformal-GHP operator in eq. (A.18) the conformal and GHP-weights cancel. Thus eq. (A.14) holds in any conformal frame, giving[14]

$$\overline{N} = -\text{þ}'\,\sigma \overset{\circ}{=} (2, -2; -2) \tag{A.19}$$

The *conformally-invariant* News tensor N_{ab} is given by

$$N_{ab} = 2N m_a m_b + 2\overline{N}\,\overline{m}_a \overline{m}_b\,, \quad 2\overline{N} = N_{ab}m^a m^b \tag{A.20}$$

Using eqs. (A.10), (A.13), (A.19) and (A.20) we reproduce the definition of the News tensor in eq. (3.4) and the relation in eq. (3.5).

The conformal-GHP-weights of the News function are $N \overset{\circ}{=} (-2, 2; -2)$, and again replacing the $\text{þ}'$ and ð in eq. (A.15) by the corresponding operators eq. (A.18), we get

$$A\psi_4 = (\text{þ}' - 2\rho')N\,, \quad A\psi_3 = \text{ð}\,N \tag{A.21}$$

Rewriting the above in terms of tensors using eqs. (A.11) and (A.20) and setting $A = 1$ we get eq. (3.6).

We can similarly obtain expressions for the BMS-supertranslations on $\mathscr{I}$ in arbitrary conformal choices. Let the BMS-supertranslation vector field for $f \in \mathfrak{s}^{\pm}$ on $\mathscr{I}$ be $\xi^a = f\nabla^a\Omega = (fA)n^a$, where $fA \overset{\circ}{=} (1, 1; 1)$. In the Bondi-Sachs frame, we have

$$\begin{aligned} \text{þ}'(fA) &= 0 \quad \text{for } f \in \mathfrak{s}^{\pm} \\ \text{ð}^2(fA) &= 0 \quad \text{for } f \in \mathfrak{t}^{\pm} \end{aligned} \quad \text{(Bondi-Sachs)} \tag{A.22}$$

Carrying out the replacements of the GHP-derivatives as before, we have in arbitrary conformal frames

$$\begin{aligned} (\text{þ}' + 2\rho')(fA) &= 0 \quad \text{for } f \in \mathfrak{s}^{\pm} \\ (\text{ð} - \tau)(\text{ð} + \tau)(fA) &= 0 \quad \text{for } f \in \mathfrak{t}^{\pm} \end{aligned} \tag{A.23}$$

Using eqs. (A.5) and (A.9) and setting $A = 1$ we reproduce eqs. (3.7) and (3.8).

Remark A.1 (Universal structure and supertranslations at $\mathscr{I}$). Note that the above procedure does not give us the boundary condition $f|_{i^0} = 0$ on the BMS-supertranslations. This boundary condition can be obtained by transforming between the Bondi conformal frames to a general conformal choice (see eq. (E.2)), or directly from the universal structure on $\mathscr{I}$ induced by asymptotic-flatness as follows: the universal structure at $\mathscr{I}$ (that is, the common structure induced on $\mathscr{I}$ by *all* physical spacetimes satisfying Definition 2.1) is the pair (n^a, q_{ab}) where two pairs (n^a, q_{ab}) and $(\omega^{-1}n^a, \omega^2 q_{ab})$ are equivalent for any ω

---

[14]Note that eq. (A.19) differs from the definition of the News function in eq. 9.8.73 [39] by terms involving the spin coefficient τ, which vanishes in the Bondi-Sachs frames. In arbitrary conformal frames we choose eq. (A.19) as the definition of the News function, since the corresponding News tensor eq. (A.20) is conformally-invariant.

satisfying the conditions in Remark 2.1. The infinitesimal diffeomorphisms generated by vector fields ξ^a on $\mathscr{I}$ which preserve this universal structure satisfy

$$\mathcal{L}_\xi n^a = -\alpha_{(\xi)} n^a\,, \quad \mathcal{L}_\xi q_{ab} = 2\alpha_{(\xi)} q_{ab} \tag{A.24}$$

for some function $\alpha_{(\xi)}$ which depends on ξ^a, is smooth on $\mathscr{I}$, and $\alpha_{(\xi)}|_{i^0} = 0$ (since $\omega|_{i^0} = 1$). For BMS-supertranslations we are interested in vector fields of the form $\xi^a = f n^a$ on $\mathscr{I}$ and evaluating eq. (A.24) for such vector fields (using eq. (2.9)) we get

$$\alpha_{(\xi)} = \mathcal{L}_n f = \Phi f \tag{A.25}$$

This immediately tells us that $(\mathcal{L}_n - \Phi)f = 0$ on $\mathscr{I}$ and $f|_{i^0} = 0$ which is eq. (3.7).

B $\quad C^{>1}$ differential structure, direction-dependent tensors and space of directions at i^0

Consider a manifold M which is smooth everywhere except at a point $p \in M$ where it is C^1 — so that the tangent space Tp at p is well-defined. A function f in M is *direction-dependent* at p if the limit of f along any curve Γ, which is C^1 at p, exists and depends only on the tangent direction to Γ at p. We write this as $\lim_{\to p} f = \boldsymbol{f}(\vec{N})$ where $\vec{N}$ is the direction of the tangent to Γ at p. Note that we can consider the limit $\boldsymbol{f}(\vec{N})$ as a function in the tangent space Tp which is constant along the rays (represented by $\vec{N}$) from p.

Definition B.1 (Regular direction-dependent function [50])**.** Let x^i denote a C^1 coordinate chart with $x^i(p) = 0$. A direction-dependent function f is *regular direction-dependent* at p (with respect to the chosen chart x^i) if for all k

$$\lim_{\to p} \left(x^{i_1} \frac{\partial}{\partial x^{i_1}}\right) \cdots \left(x^{i_k} \frac{\partial}{\partial x^{i_k}}\right) f = \left[\left(x^{i_1} \frac{\partial}{\partial x^{i_1}}\right) \cdots \left(x^{i_k} \frac{\partial}{\partial x^{i_k}}\right) \boldsymbol{f}\right](\vec{N}) \tag{B.1}$$

where on the right-hand-side we consider $\boldsymbol{f}$ as a function in Tp as mentioned above.

Eq. (B.1) ensures that, in the limit, f is smooth in its dependence on the directions $\vec{N}$ — the additional factors of x^i arise from converting the derivatives with respect to the "rectangular" coordinates x^i to derivatives with respect to the different directions (in eq. (2.18), these factors are provided instead by $\Omega^{1/2}$ [12, 14, 18]).

In general, the notion of regular direction-dependent tensors in any two coordinate charts in the same C^1-structure are not equivalent [12, 50]. Thus, we restrict the differential structure of M at p to a $C^{>1}$-structure defined as follows.

Definition B.2 ($C^{>1}$ differential structure)**.** Consider any two C^1 coordinate charts x^i and y^i, in the same C^1-structure, containing the point $p \in M$ such that for all i, j, k the *transition functions*

$$\frac{\partial^2 y^i(x)}{\partial x^j \partial x^k}\,, \quad \frac{\partial^2 x^i(y)}{\partial y^j \partial y^k} \tag{B.2}$$

are regular direction-dependent at p in their respective coordinate charts. A collection of all coordinate charts related by eq. (B.2) defines a choice of $C^{>1}$-structure on M at p.

Given such a $C^{>1}$-structure at p, any function whose derivatives upto the $(k-1)^{\text{th}}$ order vanish, whose k^{th} derivative is direction-independent, and whose $(k+1)^{\text{th}}$ derivative is regular direction-dependent will be called $C^{>k}$. By a slight abuse of notation we denote regular direction-dependent functions by $C^{>-1}$. Similarly, any tensor field is $C^{>k}$ at p if all of its components in any coordinate chart in the chosen $C^{>1}$-structure are $C^{>k}$ functions at p.

$$* * *$$

Now we consider the $C^{>1}$-structure at the point $p = i^0$ representing spatial infinity as defined in Definition 2.1 and summarise the structure on the spaces of directions $\mathscr{H}$ and $\mathscr{C}$ (for details see appendix B [31]). Since the metric g_{ab} in the Ashtekar-Hansen structure is universal at i^0, it induces a metric $\boldsymbol{g}_{ab}$ in the tangent space Ti^0 which is isometric to the Minkowski metric. Thus, we can introduce $C^{>1}$-coordinates $x^i = (t, x, y, z)$ in Ti^0 so that

$$
\begin{aligned}
\boldsymbol{g}_{ab} &\equiv -dt^2 + dx^2 + dy^2 + dz^2 \\
&= -dt^2 + dr^2 + r^2 s_{AB} d\theta^A d\theta^B
\end{aligned}
\tag{B.3}
$$

where the "polar" coordinates (t, r, θ^A) with $\theta^A = (\theta, \phi)$ are defined in the usual way from (t, x, y, z), and s_{AB} is the unit sphere metric

$$
s_{AB} d\theta^A d\theta^B \equiv d\theta^2 + \sin^2 \theta d\phi^2
\tag{B.4}
$$

Let $\alpha := t/r$, so that the spatial directions in Ti^0 correspond to $-1 < \alpha < 1$ while $\alpha = \pm 1$ corresponds to null directions. The conformal factor can be chosen to be $\Omega = r^2 - t^2$ so that

$$
n^a := \nabla^a \Omega \equiv 2r \left(\alpha \partial_t + \partial_r \right)
\tag{B.5}
$$

while the $C^{>-1}$ vector field

$$
\boldsymbol{\eta}^a := \nabla^a \Omega^{1/2} \equiv \left(1 - \alpha^2 \right)^{-1/2} \left(\alpha \partial_t + \partial_r \right)
\tag{B.6}
$$

represents the unit spatial directions $\vec{\eta}$ in Ti^0. The space of spatial directions is the unit-hyperboloid $\mathscr{H}$ in Ti^0 given by the surface $r^2 - t^2 = 1$ with the induced metric[15]

$$
\boldsymbol{h}_{ab} \equiv -\frac{1}{(1-\alpha^2)^2} d\alpha^2 + \frac{1}{1-\alpha^2} s_{AB} d\theta^A d\theta^B
\tag{B.7}
$$

A convenient choice of the rescaling function is given by.

$$
\Sigma^{-1} = r
\tag{B.8}
$$

On $\mathscr{H}$ the function $\boldsymbol{\Sigma}$ induced by $\lim\limits_{\to i^0} \Omega^{1/2} \Sigma$ in spatial directions is

$$
\boldsymbol{\Sigma} = \left(1 - \alpha^2 \right)^{1/2}
\tag{B.9}
$$

_____

[15]The forms of the metric on $\mathscr{H}$ in eqs. (2.22) and (B.7) are related by the coordinate change $\alpha = \tanh \tau$.

The rescaled null and spatial directions $\vec{N}$ are given by

$$
\begin{aligned}
N^a = \tfrac{1}{2}\Sigma n^a &\equiv (\alpha\partial_t + \partial_r) \\
&\equiv (\pm\partial_t + \partial_r) \qquad \text{in null directions} \\
&= \boldsymbol{\Sigma}\boldsymbol{\eta}^a \qquad\qquad \text{in spatial directions}
\end{aligned}
\tag{B.10}
$$

The cylinder $\mathscr{C}$ is then given by the surface $\Sigma^{-1} = r = 1$ which is diffeomorphic to a conformal-completion of $\mathscr{H}$ (with conformal factor $\boldsymbol{\Sigma}$), and the boundaries $\mathscr{N}^{\pm} \cong \mathbb{S}^2$ correspond to $\alpha = \pm 1$ representing the space of null directions at i^0. With this choice the vector field $\boldsymbol{U}^a$ and metric $\tilde{\boldsymbol{q}}_{ab}$ on $\mathscr{N}^{\pm}$ are

$$
\boldsymbol{U}^a \equiv \alpha(1-\alpha^2)^{1/2}\partial_\alpha \,, \quad \tilde{\boldsymbol{q}}_{ab} \equiv s_{AB}d\theta^A d\theta^B
\tag{B.11}
$$

The reflection conformal isometry of $\mathscr{C}$, eq. (2.39), is the map

$$
\Upsilon : (\alpha, \theta^A) \mapsto (-\alpha, -\theta^A)
\tag{B.12}
$$

with $\varsigma = 1$ since our choice of $\boldsymbol{\Sigma}$ (eq. (B.9)) is invariant under this reflection.

Remark B.1 (Logarithmic translations). Given a physical spacetime, the choice of unphysical spacetime in Definition 2.1 is ambiguous upto a 4-parameter family of *logarithmic translations* at i^0 which simultaneously change the $C^{>1}$-structure and the conformal factor at i^0 [34–36]. Let $x^i = (t, x, y, z)$ be a $C^{>1}$-chart, at i^0 such that (M, g_{ab}) is an unphysical spacetime satisfying Definition 2.1 with conformal factor Ω. Let $\rho^2 := \eta_{ij}x^i x^j$ with η_{ij} being the Minkowski metric at i^0. Define another chart $\tilde{x}^i$ at i^0 by

$$
x^i = \tilde{x}^i(1 + 2\ln\tilde{\rho}\ \tilde{x}_j\lambda^j) - \tilde{\rho}^2\ln\tilde{\rho}\ \lambda^i
\tag{B.13}
$$

where $\tilde{\rho}$ is defined in the new $\tilde{x}^i$-chart similar to ρ, and λ^i are some constants. Consider the new unphysical manifold $\tilde{M}$ which has a $C^{>1}$-structure in the $\tilde{x}^i$-chart, and the new unphysical metric $\tilde{g}_{ab}$ and conformal factor $\tilde{\Omega}$ given by

$$
\tilde{\Omega} = \tilde{\omega}\Omega\,, \quad \tilde{g}_{ab} = \tilde{\omega}^2 g_{ab} \quad \text{with} \ \ \tilde{\omega} := 1 - 2\ln\rho\ x_i\lambda^i
\tag{B.14}
$$

Then, it can be verified that $(\tilde{M}, \tilde{g}_{ab})$ with conformal factor $\tilde{\Omega}$ also satisfies Definition 2.1. Note, that $\partial x^i/\partial\tilde{x}^j$ is C^0 at i^0 but $\partial^2 x^i/\partial\tilde{x}^j\partial\tilde{x}^k$ diverges logarithmically. Thus, M and $\tilde{M}$ are equivalent as C^1-manifolds but *not* as $C^{>1}$-manifolds at i^0 (see Definition B.2). Similarly, the relative conformal factor $\tilde{\omega}$ in eq. (B.14) does not satisfy the conditions in Remark 2.1 since $\tilde{\omega}$ is C^0 at i^0 but not $C^{>0}$ (in either coordinate chart). Since M and $\tilde{M}$ are equivalent C^1-manifolds, the tangent spaces at i^0 can be naturally identified and the space of directions $\mathscr{H}$ and $\mathscr{C}$ are unaffected by these logarithmic translations. Thus, the logarithmic translations can also be parametrised by a *direction-independent* vector $\boldsymbol{\lambda}^a$ in Ti^0 with components λ^i (in either coordinate chart).

C Solutions for $\boldsymbol{E_{ab}}$ on $\mathscr{C}$

In this section we examine solutions to the Einstein equation eq. (3.12) on the hyperboloid $\mathscr{H}$ and the cylinder $\mathscr{C}$ at i^0 using the construction in appendix B.

The asymptotic unphysical Ricci tensor provides a scalar potential $\boldsymbol{E}$ for $\boldsymbol{E_{ab}}$ as follows (see [12] for details). Since, the metric g_{ab} is $C^{>0}$, the tensor S_{ab} defined in eq. (2.3) is such that $\Omega^{1/2} S_{ab}$ is $C^{>-1}$ at i^0 along spatial directions. Let $\boldsymbol{S}_{ab}(\vec{\eta}) := \lim\limits_{\to i^0} \Omega^{1/2} S_{ab}$, then,

$$\boldsymbol{E}(\vec{\eta}) := \boldsymbol{S}_{ab}(\vec{\eta})\boldsymbol{\eta}^a\boldsymbol{\eta}^b \tag{C.1}$$

defines a function $\boldsymbol{E}$ intrinsic to $\mathscr{H}$. Eq. (2.4b) in the limit to i^0 along spatial directions, implies that $\boldsymbol{E}$ acts as a scalar potential for $\boldsymbol{E_{ab}}$ so that (see [12])

$$\boldsymbol{E_{ab}} = -\tfrac{1}{4}\left(\boldsymbol{D_a D_b E} + \boldsymbol{E h_{ab}}\right) \tag{C.2}$$

Then, the second equation in eq. (3.12) is automatically satisfied and the first equation in eq. (3.12) gives

$$\boldsymbol{D^a D_a E} + 3\boldsymbol{E} = 0 \tag{C.3}$$

We can solve eq. (C.3) using a decomposition $\boldsymbol{E} = \sum\limits_{\ell,m} E_{\ell,m}(\alpha) Y_{\ell,m}(\theta^A)$ in terms of the spherical harmonic functions $Y_{\ell,m}(\theta^A)$ so that each $E_{\ell,m}(\alpha)$ satisfies (with the metric eq. (B.7))

$$(1-\alpha^2)\frac{d^2}{d\alpha^2}E_{\ell,m} + \left[\ell(\ell+1) - \frac{3}{1-\alpha^2}\right]E_{\ell,m} = 0 \tag{C.4}$$

The solutions $E_{\ell,m}(\alpha)$ are spanned by[16]

$$(1-\alpha^2)^{1/2}P_\ell^2(\alpha), \quad (1-\alpha^2)^{1/2}Q_\ell^2(\alpha) \tag{C.5}$$

where $P_\ell^2(\alpha)$ and $Q_\ell^2(\alpha)$ are the *Legendre functions* [63]. Note, these miss out the solutions

$$E_{\ell=0,m=0} \propto (1+\alpha^2)(1-\alpha^2)^{-1/2}$$
$$E_{\ell=1,m=0,\pm1} \propto \alpha(3-\alpha^2)(1-\alpha^2)^{-1/2} \tag{C.6}$$

Under the time reflection isometry $\alpha \mapsto -\alpha$ on $\mathscr{H}$, the solutions spanned by $P_\ell^2(\alpha)$ and $Q_\ell^2(\alpha)$ have a parity of $(-1)^\ell$ and $(-1)^{\ell+1}$, respectively. Combined with the parity $(-1)^\ell$ of the spherical harmonics $Y_{\ell,m}$ under $\theta^A \mapsto -\theta^A$, we get two linearly independent solutions $\boldsymbol{E}^{(\mathrm{odd})}$ and $\boldsymbol{E}^{(\mathrm{even})}$ to eq. (C.3) which are odd and even, respectively, under the reflection isometry $\Upsilon : (\alpha, \theta^A) \mapsto (-\alpha, -\theta^A)$ of $\mathscr{H}$. Thus the solutions to eq. (C.3) have the following

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[16]We can also solve eq. (C.4) in terms of the *Gauß hypergeometric functions*, $(1-\alpha^2)^{-1/2} \times {}_2F_1\left(-\frac{\ell+2}{2}, \frac{\ell-1}{2}; \frac{1}{2}; \alpha^2\right)$ and $\alpha(1-\alpha^2)^{-1/2} \times {}_2F_1\left(-\frac{\ell+1}{2}, \frac{\ell}{2}; \frac{3}{2}; \alpha^2\right)$ which do not miss the solutions in eq. (C.6) but these obscure the parity transformation under Υ. These can be related to the solutions eq. (C.5) using the transformation formulae in section 3.2 [63] or section 14.3 [64].

behaviour in α

$$\boldsymbol{E}^{(\text{odd})} = \begin{cases} \text{constant} \times \alpha(1-\alpha^2)^{-\frac{1}{2}} & \text{for } \ell = 0 \\ E_1(\theta^A)(1-\alpha^2)^{-\frac{1}{2}} & \text{for } \ell = 1 \\ E_2(\theta^A)(1-\alpha^2)^{\frac{1}{2}}Q_\ell^2(\alpha) & \text{for } \ell \geq 2 \end{cases} \tag{C.7a}$$

$$\boldsymbol{E}^{(\text{even})} = \begin{cases} \text{constant} \times (1+\alpha^2)(1-\alpha^2)^{-\frac{1}{2}} & \text{for } \ell = 0 \\ E_3(\theta^A)\alpha(3-\alpha^2)(1-\alpha^2)^{-\frac{1}{2}} & \text{for } \ell = 1 \\ E_4(\theta^A)(1-\alpha^2)^{\frac{1}{2}}P_\ell^2(\alpha) & \text{for } \ell \geq 2 \end{cases} \tag{C.7b}$$

The solutions $\boldsymbol{E}^{(\text{odd})}$ for $\ell = 0, 1$ have $\boldsymbol{E}_{ab} = 0$, and are "pure-gauge" solutions generated by logarithmic translations (see eq. (C.11)). Computing the Spi-supermomenta eq. (3.18) where $\boldsymbol{f}$ is a Spi-translation (see eq. (C.12)), the solutions $\boldsymbol{E}^{(\text{even})}$ with $\ell = 0, 1$ represent the "Schwarzschild-part" of the solution which, in general, is boosted relative to the coordinates (t, x, y, z) at i^0.

Using the diffeomorphism between $\mathscr{H}$ and $\mathscr{C} \backslash \mathscr{N}^{\pm}$ we can treat $\boldsymbol{E}$, and $\boldsymbol{E}_{ab}$ as fields on $\mathscr{C}$. For null-regular spacetimes we need $\boldsymbol{\Sigma}^{-1}\boldsymbol{E}_{ab}\boldsymbol{U}^a\boldsymbol{U}^b$ to have a finite limit to $\mathscr{N}^{\pm}$, where $\alpha = \pm 1$. As noted above the reflection-odd solutions $\boldsymbol{E} = \boldsymbol{E}^{(\text{odd})}$ with $\ell = 0, 1$ do not contribute to $\boldsymbol{E}_{ab}$, and an explicit computation, using eqs. (B.7), (B.11) and (C.7), shows that for the reflection-even solutions $\boldsymbol{E} = \boldsymbol{E}^{(\text{even})}$

$$\lim_{\to \mathscr{N}^{\pm}} \boldsymbol{\Sigma}^{-1}\boldsymbol{E}_{ab}\boldsymbol{U}^a\boldsymbol{U}^b \text{ exists}, \quad \lim_{\to \mathscr{N}^{\pm}} \tilde{\boldsymbol{q}}_a{}^b\boldsymbol{E}_{bc}\boldsymbol{U}^c = \lim_{\to \mathscr{N}^{\pm}} \tilde{\boldsymbol{q}}_a{}^c\tilde{\boldsymbol{q}}_b{}^d\boldsymbol{E}_{cd} = 0 \tag{C.8}$$

while for the reflection-odd solutions $\boldsymbol{E} = \boldsymbol{E}^{(\text{odd})}$ with $\ell \geq 2$, these limits diverge. Thus for null-regular spacetimes $\boldsymbol{E}_{ab}$ is reflection-even, and we have

$$\Upsilon \circ (\boldsymbol{\Sigma}^{-1}\boldsymbol{E}_{ab}\boldsymbol{U}^a\boldsymbol{U}^b)|_{\mathscr{N}^-} = (\boldsymbol{\Sigma}^{-1}\boldsymbol{E}_{ab}\boldsymbol{U}^a\boldsymbol{U}^b)|_{\mathscr{N}^+} \tag{C.9}$$

in our choice of the rescaling function eqs. (B.8) and (B.9); for more general choices we get eq. (4.17).

Consider the logarithmic translations as described in Remark B.1 generated by a direction-independent vector $\boldsymbol{\lambda}^a$ at i^0. Any such vector can be written in the form (see for instance Remark 3.2)

$$\boldsymbol{\lambda}^a = \boldsymbol{\lambda}\boldsymbol{\eta}^a + \boldsymbol{D}^a\boldsymbol{\lambda}$$
$$\text{where } \boldsymbol{\lambda} = \boldsymbol{\lambda}^a\boldsymbol{\eta}_a \text{ and } \boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{\lambda} + \boldsymbol{h}_{ab}\boldsymbol{\lambda} = 0 \tag{C.10}$$

Then, under the logarithmic translation generated by $\boldsymbol{\lambda}^a$ we have [35]

$$\boldsymbol{E} \mapsto \boldsymbol{E} + 4\boldsymbol{\lambda}, \quad \boldsymbol{E}_{ab} \mapsto \boldsymbol{E}_{ab} \tag{C.11}$$

Thus, the "pure-gauge" solutions $\boldsymbol{E}^{(\text{odd})}$ with $\ell = 0, 1$ in eq. (C.7) are precisely the ones generated by a logarithmic translation.

Since the Spi-translations $\boldsymbol{f} \in \mathfrak{t}^0$ also satisfy $\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{f} + \boldsymbol{f}\boldsymbol{h}_{ab} = 0$ (eq. (3.15)) and can be represented by direction-independent vectors at i^0 (Remark 3.2), the Spi-translations

$\boldsymbol{f}$ on $\mathscr{H}$ are also spanned by functions corresponding to $\boldsymbol{E}^{(\text{odd})}$ with $\ell = 0, 1$ in eq. (C.7), that is,[17]

$$\boldsymbol{f} = (1 - \alpha^2)^{-\frac{1}{2}} \left[F_{0,0} \alpha + \sum_m F_{1,m} Y_{\ell=1,m}(\theta^A) \right] \in \mathfrak{t}^0 \tag{C.12}$$

where $F_{0,0}, F_{1,m}$, are constants. Using eq. (3.16) it can be checked these correspond to the direction-independent vectors $\boldsymbol{v}^a$ at i^0 spanned by constant linear combinations of the vectors $(\partial_t, \partial_x, \partial_y, \partial_z)$ determined by the $C^{>1}$-coordinates at i^0. Note that $\boldsymbol{f}$ diverges at $\mathscr{N}^\pm$ (viewed as a function on $\mathscr{C}$) but the rescaled function $\boldsymbol{F} = \boldsymbol{\Sigma} \boldsymbol{f}$ (with $\boldsymbol{\Sigma}$ given by eq. (B.9)) is smooth at $\mathscr{N}^\pm$ and

$$\boldsymbol{F}|_{\mathscr{N}^\pm} = \pm F_{0,0} + \sum_m F_{1,m} Y_{\ell=1,m}(\theta^A) \tag{C.13}$$

Thus, the boundary values of $\boldsymbol{F}$ on $\mathscr{N}^\pm$ are precisely the $\ell = 0, 1$ functions on $\mathbb{S}^2$ and are reflection-odd i.e. $\Upsilon \circ \boldsymbol{F}|_{\mathscr{N}^-} = -\boldsymbol{F}|_{\mathscr{N}^+}$.

D Comparison to the Compère-Dehouck supermomenta on $\mathscr{H}$

Using the symplectic formalism for general relativity, Compère and Dehouck [32] derived an expression for the supermomenta at i^0, different from the one in eq. (3.18). This expression was used by Troessaert to resolve the matching of supertranslations and supermomenta for linearised perturbations off Minkowski spacetime [27]. In this section, we compare their results to the ones in the main body of the paper.

The spacetimes considered in [27, 32] are asymptotically-flat in the sense of Definition 2.1 and satisfy the additional "boundary condition" $\boldsymbol{h}^{ab} \boldsymbol{K}_{ab} = 0$, where $\boldsymbol{K}_{ab}$ is the tensor potential for the magnetic part of the asymptotic Weyl tensor on $\mathscr{H}$ [12]. This condition restricts the Spi-supertranslations $\mathfrak{s}^0$ to the subalgebra given by [32]

$$\mathfrak{s}^0_{(\text{CD})} = \left\{ \boldsymbol{f} \in \mathfrak{s}^0 \ : \ \boldsymbol{D}^a \boldsymbol{D}_a \boldsymbol{f} + 3\boldsymbol{f} = 0 \right\} \tag{D.1}$$

Following appendix C, we can obtain solutions for $\boldsymbol{f} \in \mathfrak{s}^0_{(\text{CD})}$ which again split into even and odd parity solutions $\boldsymbol{f}^{(\text{even})}$ and $\boldsymbol{f}^{(\text{odd})}$ as in eq. (C.7).

The supermomenta for any $\boldsymbol{f} \in \mathfrak{s}^0_{(\text{CD})}$ derived in [32] is then given by (in our notation)

$$\mathcal{Q}_{(\text{CD})}[\boldsymbol{f}; S] = \tfrac{1}{2} \int\limits_S \boldsymbol{\varepsilon}_2 \left(\boldsymbol{E} \boldsymbol{D}_a \boldsymbol{f} - \boldsymbol{D}_a \boldsymbol{E} \boldsymbol{f} \right) \boldsymbol{u}^a \tag{D.2}$$

where $\boldsymbol{E}$ is the potential for $\boldsymbol{E}_{ab}$ as defined in eq. (C.1). Using eqs. (C.3) and (D.1) it can be shown that the flux of the Compère-Dehouck supermomenta across any region $\Delta\mathscr{H}$ bounded by two cross-sections vanishes, that is, the Compère-Dehouck supermomenta are exactly conserved on $\mathscr{H}$.

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[17]While both the logarithmic translations and Spi-translations are spanned by the same functions on $\mathscr{H}$ and by direction-independent vectors at i^0, their transformations on the Ashtekar-Hansen structure are very different. The logarithmic translations change the $C^{>1}$-structure at i^0 (see Remark B.1) while Spi-translations preserve any chosen $C^{>1}$-structure. Similarly, the potential $\boldsymbol{E}$ transforms as in eq. (C.11) under logarithmic translations but is invariant under all Spi-supertranslations [12, 35].

The Compère-Dehouck supermomenta eq. (D.2) are related to eq. (3.18) as follows. Using eq. (C.3), for any $\boldsymbol{f} \in \mathfrak{s}^0_{(\mathrm{CD})}$, we have

$$\boldsymbol{E}_{ab}\boldsymbol{D}^b\boldsymbol{f} = \tfrac{1}{2}\left(\boldsymbol{E}\boldsymbol{D}_a\boldsymbol{f} - \boldsymbol{D}_a\boldsymbol{E}\boldsymbol{f}\right) - \tfrac{1}{4}\boldsymbol{D}^b\boldsymbol{E}(\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{f} + \boldsymbol{h}_{ab}\boldsymbol{f}) - \tfrac{1}{2}\boldsymbol{D}^b\left(\boldsymbol{D}_{[a}\boldsymbol{E}\boldsymbol{D}_{b]}\boldsymbol{f}\right) \quad \text{(D.3)}$$

Integrating over a cross-section S of $\mathscr{H}$, the last term vanishes as it is a boundary term on $S \cong \mathbb{S}^2$ and we get

$$\mathcal{Q}[\boldsymbol{f}; S] = \mathcal{Q}_{(\mathrm{CD})}[\boldsymbol{f}; S] - \tfrac{1}{4}\int_S \boldsymbol{\varepsilon}_2 \ \boldsymbol{D}^b\boldsymbol{E}(\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{f} + \boldsymbol{h}_{ab}\boldsymbol{f})\boldsymbol{u}^a \quad \text{(D.4)}$$

Now, we consider the Compère-Dehouck supermomenta on $\mathscr{C}$ and their limits to the space of null directions $\mathscr{N}^\pm$. As before, we will consider null regular spacetimes where, using the explicit solutions in eq. (C.7), it can be shown that

$$\lim_{\to \mathscr{N}^\pm} \boldsymbol{\Sigma}^{-1}\boldsymbol{E}_{ab}\boldsymbol{U}^a\boldsymbol{U}^b = \lim_{\to \mathscr{N}^\pm} -\frac{\partial^2}{\partial\alpha^2}(\boldsymbol{\Sigma}\boldsymbol{E}) \quad \text{(D.5)}$$

exists whenever $\boldsymbol{E}^{(\mathrm{odd})} = 0$ for $\ell \geq 2$, while the odd-solutions with $\ell = 0, 1$ do not contribute. In this case the only non-vanishing Compère-Dehouck supermomenta correspond to the odd-parity supertranslations $\boldsymbol{f} = \boldsymbol{f}^{(\mathrm{odd})} \in \mathfrak{s}^0_{(\mathrm{CD})}$ [27, 32]. Further, with $\boldsymbol{E} = \boldsymbol{E}^{(\mathrm{even})}$ and $\boldsymbol{f} = \boldsymbol{f}^{(\mathrm{odd})}$ it can be shown that

$$\lim_{S \to \mathscr{N}^\pm} \int_S \boldsymbol{\varepsilon}_2 \ \boldsymbol{D}^b\boldsymbol{E}(\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{f} + \boldsymbol{h}_{ab}\boldsymbol{f})\boldsymbol{u}^a = 0 \quad \text{(D.6)}$$

where, for a Spi-translation $\boldsymbol{f}$ this term identically vanishes (from eq. (3.15)), and for $\boldsymbol{f} = \boldsymbol{f}^{(\mathrm{odd})}$ with $\ell \geq 2$ the vanishing of the limit follows from the falloff of $\boldsymbol{E} = \boldsymbol{E}^{(\mathrm{even})}$ in eq. (C.7). The limit of the Compère-Dehouck supermomenta to $\mathscr{N}^\pm$ is then finite and given by [27]

$$\mathcal{Q}_{(\mathrm{CD})}[\boldsymbol{f}; \mathscr{N}^\pm] = -\int_{\mathscr{N}^\pm} \tilde{\boldsymbol{\varepsilon}}_2 \ \boldsymbol{F}\frac{\partial^2}{\partial\alpha^2}(\boldsymbol{\Sigma}\boldsymbol{E}) \quad \text{(D.7)}$$

where, as before, $\boldsymbol{F} = \boldsymbol{\Sigma}\boldsymbol{f}$. From eqs. (4.14) and (D.4)–(D.6) we see that, on $\mathscr{N}^\pm$, the Compère-Dehouck supermomenta coincide with the Spi-supermomenta used the main body of the paper, that is,

$$\mathcal{Q}_{(\mathrm{CD})}[\boldsymbol{f}; \mathscr{N}^\pm] = \mathcal{Q}[\boldsymbol{f}; \mathscr{N}^\pm] \quad \text{(D.8)}$$

Further, the odd-supertranslations represented by $\boldsymbol{F} = \boldsymbol{\Sigma}\boldsymbol{f}^{(\mathrm{odd})}$ are fluxless on $\mathscr{C}$ and satisfy eq. (4.21) and hence determine an equivalence class in the diagonal supertranslation algebra $\mathfrak{s}^\times$. Thus, the matching of the supertranslations and supermomenta can be, equivalently, derived using the Compère-Dehouck expression for the supermomenta. Thus, our result is a generalisation of the analysis by Troessaert [27] to full nonlinear general relativity.

E   Relation to some coordinate-based approaches

In this appendix we collect the relations between our covariant approach to some of the coordinate-based approaches.

Given the Ashtekar-Hansen structure of Definition 2.1, consider a different choice of conformal factor $\Omega_{(\mathrm{BS})} = \varpi\Omega$ so that the Bondi condition $\nabla_a^{(\mathrm{BS})}\nabla_b^{(\mathrm{BS})}\Omega_{(\mathrm{BS})}|_{\mathscr{I}^+} = 0$ holds. We denote quantities computed in this choice of conformal factor by a $_{(\mathrm{BS})}$ for Bondi-Sachs. The Bondi-Sachs normal to $\mathscr{I}^+$ is then $n^a_{(\mathrm{BS})}|_{\mathscr{I}^+} = \varpi^{-1}n^a$. The Bondi condition implies that $\nabla_a^{(\mathrm{BS})}n^a_{(\mathrm{BS})}|_{\mathscr{I}^+} = 0$ which in terms of the conformal-completion with Ω gives $\pounds_n\ln\varpi|_{\mathscr{I}^+} = -\frac{1}{4}\nabla_a n^a$. From Definition 2.1 we have $\nabla_a n^a|_{i^0} = 8$, and thus $\varpi|_{\mathscr{I}^+} = O(1/r)$, as we approach i^0 along $\mathscr{I}^+$, where r is the $C^{>0}$ "radial" coordinate at i^0 from appendix B. The Bondi-Sachs parameter u on $\mathscr{I}^+$ is defined by $n^a_{(\mathrm{BS})}\nabla_a u = 1$ which gives $u = O(1/r)$ with $u \to -\infty$ being the limit to spatial infinity along $\mathscr{I}^+$. Note, however, that the unphysical metric in the Bondi-Sachs completion is $g_{ab}^{(\mathrm{BS})} = \varpi^2 g_{ab} = O(u^2)g_{ab}$ which diverges in any C^1-chart at i^0 even in Minkowski spacetime.

From appendix B, any choice of the rescaling function behaves as $\Sigma = O(1/r) = O(u)$ approaching i^0 along $\mathscr{I}^+$. Converting to the conformal factor $\Omega_{(\mathrm{BS})}$, the null-regularity conditions of Definition 4.1 and eqs. (4.6) and (4.10) imply that

$$\lim_{u\to-\infty} \mathcal{P}^{(\mathrm{BS})} \text{ is a smooth function on } \mathbb{S}^2$$

$$N_{ab}^{(\mathrm{BS})} = O(1/u^{1+\epsilon})\,, \quad \mathcal{S}_a^{(\mathrm{BS})} = O(1/u^{1+\epsilon})\,, \quad \mathcal{R}_{ab}^{(\mathrm{BS})} = O(1/u^{2+\epsilon}) \tag{E.1}$$

$$\lim_{u\to-\infty} \sigma_{ab}^{(\mathrm{BS})} \text{ is bounded}$$

for some small $\epsilon > 0$. In the Newman-Penrose notation (see appendix A) as $u \to -\infty$, $\mathrm{Re}\,\psi_2^{(\mathrm{BS})}$ has a limit as a smooth function on $\mathbb{S}^2$, $\psi_4^{(\mathrm{BS})}$ fallsoff as $O(1/u^{2+\epsilon})$, $\psi_3^{(\mathrm{BS})}$ and the News function N falloff as $O(1/u^{1+\epsilon})$, and the spin-coefficient $\sigma^{(\mathrm{BS})}$ is bounded.

Consider now the vector field on $\mathscr{I}^+$, $\xi^a = f_{(\mathrm{BS})}n^a_{(\mathrm{BS})} = fn^a$ where $f_{(\mathrm{BS})}$ is a smooth function on $\mathbb{S}^2$ representing the BMS-supertranslation in a Bondi-Sachs frame. From the above discussion we have

$$f = \varpi^{-1}f_{(\mathrm{BS})} = O(1/u) \implies \lim_{u\to-\infty} f = 0 \tag{E.2}$$

The rescaled function $F = \Sigma f$ behaves as $F = O(u^0)f_{(\mathrm{BS})}$ where $O(u^0)$ denotes some smooth function on $\mathbb{S}^2$. We can use the rescaling freedom in Σ (Remark 2.3) so that $\boldsymbol{F}^+ = \lim_{u\to-\infty} F = f_{(\mathrm{BS})}$. Note that the remaining conformal freedom in $f_{(\mathrm{BS})}$ (conformal weight $w = 1$) corresponds precisely to the rescaling freedom in $\boldsymbol{F}^+$ (rescaling weight $r = 1$). Thus, the BMS-supertranslations in terms of the limiting functions $\boldsymbol{F}^+$ given in eq. (3.24) are precisely the usual BMS-supertranslation functions in the Bondi frame.

Similarly along spatial directions, we can relate the coordinates used in eq. (B.3) at i^0 to the Beig-Schmidt radial coordinate [11] by $\rho_{(\mathrm{BS})} = O(1/(r^2-t^2))$ so that i^0 lies at $\rho_{(\mathrm{BS})} \to \infty$ (now we use $_{(\mathrm{BS})}$ for Beig-Schmidt). It was shown in [11] that one can choose coordinates $(\rho_{(\mathrm{BS})}, X^a)$ where X^a are some coordinates on the hyperboloids of $\rho_{(\mathrm{BS})} = $ constant, so that

the physical metric takes the form

$$\hat{g}_{ab} \equiv \left[1 + \frac{\sigma^{(1)}}{\rho_{\text{(BS)}}}\right]^2 d\rho_{\text{(BS)}}^2 + \rho_{\text{(BS)}}^2 \left[h_{ab}^{(0)} + \frac{h_{ab}^{(1)}}{\rho_{\text{(BS)}}} + o(1/\rho_{\text{(BS)}})\right] dX^a dX^b + o(1/\rho_{\text{(BS)}}) \quad \text{(E.3)}$$

The relation to the direction-dependent quantities on $\mathscr{H}$ is given by $\boldsymbol{h}_{ab} \equiv h_{ab}^{(0)}$, $\boldsymbol{E} \equiv 4\sigma^{(1)}$, while $h_{ab}^{(1)}$ is related to the tensor potential for the magnetic part of the asymptotic Weyl tensor on $\mathscr{H}$ (see eqs. 3.30 and 3.31 [11]).

In the Beig-Schmidt formalism Spi-supertranslations are given by the coordinate transformations (see eqs. 2.3 and 2.4 [27]; note that eq. 2.18 [11] seems to have a typographical error in the transformation of $\rho_{\text{(BS)}}$)

$$\begin{aligned}
\rho_{\text{(BS)}} &\mapsto \rho_{\text{(BS)}}\left[1 - \frac{1}{\rho_{\text{(BS)}}}f(X^a) + o(1/\rho_{\text{(BS)}})\right] \\
X^a &\mapsto X^a - \frac{1}{\rho_{\text{(BS)}}}h^{(0)ab}D_b f(X^a) + o(1/\rho_{\text{(BS)}})
\end{aligned} \quad \text{(E.4)}$$

In terms of the $C^{>1}$-coordinates at i^0 these correspond to the vector fields eq. (3.13) with $\boldsymbol{f} \equiv f(X^a)$.

In Minkowski spacetime, the explicit transformations between the Bondi-Sachs and Beig-Schmidt coordinates reproduce the relation eq. (3.25) between the BMS-supertranslations and the Spi-supertranslations. For suitably regular linearised perturbations on Minkowski spacetime, these coordinate transformations also reproduce the relation eq. (4.4) between the perturbed Weyl tensor components at spatial infinity. A similar approach was used in [27] to prove the matching of supermomenta in the linearised theory on Minkowski spacetime using coordinates adapted to the formalism of Friedrich [16]. Using the coordinate transformations detailed in [51] one can also consider linearised perturbations around a Kerr background though the computations are extremely tedious. Nevertheless such explicit coordinate transformations are not available in general spacetimes, and our covariant approach is more suited to the general matching problem.

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