# Classical torus conformal block, $\mathcal{N} = 2^*$ twisted superpotential and the accessory parameter of Lamé equation

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ABSTRACT: In this work the correspondence between the semiclassical limit of the DOZZ quantum Liouville theory on the torus and the Nekrasov-Shatashvili limit of the $\mathcal{N} = 2^*$ ($\Omega$-deformed) U(2) super-Yang-Mills theory is used to propose new formulae for the accessory parameter of the Lamé equation. This quantity is in particular crucial for solving the problem of uniformization of the one-punctured torus. The computation of the accessory parameters for torus and sphere is an open longstanding problem which can however be solved if one succeeds to derive an expression for the so-called classical Liouville action. The method of calculation of the latter has been proposed some time ago by Zamolodchikov brothers. Studying the semiclassical limit of the four-point function of the quantum Liouville theory on the sphere they have derived the classical action for the Riemann sphere with four punctures. In the present work Zamolodchikovs idea is exploited in the case of the Liouville field theory on the torus. It is found that the Lamé accessory parameter is determined by the classical Liouville action on the one-punctured torus or more concretely by the torus classical block evaluated on the saddle point intermediate classical weight. Secondly, as an implication of the aforementioned correspondence it is obtained that the torus accessory parameter is related to the sum of all rescaled column lengths of the so-called "critical" Young diagrams extremizing the instanton "free energy" for the $\mathcal{N} = 2^*$ gauge theory. Finally, it is pointed out that thanks to the known relation the sum over the "critical" column lengths can be expressed in terms of a contour integral in which the integrand is built out of certain special functions.

# Contents

# 1 Introduction

The name "Lamé equation" denotes in fact a class of related ordinary second-order differential equations in the complex or real domain which contain (explicitly or implicitly) certain elliptic functions [1]. One of the most suitable forms of the Lamé equation in practical applications is the so-called *Jacobian form*:

$$\frac{d^2\Psi}{du^2} - \left[\kappa m \mathrm{sn}^2(u|m) + \mathrm{A}\right]\Psi = 0. \tag{1.1}$$

Eq. (1.1) can be looked at as a one-dimensional Schrödinger equation: $-\Psi''(u) + V(u)\Psi(u) = \mathcal{E}\Psi(u)$ with a *doubly periodic* potential $V(u) = \kappa m \mathrm{sn}^2(u|m)$ and the energy

eigenvalue $\mathcal{E} = -\mathrm{A}$. The potential is parameterized by the elliptic modular parameter[1] $m$ of the Jacobi sn-function $\mathrm{sn}(u|m)$ and the constant $\kappa$. In order to classify the solutions it is convenient to write $\kappa = \ell(\ell+1)$. In particular, in the real domain if $\ell$ is a nonnegative integer the energy spectrum consists of bands. There are $2\ell+1$ eigenfunctions called *Lamé polynomials*[2] associated with the boundaries of the energy gaps. For these solutions the values of $\mathcal{E}$ are the solutions of a certain algebraic equation [1, 2].[3]

Another important representation of the Lamé equation contains the Weierstrass $\wp$-function:

$$\frac{d^2\Psi}{dz^2} - \left[\kappa\wp(z) + \mathrm{B}\right]\Psi = 0. \tag{1.2}$$

Eq. (1.2) is known as the *Weierstrassian form* of the Lamé equation and can be achieved from eq. (1.1) by an appropriate change of the independent variable (see appendix A). The *accessory parameters* A and B appearing in eqs. (1.1) and (1.2) are related to each other in the following way[4]

$$\mathrm{B} = \mathrm{A}(e_1 - e_3) - \kappa\, e_3 \quad \Leftrightarrow \quad \mathrm{A} = \frac{\mathrm{B}}{e_1 - e_3} - \frac{1}{3}\kappa(m+1). \tag{1.3}$$

The substitution $\eta = \wp(z)$ converts eq. (1.2) to the third version of the Lamé equation, commonly encountered in the literature, the so-called *algebraic form*:

$$\frac{d^2\Psi}{d\eta^2} + \frac{1}{2}\left[\frac{1}{\eta - e_1} + \frac{1}{\eta - e_2} + \frac{1}{\eta - e_3}\right]\frac{d\Psi}{d\eta} - \left[\frac{\kappa\,\eta + \mathrm{B}}{4(\eta - e_1)(\eta - e_2)(\eta - e_3)}\right]\Psi = 0. \tag{1.4}$$

Eq. (1.4) "lives" on the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Its (regular) singular points are located at $\eta = e_1, e_2, e_3, \infty$. The algebraic form of the Lamé equation is the most appropriate form for further generalizations, cf. [2].

Historically, the Lamé equation has first been obtained (by Lamé) by applying the method of separation of variables to the Laplace equation in ellipsoidal coordinates [1]. More recently it has been noticed that the Lamé equation arises in various physical contexts. First, the Lamé potential can be considered as a good candidate for a realistic model of a one-dimensional crystal [13]. Other areas, where the Lamé equation is applicable, are superconductivity [14], certain version of the Ginzburg-Landau theory [15], and the cosmological models [16–20].[5] In mathematical physics applications the Lamé equation occurs in the so-called Lie-algebraic approaches to the Schrödinger equation [22].

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[1]We will use two notations of Jacobi elliptic functions, i.e. with $m$: $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, $\mathrm{dn}(u|m)$ and an alternative notation: $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, $\mathrm{dn}(u,k)$ which uses a parameter $k = \sqrt{m}$. For definition and properties of the Jacobi elliptic functions see appendix A.

[2]These are homogeneous polynomials of degree $\ell$ in the elliptic functions: sn, cn, dn.

[3]Note, that for $\ell \in \mathbb{Z}^+$ the spectra of the Lamé and related finite-gap periodic systems are characterized by a hidden bosonized nonlinear $\mathcal{N} = 2$ supersymmetry, cf. [3–9]. It seems to be an interesting task to study whether this observation has something to do with the so-called *Bethe/gauge* correspondence [10–12].

[4]Recall, that the Weierstrass $\wp$-function is doubly periodic on the complex plane with periods $2\omega_1$, $2\omega_2$ and points $e_1, e_2, e_3$ are images $e_k = \wp(\omega_k)$ of the points $\omega_1$, $\omega_2$ and $\omega_3 = -\omega_1 - \omega_2$.

[5]For instance, in the theory of preheating the Lamé equation has been recognized to determine quantum fluctuations of the inflaton field [17–20]. Let us note, that in several listed above applications the Lamé equation plays the same role. Indeed, regardless of the specific physical contexts one can observe that the Lamé equation serves as the *stability equation* or the *equation of small fluctuations* around classical configurations associated with a certain class of basic potentials, cf. [21].

In pure mathematics the Lamé equation arises in the uniformization theory of tori [23]. More concretely, let $\mathfrak{T}$ denotes the Teichmüller space for the one-punctured torus. It is well known that $\mathfrak{T}$ has at least two distinct models. The first one is the upper half plane $\mathbb{U} = \{\tau \in \mathbb{C} \,|\, \text{Im}\,\tau > 0\}$. The second model for $\mathfrak{T}$ is a subset of $\mathbb{R}^3$, namely $\mathbb{F} = \{(x,y,z) \in \mathbb{R}^3 \,|\, x,y,z \text{ are positive and } x^2 + y^2 + z^2 = xyz\}$. The question arises what is a relationship between $\mathbb{U}$ and $\mathbb{F}$. It turns out that the mapping $\varphi : \mathbb{U} \to \mathbb{F}$ is determined by the monodromy of the linearly independent solutions $(\Psi_1, \Psi_2)$ of the equation:

$$\frac{d^2\Psi}{dz^2} + \frac{1}{4}\left[\,\wp(z|\mathcal{L}) + \text{C}(\mathcal{L})\,\right]\Psi = 0, \tag{1.5}$$

where $z \in \mathbb{C} - \mathcal{L}$ and $\mathcal{L}$ is a period lattice. However, an explicit construction of $\varphi$ is difficult and still an open problem since the accessory parameter C in the above equation is an undetermined constant, cf. [23].

The Weierstrassian-form Lamé equation appears also in a two-dimensional conformal field theory (2$d$ CFT) as a classical limit of the null vector decoupling equation satisfied by a torus two-point correlation function with a degenerate field.[6] Moreover, one can observe that the accessory parameter B in this equation is expressed in terms of the so-called *classical Liouville action* on the one-punctured torus. The latter quantity can be computed by means of 2$d$ CFT technics and recently discovered dualities, in particular applying the correspondence between the classical limit of the quantum DOZZ Liouville theory on the torus and the Nekrasov-Shatasvili limit of the $\mathcal{N} = 2^*$ U(2) super-Yang-Mills theory.

The aim of the present work is to find an analytical expression of the Liouville classical action on the one-punctured torus employing aforementioned technology and apply it to compute the Lamé accessory parameter B ($\Leftrightarrow$ A and/or C). The main motivation for this line of research is the above mentioned monodromy problem for the Lamé equation. Its solution is crucial not only for finding the correspondence between models of the Teichmüller space for punctured torus but also for constructing a solution of the Liouville equation on such surface, cf. [29–32].

The organization of the paper is as follows. In section 2 we briefly review interrelationships between Liouville theory and the problem of computation of the accessory parameters of the Fuchsian uniformization of the punctured Riemann sphere. The principal purpose of this section is to recall the concept of the classical Liouville action and the *classical conformal block* [33]. For a long time the motivations to study classical blocks were mainly confined to applications in pure mathematics, in particular to the celebrated uniformization problem, which roughly speaking is related to the construction of conformal mappings between Riemann surfaces (RS) admitting a simply connected universal covering and the three existing simply connected RS, the sphere, the complex plane and the upper half plane. The uniformization problem is well illustrated by the example of the uniformization of the Riemann sphere with $n$ punctures. Its uniformization may be associated to a Fuchsian

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[6]Surprisingly, it seems to be possible to obtain the Lamé system (Hamiltonian and eigenfunctions) from a certain matrix model. Indeed, as has been observed in [24], in a suitable limit the so-called loop equation of certain generalized matrix model yields the equation closely related to the KZB equation [25–27] (see also [28]). The latter is known to reduces to Lamé equation in some cases.

equation whose form is known up to some constants that are called accessory parameters. Their computation is an open longstanding problem, which can however be solved if we succeed to derive an analytical expression of the classical block obtained by performing the classical limit of the $n$-point correlation function of the quantum Liouville field theory. The importance of the classical blocks is not only limited to the uniformization theorem, but gives also information about the solution of the Liouville equation on surfaces with punctures. Recently, an interesting mathematical application of classical blocks emerged in the context of Painlevé VI equation [34]. Due to the recent discoveries the classical blocks are also relevant for physics, since they are related to integrable models and to the instantonic sector of certain $\mathcal{N} = 2$ supersymmetric gauge field theories [35–38]. Moreover, lately classical conformal blocks have been of use to studies of holographic principle and AdS/CFT correspondence [39].

In section 3 we exploit the idea of brothers Zamolodchikov (see [33]) in order to propose the form of the Liouville classical action $S_{\mathrm{L}}^{\mathrm{torus}}$ on the one-punctured torus. Our conjecture is that ($i$) $S_{\mathrm{L}}^{\mathrm{torus}}$ decomposes into a sum of the three-point Liouville action on the sphere and the torus classical block; ($ii$) factorization holds on the saddle point intermediate classical conformal weight. Next, we consider the classical limit of the null vector decoupling equation satisfied by the torus two-point function with a degenerate field and find an expression for the Lamé accessory parameter. As has been already mentioned the latter is determined by the (one-punctured) torus classical action.

In section 4 we employ the AGT correspondence and express the toroidal classical block and then the eigenvalue/accessory parameter of the Lamé equation in terms of the so-called effective twisted superpotential of the $\mathcal{N} = 2^*$ U(2) supersymmetric gauge theory. In order to compute the latter quantity, i.e. the twisted superpotential, we use a straight-forward generalization of the calculation performed by Poghossian in [40]. We check that on the "classical level" the eigenvalue computed by means of the WKB method exactly coincides with that obtained from the classical torus block. Finally, using a relation between the instantonic sector of the twisted superpotential of the $\mathcal{N} = 2^*$ U(2) SYM theory and the classical toroidal block we find that the Lamé accessory parameter is related to the sum of column lengths of the so-called "critical" Young diagrams. It is shown that such sum can be rewritten in terms of a contour integral in which the integrand is built out of certain special functions.

In section 5 we present our conclusions. The problems that are still open and the possible extensions of the present work are discussed.

## 2 Liouville theory and accessory parameters

### 2.1 Quantum and classical conformal blocks

Let $C_{g,n}$ denotes the Riemann surface with genus $g$ and $n$ punctures. The basic objects of any two-dimensional conformal field theory living on $C_g$ [41, 42] are the $n$-point correlation functions of primary vertex operators defined on $C_{g,n}$. Given a marking[7] $\sigma$ of the

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[7]A marking of the Riemann surface $C_{g,n}$ (for definition see [43]) is a pants decomposition of $C_{g,n}$ together with the corresponding trivalent graph.

Riemann surface $C_{g,n}$ any correlation function can be factorized according to the pattern given by a pants decomposition of $C_{g,n}$ and written as a sum (or an integral for theories with a continuous spectrum) which includes the terms consisting of holomorphic and anti-holomorphic conformal blocks times the three-point functions of the model for each pair of pants. The Virasoro conformal block $\mathcal{F}_{c,\alpha}^{(\sigma)}[\beta](\mathsf{Z})$ on $C_{g,n}$, where $\alpha \equiv (\alpha_1, \dots, \alpha_{3g-3+n})$, $\beta \equiv (\beta_1, \dots, \beta_n)$ depends on the cross ratios of the vertex operators locations denoted symbolically by $\mathsf{Z}$ and on the $3g - 3 + n$ intermediate conformal weights $\Delta_{\alpha_i} = \alpha_i(Q - \alpha_i)$. Moreover, it depends on the $n$ external conformal weights $\Delta_{\beta_a} = \beta_a(Q - \beta_a)$ and on the central charge $c$ which can be parameterized as follows $c = 1 + 6Q^2$ with $Q = b + b^{-1}$.

Conformal blocks are fully determined by the underlying conformal symmetry. These functions possess an interesting, although not yet completely understood analytic structure. In general, they can be expressed only as a formal power series and no closed formula is known for its coefficients. Let us write down two canonical examples which illustrate this fact.

Let $q = \mathrm{e}^{2\pi i \tau}$ be the elliptic variable on the torus with modular parameter $\tau$ then the conformal block on $C_{1,1}$ is given by the following $q$-series:

$$\mathcal{F}_{c,\Delta}^{\tilde{\Delta}}(q) = q^{\Delta - \frac{c}{24}} \left( 1 + \sum_{n=1}^{\infty} \mathcal{F}_{c,\Delta}^{\tilde{\Delta},n} q^n \right), \tag{2.1}$$

$$\mathcal{F}_{c,\Delta}^{\tilde{\Delta},n} = \sum_{n=|I|=|J|} \langle \nu_{\Delta,I}, V_{\tilde{\Delta}}(1)\nu_{\Delta,J} \rangle \left[ G_{c,\Delta} \right]^{IJ}. \tag{2.2}$$

Let $x$ be the modular parameter of the four-punctured sphere then the $s$-channel conformal block on $C_{0,4}$ is defined as the following $x$-expansion:

$$\mathcal{F}_{c,\Delta}\begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}(x) = x^{\Delta - \Delta_2 - \Delta_1} \left( 1 + \sum_{n=1}^{\infty} \mathcal{F}_{c,\Delta}^n \begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix} x^n \right), \tag{2.3}$$

$$\mathcal{F}_{c,\Delta}^n \begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix} = \sum_{n=|I|=|J|} \langle \nu_{\Delta_4}, V_{\Delta_3}(1)\nu_{\Delta,I} \rangle \left[ G_{c,\Delta} \right]^{IJ} \langle \nu_{\Delta,J}, V_{\Delta_2}(1)\nu_{\Delta_1} \rangle. \tag{2.4}$$

In the above equations $\left[ G_{c,\Delta} \right]^{IJ}$ is the inverse of the Gram matrix $\left[ G_{c,\Delta} \right]_{IJ} = \langle \nu_{\Delta,I}, \nu_{\Delta,J} \rangle$ of the standard symmetric bilinear form in the Verma module $\mathcal{V}_\Delta = \bigoplus_{n=0}^{\infty} \mathcal{V}_\Delta^n$,

$$\mathcal{V}_\Delta^n = \mathrm{Span}\Big\{ \nu_{\Delta,I}^n = L_{-I}\nu_\Delta = L_{-i_k} \dots L_{-i_2} L_{-i_1} \nu_\Delta :$$
$$I = (i_k \geq \dots \geq i_1 \geq 1) \text{ an ordered set of positive integers}$$
$$\text{of the length } |I| \equiv i_1 + \dots + i_k = n \Big\}.$$

The operator $V_\Delta$ in the matrix elements is the normalized primary chiral vertex operator acting between the Verma modules

$$\langle \nu_{\Delta_i}, V_{\Delta_j}(z)\nu_{\Delta_k} \rangle = z^{\Delta_i - \Delta_j - \Delta_k}.$$

In order to calculate the matrix elements in (2.2) and (2.4) it is enough to know the covariance properties of the primary chiral vertex operator with respect to the Virasoro algebra:

$$[L_n, V_\Delta(z)] = z^n \left( z\frac{d}{dz} + (n+1)\Delta \right) V_\Delta(z), \qquad n \in \mathbb{Z}.$$

As the dimension of $\mathcal{V}_\Delta^n$ grows rapidly with $n$, the calculations of conformal blocks coefficients by inverting the Gram matrices become very laborious for higher orders. A more efficient method based on recurrence relations for the coefficients can be used [44–48].

Among the issues concerning conformal blocks which are still not fully understood there is the problem of their *semiclassical limit*. This is the limit in which all parameters of the conformal blocks tend to infinity in such a way that their ratios are fixed. It is commonly believed that such limit exists and the conformal blocks behave in this limit exponentially with respect to $Z$. This last property can be heuristically justified in the case of conformal blocks on $C_{0,4}$ and $C_{1,1}$.

Indeed, the existence of the semiclassical limit of the Liouville four-point correlation function with the projection on one intermediate conformal family implies a semiclassical limit of the quantum conformal block with *heavy* weights $\Delta = b^{-2}\delta$, $\Delta_i = b^{-2}\delta_i$, with $\delta, \delta_i = \mathcal{O}(1)$ in the following form:

$$\mathcal{F}_{1+6Q^2,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1\end{bmatrix}(x) \overset{b \to 0}{\sim} \exp\left\{\frac{1}{b^2} f_\delta\begin{bmatrix}\delta_3 & \delta_2 \\ \delta_4 & \delta_1\end{bmatrix}(x)\right\}. \tag{2.5}$$

The function $f_\delta\begin{bmatrix}\delta_3 & \delta_2 \\ \delta_4 & \delta_1\end{bmatrix}(x)$ is called the classical conformal block [33] or with some abuse of terms, the "classical action" [44, 49]. The existence of the semiclassical limit (2.5) has been postulated first in [44, 49] where it has been pointed out that the classical block is related to a certain monodromy problem of a null vector decoupling equation in a similar way in which the classical Liouville action is related to the Fuchsian uniformization. This relation has been further used to derive the $\Delta \to \infty$ limit of the four-point conformal block and its expansion in powers of the so-called elliptic variable.

Analogously, the existence of the semiclassical limit of the projected Liouville torus one-point function implies that the semiclassical limit of the torus one-point block with heavy weights $\Delta = b^{-2}\delta$, $\tilde{\Delta} = b^{-2}\tilde{\delta}$ with $\delta, \tilde{\delta} = \mathcal{O}(1)$ has the form:

$$\mathcal{F}_{1+6Q^2,\Delta}^{\tilde{\Delta}}(q) \overset{b \to 0}{\sim} \exp\left\{\frac{1}{b^2} f_\delta^{\tilde{\delta}}(q)\right\}. \tag{2.6}$$

The function $f_\delta^{\tilde{\delta}}(q)$ we shall call the classical torus (or toroidal) conformal block.

It should be stressed once again that the exponential behavior (2.5) and/or (2.6) is a nontrivial statement concerning the quantum conformal blocks. Although there is no proof of this property, it seems to be well confirmed together with its consequences by sample numerical calculations [50] and recent discoveries.

The classical conformal blocks are in general again available only as power series with coefficients calculated from the semiclassical asymptotics and the power expansions of the quantum blocks. The question arises how to sum up these series. Surprisingly, one can find closed formulae for at least the four-point spherical [38] and the one-point toroidal classical blocks employing the AGT correspondence.

Indeed, a considerable progress in the theory of conformal blocks and their applications has been achieved recently. This is mainly due to the discovery of the Liouville/$\mathcal{N} = 2$ gauge theories correspondence by Alday, Gaiotto and Tachikawa in 2009 [51]. The AGT

conjecture states that the LFT correlators on the Riemann surface $C_{g,n}$ with genus $g$ and $n$ punctures can be identified with the partition functions of a class $T_{g,n}$ of four-dimensional $\mathcal{N} = 2$ supersymmetric SU(2) quiver gauge theories. A significant part of the AGT conjecture is an exact correspondence between the Virasoro blocks on $C_{g,n}$ and the instanton sectors of the Nekrasov partition functions of the gauge theories $T_{g,n}$. Very soon after its discovery, the AGT hypothesis was extended to the SU(N)-gauge theories/conformal Toda correspondence [52–54].[8]

Let us recall that originally the Nekrasov partition functions have been introduced to calculate the low energy effective $\mathcal{N} = 2$ SUSY gauge theories prepotentials [64, 65]. The Seiberg-Witten prepotentials [66, 67] determine the low energy effective dynamics of the four-dimensional $\mathcal{N} = 2$ super-Yang-Mills theories and can be recovered from the Nekrasov partition functions in the appropriate limit, i.e. when the so-called $\Omega$-background parameters: $\epsilon_1$, $\epsilon_2$ appearing in the Nekrasov functions tend to zero.

The Nekrasov functions lead also to an interesting application when one of the two $\Omega$-background parameters is non-zero. Such situation has been considered by Nekrasov and Shatashvili in [10]. They observed that in the limit when one of the $\Omega$-background parameters, say $\epsilon_2$, is zero and the second one, i.e. $\epsilon_1$, is kept finite (*Nekrasov-Shatashvili limit*) then one can extract from the Nekrasov functions in this limit the so-called *effective twisted superpotentials*. These quantities determine the low energy effective dynamics of the two-dimensional ($\Omega$-deformed) supersymmetric gauge theories. Twisted superpotentials play also a prominent role in another context, namely in the so-called *Bethe/gauge correspondence* [10–12] which maps supersymmetric vacua of the $\mathcal{N} = 2$ two-dimensional theories to Bethe states of quantum integrable systems (QIS's). A result of that duality is that twisted superpotentials are identified with *Yang's functionals* [68] which describe a spectrum of QIS's.

Looking at the AGT relations it is not difficult to realize that the Nekrasov-Shatashvili limit of the Nekrasov instanton partition functions corresponds to the classical limit of conformal blocks. Let us note that by combining the AGT duality and the Bethe/gauge correspondence it is possible to link classical blocks to Yang's functionals, cf. [35–37].

Using correspondence identifying classical blocks and twisted superpotentials one can find a closed formulae for at least spherical and toroidal classical blocks. As has been already mentioned, the relevant technical problem of this strategy consists in the summation of the series defining the classical block. This problem can be tackled on the gauge theory side by means of the saddle point method [40, 64, 65, 69].

## 2.2   Classical Liouville action

Let $C_{0,n}$ be the $n$-punctured Riemann sphere with complex coordinates chosen in such a way that $z = \infty$. Consider the Liouville equation

$$\partial_z \partial_{\bar{z}} \phi(z, \bar{z}) = \frac{\varrho}{2} \, \mathrm{e}^{\phi(z,\bar{z})} \tag{2.7}$$

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[8]Of course, there have been made attempts to prove the AGT conjecture and its generalizations soon after its discovery. Active studies of this duality have first led to proofs of the AGT relations in certain special cases [46, 55, 56]. For more recent and more general achievements in this field, see [57–63].

with one of the following asymptotic behaviors of the Liouville field $\phi(z, \bar{z})$ near the punctures:

1. case of *elliptic singularities*:

$$\phi(z, \bar{z}) = \begin{cases} -2\left(1 - \xi_j\right) \log|z - z_j| + O(1) & \text{as } z \to z_j, \quad j = 1, \ldots, n-1, \\ -2\left(1 + \xi_n\right) \log|z| + O(1) & \text{as } z \to \infty, \end{cases} \tag{2.8}$$

$$\xi_i \in \mathbb{R}_{>0} \quad \text{for all } i = 1, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} \xi_i < n - 2;$$

2. case of *parabolic singularities* ($\xi_i \to 0$):

$$\phi(z, \bar{z}) = \begin{cases} -2 \log|z - z_j| - 2 \log|\log|z - z_j|| + O(1) & \text{as } z \to z_j, \\ -2 \log|z| - 2 \log|\log|z|| + O(1) & \text{as } z \to \infty. \end{cases} \tag{2.9}$$

It is known that it exists a unique solution of eq. (2.7) if one of the conditions (2.8) [70–72] or (2.9) [73] is satisfied.

One can define the Liouville action $S_{\mathrm{L}}[\phi]$ on $C_{0,n}$. Because of the singular nature of the Liouville field at the punctures such action has to be properly regularized:

$$S_{\mathrm{L}}[\phi] = \frac{1}{4\pi} \lim_{\epsilon \to 0} S_{\mathrm{L}}^{\epsilon}[\phi], \tag{2.10}$$

$$S_{\mathrm{L}}^{\epsilon}[\phi] = \int_{X_\epsilon} d^2 z \left[ |\partial \phi|^2 + \varrho \, \mathrm{e}^{\phi} \right] + \sum_{j=1}^{n-1} \left(1 - \xi_j\right) \int_{|z-z_j|=\epsilon} |dz| \; \kappa_z \phi + \left(1 + \xi_n\right) \int_{|z|=\frac{1}{\epsilon}} |dz| \; \kappa_z \phi$$

$$- 2\pi \sum_{j=1}^{n-1} \left(1 - \xi_j\right)^2 \log \epsilon - 2\pi \left(1 + \xi_n\right)^2 \log \epsilon, \tag{2.11}$$

$X_\epsilon = \mathbb{C} \backslash \left\{ \left( \bigcup_{j=1}^{n} |z - z_j| < \epsilon \right) \cup \left( |z| > \frac{1}{\epsilon} \right) \right\}$. The prescription given in eqs. (2.10) and (2.11) is valid for parabolic singularities (corresponding to $\xi_j = 0$) as well.

It is well known mathematical fact that the critical value $S_{\mathrm{L}}^{\mathrm{cl}}[\phi]$ of the Liouville action functional $S_{\mathrm{L}}[\phi]$ on $C_{0,n}$ (the classical Liouville action) is the generating function for the accessory parameters $c_j$ of the Fuchsian uniformization of the punctured Riemann sphere, i.e.:

$$c_j = -\frac{\partial S_{\mathrm{L}}^{\mathrm{cl}}[\phi]}{\partial z_j}. \tag{2.12}$$

Primarily, this formula has been derived within the so-called *geometric path integral approach* to the quantum Liouville theory by analyzing the quasi-classical limit of the conformal Ward identity [74]. Then, for parabolic singularities formula (2.12) has been proved by Takhtajan and Zograf. The details can be found in [75]. In ref. [76] the extension of [75] to compact Riemann surfaces has been presented. For general elliptic singularities eq. (2.12) has been proved in [77] and non rigorously derived in [78]. It is also possible to construct the Liouville action functional satisfying (2.12) for the so-called hyperbolic singularities (holes) on the Riemann sphere, see [79].

On the other hand one can observe that in the case of the four-punctured sphere with singularities located at $z_4 = \infty, z_3 = 1, z_2 = x, z_1 = 0$ the Fuchsian differential equation

$$\partial_z^2 \Psi(z) + \left[ \frac{\delta_1}{z^2} + \frac{\delta_2}{(z-x)^2} + \frac{\delta_3}{(1-z)^2} + \frac{\delta_1 + \delta_2 + \delta_3 - \delta_4}{z(1-z)} + \frac{x(1-x)c_2(x)}{z(z-x)(1-z)} \right] \Psi(z) = 0$$

with an accessory parameter $c_2(x)$ given by the derivative w.r.t. $x$ of the four-point classical action can be obtained from the classical limit $b \to 0$ of certain null vector decoupling equation. Concretely, from the equation

$$\left[ \frac{\partial^2}{\partial z^2} - b^2 \left( \frac{1}{z} - \frac{1}{1-z} \right) \frac{\partial}{\partial z} \right] G(z,x) =$$
$$-b^2 \left[ \frac{\Delta_1}{z^2} + \frac{\Delta_2}{(z-x)^2} + \frac{\Delta_3}{(1-z)^2} + \frac{\Delta_1 + \Delta_2 + \Delta_3 + \Delta_{-\frac{b}{2}} - \Delta_4}{z(1-z)} + \frac{x(1-x)}{z(z-x)(1-z)} \frac{\partial}{\partial x} \right] G(z,x)$$

satisfied by the five-point function

$$G(z,x) \equiv \left\langle \mathsf{V}_4(\infty,\infty)\mathsf{V}_3(1,1)\mathsf{V}_{-\frac{b}{2}}(z,\bar{z})\mathsf{V}_2(x,\bar{x})\mathsf{V}_1(0,0) \right\rangle$$

with a degenerate field

$$\mathsf{V}_{\alpha = -\frac{b}{2}}, \qquad \Delta_{-\frac{b}{2}} = \Delta_{\alpha = -\frac{b}{2}} = \alpha(Q-\alpha) = -\frac{1}{2} - \frac{3}{4}b^2, \qquad Q = b + \frac{1}{b}$$

and four heavy primary operators $\mathsf{V}_{\Delta_i}$, $\Delta_i = b^{-2}\delta_i$, $\delta_i = \mathcal{O}(1)$.

Analogously, the Weierstrassian-form Lamé equation with the accessory parameter determined by the torus classical one-point action can be recovered from the null vector decoupling equation satisfied by the torus two-point correlation function with one degenerate field (see subsection 3.2).

## 2.3 Zamolodchikov's conjecture

Hence, one can compute the accessory parameters once the classical action is known. The latter can be derived by performing the classical limit of the DOZZ quantum LFT correlation functions. In particular, Zamolodchikov brothers [33] studying the classical limit of the four-point function of the quantum Liouville theory on the sphere argued that the classical Liouville action with four elliptic/parabolic singularities located at $z_4 = \infty, z_3 = 1, z_2 = x, z_1 = 0$ can be expressed as follows:

$$S_{\mathrm{L}}^{\mathrm{cl}}(\delta_4, \delta_3, \delta_2, \delta_1; x) = \tag{2.13}$$
$$S_{\mathrm{L}}^{\mathrm{cl}}(\delta_4, \delta_3, \delta_s(x)) + S_{\mathrm{L}}^{\mathrm{cl}}(\delta_s(x), \delta_2, \delta_1) - f_{\delta_s(x)} \begin{bmatrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{bmatrix}(x) - \bar{f}_{\delta_s(x)} \begin{bmatrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{bmatrix}(\bar{x}).$$

Indeed, the four-point function of the DOZZ theory can be defined as an integral of $s$-channel conformal blocks and DOZZ couplings over the continuous spectrum of the theory. In the semiclassical limit $b \to 0$ the integrand can be expressed in terms of three-point classical Liouville actions and the classical block, and the integral itself is dominated by the saddle point $\Delta_s = \frac{1}{b^2}\delta_s(x)$. One thus gets the factorization (2.13).

Concluding, in order to construct the four-point classical action via Zamolodchikov's prescription one needs the following data:

(a) the classical three-point Liouville action $S_{\mathrm{L}}^{\mathrm{cl}}(\delta_3, \delta_2, \delta_1)$ for the classical weights $\delta_1, \delta_2, \delta_3$ at the locations $0, 1, \infty$;

(b) the four-point classical conformal block on the sphere $f_\delta \begin{bmatrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{bmatrix}(x)$ ;

(c) the $s$-channel saddle point conformal weight $\delta_s(x) = \frac{1}{4} + p_s^2(x)$ where the $s$-channel saddle point momentum $p_s(x)$ is determined by the saddle point condition ($p \in \mathbb{R}$):

$$\left( \frac{\partial}{\partial p} S_{\mathrm{L}}^{\mathrm{cl}}(\delta_4, \delta_3, \tfrac{1}{4} + p^2) + \frac{\partial}{\partial p} S_{\mathrm{L}}^{\mathrm{cl}}(\tfrac{1}{4} + p^2, \delta_2, \delta_1) - 2 \mathrm{Re} \frac{\partial}{\partial p} f_{\frac{1}{4}+p^2} \begin{bmatrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{bmatrix}(x) \right)\Bigg|_{p=p_s} = 0.$$

Let us stress that we will exploit the above idea in the present work in order to compute the classical action on the one-punctured torus.

The semiclassical limit should be independent of the choice of the channel in the representation of the DOZZ four-point function. Therefore, one gets the consistency conditions known as the *classical bootstrap equations* [50]:

$$\begin{aligned} S_{\mathrm{L}}^{\mathrm{cl}}(\delta_4, \delta_3, \delta_s(x)) &+ S_{\mathrm{L}}^{\mathrm{cl}}(\delta_s(x), \delta_2, \delta_1) - f_{\delta_s(x)} \begin{bmatrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{bmatrix}(x) - \bar{f}_{\delta_s(x)} \begin{bmatrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{bmatrix}(\bar{x}) \\ &= S_{\mathrm{L}}^{\mathrm{cl}}(\delta_4, \delta_1, \delta_t(x)) + S_{\mathrm{L}}^{\mathrm{cl}}(\delta_t(x), \delta_2, \delta_3) \\ &\quad - f_{\delta_t(x)} \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_4 & \delta_3 \end{bmatrix}(1 - x) - \bar{f}_{\delta_t(x)} \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_4 & \delta_3 \end{bmatrix}(1 - \bar{x}) \\ &= 2\delta_2 \log x\bar{x} + S_{\mathrm{L}}^{\mathrm{cl}}(\delta_1, \delta_3, \delta_u(x)) + S_{\mathrm{L}}^{\mathrm{cl}}(\delta_u(x), \delta_2, \delta_4) \\ &\quad - f_{\delta_u(x)} \begin{bmatrix} \delta_3 & \delta_2 \\ \delta_1 & \delta_4 \end{bmatrix}\left(\frac{1}{x}\right) - \bar{f}_{\delta_u(x)} \begin{bmatrix} \delta_3 & \delta_2 \\ \delta_1 & \delta_4 \end{bmatrix}\left(\frac{1}{\bar{x}}\right). \end{aligned} \tag{2.14}$$

The saddle weights $\delta_t(x), \delta_u(x)$ in the $t$- and $u$-channel are simply related to the $s$-channel saddle point classical weight:

$$\delta_t(x) = \delta_s(1 - x), \qquad \delta_u(x) = \delta_s\left(\frac{1}{x}\right).$$

There is a nice geometric interpretation of the saddle point conformal weight $\delta_i(x)$. Let us recall that the classical solution describes a unique hyperbolic geometry with singularities at the locations of conformal weights. For elliptic, parabolic and hyperbolic weights one gets conical singularities, punctures and holes with geodesic boundaries respectively [79–81]. In the latter case the classical conformal weight $\delta$ is related to the length $\ell$ of the corresponding hole by

$$\delta = \frac{1}{4} + \frac{\varrho}{4}\left(\frac{\ell}{2\pi}\right)^2, \tag{2.15}$$

where the scale of the classical configuration is set by the condition $R = -\varrho/2$ imposed on the constant scalar curvature $R$.

In the case of four singularities at the standard locations $0, x, 1, \infty$ there are three closed geodesics $\Gamma_s, \Gamma_t, \Gamma_u$ separating the singular points into pairs $(x, 0|1, \infty)$, $(x, 1|0, \infty)$ and $(x, \infty|0, 1)$ respectively. Since the spectrum of DOZZ theory is hyperbolic the singularities corresponding to the saddle point weights $\delta_i(x)$ are geodesic holes. One may expect that

these weights are related to the lengths $\ell_i$ of the closed geodesics $\Gamma_i$ in corresponding channels [50]:

$$\delta_i(x) = \frac{1}{4} + \frac{\varrho}{4}\left(\frac{\ell_i(x)}{2\pi}\right)^2, \quad i = s, t, u. \tag{2.16}$$

## 3 Lamé accessory parameter from Liouville theory

### 3.1 Semiclassical Liouville one-point function on the torus

Let $\tau$ be the torus modular parameter and $q = \mathrm{e}^{2\pi i\tau}$, $\bar{q} = \mathrm{e}^{-2\pi i\bar{\tau}}$. The Liouville one-point function on the torus (expressed in terms of $2d$ CFT quantities defined on the complex plane) reads as follows

$$\langle \mathsf{V}_\beta(1)\rangle_\tau \equiv \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - c/24}\mathsf{V}_\beta(z, \bar{z})\right)\Big|_{z=1} \tag{3.1}$$

$$= \int_{\mathbb{R}^+} d\mathrm{P}\, C(\bar{\alpha}_\mathrm{P}, \beta, \alpha_\mathrm{P})\left|\mathcal{F}_{c,\Delta_{\alpha_\mathrm{P}}}^{\Delta_\beta}(q)\right|^2, \tag{3.2}$$

where

$$\alpha_\mathrm{P} = \frac{Q}{2} + i\mathrm{P}, \qquad \bar{\alpha}_\mathrm{P} = \frac{Q}{2} - i\mathrm{P} = Q - \alpha_\mathrm{P}, \qquad \mathrm{P} \in \mathbb{R}^+; \tag{3.3}$$

$$\beta = \frac{Q}{2}(1 + \xi), \qquad \xi \in [0, 1) \tag{3.4}$$

and

$$\Delta_{\alpha_\mathrm{P}} = \alpha_\mathrm{P}(Q - \alpha_\mathrm{P}) = \frac{Q^2}{4} + \mathrm{P}^2 \equiv \Delta(\mathrm{P}) \equiv \Delta, \tag{3.5}$$

$$\Delta_\beta = \beta(Q - \beta), \tag{3.6}$$

$$c = 1 + 6Q^2, \qquad\qquad Q = b + b^{-1}. \tag{3.7}$$

The trace in (3.1) is taken over the basis of the Liouville Hilbert space [82]:

$$\mathcal{H} = \int_{\mathbb{R}^+}^{\oplus} d\mathrm{P}\, \mathcal{V}_{\Delta(\mathrm{P})} \otimes \mathcal{V}_{\Delta(\mathrm{P})}.$$

The operator $\mathsf{V}_\beta$ in (3.1) is the primary Liouville vertex operator with the conformal weight $\Delta_\beta$. It has been assumed that $\mathsf{V}_\beta$ is a heavy field ($\Delta_\beta \overset{b\to 0}{\sim} b^{-2} \cdot \mathrm{const.} \Leftrightarrow \beta \overset{b\to 0}{\sim} b^{-1} \cdot \mathrm{const.}$). Moreover, the operator $\mathsf{V}_\beta$ corresponds to the so-called elliptic or parabolic ($\xi = 0$) singularity (cf. condition (3.4)). The integrand in (3.2) is built out of the DOZZ structure constant [33, 83]:

$$C(\alpha_1, \alpha_2, \alpha_3) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{(Q-\alpha_1-\alpha_2-\alpha_3)/b} \times$$

$$\frac{\Upsilon_0\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\sum \alpha_i - Q)\Upsilon(Q + \alpha_1 - \alpha_2 - \alpha_3)\Upsilon(\alpha_1 + \alpha_2 - \alpha_3)\Upsilon(\alpha_1 - \alpha_2 + \alpha_3)},$$

$$\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}, \qquad \Upsilon_0 = \frac{d\Upsilon(x)}{dx}\Big|_{x=0}$$

and the torus one-point conformal block $\mathcal{F}_{c,\Delta}^{\Delta_\beta}(q)$ defined in (2.1)–(2.2).

Now, we want to find the limit $b \to 0$ of the one-point function (3.1)–(3.2) in the case when all the conformal weights $\Delta_{\alpha_{\mathrm{P}}}$, $\Delta_\beta$ are heavy [33], i.e.:

$$\Delta_{\alpha_{\mathrm{P}}} \equiv \Delta \overset{b \to 0}{\sim} \frac{1}{b^2}\, \delta, \qquad \Delta_\beta \overset{b \to 0}{\sim} \frac{1}{b^2}\, \tilde{\delta}, \qquad \delta, \tilde{\delta} = \mathcal{O}(1). \tag{3.8}$$

As has been already assumed the external weight $\Delta_\beta$ is heavy,

$$\beta \overset{b \to 0}{\sim} \frac{1}{2b}(1 + \xi). \tag{3.9}$$

The corresponding classical conformal weight $\tilde{\delta}$ is defined as follows

$$\tilde{\delta} = \lim_{b \to 0} b^2 \Delta_\beta = \frac{1}{4}\left(1 - \xi^2\right), \qquad \xi \in [0, 1) \quad \Leftrightarrow \quad 0 < \tilde{\delta} \leq \frac{1}{4}. \tag{3.10}$$

In the case of the intermediate weight $\Delta_{\alpha_{\mathrm{P}}}$, if we rescale the integration variable $\mathrm{P} = \frac{p}{b}$, then

$$\alpha_{\mathrm{P}} = \frac{Q}{2} + i\frac{p}{b} \overset{b \to 0}{\sim} \frac{1}{2b}(1 + 2ip) \quad \Leftrightarrow \quad \delta = \lim_{b \to 0} b^2 \Delta_{\alpha_{\mathrm{P}}} = \frac{1}{4} + p^2, \quad p \in \mathbb{R}^+. \tag{3.11}$$

Let us recall that the elliptic/parabolic ($\xi = 0$) classical weights $0 < \tilde{\delta} \leq \frac{1}{4}$ are related to the parabolic/elliptic singularities, i.e. conical singularities with an opening angle $2\pi\xi$. The hyperbolic weights $\delta > \frac{1}{4}$ correspond to the hyperbolic singularities — holes with geodesic boundary (as has been already mentioned the classical hyperbolic weight is related to the length of the corresponding hole).

Let us turn to the problem of finding the $b \to 0$ limit of the one-point function (3.19)–(3.20). First let us determine the asymptotical behavior of the integrand in (3.2) when $b \to 0$.

In [81] it has been found that for the hyperbolic spectrum:

$$\alpha_j = \frac{Q}{2}(1 + i\lambda_j) \overset{b \to 0}{\sim} \frac{1}{2b}(1 + i\lambda_j), \qquad \lambda_j \in \mathbb{R}, \quad j = 1, 2, 3 \tag{3.12}$$

the DOZZ three-point function in the limit $b \to 0$ behaves as follows

$$\begin{aligned}
C(\alpha_1, \alpha_2, \alpha_3) \sim \exp\Bigg\{ &-\frac{1}{b^2}\Bigg[ \sum_{\sigma_1, \sigma_2 = \pm} F\left(\frac{1 + i\lambda_1}{2} + \sigma_1\frac{i\lambda_2}{2} + \sigma_2\frac{i\lambda_3}{2}\right) \\
&+ \sum_{j=1}^3 \left(H(i\lambda_j) + \frac{1}{2}\pi|\lambda_j|\right) + \frac{1}{2}\log(\pi\mu b^2) \\
&- i\sum_{j=1}^3 \lambda_j\left(1 - \log|\lambda_j| + \frac{1}{2}\log(\pi\mu b^2)\right) + \text{const.}\Bigg]\Bigg\},
\end{aligned} \tag{3.13}$$

where

$$F(x) = \int\limits_{\frac{1}{2}}^{x} dy\, \log\frac{\Gamma(y)}{\Gamma(1 - y)}, \qquad H(x) = \int\limits_{0}^{x} dy\, \log\frac{\Gamma(-y)}{\Gamma(y)}.$$

At this point, a few comments are in order. The expression in the square brackets should correspond to the known expression for the classical Liouville action $S_{\mathrm{L}}^{(3)}[\varphi]$ on the sphere

with three hyperbolic singularities (holes). Such classical action has been constructed in [81] (see also [79]). The construction of $S_{\mathrm{L}}^{(3)}[\varphi]$ relies on a solution of a certain monodromy problem for the Fuchsian differential equation:

$$\frac{d^2\Phi}{dz^2} + \sum_{k=1}^{n}\left[\frac{\delta_k}{(z-z_k)^2} + \frac{c_k}{z-z_k}\right]\Phi = 0$$

with hyperbolic singularities ($\delta_k$'s are hyperbolic). In this way one can find the form of the $n$-point classical action $S_{\mathrm{L}}^{(n)}[\varphi]$ up to of at most $n-3$ undetermined constants $c_k$. $S_{\mathrm{L}}^{(n)}[\varphi]$ satisfies Polyakov's formula [79]:

$$\frac{\partial}{\partial z_j}S_{\mathrm{L}}^{(n)}[\varphi] = -c_j. \tag{3.14}$$

In the case when $n = 3$ the *Fuchsian accessory parameters* $c_k$ are known and the classical action can be determined from eq. (3.14). For the standard locations of singularities $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ this yields [81]:

$$Q^2 S_{\mathrm{L}}^{(3)}[\varphi] = Q^2\left[\sum_{\sigma_1,\sigma_2=\pm} F\left(\frac{1+i\lambda_1}{2} + \sigma_1\frac{i\lambda_2}{2} + \sigma_2\frac{i\lambda_3}{2}\right)\right.$$
$$\left. + \sum_{j=1}^{3}\left(H(i\lambda_j) + \frac{1}{2}\pi|\lambda_j|\right) + \frac{1}{2}\log(\pi\mu b^2) + \frac{1}{Q^2}\,\mathrm{const.}\right], \tag{3.15}$$

where the constant on the r.h.s. is independent of $z_j$, $\lambda_j$ and $\pi\mu b^2$. Comparing (3.13) and (3.15) we see that the classical limit of the DOZZ structure constant differs from the classical three-point action by an additional imaginary term. As has been observed in [81] this inconsistency occurs due to the fact that the classical Liouville action is by construction symmetric with respect to the reflection $\alpha \to Q - \alpha$, ($\lambda \to -\lambda$) whereas the DOZZ three-point function is not. Under this reflection the DOZZ three-point function changes according to the formula [33]:

$$C\left(Q - \alpha_1, \alpha_2, \alpha_3\right) = \mathcal{S}(i\alpha_1 - iQ/2)C\left(\alpha_1, \alpha_2, \alpha_3\right),$$

where

$$\mathcal{S}(x) = -\left(\pi\mu\gamma(b^2)\right)^{-2ix/b}\frac{\Gamma(1+2ix/b)\Gamma(1+2ixb)}{\Gamma(1-2ix/b)\Gamma(1-2ixb)} \tag{3.16}$$

is the so-called *reflection amplitude* [33]. The discrepancy between (3.13) and (3.15) can be overcome if we consider the *symmetric* three-point function $\tilde{C}\left(\alpha_1, \alpha_2, \alpha_3\right)$ [81]:

$$\tilde{C}\left(\alpha_1, \alpha_2, \alpha_3\right) \equiv \left[\prod_{j=1}^{3}\sqrt{\mathcal{S}(i\alpha_j - iQ/2)}\right]C\left(\alpha_1, \alpha_2, \alpha_3\right) \tag{3.17}$$

instead of $C\left(\alpha_1, \alpha_2, \alpha_3\right)$. Indeed, taking into account the classical limit of the reflection amplitude for $\lambda \in \mathbb{R}$:

$$\log \mathcal{S}\left(-\frac{\lambda}{2b}\right) \sim \frac{2i}{b^2}\lambda\left(1 - \log|\lambda| + \frac{1}{2}\log(\pi\mu b^2)\right)$$

one can easily verify that the symmetric three-point function in the limit $b \to 0$ behaves as follows [81]

$$
\begin{aligned}
\tilde{C}(\alpha_1, \alpha_2, \alpha_3) \sim \exp\Bigg\{ &- \frac{1}{b^2}\Bigg[ \sum_{\sigma_1,\sigma_2=\pm} F\left( \frac{1+i\lambda_1}{2} + \sigma_1 \frac{i\lambda_2}{2} + \sigma_2 \frac{i\lambda_3}{2} \right) \\
&+ \sum_{j=1}^{3} \left( H(i\lambda_j) + \frac{1}{2}\pi|\lambda_j| \right) + \frac{1}{2}\log(\pi\mu b^2) + \text{const.} \Bigg]\Bigg\} \\
&= \exp\left\{ -\frac{1}{b^2} S_{\mathrm{L}}^{(3)}(\lambda_1, \lambda_2, \lambda_3) \right\},
\end{aligned} \tag{3.18}
$$

where $\alpha_j$, $j = 1, 2, 3$ are given by (3.12).

Hence, in order to obtain consistent semiclassical one-point function from the quantum one it is more convenient to take as a starting point the quantum one-point function with the *symmetric* DOZZ structure constant:

$$
\langle \mathbf{V}_\beta(1) \rangle_\tau^{\mathrm{sym}} \equiv \mathrm{Tr}_{\mathcal{H}} \left( q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} \, \mathbf{V}_\beta(z, \bar{z}) \right)\Big|_{z=1} \tag{3.19}
$$

$$
= \int_{\mathbb{R}^+} d\mathrm{P}\, \tilde{C}(Q - \alpha_{\mathrm{P}}, \beta, \alpha_{\mathrm{P}}) \left| \mathcal{F}_{c,\Delta_{\alpha_{\mathrm{P}}}}^{\Delta_\beta}(q) \right|^2, \tag{3.20}
$$

where $c$, $\Delta_\beta$, $\Delta_{\alpha_{\mathrm{P}}}$ are given by (3.5)–(3.7). The operator:

$$
\mathbf{V}_\beta = \sqrt{\mathcal{S}(i\beta - iQ/2)}\, \mathsf{V}_\beta \tag{3.21}
$$

in (3.19) is the primary Liouville vertex operator $\mathsf{V}_\beta$ rescaled by the square root of the reflection amplitude (3.16). Also the primary vertex operators $\mathsf{V}_{\alpha_{\mathrm{P}}}$'s which generate intermediate highest weight states are assumed to be rescaled by the square root of the reflection amplitude, in accordance with (3.21).

Now, for heavy insertions (3.9)–(3.11) using (3.18) one gets

$$
\tilde{C}(Q - \alpha_{\mathrm{P}}, \beta, \alpha_{\mathrm{P}}) = \tilde{C}(\alpha_{\mathrm{P}}, \beta, \alpha_{\mathrm{P}}) \overset{b \to 0}{\sim} \mathrm{e}^{-\frac{1}{b^2} S_{\mathrm{L}}^{(3)}(2p, -i\xi, 2p)}.
$$

It is reasonable to assume that the torus one-point block has the $b \to 0$ asymptotic similar to the DOZZ three-point function, i.e. given by the eq. (2.6).

Finally, in the limit $b \to 0$ the integral (3.20) with heavy insertions is determined by the saddle point approximation, i.e., by the critical value of the function:

$$
\begin{aligned}
\widehat{S}(\xi, p; q) &= S_{\mathrm{L}}^{(3)}(2p, -i\xi, 2p) - f_\delta^{\tilde{\delta}}(q) - \bar{f}_\delta^{\tilde{\delta}}(\bar{q}) \\
&= S_{\mathrm{L}}^{(3)}(2p, -i\xi, 2p) - 2\mathrm{Re}f_\delta^{\tilde{\delta}}(q),
\end{aligned} \tag{3.22}
$$

where $\delta = \frac{1}{4} + p^2$ and $\tilde{\delta} = \frac{1}{4}\left(1 - \xi^2\right)$. Indeed, when $b \to 0$ then the symmetric one-point function behaves as follows

$$
\langle \mathbf{V}_\beta(1) \rangle_\tau^{\mathrm{sym}} \sim \mathrm{e}^{-\frac{1}{b^2} S_{\mathrm{L}}^{\mathrm{torus}}(\xi; q)},
$$

where $S_L^{\text{torus}}(\xi; q) = \widehat{S}(\xi, p_*; q)$ and the *saddle point momentum* $p_* = p_*(\xi, q, \bar{q})$ is determined by the equation:

$$\frac{\partial}{\partial p} \widehat{S}(\xi, p; q)\Big|_{p=p_*} = 0 \qquad \overset{(3.22)}{\Longrightarrow}$$

$$\frac{\partial}{\partial p} S_L^{(3)}(2p, -i\xi, 2p)\Big|_{p=p_*} = 2\text{Re}\frac{\partial}{\partial p} f_{\frac{1}{4}+p^2}^{\tilde{\delta}}(q)\Big|_{p=p_*}. \tag{3.23}$$

One thus gets the factorization

$$\begin{aligned} S_L^{\text{torus}}(\xi; q) &= \widehat{S}(\xi, p_*; q) \\ &= S_L^{(3)}(2p_*, -i\xi, 2p_*) - 2\text{Re}f_{\frac{1}{4}+p_*^2}^{\tilde{\delta}}(q). \end{aligned} \tag{3.24}$$

As a final remark in this paragraph let us note that the modular invariance of the torus Liouville one-point function [84, 85] implies the *classical modular bootstrap* equation:

$$S_L^{\text{torus}}\left(\xi; e^{2\pi i\tau}\right) = S_L^{\text{torus}}\left(\xi; e^{-2\pi i\frac{1}{\tau}}\right) - 2\tilde{\delta}\log|\tau|.$$

## 3.2 Accessory parameter from torus one-point classical action

Consider the null fields [41]

$$\chi_\pm(z) = \left(\widehat{L}_{-2}(z) - \frac{3}{2(2\Delta_\pm + 1)}\widehat{L}_{-1}^2(z)\right)\mathsf{V}_\pm(z) \tag{3.25}$$

which correspond to the null vectors

$$|\chi_\pm\rangle = \left(L_{-2} - \frac{3}{2(2\Delta_\pm + 1)}L_{-1}^2\right)|\Delta_\pm\rangle \tag{3.26}$$

appearing on the second level of the Verma module. The operators $\mathsf{V}_\pm$ in (3.25) are the primary degenerate fields with the following conformal weights:

$$\Delta_+ = -\frac{1}{2} - \frac{3}{4}b^2, \qquad \Delta_- = -\frac{1}{2} - \frac{3}{4b^2}.$$

The correlation functions with null fields must vanish. In particular, for the two-point function on a torus[9] with the null field $\chi_+ \equiv \chi_{\alpha_+=-\frac{b}{2}}$ one has

$$\begin{aligned} \langle\chi_+(z)\mathsf{V}_\beta(w)\rangle_\tau &= \left\langle\widehat{L}_{-2}(z)\mathsf{V}_+(z)\mathsf{V}_\beta(w)\right\rangle_\tau \\ &+ \frac{1}{b^2}\left\langle\widehat{L}_{-1}^2(z)\mathsf{V}_+(z)\mathsf{V}_\beta(w)\right\rangle_\tau = 0. \end{aligned} \tag{3.27}$$

Using the torus Ward identities [42] one can convert the condition (3.27) to the second order differential equation:

$$\left[\frac{1}{b^2}\frac{\partial^2}{\partial z^2} + \left(2\Delta_+\eta_1 + 2\eta_1 z\frac{\partial}{\partial z}\right) + \Delta_\beta\left(\wp(z-w) + 2\eta_1\right)\right. \tag{3.28}$$

$$\left. + (\zeta(z-w) + 2\eta_1 w)\frac{\partial}{\partial w}\right]\langle\mathsf{V}_+(z)\mathsf{V}_\beta(w)\rangle_\tau = -\frac{2\pi i}{Z(\tau)}\frac{\partial}{\partial\tau}\left[Z(\tau)\langle\mathsf{V}_+(z)\mathsf{V}_\beta(w)\rangle_\tau\right],$$

---

[9]With periods 1 and $\tau$.

where $Z(\tau)$ is a partition function and

$$\zeta(z|\tau) = \partial_z \log \theta_1(z|\tau) + 2\eta_1 z,$$
$$\wp(z) = -\partial_z \zeta(z|\tau),$$
$$\eta(\tau) = \mathrm{e}^{2i\pi\tau/24} \prod_{n>0}(1 - \mathrm{e}^{2i\pi\tau n}) = q^{\frac{1}{24}} \prod_{n>0}(1 - q^n),$$
$$\eta_1 = (2\pi)^2 \left( \frac{1}{24} - \sum_{n=1}^{\infty} \frac{n\mathrm{e}^{2\pi in\tau}}{1 - \mathrm{e}^{2\pi in\tau}} \right) = -2\pi i \partial_\tau \log \eta(\tau).$$

Let us introduce

$$\zeta_*(z|\tau) = \zeta(z|\tau) - 2\eta_1 z = \partial_z \log \theta_1(z|\tau),$$
$$\wp_*(z) = -\partial_z \zeta_*(z|\tau) = \wp(z) + 2\eta_1.$$

For $w = 0$ from (3.28) one gets

$$\left[ -\frac{1}{b^2}\frac{\partial^2}{\partial z^2} + \zeta_*(z)\frac{\partial}{\partial z} - \Delta_\beta\,\wp_*(z) - 2\Delta_+\eta_1 \right] \langle \mathsf{V}_+(z)\mathsf{V}_\beta(0) \rangle_\tau$$
$$= 2\pi i \left( \frac{\partial}{\partial \tau} \log Z(\tau) \right) \langle \mathsf{V}_+(z)\mathsf{V}_\beta(0) \rangle_\tau - 4\pi^2 q \frac{\partial}{\partial q} \langle \mathsf{V}_+(z)\mathsf{V}_\beta(0) \rangle_\tau. \qquad (3.29)$$

Recall, that the primary vertex operators in the two-point function above are defined on the cylinder $\mathsf{V}_\sigma \equiv \mathsf{V}_\sigma^{\mathrm{cyl}}$. These operators are related to the operators living on the plane (used in the previous paragraph) in the standard way $\mathsf{V}_\sigma^{\mathrm{cyl}}(z, \bar{z}) = \mathrm{e}^{z\Delta_\sigma}\mathrm{e}^{\bar{z}\Delta_\sigma}\mathsf{V}_\sigma^{\mathbb{C}}(\mathrm{e}^z, \mathrm{e}^{\bar{z}})$.

One can think of (3.29) as the equation obeyed by the Liouville two-point function on the torus where the "alpha" $\alpha_+$ or equivalently the conformal weight $\Delta_{\alpha_+}$ of the operator located at $z$ "has been continued" to the degenerate value.[10] If so, let us multiply both sides of the eq. (3.29) by $[\mathcal{S}(i\bar{\alpha}_{\mathrm{P}} - iQ/2)\,\mathcal{S}(i\beta - iQ/2)\,\mathcal{S}(i\alpha_{\mathrm{P}} - iQ/2)]^{\frac{1}{2}} =: \mathcal{R}$. Next, let us assume that $\alpha_{\mathrm{P}}$ and $\beta$ are heavy. On the other hand for $b \to 0$ the operator $\mathsf{V}_+$ remains *light* ($\Delta_+ = \mathcal{O}(1)$) and its presence in the correlation function has no influence on the classical dynamics. Also $\log Z(\tau)$ is of order $\mathcal{O}(1)$. Hence, one can expect that for $b \to 0$,[11]

$$\mathcal{R} \left\langle \mathsf{V}_+^{\mathrm{cyl}}(z)\mathsf{V}_\beta^{\mathrm{cyl}}(0) \right\rangle_\tau \sim \Psi(z)\,\mathcal{R} \left\langle \mathsf{V}_\beta^{\mathrm{cyl}}(0) \right\rangle_\tau = \Psi(z)\,\mathcal{R} \left\langle \mathsf{V}_\beta^{\mathbb{C}}(1) \right\rangle_\tau$$
$$= \Psi(z) \left\langle \mathbf{V}_\beta^{\mathbb{C}}(1) \right\rangle_\tau^{\mathrm{sym}} \sim \Psi(z)\,\mathrm{e}^{-\frac{1}{b^2}S_{\mathrm{L}}^{\mathrm{torus}}(\xi; q)}. \qquad (3.30)$$

After substituting (3.30) into the eq. (3.29) and taking the limit $b \to 0$ one gets

$$\frac{\partial^2}{\partial z^2}\Psi(z) + \left( \tilde{\delta}\,\wp_*(z) + 4\pi^2 q \frac{\partial}{\partial q} S_{\mathrm{L}}^{\mathrm{torus}}(\xi; q) \right) \Psi(z) = 0.$$

---

[10]The idea which makes use of the degenerate representation of the Virasoro algebra in the Liouville field theory is not new. In particular, such trick has been used by Teschner in order to re-derive the DOZZ formula [86] (for reviews, see [82, 87]).

[11]Eq. (3.30) is justified by the well known semiclassical behavior of Liouville correlators with heavy and light vertices on the sphere, see for instance [88]. It is reasonable to expect that the same holds on the cylinder, cf. [89].

From eqs. (3.23) and (3.24) we have

$$\frac{\partial}{\partial q} S_{\mathrm{L}}^{\mathrm{torus}}(\xi; q) = \frac{\partial}{\partial q} \widehat{S}(\xi, p_*(q); q)$$

$$= \frac{\partial}{\partial p} \widehat{S}(\xi, p; q)\Big|_{p=p_*(q)} \frac{\partial p_*(q)}{\partial q} + \frac{\partial}{\partial q} \widehat{S}(\xi, p_*; q)$$

$$= -\frac{\partial}{\partial q} f_{\frac{1}{4}+p_*^2}^{\tilde{\delta}}(q).$$

Therefore,

$$\frac{\partial^2}{\partial z^2} \Psi(z) - \left[ -\tilde{\delta}\wp(z) + 4\pi^2 q \frac{\partial}{\partial q} f_{\frac{1}{4}+p_*^2}^{\tilde{\delta}}(q) - 2\tilde{\delta}\eta_1 \right] \Psi(z) = 0.$$

Now, one can identify the parameters $\kappa$ and B appearing in the Lamé equation (1.2) as follows

$$\kappa = -\tilde{\delta}, \quad \mathrm{B} = 4\pi^2 q \frac{\partial}{\partial q} f_{\frac{1}{4}+p_*^2}^{\tilde{\delta}}(q) - 2\tilde{\delta}\eta_1 . \tag{3.31}$$

Then, for $-\kappa = \tilde{\delta} = \frac{1}{4}$ (parabolic singularity) the accessory parameter C which occurs in the version (1.5) of the Lamé equation explicitly reads as follows

$$\mathrm{C} = -4\mathrm{B} = -16\pi^2 q \frac{\partial}{\partial q} f_{\frac{1}{4}+p_*^2}^{\frac{1}{4}}(q) + 2\eta_1 . \tag{3.32}$$

More in general, i.e. for the elliptic singularities $-\kappa = \tilde{\delta} \in \left(0, \frac{1}{4}\right)$ one gets

$$\mathrm{C}_{\mathrm{ell}} = \frac{\mathrm{B}}{\kappa} = -4\pi^2 \tilde{\delta}^{-1} q \frac{\partial}{\partial q} f_{\frac{1}{4}+p_*^2}^{\tilde{\delta}}(q) + 2\eta_1 . \tag{3.33}$$

Let us note that $\eta_1$ depends on the modular parameter $\tau$ according to the formula $\eta_1 = \frac{4\pi^2}{24} E_2(\tau)$ where $E_2(\tau)$ is the second Eisenstein series. Hence, finally one can express B (and C, $\mathrm{C}_{\mathrm{ell}}$) in terms of functions depending on $\tau$:

$$\frac{\mathrm{B}(\tau)}{4\pi^2} = q \frac{\partial}{\partial q} f_{\frac{1}{4}+p_*^2}^{\tilde{\delta}}(q) - \frac{\tilde{\delta}}{12} E_2(\tau) . \tag{3.34}$$

## 4  Lamé accessory parameter from $\mathcal{N} = 2^*$ gauge theory

### 4.1  Accessory parameter from twisted superpotential

In order to compute the torus classical block $f_{\tilde{\delta}}^{\tilde{\delta}}(q)$ entering the expressions for B and/or C, $\mathrm{C}_{\mathrm{ell}}$ one can exploit the "chiral" AGT relation on the torus and the correspondence between the classical limit of the conformal blocks and the Nekrasov-Shatashvili limit of the Nekrasov instanton partition functions.

The "chiral" AGT relation on the torus identifies the torus quantum block with the Nekrasov instanton partition function [64, 65] of the $\mathcal{N} = 2^*$, SU(2) gauge theory (which equals to $[\mathcal{Z}_{\mathrm{inst}}^{\mathrm{U}(1)}]^{-1} \times \mathcal{Z}_{\mathrm{inst}}^{\mathrm{U}(2)}$ as it is written in the second line of the equation below):

$$q^{\frac{c}{24}-\Delta} \mathcal{F}_{c,\Delta}^{\Delta_\beta}(q) = \mathcal{Z}_{\mathrm{inst}}^{\mathcal{N}=2^*,\mathrm{SU}(2)}(q, a, \mu, \epsilon_1, \epsilon_2)$$

$$= \left(\frac{\eta(q)}{q^{\frac{1}{24}}}\right)^{1-2\Delta_\beta} \mathcal{Z}_{\mathrm{inst}}^{\mathcal{N}=2^*,\mathrm{U}(2)}(q, a, \mu, \epsilon_1, \epsilon_2). \tag{4.1}$$

In eq. (4.1) $\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n)$ is the Dedekind $\eta$-function. The torus block parameters, namely the external conformal weight $\Delta_\beta$, the intermediate weight $\Delta$ and the Virasoro central charge $c$ can be expressed in terms of the $\mathcal{N} = 2^*$, SU(2) super-Yang-Mills theory parameters as follows

$$c = 1 + 6\frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2} \equiv 1 + 6Q^2 \quad \Leftrightarrow \quad b = \sqrt{\frac{\epsilon_2}{\epsilon_1}}, \tag{4.2}$$

$$\Delta_\beta = -\frac{\mu(\epsilon_1 + \epsilon_2 + \mu)}{\epsilon_1 \epsilon_2} \quad \Leftrightarrow \quad \beta = -\frac{\mu}{\sqrt{\epsilon_1 \epsilon_2}}, \tag{4.3}$$

$$\Delta = \frac{(\epsilon_1 + \epsilon_2)^2 - 4a^2}{4\epsilon_1 \epsilon_2} \quad \Leftrightarrow \quad \mathrm{P} = \frac{ia}{\sqrt{\epsilon_1 \epsilon_2}}. \tag{4.4}$$

Above $\mu$ is the mass of the adjoint hypermultiplet, $a$ is the vacuum expectation value of the complex scalar of the gauge multiplet and $\epsilon_1, \epsilon_2$ are $\Omega$-background parameters. The relation (4.1) is understood as an equality between the coefficients of the expansions of both sides in powers of $q$. For the torus conformal block such expansion has been introduced in eqs. (2.1)–(2.2). For the definition of the instanton partition function appearing in (4.1), see subsection 4.3. The identity (4.1) has been proved by Fateev and Litvinov [46]. They have shown that the coefficients of the expansions of both sides of (4.1) obey the same recurrence relation.

Let us note that the relation (4.1) holds for the heavy conformal weights. Indeed, from (4.2)–(4.4) we have

$$\tilde{\delta} = \lim_{b \to 0} b^2 \Delta_\beta = -\lim_{\epsilon_2 \to 0} \frac{\epsilon_2}{\epsilon_1} \frac{\mu(\epsilon_1 + \epsilon_2 + \mu)}{\epsilon_1 \epsilon_2} = -\frac{\mu}{\epsilon_1}\left(\frac{\mu}{\epsilon_1} + 1\right), \tag{4.5}$$

$$\delta = \lim_{b \to 0} b^2 \Delta = \lim_{\epsilon_2 \to 0} \frac{\epsilon_2}{\epsilon_1} \frac{(\epsilon_1 + \epsilon_2)^2 - 4a^2}{4\epsilon_1 \epsilon_2} = \frac{1}{4} - \frac{a^2}{\epsilon_1^2}. \tag{4.6}$$

Hence, one can consider the limit $b \to 0$ of the AGT relation (4.1). The limit $b \to 0$ corresponds to $\epsilon_2 \to 0$ ($\epsilon_1 = $ const.), in accordance with (4.2). As has been observed by Nekrasov and Shatashvili [10] if $\epsilon_2 \to 0$ while $\epsilon_1$ is kept finite the Nekrasov instanton partition function has the following asymptotical behavior:

$$\mathcal{Z}_{\mathrm{inst}}(\,\cdot\,, \epsilon_1, \epsilon_2) \overset{\epsilon_2 \to 0}{\sim} \exp\left\{\frac{1}{\epsilon_2} \mathcal{W}_{\mathrm{inst}}(\,\cdot\,, \epsilon_1)\right\}. \tag{4.7}$$

$\mathcal{W}_{\mathrm{inst}}(\,\cdot\,, \epsilon_1)$ is the instanton contribution to the so-called effective twisted superpotential of the corresponding two-dimensional gauge theory restricted to the two-dimensional $\Omega$-background. Twisted superpotentials play also a prominent role in already mentioned Bethe/gauge correspondence [10–12].

Therefore, taking into account the semiclassical asymptotic (2.6) of the torus quantum block and (4.7) one can get from (4.1) the "classical version" of the torus AGT relation:

$$f_\delta^{\tilde{\delta}}(q) = \left(\delta - \frac{1}{4}\right)\log q - 2\tilde{\delta} \log\left(\frac{\eta(q)}{q^{\frac{1}{24}}}\right) + \frac{1}{\epsilon_1} \mathcal{W}_{\mathrm{inst}}^{\mathcal{N}=2^*, \mathrm{U}(2)}(q, a, \mu, \epsilon_1). \tag{4.8}$$

Using (4.8) and $\partial_\tau \log \eta(\tau) = \frac{i\pi}{12} E_2(\tau)$ one can rewrite the expression (3.34) for the parameter B to the following form:

$$\frac{\mathrm{B}(\tau)}{4\pi^2} = p_*^2 + \frac{\tilde{\delta}}{12} \left(1 - 2E_2(\tau)\right) + \frac{1}{\epsilon_1} q \frac{\partial}{\partial q} \mathcal{W}_{\mathrm{inst}}^{\mathcal{N}=2^*,\mathrm{U}(2)}(q, a, \mu, \epsilon_1) \tag{4.9}$$

where

$$p_*(\xi, q, \bar{q}) = \frac{ia}{\epsilon_1}, \qquad \tilde{\delta} = -\kappa = -\frac{\mu}{\epsilon_1}\left(\frac{\mu}{\epsilon_1} + 1\right) = \frac{1}{4}\left(1 - \xi^2\right). \tag{4.10}$$

Let us stress that the twisted superpotential $\mathcal{W}_{\mathrm{inst}}^{\mathcal{N}=2^*,\mathrm{U}(2)}$ is a computable quantity. The derivation of $\mathcal{W}_{\mathrm{inst}}^{\mathcal{N}=2^*,\mathrm{U}(2)}$ directly from the Nekrasov instanton partition function shall be presented in the third subsection. In the next subsection we confront the calculation method of the eigenvalue, which employs the idea of the classical block, with another procedure based on the WKB analysis, cf. [46, 89–92].

### 4.2 WKB analysis

Taking into account (4.10) one can rewrite the eq. (1.2) to the following Schrödinger-like form:

$$-\epsilon_1^2 \Psi''(z) + V(z, \epsilon_1)\Psi(z) = \mathrm{E}\,\Psi(z), \tag{4.11}$$

where

$$V(z, \epsilon_1) = \mu\left(\mu + \epsilon_1\right)\wp(z) = \mu^2\wp(z) + \epsilon_1\mu\wp(z) \equiv V_0(z) + \epsilon_1 V_1(z)$$

and

$$\mathrm{E} = -\epsilon_1^2\,\mathrm{B}. \tag{4.12}$$

Substituting

$$\Psi(z) = \exp\left\{-\frac{1}{\epsilon_1}\int^z P(x, \epsilon_1)\,dx\right\}$$

into the eq. (4.11) one finds

$$-P^2(z, \epsilon_1) + \epsilon_1 P'(z, \epsilon_1) + V(z, \epsilon_1) = \mathrm{E}.$$

Above equation can be solved iteratively by expansions in $\epsilon_1$:

$$P(z, \epsilon_1) = \sum_{k=0}^\infty \epsilon_1^k P_k(z), \qquad\qquad V(z, \epsilon_1) = \sum_{k=0}^\infty \epsilon_1^k V_k(z).$$

In particular, for lower orders this yields

$$-P_0^2 + V_0 = \mathrm{E},$$
$$-2P_0P_1 + P_0' + V_1 = 0,$$
$$-2P_0P_2 - P_1^2 + P_1' + V_2 = 0.$$

Note, that in our case $V_0(z) = \mu^2\wp(z)$, $V_1(z) = \mu\wp(z)$ and $V_k(z) = 0$ for all $k > 1$.

The quasiclassical approximation $\mathrm{E} \equiv \mathrm{E}|_{\text{zero order in } \epsilon_1}$ to the energy eigenvalue is determined by the $\mathcal{A}$-cycle integral [46, 90, 91]:

$$2\pi i a = \oint_{\mathcal{A}} P_0 dz = \oint_{\mathcal{A}} \sqrt{V_0 - \mathrm{E}}\, dz = \oint_{\mathcal{A}} \sqrt{\mu^2 \wp(z) - \mathrm{E}}\, dz. \tag{4.13}$$

Physically, the equation above is nothing but the Bohr-Sommerfeld quantization condition [90]. It is convenient to introduce

$$\mathbb{E} = \frac{\mathrm{E}}{4\pi^2 a^2}, \qquad\qquad \nu = \frac{\mu^2}{4\pi^2 a^2} \tag{4.14}$$

and rewrite the eq. (4.13) to the form

$$1 = \oint_{\mathcal{A}} \sqrt{\mathbb{E} - \nu \wp(z)}\, dz. \tag{4.15}$$

Let us recall that we are working on the torus with periods 1 and $\tau$ parameterized by the complex coordinate $z \equiv z + 1 \equiv z + \tau$. The $\mathcal{A}$-cycle here is just the interval $[0, 1]$.

Eq. (4.15) allows to compute $\mathbb{E}$ as an expansion in $\nu$ with coefficients depending on $q = \mathrm{e}^{2\pi i \tau}$:

$$\mathbb{E} = 1 + \mathbb{E}_1(q)\,\nu + \mathbb{E}_2(q)\,\nu^2 + \mathbb{E}_3(q)\,\nu^3 + \mathbb{E}_4(q)\,\nu^4 + \dots. \tag{4.16}$$

Indeed, after an expansion of the square root the eq. (4.15) becomes

$$1 = \oint_{\mathcal{A}} dz + \oint_{\mathcal{A}} \frac{1}{2}\left(\mathbb{E}_1 - \wp(z)\right)\nu\, dz + \oint_{\mathcal{A}} \frac{1}{2}\left(\mathbb{E}_2 - \frac{1}{4}\left(\mathbb{E}_1 - \wp(z)\right)^2\right)\nu^2\, dz + \dots. \tag{4.17}$$

Since

$$\oint_{\mathcal{A}} dz = 1,$$

then, the higher terms on the r.h.s. of the eq. (4.17) must vanish. Therefore, up to $\nu^4$ one finds [89]:

$$\mathbb{E}_1 = \oint_{\mathcal{A}} \wp(z) dz = -\frac{\pi^2}{3} E_2, \tag{4.18}$$

$$\mathbb{E}_2 = \oint_{\mathcal{A}} \left[\frac{1}{4}\wp(z)^2 - \frac{1}{2}\mathbb{E}_1 \wp(z) + \frac{1}{4}\mathbb{E}_1^2\right] dz = \frac{\pi^4}{36}\left(E_4 - E_2^2\right), \tag{4.19}$$

$$\mathbb{E}_3 = \oint_{\mathcal{A}} \left[\frac{1}{8}\wp(z)^3 - \frac{3}{8}\mathbb{E}_1 \wp(z)^2 + \frac{1}{8}\left(3\mathbb{E}_1^2 - 4\mathbb{E}_2\right)\wp(z) + \frac{1}{8}\left(-\mathbb{E}_1^3 + 4\mathbb{E}_1 \mathbb{E}_2\right)\right] dz$$

$$= \frac{\pi^6}{540}\left(2E_6 + 3E_2 E_4 - 5E_2^3\right), \tag{4.20}$$

$$\mathbb{E}_4 = \oint_{\mathcal{A}} \left[\frac{5}{64}\wp(z)^4 - \frac{5}{16}\mathbb{E}_1 \wp(z)^3 + \frac{1}{64}\left(30\mathbb{E}_1^2 - 24\mathbb{E}_2\right)\wp(z)^2\right.$$

$$\left. + \frac{1}{64}\left(-20\mathbb{E}_1^3 + 48\mathbb{E}_1 \mathbb{E}_2 - 32\mathbb{E}_3\right)\wp(z) + \frac{1}{64}\left(5\mathbb{E}_1^4 - 24\mathbb{E}_1^2 \mathbb{E}_2 + 16\mathbb{E}_2^2 + 32\mathbb{E}_1 \mathbb{E}_3\right)\right] dz$$

$$= \frac{\pi^8}{9072}\left(-35E_2^4 + 7E_2^2 E_4 + 10E_4^2 + 18E_2 E_6\right), \tag{4.21}$$

where $E_{2n}$, $n = 1, 2, 3$ are the Eisenstein series. In above calculations appear integrals of the form

$$\oint_{\mathcal{A}} \wp(z)^n dz =: \mathscr{K}_n.$$

For their explicit computation see appendix B. Finally, using (4.14), (4.16), (4.18)–(4.21) and the $q$-expansions of the Eisenstein series (A.4)–(A.6) one gets

$$\frac{1}{4\pi^2} \left. \mathrm{E} \right|_{\text{zero order in } \epsilon_1} = a^2 - \frac{\mu^2}{12} + \frac{\mu^2 \left(4a^2 + \mu^2\right)}{2a^2} q$$

$$+ \frac{\mu^2 \left(192a^6 + 96a^4\mu^2 - 48a^2\mu^4 + 5\mu^6\right)}{32a^6} q^2 + \dots . \quad (4.22)$$

On the other hand from (3.34) and (4.12) we have

$$\frac{1}{4\pi^2} \left. \mathrm{E} \right|_{\text{zero order in } \epsilon_1} = -\lim_{\epsilon_1 \to 0} \epsilon_1^2 \frac{\mathrm{B}}{4\pi^2} = -\lim_{\epsilon_1 \to 0} \epsilon_1^2 \left[ q \frac{\partial}{\partial q} f_{\delta_*}^{\tilde{\delta}}(q) - \frac{\tilde{\delta}}{12} E_2(q) \right]. \quad (4.23)$$

The torus classical block $f_\delta^{\tilde{\delta}}(q)$ appearing above has the following expansion

$$f_\delta^{\tilde{\delta}}(q) = \left(\delta - \frac{1}{4}\right) \log q + \mathbf{f}_\delta^{\tilde{\delta}}(q) = \left(\delta - \frac{1}{4}\right) \log q + \sum_{n=1}^{\infty} \mathbf{f}_\delta^{\tilde{\delta},n} q^n \quad (4.24)$$

with coefficients $\mathbf{f}_\delta^{\tilde{\delta},n}$ determined by the semiclassical asymptotic (2.6) of the quantum block:

$$\sum_{n=1}^{\infty} \mathbf{f}_\delta^{\tilde{\delta},n} q^n = \lim_{b \to 0} b^2 \log \left[ 1 + \sum_{n=1}^{\infty} \mathcal{F}_{1+6Q^2,\Delta}^{\Delta_\beta,n} q^n \right].$$

For instance,

$$\mathbf{f}_\delta^{\tilde{\delta},1} = \frac{\tilde{\delta}^2}{2\delta}, \qquad \mathbf{f}_\delta^{\tilde{\delta},2} = \frac{\tilde{\delta}^2[24\delta^2\left(4\delta + 1\right) + \tilde{\delta}^2\left(5\delta - 3\right) - 48\tilde{\delta}\delta^2]}{16\delta^3\left(4\delta + 3\right)}. \quad (4.25)$$

Using (4.5)–(4.6), (A.4) and (4.25) one can check that (4.23) exactly agrees with (4.22).[12]

## 4.3 Nekrasov-Shatashvili limit

A goal of this subsection is to compute an instanton contribution to the so-called effective twisted superpotential of the $\mathcal{N} = 2^*$ U(2) gauge theory. We closely follow here the method of the calculation developed by Poghossian in ref. [40].

Consider the instanton part of the Nekrasov partition function of the $\mathcal{N} = 2$ supersymmetric U(2) gauge theory with an adjoint hypermultiplet (the $\mathcal{N} = 2^*$ theory) [10, 64]:

$$\mathcal{Z}_{\text{inst}}^{\mathcal{N}=2^*,\text{U}(2)} = 1 + \sum_{k=1}^{\infty} \frac{q^k}{k!} \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1\epsilon_2}\right)^k \mathcal{Z}_k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{q^k}{k!} \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1\epsilon_2}\right)^k \oint \frac{d\phi_1}{2\pi i} \cdots \oint \frac{d\phi_k}{2\pi i} \Omega_k, \quad (4.26)$$

---

[12]Here $\delta_* = \frac{1}{4} + p_*^2$ and $p_*^2 = -\frac{a^2}{\epsilon_1^2}$.

where

$$\Omega_k = \prod_{I=1}^{k} \frac{P(\phi_I - \mu)P(\phi_I + \mu + \epsilon_1 + \epsilon_2)}{P(\phi_I)P(\phi_I + \epsilon_1 + \epsilon_2)}$$

$$\times \prod_{\substack{I,J=1 \\ I \neq J}}^{k} \frac{\phi_{IJ}\left(\phi_{IJ} + \epsilon_1 + \epsilon_2\right)\left(\phi_{IJ} + \mu + \epsilon_1\right)\left(\phi_{IJ} + \mu + \epsilon_2\right)}{\left(\phi_{IJ} + \epsilon_1\right)\left(\phi_{IJ} + \epsilon_2\right)\left(\phi_{IJ} + \mu\right)\left(\phi_{IJ} + \mu + \epsilon_1 + \epsilon_2\right)},$$

$\phi_{IJ} = \phi_I - \phi_J$ and $P(x) = (x - a_1)(x - a_2)$. We will assume that $\mu, a_u, \epsilon_1, \epsilon_2 \in \mathbb{R}$. The poles which contribute to the integral (4.26) are at

$$\phi_I = \phi_{uij} = a_u + (i-1)\epsilon_1 + (j-1)\epsilon_2, \qquad u = 1, 2. \tag{4.27}$$

Recall, that these poles are in correspondence with pairs $\mathsf{Y} = (Y_1, Y_2)$ of Young diagrams with total number of boxes $|\mathsf{Y}| = |Y_1| + |Y_2| = k$. The index $i$ parameterizes the columns whereas $j$ runs over the rows of the diagram $Y_u$. The parameters $\epsilon_1, \epsilon_2$ describe a size of a box $(i, j) \in Y_u$ in horizontal, vertical direction respectively. The instanton sum over $k$ in (4.26) can be rewritten as a sum over a pairs of Young diagrams as follows:

$$\mathcal{Z}_k = \sum_{\substack{\mathsf{Y} \\ |\mathsf{Y}|=k}} \mathcal{Z}_{\mathsf{Y}}.$$

The contributions $\mathcal{Z}_{\mathsf{Y}}$ to the instanton sum correspond to those obtained by performing (in some specific order) the contour integrals in (4.26).

Now we want to calculate the Nekrasov-Shatashvili limit [10] $\epsilon_2 \to 0$ ($\epsilon_1$ is kept finite) of the instanton partition function (4.26). Based on the arguments developed by Nekrasov and Okounkov in ref. [65] it is reasonable to expect that for vanishingly small values of $\epsilon_2$ the dominant contribution to the instanton partition function (4.26) will occur when $k \sim \frac{1}{\epsilon_2}$. Unfortunately, we have found no proof of that mechanism in the general case. Let us note only that this statement becomes evident in the trivial case in which $\mathcal{Z}_k = 1$ for all $k = 1, 2, \ldots$. Indeed, for $\epsilon_2 \to 0$ and $x = \frac{q}{\epsilon_2} \in \mathbb{R}_{>0}$ we have then from eq. (4.26):

$$\mathcal{Z}_{\mathrm{inst}} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{q}{\epsilon_2}\right)^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \mathrm{e}^x = \frac{\mathrm{e}^{x \log x}}{\mathrm{e}^{x \log x - x}} \sim \frac{\mathrm{e}^{x \log x}}{\mathrm{e}^{\log x!}} = \frac{x^x}{x!}.$$

This means that the whole sum is dominated by a single term with $k \sim x \to \infty$.

Hence, in order to compute the limit $\epsilon_2 \to 0$ of $\mathcal{Z}_{\mathrm{inst}}^{\mathcal{N}=2^*, \mathrm{U}(2)}$ let us find first the leading behavior of $\log\left|q^k \Omega_k\right|$ for large $k$ (i.e. small values of $\epsilon_2$ and finite $\epsilon_1$). After simple calculations, using $\log(x \pm \epsilon_2) = \log(x) \pm \frac{\epsilon_2}{x} + O(\epsilon_2^2)$, one gets

$$\log\left|q^k \Omega_k\right| \sim \frac{1}{\epsilon_2} \mathcal{H}_{\mathrm{inst}}^{\mathcal{N}=2^*, \mathrm{U}(2)},$$

where

$$\mathcal{H}_{\text{inst}}^{\mathcal{N}=2^*,\text{U}(2)} = \epsilon_2 k \log|q| + \epsilon_2 \sum_{I=1}^{k} \log\left|\frac{P(\phi_I - \mu)P(\phi_I + \mu + \epsilon_1)}{P(\phi_I)P(\phi_I + \epsilon_1)}\right|$$

$$+ \ \epsilon_2^2 \sum_{\substack{I,J=1 \\ I \neq J}}^{k} \left[\frac{1}{\phi_{IJ} + \epsilon_1} - \frac{1}{\phi_{IJ}} + \frac{1}{\phi_{IJ} + \mu} - \frac{1}{\phi_{IJ} + \mu + \epsilon_1}\right]. \quad (4.28)$$

In eq. (4.28) it is implicitly understood that the poles $\phi_I$ are obtained from eq. (4.27) in the limit $\epsilon_2 \to 0$. Note that in the limit $\epsilon_2 \to 0$ the poles form a continuous distribution (cf. [69]):

$$\phi_I = \phi_{u,i} \in \left[x_{ui}^0, x_{ui}\right] \quad (4.29)$$

where

$$x_{ui}^0 = a_u + (i-1)\epsilon_1, \qquad u = 1, 2, \qquad i = 1, \ldots, \infty,$$
$$x_{ui} = a_u + (i-1)\epsilon_1 + \omega_{ui} \ .$$

In terms of Young diagrams the situation can be explained heuristically as follows. When $\epsilon_2$ is very small then the number of boxes $k_{ui}$ in the vertical direction (the number of rows) is very large, however this number multiplied by $\epsilon_2$, i.e.: $\epsilon_2 k_{ui} = \omega_{ui}$ is expected to be finite. In other words we obtain a continuous distribution of rows in the limit under consideration. Then, in order to evaluate (4.28) at the values (4.29) one can assume that the summations "over instantons" in (4.28) become continuous in the row index:

$$\epsilon_2 \sum_I \longrightarrow \sum_{u,i} \int_{x_{ui}^0}^{x_{ui}} d\phi_{ui}. \quad (4.30)$$

The limits of integration $x_{ui}^0$ and $x_{ui}$ are the bottom and the top ends of the $i$-th column in $Y_u$ respectively. Applying eq. (4.30) to eq. (4.28) one gets

$$\mathcal{H}_{\text{inst}}^{\mathcal{N}=2^*,\text{U}(2)}(x_{ui}) = \sum_{u,v=1}^{2} \sum_{i,j=1}^{\infty} \Big[ -F\left(x_{ui} - x_{vj} + \epsilon_1\right) + F\left(x_{ui} - x_{vj}^0 + \epsilon_1\right)$$

$$+ F\left(x_{ui}^0 - x_{vj} + \epsilon_1\right) - F\left(x_{ui}^0 - x_{vj}^0 + \epsilon_1\right) + F(x_{ui} - x_{vj}) - F(x_{ui} - x_{vj}^0)$$

$$- F(x_{ui}^0 - x_{vj}) + F(x_{ui}^0 - x_{vj}^0) - F(x_{ui} - x_{vj} + \mu) + F(x_{ui} - x_{vj}^0 + \mu)$$

$$+ F(x_{ui}^0 - x_{vj} + \mu) - F(x_{ui}^0 - x_{vj}^0 + \mu) + F(x_{ui} - x_{vj} + \mu + \epsilon_1) \quad (4.31)$$

$$- F(x_{ui} - x_{vj}^0 + \mu + \epsilon_1) - F(x_{ui}^0 - x_{vj} + \mu + \epsilon_1) + F(x_{ui}^0 - x_{vj}^0 + \mu + \epsilon_1) \Big]$$

$$+ \sum_{u,v=1}^{2} \sum_{i=1}^{\infty} \Big[ F\left(x_{ui} - \mu - a_v\right) - F\left(x_{ui}^0 - \mu - a_v\right) + F\left(x_{ui} + \mu - a_v + \epsilon_1\right)$$

$$- F\left(x_{ui}^0 + \mu - a_v + \epsilon_1\right) - F\left(x_{ui} - a_v\right) + F\left(x_{ui}^0 - a_v\right)$$

$$- F\left(x_{ui} - a_v + \epsilon_1\right) + F\left(x_{ui}^0 - a_v + \epsilon_1\right) \Big] + \sum_{u=1}^{2} \sum_{i=1}^{\infty} (x_{ui} - (i-1)\epsilon_1 - a_u) \log|q| \ ,$$

where $F(x) = x(\log|x| - 1)$.

Finally, the Nekrasov instanton partition function in the limit $\epsilon_2 \to 0$ can be represented as follows:

$$\mathcal{Z}_{\text{inst}}^{\mathcal{N}=2^*,\text{U}(2)} \sim \int \left[ \prod_{u,i} dx_{ui} \right] \exp \left\{ \frac{1}{\epsilon_2} \mathcal{H}_{\text{inst}}^{\mathcal{N}=2^*,\text{U}(2)}(x_{ui}) \right\}, \tag{4.32}$$

where the "integral" is over the infinite set of variables $\{x_{ui} : u = 1, 2; i = 1, \ldots, \infty\}$. As a consequence, the Nekrasov-Shatashvili limit of $\mathcal{Z}_{\text{inst}}^{\mathcal{N}=2^*,\text{U}(2)}$ is nothing but the critical value of $\mathcal{H}_{\text{inst}}^{\mathcal{N}=2^*,\text{U}(2)}$:

$$\mathcal{W}_{\text{inst}}^{\mathcal{N}=2^*,\text{U}(2)} \equiv \lim_{\epsilon_2 \to 0} \epsilon_2 \log \mathcal{Z}_{\text{inst}}^{\mathcal{N}=2^*,\text{U}(2)} = \mathcal{H}_{\text{inst}}^{\mathcal{N}=2^*,\text{U}(2)}(\hat{x}_{ui}), \tag{4.33}$$

where $\hat{x}_{ui}$ denotes the "critical configuration" extremizing the "free energy" (4.31).

## 4.4 Saddle point equation

The extremality condition for the "action" $\mathcal{H}_{\text{inst}}^{\mathcal{N}=2^*,\text{U}(2)}$ given by (4.31) reads as follows:

$$\left| q \left( \prod_{v,j} \frac{(x_{ui} - x_{vj} - \epsilon_1)(x_{ui} - x_{vj}^0 + \epsilon_1)}{(x_{ui} - x_{vj}^0 - \epsilon_1)(x_{ui} - x_{vj} + \epsilon_1)} \frac{(x_{ui} - x_{vj} - \mu)(x_{ui} - x_{vj}^0 + \mu)}{(x_{ui} - x_{vj}^0 - \mu)(x_{ui} - x_{vj} + \mu)} \right. \right.$$
$$\left. \left. \times \frac{(x_{ui} - x_{vj} + \mu + \epsilon_1)(x_{ui} - x_{vj}^0 - \mu - \epsilon_1)}{(x_{ui} - x_{vj}^0 + \mu + \epsilon_1)(x_{ui} - x_{vj} - \mu - \epsilon_1)} \right) \left( \prod_v \frac{(x_{ui} - \mu - a_v)(x_{ui} + \mu + \epsilon_1 - a_v)}{(x_{ui} - a_v)(x_{ui} + \epsilon_1 - a_v)} \right) \right| = 1,$$

where $u, v = 1, 2$ and $i, j = 1, \ldots, \infty$. This implies that either the following eq.:

$$-q \left( \prod_{v,j} \frac{(x_{ui} - x_{vj} - \epsilon_1)(x_{ui} - x_{vj}^0 + \epsilon_1)}{(x_{ui} - x_{vj}^0 - \epsilon_1)(x_{ui} - x_{vj} + \epsilon_1)} \frac{(x_{ui} - x_{vj} - \mu)(x_{ui} - x_{vj}^0 + \mu)}{(x_{ui} - x_{vj}^0 - \mu)(x_{ui} - x_{vj} + \mu)} \right. \tag{4.34}$$
$$\left. \times \frac{(x_{ui} - x_{vj} + \mu + \epsilon_1)(x_{ui} - x_{vj}^0 - \mu - \epsilon_1)}{(x_{ui} - x_{vj}^0 + \mu + \epsilon_1)(x_{ui} - x_{vj} - \mu - \epsilon_1)} \right) \left( \prod_v \frac{(x_{ui} - \mu - a_v)(x_{ui} + \mu + \epsilon_1 - a_v)}{(x_{ui} - a_v)(x_{ui} + \epsilon_1 - a_v)} \right) = 1$$

or its analog in which $-q$ is replaced by $+q$ are holding.[13] Eq. (4.34) can be regularized assuming that there is an integer $L$ such that the length of the column $\omega_{ui}$ is equal to zero for $i > L$. Analyzing eq. (4.34) in such a case, i.e. when $j = 1, \ldots, L$, one can observe that the column lengths extremizing the "free energy" are of the order $\omega_{ui} \sim \mathcal{O}(q^i)$.[14] For

---

[13]In eq. (4.34) and below $q \equiv |q|$.

[14]Note that the saddle point equation (or in fact a system of equations) must be solved subject to the condition that the solution is consistent with the $q$-expansion of the twisted superpotential obtained from the instanton partition function. This implies that the order of $q$ in the expansions of rescaled column lengths must correlate with the index $i$. Indeed, first, we have $\mathcal{W}_{\text{inst}} \equiv \lim_{\epsilon_2 \to 0} \epsilon_2 \log \mathcal{Z}_{\text{inst}} = \sum_{i=1} \mathcal{W}_i \, q^i$. Then,

$$q \frac{d}{dq} \mathcal{W}_{\text{inst}} = \mathcal{W}_1 \, q + 2\mathcal{W}_2 \, q^2 + \ldots = \widetilde{\mathcal{W}}_1 \, q + \widetilde{\mathcal{W}}_2 \, q^2 + \ldots . \tag{4.35}$$

On the other hand

$$q \frac{d}{dq} \mathcal{W}_{\text{inst}} = \sum_{i=1} \sum_u \omega_{ui} = \sum_u \omega_{u1} + \sum_u \omega_{u2} + \ldots , \tag{4.36}$$

where $\omega_{ui}$'s are the column lengths of the critical diagrams (see the calculation at the beginning of the next subsection). Hence, in order to get from (4.36) the $q$-expansion consistent with (4.35) the expansions of $\omega_{ui}$'s must start from $q^i$.

example, at order $q^L$ one can write

$$\hat{x}_{ui} \equiv x_{ui} = a_u + (i-1)\epsilon_1 + \omega_{ui}(q) = a_u + (i-1)\epsilon_1 + \sum_{n=i}^{L} \omega_{uin}\, q^n. \qquad (4.37)$$

Here the symbols $\omega_{uin}$ denote the contributions to the coefficients $\omega_{ui}$ at the $n$-th order in $q$. Now it is possible to solve equation (4.34) starting from $L = 1$ and deriving recursively the $\omega_{uin}$'s step by step up to desired order. For instance,

$$\omega_{111} = \frac{\mu(2a-\mu)\,(\epsilon_1+\mu)\,(2a+\epsilon_1+\mu)}{2a\epsilon_1\,(2a+\epsilon_1)},$$

$$\omega_{211} = -\frac{\mu(2a+\mu)\,(\epsilon_1+\mu)\,(-2a+\epsilon_1+\mu)}{2a\epsilon_1\,(2a-\epsilon_1)}.$$

In order to investigate the solution of the saddle point equation in the case when $L \to \infty$ it is helpful to convert it to other equivalent form. Indeed, as has been observed in [69] (see also [40]) the eq. (4.34) can be rewritten in terms of certain "Y-system":

$$-q\frac{Y(x_{ui}-\epsilon_1)Y(x_{ui}-\mu)Y(x_{ui}+\mu+\epsilon_1)}{Y(x_{ui}+\epsilon_1)Y(x_{ui}+\mu)Y(x_{ui}-\mu-\epsilon_1)} = 1, \qquad (4.38)$$

where

$$Y(z) = \prod_{u=1}^{2} \exp\left\{\frac{z}{\epsilon_1}\,\psi\left(\frac{a_u}{\epsilon_1}\right)\right\} \prod_{i=1}^{\infty} \left(1 - \frac{z}{x_{ui}}\right) \exp\left\{\frac{z}{x_{ui}^0}\right\} \qquad (4.39)$$

and $\psi(z) = \partial_z \log\Gamma(z)$. The product in (4.39) is convergent for arbitrary $z \in \mathbb{C}$ provided that the column lengths tend to zero for large enough $i$, which is equivalent to the assumption that $x_{ui} \to x_{ui}^0$. If $\omega_{ui} = 0$ for all $i$, i.e. all column lengths are zero, then $Y(z)$ becomes

$$Y_0(z) = \prod_{u=1}^{2} \exp\left\{\frac{z}{\epsilon_1}\,\psi\left(\frac{a_u}{\epsilon_1}\right)\right\} \prod_{i=1}^{\infty} \left(1 - \frac{z}{x_{ui}^0}\right) \exp\left\{\frac{z}{x_{ui}^0}\right\}. \qquad (4.40)$$

The functions $Y(z)$, $Y_0(z)$ have zeros located at $x_{ui}$ and $x_{ui}^0$ respectively.

## 4.5 Twisted superpotential, classical block, accessory parameter

Now we are ready to compute the critical value of the "free energy" (4.33), i.e. the so-called twisted superpotential. It is convenient first to calculate the derivative of $\mathcal{W}_{\text{inst}}^{\mathcal{N}=2^*,\text{U}(2)}(q,a,m_i;\epsilon_1)$ with respect to $q$:

$$\frac{\partial}{\partial q}\,\mathcal{W}_{\text{inst}}^{\mathcal{N}=2^*,\text{U}(2)}(q,a,\mu;\epsilon_1) = \left(\frac{\partial\mathcal{H}_{\text{inst}}}{\partial x_{ui}}\frac{\partial x_{ui}}{\partial q} + \frac{\partial\mathcal{H}_{\text{inst}}}{\partial q}\right)\Bigg|_{x_{ui}=\hat{x}_{ui}} = \frac{1}{q}\sum_{u,i}\omega_{ui}. \quad (4.41)$$

Above we have used the fact that $\partial\mathcal{H}_{\text{inst}}/\partial x_{ui}|_{x_{ui}=\hat{x}_{ui}} = 0$. Hence, it is easy to realize that the last term in (4.41) coincides with the sum over the column lengths of the "critical" Young diagrams. More explicitly, the eq. (4.41) reads as follows

$$q\frac{\partial}{\partial q}\,\mathcal{W}_{\text{inst}}^{\mathcal{N}=2^*,\text{U}(2)} = \sum_i \left(\omega_{1i}(q) + \omega_{2i}(q)\right) = \sum_i \left[\sum_{n=i}^{\infty}\left(\omega_{1in} + \omega_{2in}\right)q^n\right]$$

$$= \left[\left(\omega_{111} + \omega_{211}\right)q + \left(\omega_{112} + \omega_{212}\right)q^2 + \ldots\right]$$

$$+ \left[\left(\omega_{122} + \omega_{222}\right)q^2 + \left(\omega_{123} + \omega_{223}\right)q^3 + \ldots\right] + \ldots. \quad (4.42)$$

Then,

$$\mathcal{W}^{\mathcal{N}=2^*,\mathrm{U}(2)}_{\mathrm{inst}} = (\omega_{111} + \omega_{211})\,q \,+\, (\omega_{112} + \omega_{212} + \omega_{122} + \omega_{222})\,\frac{q^2}{2} \,+\, \dots \,. \qquad (4.43)$$

Knowing the coefficients of the extremal lengths of the columns from a solution of the system of equations (4.34) one gets

$$\frac{1}{\epsilon_1}\,\mathcal{W}^{\mathcal{N}=2^*,\mathrm{U}(2)}_{\mathrm{inst}} = \frac{1}{\epsilon_1}\,(\omega_{111} + \omega_{211})\,q \,+\, \dots = \frac{1}{\epsilon_1}\,\left(\mathbf{f}^{\tilde{\delta},1}_{\delta} - 2\tilde{\delta}\right)\,q + \dots \,, \qquad (4.44)$$

where $\delta$, $\tilde{\delta}$ are given by (4.5)–(4.6) and $\mathbf{f}^{\tilde{\delta},n}_{\delta}$, $n = 1, \dots$ denote coefficients of the torus classical block (see (4.24)–(4.25)). Concluding, the eq. (4.44) is nothing but the expansion of both sides of the "classical" AGT relation:[15]

$$\mathbf{f}^{\tilde{\delta}}_{\delta}(q) = -2\tilde{\delta}\log\left(\frac{\eta(q)}{q^{\frac{1}{24}}}\right) \,+\, \frac{1}{\epsilon_1}\,\mathcal{W}^{\mathcal{N}=2^*,\mathrm{U}(2)}_{\mathrm{inst}}(q,a,\mu;\epsilon_1). \qquad (4.45)$$

As a final conclusion of this subsection let us write down the main result of the present work. Knowing the classical torus one-point block from (4.8) and applying eqs. (4.9) and (4.41), one arrives at the following expression of the Lamé accessory parameter:

$$\frac{\mathrm{B}(\tau)}{4\pi^2} = p_*^2(\xi,q,\bar{q}) + \frac{\tilde{\delta}}{12}\,(1 - 2E_2(\tau)) + \frac{1}{\epsilon_1}\sum_{u,i}\omega_{ui}(q,a,\mu,\epsilon_1) \qquad (4.46)$$

where

$$p_*(\xi,q,\bar{q}) = \frac{ia}{\epsilon_1}, \qquad \tilde{\delta} = -\kappa = -\frac{\mu}{\epsilon_1}\left(\frac{\mu}{\epsilon_1} + 1\right) = \frac{1}{4}\left(1 - \xi^2\right).$$

Hence, we have found that the accessory parameter B is related to the sum of column lengths of the "critical" Young diagrams. The latter can be rewritten using the contour integral representation. Indeed, let $\gamma$ denotes the contour which encloses all the points $\hat{x}_{ui}$, $x^0_{ui}$, $u = 1, 2$, $r = 1, \dots, \infty$. Then, as has been noticed by Poghossian in ref. [40], the sum $\sum_{u,i}\omega_{ui}$ can be expressed as follows

$$\sum_{u,i}\omega_{ui} = \sum_{u,i}\left(\hat{x}_{ui} - x^0_{ui}\right) = \oint_{\gamma}\frac{dz}{2\pi i}\,z\frac{\partial}{\partial z}\log\frac{Y(z)}{Y_0(z)}.$$

## 5 Concluding remarks and open problems

The main result of the present work is the expression of the Lamé accessory parameter B (or equivalently A, C, $C_{\mathrm{ell}}$) in terms of the solution of the TBA-like eq. (4.34). This equation has been solved by a power expansion in the parameter $q$. It has been noticed that obtained expression for B can be rewritten in terms of a contour integral in which the integrand is built out of the functions $Y$ and $Y_0$ introduced in eqs. (4.39)–(4.40). As has been observed in [40, 69] also the TBA-like eq. (4.34) can be rewritten in terms of the

---

[15]Of course, eqs. (4.45) and (4.8) are exactly the same.

function $Y$. Hence, one can relate the problem of finding the Lamé accessory parameter to the problem of searching a solution of the functional equation (4.38).

Another result presented in this paper is a check that the classical limit of the torus quantum conformal block exists and yields a consistent definition of the torus classical block. Moreover, it has been verified that the torus classical block corresponds to the twisted superpotential of the $\mathcal{N} = 2^*$ U(2) gauge theory.

Finally, in this paper has been proposed an expression for the Liouville classical action $S_{\mathrm{L}}^{\mathrm{torus}}$ on the one-punctured torus. It has been conjectured that $S_{\mathrm{L}}^{\mathrm{torus}}$ can be calculated in terms of the torus classical conformal block, classical three-point Liouville action and the saddle point classical intermediate weight (the saddle point momentum). We leave as an open question (to which we return very soon) a comparison of this result with the results obtained by Menotti in [29–32].

Work is in progress in order to verify formulae (3.31)–(3.34) and (4.46). First of all let us note that these formulae pave the way for numerical studies of the function $B(\tau)$ and the results of such investigation can be compared with that obtained by Keen, Rauch and Vasquez in [23]. It is well known that in a certain limit the Lamé equation becomes the Mathieu equation. Hence, other possible check beyond the WKB analysis performed in subsection 4.2 is to verify whether our candidate for the Lamé eigenvalue correctly reduces to the Mathieu eigenvalue, cf. [92].[16] As a final remark concerning possible combinatorial tests of our main result let us note that it would be valuable to check whether one can recover from eqs. (3.34), (4.9) the expansions of the Lamé eigenvalue worked out by Müller-Kirsten [93] (see also [94]) and Longmann [95, 96].[17]

It is a well known fact [23] that the Lamé accessory parameter is a modular form of weight 2. In order to answer the question whether proposed candidate for the Lamé accessory parameter has correct modular transformation properties one has to put a heuristic derivation of eqs. (3.34), (4.9), (4.46) on a more rigorous mathematical level. First, it has to be proved that in the classical limit the quantum conformal blocks behave exponentially.[18] Let us stress that there is still no rigorous proof of convergency of the expansions defining generic quantum conformal blocks. Secondly, it seems to be possible to derive the "classical" torus AGT relation (4.8) more rigorously exploiting the Teichmüller theory approach to Liouville theory. Indeed, relations such as (4.8) implicitly appear as a byproduct in the proof of the AGT correspondence recently proposed by Teschner and Vartanov [58].

---

[16]Let us stress that we have found an agreement between $B(\tau)$ and the energy eigenvalue E computed by means of the WKB method so far only for $\epsilon_1 = 0$. It remains still to calculate the "quantum corrections" to E in the higher powers of $\epsilon_1$ and compare the result with $B(\tau)$ obtained from the classical block.

[17]Note, that since the modular properties of the Lamé accessory parameter are known [23], then the relation (4.9), if it is true, encodes some information about the modular transformation properties of the $\mathcal{N} = 2^*$ U(2) twisted superpotential. This observation fits in an interesting line of research which aims to answer the question how the S-duality is realized in the $\Omega$-deformed $\mathcal{N} = 2$ gauge theories, see for instance [97, 98].

[18]It seems to be possible to prove the classical asymptotics of conformal blocks representing them as the Coulomb gas/Dotsenko-Fateev/$\beta$-ensemble integrals [99–102] and applying matrix models technics [24, 103–112].

## A   Special functions

**Jacobi and Weierstrass elliptic functions.**   Let $\omega_1$ and $\omega_2$ be two complex numbers whose ratio is not real. A function which satisfies $f(z) = f(z + 2\omega_1) = f(z + 2\omega_2)$ for all $z \in D(f(z))$ is called a *doubly periodic* function of $z$ with periods $2\omega_1$ and $2\omega_2$. A doubly periodic function that is analytic except at its poles and which has no singularities other than these poles in a finite part of the complex plane is called an *elliptic function*.

Let $\mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right) \in \mathbb{R}$, then the points $0$, $2\omega_1$, $2\omega_1 + 2\omega_2$, $2\omega_2$ when taken in order are the vertices of a parallelogram, known as the *fundamental parallelogram*. The behaviour of an elliptic function is completely determined by its values in fundamental parallelogram.

If we consider the points of the period lattice defined as $\mathcal{L} = \{2m\omega_1 + 2n\omega_2\}$, then the four points $2m\omega_1 + 2n\omega_2$, $2(m + 1)\omega_1 + 2n\omega_2$, $2(m + 1)\omega_1 + 2(n + 1)\omega_2$, $2m\omega_1 + 2(n + 1)\omega_2$ are vertices of a similar parallelogram, obtained from the fundamental parallelogram by a translation without rotations. This parallelogram is called a *period parallelogram*. The complex plane is covered by a system of non-overlapping period parallelograms.

If we wish to count the number of poles or zeros of an elliptic function in a given period parallelogram and it happens that certain poles or zeros lie on the boundaries of this parallelogram, then one can translate the period parallelogram without rotation until no pole or zero lies on its boundary. Such obtained parallelogram is called a *cell*.

In [1] one can find a list of general properties of elliptic functions. In particular, one can prove, that (i) the number of poles of an elliptic function in any cell is finite; (ii) the sum of the residues of an elliptic function at its poles in any cell is zero.

The number of poles of an elliptic function in any cell, counted with multiplicity, is called the *order* of the function. The statements (i)-(ii) written down above imply, that the order of an elliptic function is necessarily at least equal to 2. Indeed, an elliptic function of order 1 would have a single irreducible pole. If this were actually a pole its residue would not be zero. Hence, in terms of singularities, the simplest elliptic functions are those of order 2. These can be divided into two classes: (I) those which have a single irreducible double pole in each cell at which the residue is zero; (II) those which have two simple poles in each cell at which the two residues are equal in absolute value, but of opposite sign.

The *Jacobi elliptic functions* are examples of the second class of elliptic functions of order 2. The Jacobi function $\mathrm{sn}\, u$ is defined by means of the integral

$$u = \int\limits_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}},$$

where $k$ is a constant. By inversion of the integral we have $x = \mathrm{sn}\, u$. From definition follows that $\mathrm{sn}\, 0 = 0$.

The functions $\mathrm{cn}\, u$ and $\mathrm{dn}\, u$ are defined by the identities:

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1, \qquad k^2 \mathrm{sn}^2 u + \mathrm{dn}^2 u = 1.$$

It follows that $\mathrm{cn}\, 0 = \mathrm{dn}\, 0 = 1$.

Each of the Jacobi elliptic functions depends on a parameter $k$, called the *modulus*. In order to emphasize this dependence one can write the three functions as $\operatorname{sn}(u, k)$, $\operatorname{cn}(u, k)$, $\operatorname{dn}(u, k)$. An alternative notation: $\operatorname{sn}(u|m)$, $\operatorname{cn}(u|m)$, $\operatorname{dn}(u|m)$ uses a parameter $m = k^2$.

In accordance with the definition of an elliptic function the Jacobi elliptic functions are doubly periodic:

$$
\begin{aligned}
\operatorname{sn} u &= \operatorname{sn}(u + 4\mathcal{K}) = \operatorname{sn}\big(u + 4\mathcal{K} + 4i\mathcal{K}'\big) = \operatorname{sn}\big(u + 2i\mathcal{K}'\big), \\
\operatorname{cn} u &= \operatorname{cn}(u + 4\mathcal{K}) = \operatorname{cn}\big(u + 2\mathcal{K} + 2i\mathcal{K}'\big) = \operatorname{cn}\big(u + 4i\mathcal{K}'\big), \\
\operatorname{dn} u &= \operatorname{dn}(u + 2\mathcal{K}) = \operatorname{dn}\big(u + 4\mathcal{K} + 4i\mathcal{K}'\big) = \operatorname{dn}\big(u + 4i\mathcal{K}'\big).
\end{aligned}
$$

Periods are expressed in terms of the constants $\mathcal{K}$ and $\mathcal{K}'$ defined as follows

$$
\mathcal{K} \equiv K(k) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}, \qquad \mathcal{K}' \equiv K(k'),
$$

where $k'$ is the so-called *complementary modulus* defined by the relation $k^2 + k'^2 = 1$. The integral $K(k)$ is the complete elliptic integral of the first kind.

The derivatives of the Jacobi elliptic functions are

$$
\frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u, \qquad \frac{d}{du} \operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u, \qquad \frac{d}{du} \operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u.
$$

For practical reasons it is convenient to introduce a shortened notation to express reciprocals and quotients of the Jacobi elliptic functions. The reciprocals are denoted by reversing the orders of the letters of the function:

$$
\operatorname{ns} u = \frac{1}{\operatorname{sn} u}, \qquad \operatorname{nc} u = \frac{1}{\operatorname{cn} u}, \qquad \operatorname{nd} u = \frac{1}{\operatorname{dn} u}.
$$

Quotients are denoted by writing in order the first letters of the numerator and denominator functions:

$$
\begin{aligned}
\operatorname{sc} u &= \frac{\operatorname{sn} u}{\operatorname{cn} u}, & \operatorname{sd} u &= \frac{\operatorname{sn} u}{\operatorname{dn} u}, & \operatorname{cd} u &= \frac{\operatorname{cn} u}{\operatorname{dn} u}, \\
\operatorname{cs} u &= \frac{\operatorname{cn} u}{\operatorname{sn} u}, & \operatorname{ds} u &= \frac{\operatorname{dn} u}{\operatorname{sn} u}, & \operatorname{dc} u &= \frac{\operatorname{dn} u}{\operatorname{cn} u}.
\end{aligned}
$$

The *Weierstrass elliptic function* $\wp(z)$ belongs to the first class of elliptic functions of order 2, those with a single irreducible double pole in each cell with residue equal to zero. The function $\wp(z)$ is defined by the infinite sum

$$
\wp(z) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 - 2n\omega_2)^2} \right],
$$

where $2\omega_1$ and $2\omega_2$ are periods and the summation is taken over all (positive, negative and zero) integer values of $m$ and $n$, except for when $m, n$ are both equal to zero.

The function $\wp(z)$ satisfies the differential equation $(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3$, where the *elliptic invariants* $g_2$ and $g_3$ are given by

$$g_2 = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(2m\omega_1 + 2n\omega_2)^4}, \qquad g_3 = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(2m\omega_1 + 2n\omega_2)^6}.$$

Conversely, given $(dy/dz)^2 = 4y^3 - g_2y - g_3$, and if numbers $\omega_1$ and $\omega_2$ can be determined such that

$$g_2 = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(2m\omega_1 + 2n\omega_2)^4}, \qquad g_3 = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(2m\omega_1 + 2n\omega_2)^6}.$$

then the general solution of the differential equation is $y = \wp(z + \text{const.})$.

Let $e_1 = \wp(\omega_1)$, $e_2 = \wp(\omega_2)$, $e_3 = \wp(\omega_3)$, where $\omega_3 = -(\omega_1 + \omega_2)$. The constants $e_1$, $e_2$ and $e_3$ are mutually distinct and are roots of the equation $4y^3 - g_2y - g_3 = 0$. It follows that

$$e_1 + e_2 + e_3 = 0, \qquad e_2e_3 + e_3e_1 + e_1e_2 = -\frac{1}{4}g_2, \qquad e_1e_2e_3 = \frac{1}{4}g_3.$$

Let us write

$$y = e_3 + \frac{e_1 - e_3}{\operatorname{sn}^2(\lambda z, k)}.$$

Then, we have

$$\begin{aligned}
\left(\frac{dy}{dz}\right)^2 &= 4\lambda^2(e_1 - e_3)^2 \operatorname{ns}^2 \lambda z \operatorname{cs}^2 \lambda z \operatorname{ds}^2 \lambda z \\
&= 4\lambda^2(e_1 - e_3)^2 \operatorname{ns}^2 \lambda z \left(\operatorname{ns}^2 \lambda z - 1\right)\left(\operatorname{ns}^2 \lambda z - k^2\right) \\
&= 4\lambda^2(e_1 - e_3)^{-1}(y - e_3)(y - e_1)\left[y - k^2(e_1 - e_3) - e_3\right].
\end{aligned}$$

Hence, if $\lambda^2 = e_1 - e_3$ and $k^2 = (e_2 - e_3)/(e_1 - e_3)$, then $y$ satisfies the equation $(dy/dz)^2 = 4y^3 - g_2y - g_3$. Therefore,

$$e_3 + (e_1 - e_3)\operatorname{ns}^2\left(z(e_1 - e_3)^{\frac{1}{2}}, \left(\frac{e_2 - e_3}{e_1 - e_3}\right)^{\frac{1}{2}}\right) = \wp(z + h),$$

where $h$ is a constant. When $z \to 0$, it is seen that $h$ is a period, and so finally one gets the following relation between Weierstrass and Jacobi elliptic functions:

$$\wp(z) = e_3 + (e_1 - e_3)\operatorname{ns}^2\left(z(e_1 - e_3)^{\frac{1}{2}}, \left(\frac{e_2 - e_3}{e_1 - e_3}\right)^{\frac{1}{2}}\right). \tag{A.1}$$

The identity (A.1) allows to pass from the Jacobian form of the Lamé equation (1.1) to the Weierstrassian form (1.2). Indeed, the change of independent variable is given by $z = (u - i\mathcal{K}')(e_1 - e_3)^{-\frac{1}{2}}$. Then, we have[19]

$$\begin{aligned}
m\operatorname{sn}^2(u|m) &= m\operatorname{sn}^2\left(z(e_1 - e_3)^{\frac{1}{2}} + i\mathcal{K}'|m\right) \\
&= \operatorname{ns}^2\left(z(e_1 - e_3)^{\frac{1}{2}}|m\right) = \frac{\wp(z) - e_3}{e_1 - e_3}.
\end{aligned}$$

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[19] $k\operatorname{sn}(\alpha + i\mathcal{K}', k) = \operatorname{ns}(\alpha, k)$, see [1] (pg. 503, §22·34).

where $m \equiv k^2 = (e_2 - e_3)/(e_1 - e_3)$. Then, the transformed accessory parameter B in eq. (1.2) is given by

$$\mathrm{B} = \mathrm{A}(e_1 - e_3) - \kappa\, e_3 \quad \Leftrightarrow \quad \mathrm{A} = \frac{\mathrm{B}}{e_1 - e_3} - \frac{1}{3}\kappa(m+1).$$

**Weierstrass $\zeta$-function.** The *Weierstrass zeta function* $\zeta(z)$ is defined by the equation

$$\zeta'(z) = -\wp(z), \tag{A.2}$$

along with the condition $\lim_{z \to \infty}\left(\zeta(z) - \frac{1}{z}\right) = 0$.

In terms of the theta function $\theta_1(z|\tau)$ and the constant $\eta_1$ the Weierstrass zeta function can be expressed as follows

$$\zeta(z) = \frac{\partial_z \theta_1(z|\tau)}{\theta_1(z|\tau)} + 2\eta_1 z. \tag{A.3}$$

**Eisenstein series.** The first few *Eisenstein series* can be expressed as follows

$$E_2(q) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - \dots, \tag{A.4}$$

$$E_4(q) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n = 1 + 240q + 2160q^2 + 6720q^3 + \dots, \tag{A.5}$$

$$E_6(q) = 1 - 504\sum_{n=1}^{\infty} \sigma_5(n)q^n = 1 - 504q - 16632q^2 - 122976q^3 - \dots, \tag{A.6}$$

where $q = \mathrm{e}^{2\pi i \tau}$ and $\sigma_k(n)$ is the so-called *divisor function* defined as the sum of the $k$-th powers of the divisors of $n$, $\sigma_k(n) = \sum_{d|n} d^k$.

**Dedekind $\eta$-function.**

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n), \qquad q = \mathrm{e}^{2\pi i \tau}.$$

# B  Integrals $\mathscr{K}_n$

The integral

$$\mathscr{K}_1 = \oint_{\mathcal{A}} \wp(z)dz$$

one can compute directly using (A.2) and (A.3). The result is $\mathscr{K}_1 = -2\eta_1$. In order to compute the integrals

$$\mathscr{K}_n = \oint_{\mathcal{A}} \wp(z)^n dz, \qquad n > 1$$

one can employ the following relation between powers of $\wp$ and its even derivatives (see [113] and refs. therein):

$$\wp^n = B_n^{(n)} + \sum_{r=0}^{n-1} \frac{B_r^{(n)}}{(2n - 2r - 1)!}\, \wp^{(2n-2-2r)}. \tag{B.1}$$

Quantities $B_r^{(n)}$ above — the so-called *Halphen coefficients* — are given by the recurrence relation:

$$B_r^{(n)} = \frac{(2n - 2r - 2)(2n - 2r - 1)}{(2n - 2)(2n - 1)} B_r^{(n-1)} + \frac{2n - 3}{4(2n - 1)} B_{r-2}^{(n-2)} g_2 + \frac{n - 2}{2(2n - 1)} B_{r-3}^{(n-3)} g_3$$

with $n > 0$, $r = 0, \ldots, n$; $B_r^{(n)} = 0$ for $r < 0$ or $r > n$; $B_0^{(n)} = 1$ and $B_1^{(n)} = 0$ for any $n$. $B_r^{(n)}$'s are then polynomials in $g_2$, $g_3$ with rational positive coefficients.

For $n = 2, 3, 4$ the eq. (B.1) yields

$$\wp^2 = B_2^{(2)} + \frac{1}{6} B_0^{(2)} \wp^{(2)} + B_1^{(2)} \wp = \frac{1}{12} g_2 + \frac{1}{6} \wp^{(2)},$$

$$\wp^3 = B_3^{(3)} + \frac{1}{120} B_0^{(3)} \wp^{(4)} + \frac{1}{6} B_1^{(3)} \wp^{(2)} + B_2^{(3)} \wp = \frac{1}{10} g_3 + \frac{1}{120} \wp^{(4)} + \frac{3}{20} g_2 \wp,$$

$$\wp^4 = B_4^{(4)} + \frac{1}{5040} B_0^{(4)} \wp^{(6)} + \frac{1}{120} B_1^{(4)} \wp^{(4)} + \frac{1}{6} B_2^{(4)} \wp^{(2)} + B_3^{(4)} \wp$$

$$= \frac{5}{336} g_2^2 + \frac{1}{5040} \wp^{(6)} + \frac{1}{30} g_2 \wp^{(2)} + \frac{1}{7} g_3 \wp.$$

Then,

$$\mathscr{K}_2 = \frac{1}{12} g_2,$$

$$\mathscr{K}_3 = \frac{1}{10} g_3 + \frac{3}{20} g_2 \mathscr{K}_1 = \frac{1}{10} g_3 - \frac{3}{10} g_2 \eta_1,$$

$$\mathscr{K}_4 = \frac{5}{336} g_2^2 + \frac{1}{7} g_3 \mathscr{K}_1 = \frac{5}{336} g_2^2 - \frac{2}{7} g_3 \eta_1.$$

Now, in order to get the coefficients (4.18)–(4.21) of the expansion of the energy eigenvalue one has to use the following relations:

$$\eta_1 = \frac{4\pi^2}{24} E_2, \qquad g_2 = \frac{4\pi^4}{3} E_4, \qquad g_3 = \frac{8\pi^6}{27} E_6.$$

## C  Coefficients of the torus quantum conformal block

$$\mathcal{F}_{c,\Delta}^{\Delta_\beta, 1} = \frac{(\Delta_\beta - 1) \Delta_\beta}{2\Delta} + 1,$$

$$\mathcal{F}_{c,\Delta}^{\Delta_\beta, 2} = \left[ 4\Delta \left( 2c\Delta + c + 16\Delta^2 - 10\Delta \right) \right]^{-1}$$
$$\left[ \left( 8c\Delta + 3c + 128\Delta^2 + 56\Delta \right) \Delta_\beta^2 + \left( -8c\Delta - 2c - 128\Delta^2 \right) \Delta_\beta \right.$$
$$\left. + (c + 8\Delta)\Delta_\beta^4 + (-2c - 64\Delta)\Delta_\beta^3 + 16c\Delta^2 + 8c\Delta + 128\Delta^3 - 80\Delta^2 \right].$$

$$\cdots \qquad .$$

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