

Ground states of S-duality twisted $N = 4$ super Yang-Mills theory

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ABSTRACT: We study the low-energy limit of a compactification of $N = 4$ $U(n)$ super Yang-Mills theory on S^1 with boundary conditions modified by an S-duality and R-symmetry twist. This theory has $N = 6$ supersymmetry in 2+1D. We analyze the T^2 compactification of this 2+1D theory by identifying a dual weakly coupled type-IIA background. The Hilbert space of normalizable ground states is finite-dimensional and appears to exhibit a rich structure of sectors. We identify most of them with Hilbert spaces of Chern-Simons theory (with appropriate gauge groups and levels). We also discuss a realization of a related twisted compactification in terms of the $(2, 0)$ -theory, where the recent solution by Gaiotto and Witten of the boundary conditions describing D3-branes ending on a (p, q) 5-brane plays a crucial role.

KEYWORDS: Duality in Gauge Field Theories, String Duality, Brane Dynamics in Gauge Theories, Chern-Simons Theories

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1 Introduction

In this paper we explore a new connection between S-duality and pure Chern-Simons theory. In the context of S-duality, Chern-Simons theory has already appeared in the work of Gaiotto and Witten [1] on the action of S-duality on boundary conditions. Gaiotto and Witten studied four dimensional $\mathcal{N} = 4$ U(n) Super Yang-Mills theory (SYM) formulated on a manifold with a boundary. They allowed additional degrees of freedom to be localized on the boundary and to couple to the bulk $\mathcal{N} = 4$ SYM fields, thereby generating a rich class of possibilities for boundary conditions, generalizing the standard Dirichlet and Neumann ones [2]. Chern-Simons couplings (either involving the bulk gauge fields or boundary gauge fields) are an optional additional ingredient that was included in their discussion, and S-duality can generate such couplings.

In this paper we will also study $\mathcal{N} = 4$ SYM with a novel type of boundary conditions, but these will be periodic boundary conditions that involve S-duality at the outset. We formulate $\mathcal{N} = 4$ U(n) SYM on $S^1 \times \mathbb{R}^{2,1}$ but include an S-duality and R-symmetry twist along S^1 . The S-duality twist is the novel feature, which is allowed for the special value $\tau = i$ of the coupling constant

$$\tau \equiv \frac{4\pi i}{g_{\text{YM}}^2} + \frac{\theta}{2\pi}.$$

($\tau = i$ is the only value invariant under $\tau \rightarrow -1/\tau$.) The S-duality twist is inspired by similar exotic twists that have appeared in different contexts before [3–8]. (The authors of [9] coined the term *monodrofolds* for such twists.)

We are interested in the low-energy limit of this setting, as the S^1 shrinks to zero. In this limit, roughly speaking, all that's left is the S-duality and R-symmetry twists. For example, in Euclidean signature (replacing $\mathbb{R}^{2,1} \rightarrow \mathbb{R}^3$) we can think of the S^1 direction as Euclidean time and define correlation functions of operators $\mathcal{O}_1, \mathcal{O}_2, \dots$, in the theory as

$$\langle \mathcal{O}_1 \mathcal{O}_2 \dots \rangle \equiv \text{tr}((-1)^F \widehat{\mathcal{S}} \widehat{\mathcal{R}} e^{-2\pi R H} \mathcal{O}_1 \mathcal{O}_2 \dots),$$

where $\widehat{\mathcal{S}}$ is the S-duality operator, $\widehat{\mathcal{R}}$ is the R-twist operator, H is the Hamiltonian of $\mathcal{N} = 4$ SYM, $2\pi R$ is the circumference of S^1 , and F is the fermion number [and $(-1)^F$ is a central element of the R-symmetry group $SU(4)$].¹ In the limit $R \rightarrow 0$ (with an appropriate treatment of zero-modes as will be discussed later), $\text{tr}((-1)^F \widehat{\mathcal{S}} \widehat{\mathcal{R}} \mathcal{O}_1 \mathcal{O}_2 \dots)$ is all that remains, and the theory probes the S-duality operator through $(-1)^F \widehat{\mathcal{S}} \widehat{\mathcal{R}}$. This is the main reason why we are interested in this problem.

Since abelian S-duality is completely understood (see, e.g., [10]), the solution of our problem for $U(1)$ gauge group is straightforward. As we explain in section 5, the resulting low-energy description is a pure Chern-Simons theory with gauge group $U(1)$ at level $k = 2$. (We can get other levels, $k = 1, 3$, if we replace the S-duality twist that realizes $\tau \rightarrow -1/\tau$ with other elements \mathbf{g} of the duality group $SL(2, \mathbb{Z})$ for which a self-dual coupling constant exists.) The question that we would like to raise at this point is: how does this statement generalize to nonabelian gauge groups?

Given the results for $U(1)$ gauge group, a naïve conjecture would suggest that the low-energy theory is the nonabelian Chern-Simons theory at the same level as in the abelian case [11]. We find, however, that the nonabelian theories present a somewhat richer picture than their abelian counterparts.

The main tool that we will use in this paper is a weakly coupled type-IIA dual of the problem. To arrive at this dual, we start with type-IIB string theory, where $\mathcal{N} = 4$ $U(n)$ SYM is naturally realized as the low-energy description of coincident D3-branes [12], and S-duality of the gauge theory descends from S-duality of the full string theory. The latter can be realized as a geometrical symmetry in a dual string theory [13, 14]. In order to utilize this geometrical description of S-duality, we compactify the theory on T^2 (replacing $\mathbb{R}^{2,1} \rightarrow T^2 \times \mathbb{R}$) and look for the ground states. Realizing the theory on n D3-branes in type-IIB string theory, we can map the theory to a type-IIA setting where the question of identifying the ground states reduces to an easily solvable geometrical problem. With sufficient supersymmetry, the solution of the geometrical problem after duality also solves our original problem. This allows us to calculate the Witten Index and analyze the space of ground states and its symmetries in terms of the type-IIA dual background.

Based on this analysis, we argue that (for low enough rank of the gauge group) the Hilbert space of ground states decomposes into a direct sum of Hilbert spaces of Chern-Simons theories with appropriate gauge groups and levels. In particular, there exists a

¹We are grateful to E. Witten for pointing out the missing $(-1)^F$ in a previous version.

distinguished sector (which we call the $[\sigma]$ -untwisted sector) that is equivalent to the Hilbert space of nonabelian Chern-Simons theory with gauge group $U(n)$ and level $(2n, 2)$, where $2n$ refers to the $U(1)$ center, and 2 refers to $SU(n)$. By “equivalent Hilbert spaces” we mean that their symmetry operators and their behavior under modular transformations of T^2 match. Our results then suggest that in the decompactification limit $T^2 \times \mathbb{R} \rightarrow \mathbb{R}^{2,1}$ the Hilbert space of the low-energy theory decomposes into different superselection sectors, each described by an appropriate Chern-Simons theory. We were also able to extend much of this picture to the compactifications with twists by other elements \mathbf{g} of the duality group $SL(2, \mathbb{Z})$, except for a certain problematic issue that arises for $n \geq 4$ and remains unresolved.

The paper is organized as follows. In section 2 we explain the problem in detail. We discuss the S-duality twist, the various other $SL(2, \mathbb{Z})$ elements that can be used to construct twists, the R-symmetry twist, the amount of supersymmetry that is preserved, elimination of zero-modes, and restrictions on the rank n of the gauge group $U(n)$.

In section 3 we compactify the theory on T^2 and find the weakly coupled type-IIA dual. We describe in detail the U-duality element that maps the problem to a geometrical one, and discuss various conserved quantum numbers that can be defined in the geometrical setting.

In section 4, we study as a warm-up exercise a simpler problem of compactification with charge-conjugation twist (C-twist). This serves as an illustration of ideas developed in previous sections as well as methods that we will employ in later sections when we attack our main problem, the S-duality twist.

In section 5 we solve the problem for $U(1)$ gauge group explicitly, and calculate the level k of the low-energy (pure) Chern-Simons theory. We then compactify on T^2 and compare the Hilbert space of ground states of abelian Chern-Simons theory to the Hilbert space of ground states of the type-IIA dual. We identify the type-IIA dual of Wilson loop operators as well as other symmetries of the ground states.

In section 6 we study the ground states of the nonabelian problem [with $U(n)$ gauge group] on T^2 , using the type-IIA dual theory. We show that the Hilbert space of ground states decomposes into a direct sum of Hilbert spaces, which in most cases we are able to identify as the Hilbert spaces of Chern-Simons theory with appropriate gauge groups [subgroups of $U(n)$] and appropriate Chern-Simons levels.

In section 7 we take another look at our problem in terms of the $(2, 0)$ theory. We argue that the solution can be constructed from ingredients that recently appeared in the work of Gaiotto and Witten [1] in connection with the low-energy description of D3-branes that end on (p, q) 5-branes. We show how to recover the $U(1)$ result from these ingredients.

We conclude with a discussion of the results and open problems in section 8.

2 The problem

We wish to learn new facts about the $SL(2, \mathbb{Z})$ S-duality of $\mathcal{N} = 4$ super Yang-Mills theory by studying a circle compactification of the theory with unconventional boundary conditions as follows. Realizing the circle as the segment $[0, 2\pi R]$ with endpoints 0 and $2\pi R$ identified, we require the configuration at $2\pi R$ to be an S-dual of the configuration at

0. We will refer to this kind of boundary conditions as an *S-twist*. To be specific, we need to pick an element $\mathbf{g} \in \text{SL}(2, \mathbb{Z})$, and we need the coupling constant to be invariant under \mathbf{g} . There are only a small number of possibilities of this kind, which we will list in section 2.2.

The S-twist would be easy to describe if we knew a formulation of $\mathcal{N} = 4$ super Yang-Mills theory for which S-duality is manifest. Nevertheless, it is not hard to argue that the S-twist is consistent. For example, in Euclidean signature we can take the direction of the circle to be Euclidean time, and the S-twist then corresponds to an insertion of the operator corresponding to \mathbf{g} on the Hilbert space of states, thus obtaining $\text{tr}((-1)^F \mathbf{g} e^{-2\pi R H} \dots)$, where H is the Hamiltonian, (\dots) represents additional insertions of local operators if desired, and by a slight abuse of notation we used the same \mathbf{g} to denote the action of \mathbf{g} on the Hilbert space at the self-dual coupling constant. In section 3 we bring more evidence for the consistency of the S-twist: we present a string-theoretic construction with an S-twist, and show that it is dual to a conventional type-IIA string compactification.

In order to preserve some amount of supersymmetry, we also need to pick an appropriate nontrivial element γ of the R-symmetry group and identify the configuration at $2\pi R$ with the γ -transformed \mathbf{g} -dual of the configuration at 0. For a suitable choice of γ we can preserve 12 supersymmetry generators, which corresponds to $\mathcal{N} = 6$ in three dimensions.

Our problem is to find the effective three-dimensional low-energy description of the theory in the limit $R \rightarrow 0$. We propose that for a sufficiently low rank n (how low depends on \mathbf{g}), the requisite three-dimensional field theory is topological, and in the next sections we will study it in special cases.

The rest of this section provides more details on the construction above. In section 2.1 we introduce the notation for the rest of this paper; in section 2.2 we discuss the various choices for \mathbf{g} (there are only three) and the corresponding self-dual coupling constants. In section 2.3 we discuss the associated R-symmetry twist γ ; and in section 2.4 we introduce restrictions on the rank n of the gauge group that are necessary to eliminate unwanted low-energy moduli. These details are a condensed version of the discussion that can be found in [11].

2.1 $\mathcal{N} = 4$ super Yang-Mills: notation

Our starting point is four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $U(n)$.

We denote the complex coupling constant by

$$\tau \equiv \frac{4\pi i}{g_{\text{YM}}^2} + \frac{\theta}{2\pi}.$$

It transforms under an element

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \tag{2.1}$$

as

$$\tau \rightarrow \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}}.$$

Our notation for the fields of $U(n)$ $\mathcal{N} = 4$ super Yang-Mills theory is summarized below:

A_μ	gauge field	$\mu = 0, \dots, 3,$
Φ^I	adjoint-valued scalars	$I = 1, \dots, 6,$
ψ_α^a	adjoint-valued spinors	$a = 1, \dots, 4$ and $\alpha = 1, 2,$
$\overline{\psi}_{a\dot{\alpha}}$	complex conjugate spinors	$a = 1, \dots, 4$ and $\dot{\alpha} = \dot{1}, \dot{2},$
$Q_{a\alpha}$	SUSY generators	$a = 1, \dots, 4$ and $\alpha = 1, 2,$
$\overline{Q}_{\dot{\alpha}}^a$	complex conjugate generators	$a = 1, \dots, 4$ and $\dot{\alpha} = \dot{1}, \dot{2}.$

We also define the complex combinations of scalar fields

$$Z^j \equiv \Phi^j + i\Phi^{3+j}, \quad j = 1, 2, 3. \quad (2.2)$$

The S^1 on which we compactify is in direction 3.

2.2 S-duality twist

To define the S-duality twist we need a pair (\mathbf{g}, τ) comprising of an element $\mathbf{g} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ and a self-dual coupling constant τ , satisfying

$$\tau = \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}}. \quad (2.3)$$

Assuming $\mathbf{c} \neq 0$, (2.3) is equivalent to the quadratic equation $\mathbf{c}\tau^2 + (\mathbf{d} - \mathbf{a})\tau - \mathbf{b} = 0$, and if it has solutions away from the real axis, they must satisfy $|\tau|^2 = -\mathbf{b}/\mathbf{c}$ and $\tau + \overline{\tau} = (\mathbf{a} - \mathbf{d})/\mathbf{c}$, which implies that $|\mathbf{c}\tau + \mathbf{d}|^2 = 1$. We can therefore set

$$\mathbf{c}\tau + \mathbf{d} = e^{iv} \quad (2.4)$$

for some real phase v . It follows that $\cos v = \mathbf{d} + \mathbf{c}(\tau + \overline{\tau})/2 = (\mathbf{a} + \mathbf{d})/2$ can only take the values 0 or $\pm 1/2$, and so v is one of $\pm \frac{1}{2}\pi, \pm \frac{1}{3}\pi, \pm \frac{2}{3}\pi$. Furthermore, it is easy to check that the eigenvalues of \mathbf{g} are $e^{\pm iv}$, and thus \mathbf{g} has finite order, which can be of the three possibilities $\mathbf{r} = 3, 4, 6$. Thus,

$$|v| = \frac{2\pi}{\mathbf{r}}. \quad (2.5)$$

Up to conjugation $[\mathbf{g} \rightarrow \mathbf{g}_0^{-1} \mathbf{g} \mathbf{g}_0$ for some $\mathbf{g}_0 \in \text{SL}(2, \mathbb{Z})]$ and inversion $(\mathbf{g} \rightarrow \mathbf{g}^{-1})$, we are left with the following three choices for (\mathbf{g}, τ) :

1. $\tau = i$ and $\mathbf{g} = \mathbf{g}' \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of order $\mathbf{r} = 4$ ($v = \frac{1}{2}\pi$);
2. $\tau = e^{\pi i/3}$ and $\mathbf{g} = \mathbf{g}'' \equiv \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ of order $\mathbf{r} = 6$ ($v = \frac{1}{3}\pi$);
3. $\tau = e^{\pi i/3}$ and $\mathbf{g} = -\mathbf{g}''^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ of order $\mathbf{r} = 3$ ($v = \frac{2}{3}\pi$).

All other possible \mathbf{g} 's are $\text{SL}(2, \mathbb{Z})$ -conjugate to those in the list, or their inverses (which, as we will see, give theories that are physically equivalent after a parity transformation).

We recall [15] that \mathbf{g} acts nontrivially on the supercharges

$$\mathbf{g} : Q_{a\alpha} \rightarrow \left(\frac{\mathbf{c}\tau + \mathbf{d}}{|\mathbf{c}\tau + \mathbf{d}|} \right)^{-\frac{1}{2}} Q_{a\alpha} = e^{-\frac{i\nu}{2}} Q_{a\alpha}. \quad (2.6)$$

In order to get a supersymmetric theory, we therefore need to supplement the S-twist with an R-symmetry twist so that the phase in (2.6) is cancelled.

2.3 R-symmetry twist

An R-symmetry twist modifies the periodic boundary conditions by introducing additional phases for R-charged fields. It is a useful tool to eliminate unwanted zero modes while preserving some amount of supersymmetry (see for instance [16–18]). In our context it also allows us to restore some of the supersymmetry that was lost by the S-twist.

We pick a basis of the R-symmetry group $\text{SU}(4)$ so that a diagonal element

$$\gamma \equiv \begin{pmatrix} e^{i\varphi_1} & & & \\ & e^{i\varphi_2} & & \\ & & e^{i\varphi_3} & \\ & & & e^{i\varphi_4} \end{pmatrix} \in \text{SU}(4)_R, \quad \left(\sum_a \varphi_a = 0 \right), \quad (2.7)$$

acts on the fermionic fields from section 2.1 as

$$\gamma(\psi_\alpha^a) = e^{i\varphi_a} \psi_\alpha^a, \quad \gamma(\bar{\psi}_{a\dot{\alpha}}) = e^{-i\varphi_a} \bar{\psi}_{a\dot{\alpha}}, \quad a = 1, \dots, 4,$$

and on the bosonic fields as

$$\gamma(A_\mu) = A_\mu, \quad \gamma(Z^j) = e^{i(\varphi_j + \varphi_4)} Z^j, \quad j = 1, 2, 3.$$

A γ -twist, on its own, modifies the boundary conditions to

$$\psi_\alpha^a(x_0, x_1, x_2, x_3 + 2\pi R) = e^{i\varphi_a} \Lambda^{-1} \psi_\alpha^a(x_0, x_1, x_2, x_3) \Lambda, \quad (2.8)$$

$$Z^j(x_0, x_1, x_2, x_3 + 2\pi R) = e^{i(\varphi_j + \varphi_4)} \Lambda^{-1} Z^j(x_0, x_1, x_2, x_3) \Lambda, \quad (2.9)$$

$$A_\mu(x_0, x_1, x_2, x_3 + 2\pi R) = \Lambda^{-1} A_\mu(x_0, x_1, x_2, x_3) \Lambda + \Lambda^{-1} \partial_\mu \Lambda, \quad (2.10)$$

where Λ is an arbitrary gauge transformation. We combine the R-symmetry twist by γ with the S-twist from section 2.2 to get an S-R-twist. It can be formally defined by switching to Euclidean signature, considering the direction x_3 as Euclidean time, and defining correlation functions of operators, similarly to the discussion at the top of section 2, by $\text{tr}((-1)^F \gamma \mathbf{g} e^{-2\pi R H} \dots)$, where F is the fermion number, H is the Hamiltonian, (\dots) represents insertions of local operators if desired, and γ is the R-symmetry operator (in a slight abuse of notation we here denote the representation of γ on the Hilbert space by the same letter).

Combining γ with (2.6), we find the action of $\gamma \mathbf{g}$ on the supercharges:

$$\gamma \mathbf{g}(Q_{a\alpha}) = e^{i(\varphi_a - \frac{\nu}{2})} Q_{a\alpha}.$$

Therefore, to preserve $\mathcal{N} = 2$ supersymmetry in three-dimensions, for example, we need to set one of the φ_a , say φ_1 , to $\frac{1}{2}v$. The maximal amount of supersymmetry that we can preserve is $\mathcal{N} = 6$ with

$$\gamma = \begin{pmatrix} e^{\frac{i}{2}v} & & & \\ & e^{\frac{i}{2}v} & & \\ & & e^{\frac{i}{2}v} & \\ & & & e^{-\frac{3i}{2}v} \end{pmatrix} \in \text{SU}(4)_R. \tag{2.11}$$

We will work with that choice of γ from now on.

2.4 Low-energy limit

Our goal is to study the low-energy description of the compactification of $\mathcal{N} = 4$ $\text{U}(n)$ SYM on S^1 with a combination of S-duality twisted boundary conditions as in section 2.2 and R-symmetry twisted boundary conditions as in section 2.3. With the choice of γ as in (2.11), the theory has $\mathcal{N} = 6$ supersymmetry in 2+1D. In this paper we further wish to restrict the parameters so as to get a *topological* QFT in 2+1D, for which the supersymmetry is realized trivially — all generators are identically zero at low-energy (which is only possible for a topological theory for which the momentum and Hamiltonian are also zero).

This restriction requires that no massless propagating fields shall survive at low energy. For a $\text{U}(1)$ gauge group we will see in section 5 that the low-energy limit is $\text{U}(1)$ Chern-Simons theory, and indeed no low-energy propagating degrees of freedom survive; the mass gap of our setting is of the order of the Kaluza-Klein scale $1/R$. However, in the nonabelian case, $n > 1$, the S-duality twist is poorly understood, and it is less clear whether our setting has a mass gap or not. In fact, we will argue in section 6 that in general our S^1 compactification has several discrete choices leading to separate superselection sectors, each defining a different low-energy limit. Some superselection sectors come with a mass gap, while others do not.

In this section, however, we will introduce a necessary requirement — that no non-compact moduli survive the compactification to 2+1D. This requirement seems sufficient to ensure that the additional compactification on T^2 (to 0+1D), which we will study later on, leads to a discrete spectrum. So, we must start by eliminating the potential zero modes arising from the dimensional reduction of the scalar fields.

To see what that entails, let us attempt to construct a massless degree of freedom by starting at a generic point on the Coulomb branch of $\mathcal{N} = 4$ SYM in 3+1D, where the gauge group is broken to $\text{U}(1)^n$. The 3+1D low energy physics is described by n free $\mathcal{N} = 4$ vector multiplets, and the residual gauge symmetry is the permutation group S_n that permutes the n vector multiplets. If the energy scale at which the $\text{U}(n)$ gauge symmetry is broken (which is determined by the differences between the VEVs of the scalar components of the vector multiplets) is much larger than the compactification scale $1/R$, we can approximate the low-energy theory by simply compactifying the n free vector multiplets on S^1 with the R-symmetry and S-duality twists.

Compactification of a single vector multiplet with R-symmetry and S-duality twists leaves no massless fields in 2+1D. To see this, consider the gauge field and the scalars and

fermions separately. Only the S-twist affects the gauge field, and only the R-twist affects the scalars and fermions. The gauge fields will, at best, give rise to compact moduli, but for a single vector multiplet they do not produce any moduli at all. This is because for a massless mode to exist in three dimensions, we need a solution² where the electric and magnetic fields E_i, B_i (both three-dimensional vectors with $i = 1, 2, 3$) are independent of x_3 and satisfy

$$\begin{pmatrix} E_i \\ B_i \end{pmatrix} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \begin{pmatrix} E_i \\ B_i \end{pmatrix}, \tag{2.12}$$

as required by the S-twist. But since $\mathbf{g} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$ has nontrivial eigenvalues $e^{\pm i\nu}$, there is no nonzero solution to (2.12), and no massless fields arise from the gauge fields. For scalar zero modes we would look for solutions to [see (2.9) and (2.11)]:

$$Z^j = e^{i(\varphi_j + \varphi_4)} Z^j = e^{-i\nu} Z^j, \quad j = 1, 2, 3, \tag{2.13}$$

which has no nonzero solutions. Similarly, there are no fermion zero modes, which of course follows from supersymmetry. So, a single vector multiplet compactified with an S-R-twist does not have any low-energy propagating degrees of freedom.

However, as we shall now see, for $n \geq \mathbf{r}$ (where \mathbf{r} was defined in (2.5) as $2\pi/\nu$) we do get massless propagating degrees of freedom. To see this, note that the boundary conditions in (2.8)–(2.10) have an optional $U(n)$ gauge transformation Λ . Once the gauge group is broken as $U(n) \rightarrow U(1)^n$, we are only allowed to take Λ in the normalizer of $U(1)^n$ in $U(n)$, which is the permutation group S_n . We thus identify Λ with some permutation $\sigma \in S_n$ and modify the conditions for zero modes (2.12)–(2.13) to

$$\begin{pmatrix} E_i^{(\sigma(l))} \\ B_i^{(\sigma(l))} \end{pmatrix} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \begin{pmatrix} E_i^{(l)} \\ B_i^{(l)} \end{pmatrix}, \quad i = 1, \dots, 3, \quad l = 1, \dots, n, \tag{2.14}$$

and

$$Z^{j, \sigma(l)} = e^{-i\nu} Z^{j, l}, \quad j = 1, \dots, 3, \quad l = 1, \dots, n, \tag{2.15}$$

where the superscript l corresponds to the l^{th} $U(1)$ factor in $U(1)^n$ and the permutation σ maps $\{1, 2, \dots, n\}$ to $\{\sigma(1), \sigma(2), \dots, \sigma(n)\}$. Equations (2.14)–(2.15) have nonzero solutions if and only if σ has a cycle of length divisible by \mathbf{r} , and such a $\sigma \in S_n$ exists if and only if $n \geq \mathbf{r}$.

At the end of section 2.2 we listed various possible values of \mathbf{g} and \mathbf{r} . The corresponding restrictions on the rank n of the gauge group are therefore: $n \leq 2$ for the case with $\mathbf{r} = 3$; $n \leq 3$ for $\mathbf{r} = 4$; and $n \leq 5$ for $\mathbf{r} = 6$. For these cases there are no obvious zero modes, and we are going to assume that the low-energy theory has no noncompact moduli for $n < \mathbf{r}$.

²We will study in greater detail the resulting low-energy limit later in section 5, but for the purposes of the discussion in this section it suffices to look for classical solutions.

Type	Brane	1	2	3	4	5	6	7	8	9	10	Apply:
IIB	D3	–	–	÷							×	T-duality on 1.
IIA	D2		–	÷							×	Lift to M-theory.
M	M2		–	÷								Reduction to IIA on 2.
IIA	F1		×	÷								

Table 1. The sequence of dualities from n D3-branes in type-IIB to n fundamental strings in type-IIA. A direction that the corresponding brane or string wraps with periodic boundary conditions is represented by –, a direction that the object wraps with twisted boundary conditions is represented by ÷, and a dimension that doesn't exist in the particular string theory is represented by ×. All the branes in the table are at the origin of directions 4, . . . , 9.

3 Type-IIA dual

The setting of section 2 has a string-theoretic realization in terms of D3-branes of type-IIB string theory. We start with the background $\mathbb{R}^{9,1}$ with Cartesian coordinates x_0, \dots, x_9 , and place n D3-branes at $x_4 = x_5 = \dots = x_9 = 0$. The type-IIB coupling constant is denoted by $\tau = \chi + \frac{i}{g_{\text{IIB}}}$, where g_{IIB} is the string coupling constant, and χ is the R-R scalar. The S-duality transformation \mathbf{g} of section 2.2 then lifts to an S-duality transformation of the full type-IIB string theory (that we also denote by \mathbf{g}), and the R-symmetry rotation γ of section 2.3 lifts to a geometrical rotation in the 6 directions transverse to the D3-branes. We will now transform this background, using string dualities, to one where S-duality is realized geometrically.

We first compactify the x_3 -direction on a circle of radius $2\pi R$ with boundary conditions given by a simultaneous S-duality twist \mathbf{g} and a $\gamma \in \text{Spin}(6)$ geometrical twist in directions x_4, \dots, x_9 , where γ is given by (2.11) in terms of ν , and ν by (2.4). This means that as we traverse the x_3 circle once, we also apply a $\gamma \in \text{Spin}(6)$ rotation in the transverse directions before gluing $x_3 = 0$ to $x_3 = 2\pi R$, similarly to the discussion in section 2.3. We then compactify directions x_1, x_2 , so that $0 \leq x_1 < 2\pi L_1$ and $0 \leq x_2 < 2\pi L_2$ are periodic. This puts the 2+1D field theory on T^2 with area $4\pi^2 L_1 L_2$ and complex structure iL_2/L_1 .

Now we can study different limits of the parameters L_1, L_2, R . First, to reproduce the field-theory problem of section 2 we need to take the limit

$$L_1, L_2, R \gg \alpha'^{1/2}, \tag{3.1}$$

where $\alpha'^{1/2}$ is the type-IIB string scale. In the limit (3.1), we can first reduce the description of the D3-branes to $\mathcal{N} = 4$ U(n) SYM at low energy, and then compactify $\mathcal{N} = 4$ SYM with an S-duality and R-symmetry twist.

We now consider the opposite limit $L_1, L_2 \rightarrow 0$ with $R \rightarrow \infty$ (in the order to be specified below). In this limit, the type-IIB description is strongly-coupled, but we will perform a U-duality transformation, in a series of steps described below and summarized in table 1, to transform the setting to a weakly coupled type-IIA background. This will also allow us to easily study the ground states of the field theory.

Type-IIB	Type-IIA
T^2 is in directions 1, 2	(Dual) T^2 is in directions 1, 10
n D3-branes (directions 1, 2, 3)	n F1-strings (direction 3)
$SL(2, \mathbb{Z})$ diffeomorphisms of T^2	$SL(2, \mathbb{Z})$ T-duality group of (dual) T^2
Momentum P_1	D2-brane (wrapping directions 1, 10) charge
Momentum P_2	D0-brane charge
String winding in direction 1	Momentum P_1
String winding in direction 2	String winding in direction 10
D1 winding in direction 1	Momentum P_{10}
D1 winding in direction 2	String winding in direction 1

Table 2. Mapping between the quantum numbers and other notions on the type-IIB side to those on the type-IIA side.

The U-duality transformation proceeds as follows. We first replace type-IIB on a circle of radius L_1 with M-theory on T^2 with complex structure τ and area $\mathcal{A} = (2\pi)^2 \alpha'^2 \tau_2^{-1} L_1^{-2} = (2\pi)^2 M_p^{-3} L_1^{-1}$, where M_p is the 11-dimensional Planck scale. We now reduce from M-theory to type-IIA on the circle of radius L_2 to get a theory with string coupling constant

$$g_{\text{IIA}} = (M_p L_2)^{3/2} = \tau_2^{1/2} L_1^{1/2} L_2^{3/2} \alpha'^{-1},$$

and new string scale

$$\alpha'_{\text{IIA}} = M_p^{-3} L_2^{-1} = \alpha'^2 \tau_2^{-1} L_1^{-1} L_2^{-1}.$$

After these dualities, the D3-branes become fundamental type-IIA strings with a total winding number n in the x_3 direction. The S-duality twist \mathbf{g} is now a diffeomorphism of the type-IIA T^2 , which can be realized as a rotation by an angle v . To make this type-IIA background weakly coupled, we assume that the limits are taken in such a way that

$$\mathcal{A} \gg \alpha'_{\text{IIA}}, \quad g_{\text{IIA}} \ll 1, \quad R \gg \alpha'^{1/2}_{\text{IIA}}. \tag{3.2}$$

This is a different limit than (3.1), but we can use the weakly coupled type-IIA background to study the Hilbert space of (supersymmetric) ground states. Since the type-IIA setting is described by n fundamental strings on a weakly coupled background, the question of the Hilbert space of ground states reduces to a simple calculation in string theory. For quick reference, we have summarized in table 2 the dual type-IIA description of various charges of the original type-IIB setting.

3.1 The dual geometry

After the series of dualities summarized in table 1, we end-up with a type-IIA string theory that we will now describe in detail. The 9+1D geometry is flat and free of singularities, and the spatial part is a free orbifold of \mathbb{R}^9 . It is convenient to divide the 9 directions in

three groups and describe the geometry as an orbifold of $T^2 \times \mathbb{R} \times \mathbb{C}^3$. We regard the T^2 as the complex plane modded out by a lattice, $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, and take

$$z \sim z + 1 \sim z + \tau$$

as its coordinate. On \mathbb{R} , the coordinate takes

$$-\infty < x_3 < \infty,$$

and on $\mathbb{C}^3 \simeq \mathbb{R}^6$, we take the coordinates to be

$$(\zeta_1, \zeta_2, \zeta_3), \quad \zeta_1, \zeta_2, \zeta_3 \in \mathbb{C}.$$

The orbifold is then represented by the identification

$$(z, x_3, \zeta_1, \zeta_2, \zeta_3) \sim (e^{iv}z, x_3 + 2\pi R, e^{iv}\zeta_1, e^{iv}\zeta_2, e^{iv}\zeta_3). \quad (3.3)$$

Note that, because of the shift $x_3 \rightarrow x_3 + 2\pi R$, the orbifold has no fixed points, and the geometry is smooth. From now until the rest of this section, (3.3) will be our background.

It is also convenient to give a separate name for the $\zeta_1 = \zeta_2 = \zeta_3 = 0$ subspace. We will denote this smooth, flat, and compact 3-dimensional manifold by W . It is represented by the coordinates (z, x_3) with identifications

$$W : \quad (z, x_3) \sim (z + \tau, x_3) \sim (z + 1, x_3) \sim (e^{iv}z, x_3 + 2\pi R). \quad (3.4)$$

This manifold is a T^2 -fibration over S^1 with structure group \mathbb{Z}_r .

3.2 Ground states

The states that are relevant to our problem are those with a total string winding number n along direction x_3 . A state with string winding number n is a p -particle (that is, p -string) state comprising of 1-particle states of winding numbers $n_1 \geq n_2 \geq \dots \geq n_p > 0$ with $n_1 + n_2 + \dots + n_p = n$. Thus, the Hilbert space of ground states decomposes as a direct sum:

$$\mathcal{H}(n, v) = \bigoplus_{\substack{n_1 \geq n_2 \geq \dots \geq n_p > 0 \\ n_1 + n_2 + \dots + n_p = n}} \mathcal{H}_{(n_1, \dots, n_p)}(v). \quad (3.5)$$

We can recast the partition $n = n_1 + \dots + n_p$ as a conjugacy class $[\sigma]$ of a permutation $\sigma \in S_n$, so that when σ is decomposed into cycles the integers n_1, \dots, n_p denote the lengths of the cycles. So, for example $n = 1 + 1 + \dots + 1$ (i.e., $p = n$) corresponds to the identity permutation $\sigma = 1$. We therefore set

$$\mathcal{H}(n, v) = \bigoplus_{[\sigma]} \mathcal{H}_{[\sigma]}(v), \quad (\sigma \in S_n). \quad (3.6)$$

We will refer to $\mathcal{H}_{[1]}$ as the $[\sigma]$ -*untwisted sector*, and to $\mathcal{H}_{[\sigma]}$ with $\sigma \neq 1$ as the $[\sigma]$ -*twisted sectors*.

Understanding the multi-particle Hilbert spaces $\mathcal{H}_{[\sigma]}(v)$ requires analysis of the single-particle states of which the multi-particle states are constructed, so let us first

discuss the single-particle string states. The problem of superstring quantization in the flat background (3.3) was studied in detail in [19].³ In the RNS formalism, in the sector with winding number \tilde{n} in direction x_3 , the mode expansion of worldsheet fields contains modes shifted from the standard integer or half-integer values by $\pm(\tilde{n}/\mathbf{r})$. This is fractional if \tilde{n} is not divisible by \mathbf{r} .

For the purposes of the present paper, we do not need the details of the worldsheet quantization or the full string spectrum — we only need the ground states. It turns out that (for $\tilde{n} \neq 0$) the ground states are bosonic and in the R-R sector. In fact, the problem of finding the ground states can be solved using essentially classical geometry: we simply need to find classical string configurations of minimal length. For \tilde{n} that is not divisible by \mathbf{r} , there is a basis of ground states that are in one-to-one correspondence with loops of minimal length and winding number \tilde{n} in the geometry (3.3). In the limit $\alpha' \rightarrow 0$, these states reduce to the classical string configurations, but even for finite α' these classical string configurations are the minima of the worldsheet energy, and fluctuations around these classical configurations correspond to massive worldsheet modes, and there is a single ground state for each classical configuration.

To describe the classical configurations, we can fix an x_3 coordinate and specify the points where the classical string intersects the transverse coordinates $T^2 \times \mathbb{C}^3$ in the geometry (3.3). At winding number \tilde{n} , the string intersects $T^2 \times \mathbb{C}^3$ at \tilde{n} (not necessarily distinct) points, and in order to be of minimal length the coordinates of these points should be independent of x_3 . The classical configurations are thus characterized by a set of \tilde{n} points in $T^2 \times \mathbb{C}^3$ that is invariant, as a set, under the orbifold operation

$$(z, \zeta_1, \zeta_2, \zeta_3) \sim (e^{iv}z, e^{iv}\zeta_1, e^{iv}\zeta_2, e^{iv}\zeta_3).$$

For \tilde{n} that is not divisible by \mathbf{r} , there is a finite number of such sets, and they are all localized at the origin of \mathbb{C}^3 , i.e., $\zeta_1 = \zeta_2 = \zeta_3 = 0$. They are therefore entirely described by the z -coordinates of where the string intersects T^2 : $z, e^{iv}z, e^{2iv}z, \dots, e^{i(\tilde{n}-1)v}z$, since as we go once around the x_3 direction the coordinate z switches to $e^{iv}z$. After \tilde{n} loops, z becomes $e^{i\tilde{n}v}z$ which, in order to close the loop, should be identified with z , up to a shift in $\mathbb{Z} + \mathbb{Z}\tau$. The classical string configurations are then described by solutions $z = \zeta_{M_a, M_b}$ of

$$e^{i\tilde{n}v}\zeta_{M_a, M_b} = \zeta_{M_a, M_b} + M_a + M_b\tau, \tag{3.7}$$

and we consider two solutions ζ_{M_a, M_b} and $\zeta_{M_a', M_b'}$ as equivalent if they differ by a lattice element, i.e., if $\zeta_{M_a, M_b} - \zeta_{M_a', M_b'} \in \mathbb{Z} + \mathbb{Z}\tau$. In addition, ζ_{M_a, M_b} and $e^{iv}\zeta_{M_a, M_b}$ give equivalent solutions, since the intersection points of the string with T^2 are unordered. There is then only a finite number of inequivalent solutions to (3.7), and we will describe them in detail at the end of this subsection. We conclude that the full single-particle string spectrum (including excited states) decomposes into a finite sum of distinct sectors, labeled by M_a, M_b , and the solution ζ_{M_a, M_b} , which is a point on T^2 , describes the center of mass of the string in the directions of T^2 .

³We are grateful to Aki Hashimoto for pointing out this reference.

We will denote a single-particle ground state with winding number \tilde{n} by the location of the intersection of the classical string configuration with any particular T^2 fiber at a constant x_3 . In other words, we denote a single-particle state corresponding to a solution z of (3.7) by

$$\left| [z, e^{iv} z, \dots, e^{(\tilde{n}-1)iv} z] \right\rangle, \quad (3.8)$$

where z coordinates are always taken modulo the lattice $\mathbb{Z} + \mathbb{Z}\tau$. Multi-string states are denoted by

$$\left| \{ [z_1, e^{iv} z_1, \dots, e^{(\tilde{n}_1-1)iv} z_1], [z_2, e^{iv} z_2, \dots, e^{(\tilde{n}_2-1)iv} z_2], \dots, [z_p, e^{iv} z_p, \dots, e^{(\tilde{n}_p-1)iv} z_p] \} \right\rangle,$$

where z_i is a solution $\zeta_{M_{a_i}, M_{b_i}}$ of (3.7) with $\tilde{n} \rightarrow \tilde{n}_i$, and $n = \sum_1^p \tilde{n}_i$ is the total winding number.

The number of inequivalent solutions of (3.7) for $\tilde{n} = 1$ will be denoted by k . It is a function of v alone. As we will see below, in our three cases we get the following three values:

$$\left. \begin{aligned} k = 1 \text{ when } \mathbf{r} = 6, v = \frac{\pi}{3}, \tau = e^{\pi i/3}, \mathbf{g} &= \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}; \\ k = 2 \text{ when } \mathbf{r} = 4, v = \frac{\pi}{2}, \tau = i, \mathbf{g} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \\ k = 3 \text{ when } \mathbf{r} = 3, v = \frac{2\pi}{3}, \tau = e^{\pi i/3}, \mathbf{g} &= \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}. \end{aligned} \right\} \quad (3.9)$$

As promised earlier, we conclude this subsection with a full description of the single-particle ground states. For additional clarity, we found it convenient to use a pictorial notation. We draw a fundamental cell of the lattice $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ as a parallelogram and explicitly mark the location of the solutions for z on it. We denote the solutions z by a dot surrounded by a circle, and if a solution z appears with multiplicity m , we surround it with m concentric circles. Below, we explicitly present all the solutions.

Single-particle states for $v = \frac{\pi}{2}$ ($\tau = i$ and $\mathbf{r} = 4$). For $n = 1$ we get two fixed points:

$$\left| \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \right\rangle = |[0]\rangle, \quad \left| \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \right\rangle = \left| \left[\frac{1}{2} + \frac{1}{2}i \right] \right\rangle.$$

There are only two distinct solutions to (3.7), up to a lattice element in $\mathbb{Z} + \mathbb{Z}\tau$, which can be taken as $\zeta_{0,0} = 0$ and $\zeta_{0,1} = \frac{1}{2} + \frac{1}{2}i$. Two solutions ζ_{M_a, M_b} and $\zeta_{M_a', M_b'}$ are equivalent if $M_a + M_b \equiv M_a' + M_b' \pmod{2}$.

For $n = 2$ we get three fixed points:

$$\left| \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \right\rangle = |[0, 0]\rangle, \quad \left| \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \right\rangle = \left| \left[\frac{1}{2}, \frac{1}{2}i \right] \right\rangle, \quad \left| \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \right\rangle = \left| \left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i \right] \right\rangle.$$

They are constructed from

$$\zeta_{0,0} = 0, \quad \zeta_{1,0} = \frac{1}{2}, \quad \zeta_{0,1} = \frac{1}{2}i, \quad \zeta_{1,1} = \frac{1}{2} + \frac{1}{2}i, \quad (\text{mod } \mathbb{Z} + \mathbb{Z}i).$$

For $n = 3$ we get two fixed points:

$$|\square_{\odot}\rangle = |[0, 0, 0]\rangle, \quad |\square_{\odot\odot}\rangle = \left| \left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i \right] \right\rangle.$$

Single-particle states for $v = \frac{\pi}{3}$ ($\tau = e^{\pi i/3}$ and $\mathbf{r} = \mathbf{6}$). For $n = 1$ we get a single fixed point:

$$|\square\rangle = |[0]\rangle.$$

For $n = 2$ we get two fixed points:

$$|\square_{\odot}\rangle = |[0, 0]\rangle, \quad |\square_{\odot\odot}\rangle = \left| \left[\frac{1}{3} + \frac{1}{3}\tau, \frac{2}{3} + \frac{2}{3}\tau \right] \right\rangle.$$

For $n = 3$ we also get two fixed points:

$$|\square_{\odot\odot}\rangle = |[0, 0, 0]\rangle, \quad |\square_{\odot\odot\odot}\rangle = \left| \left[\frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2} + \frac{1}{2}\tau \right] \right\rangle.$$

For $n = 4$ we again get two fixed points:

$$|\square_{\odot\odot\odot}\rangle = |[0, 0, 0, 0]\rangle, \quad |\square_{\odot\odot\odot\odot}\rangle = \left| \left[\frac{1}{3} + \frac{1}{3}\tau, \frac{1}{3} + \frac{1}{3}\tau, \frac{2}{3} + \frac{2}{3}\tau, \frac{2}{3} + \frac{2}{3}\tau \right] \right\rangle.$$

For $n = 5$ we get one fixed point:

$$|\square_{\odot\odot\odot\odot\odot}\rangle = |[0, 0, 0, 0, 0]\rangle.$$

Single-particle states for $v = \frac{2\pi}{3}$ ($\tau = e^{\pi i/3}$ and $\mathbf{r} = \mathbf{3}$). For $n = 1$ we get three fixed points:

$$|\square_{\odot}\rangle = |[0]\rangle, \quad |\square_{\odot\odot}\rangle = \left| \left[\frac{1}{3} + \frac{1}{3}\tau \right] \right\rangle, \quad |\square_{\odot\odot\odot}\rangle = \left| \left[\frac{2}{3} + \frac{2}{3}\tau \right] \right\rangle.$$

For $n = 2$ we also get three fixed points:

$$|\square_{\odot\odot}\rangle = |[0, 0]\rangle, \quad |\square_{\odot\odot\odot}\rangle = \left| \left[\frac{1}{3} + \frac{1}{3}\tau, \frac{1}{3} + \frac{1}{3}\tau \right] \right\rangle, \quad |\square_{\odot\odot\odot\odot}\rangle = \left| \left[\frac{2}{3} + \frac{2}{3}\tau, \frac{2}{3} + \frac{2}{3}\tau \right] \right\rangle.$$

The single-particle states are summarized in figure 1, and the complete basis of ground states (i.e., including multi-particle states) is depicted in figure 2.

Notation for multi-particle states. The multi-particle states are states in the Fock space of identical bosons. We denote multi-particle states by combining inside a single ket the pictorial representations of the individual single-particle states which make up the multi-particle state. For example, for $v = \frac{\pi}{2}$ ($\tau = i$ and $\mathbf{r} = 4$) and $n = 2$ we get the following 2-particle states:

$$|\square\square\rangle, \quad |\square_{\odot}\square\rangle, \quad |\square_{\odot\odot}\square\rangle.$$



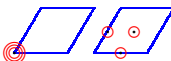


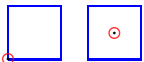
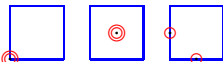
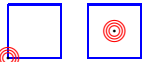
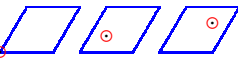
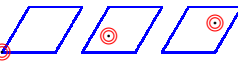
$v = \frac{\pi}{3}$	$n = 1$ 	$n = 2$ 	$n = 3$ 	
	$n = 4$ 	$n = 5$ 		
$v = \frac{\pi}{2}$	$n = 1$ 		$n = 2$ 	
	$n = 3$ 			
$v = \frac{2\pi}{3}$	$n = 1$ 		$n = 2$ 	

Figure 1. Single-particle ground states. Each ground state is depicted by the intersection of the strings with the T^2 fiber at a fixed x_3 . The string can intersect the fiber more than once at the same point, and the number of small circles surrounding the intersection point represents the number of times that the string intersects the fiber at that point.

In the middle state, the two particles occupy different single-particle states, while in the leftmost and rightmost states the two particles occupy the same single-particle state. Note that, by definition, the corresponding wavefunctions are symmetric, so for example: $|\square \circ \square\rangle \equiv |\circ \square \square\rangle$.

Next, we will discuss symmetries of the string background. We will identify two \mathbb{Z}_k symmetries, which act on the full spectrum, but in what follows we will only need their action on ground states.

3.3 \mathbb{Z}_k momentum

The space W defined in (3.4) possesses an isometry

$$\mathcal{U} : (z, x_3) \mapsto \left(z + \frac{1}{k} + \frac{1}{k}\tau, x_3 \right), \tag{3.10}$$

where k is the number of ground states of the $\tilde{n} = 1$ problem, listed in (3.9). It is not hard to check that the isometry is compatible with the structure group of the fibration since, for all three cases $k = 1, 2, 3$, the T^2 point with coordinate $z = \frac{1}{k}(1 + \tau)$ is a solution to the $\tilde{n} = 1$ version of (3.7), and so $\frac{1}{k}(1 + \tau)e^{i\nu}$ and $\frac{1}{k}(1 + \tau)$ differ by an element of the lattice $\mathbb{Z} + \mathbb{Z}\tau$. Thus, \mathcal{U} defines an operator on the Hilbert space of states, and since $\mathcal{U}^k = 1$ it follows that the eigenvalues of \mathcal{U} take the form $e^{2\pi ij/k}$ with $j \in \mathbb{Z}_k$. We interpret this j as a discrete \mathbb{Z}_k momentum.

The operator \mathcal{U} takes single-particle states to single-particle states with the same winding number \tilde{n} , and its action on any ground state can be computed from its action on the single-particle states. For future reference, we list the action explicitly below.


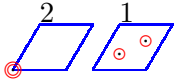
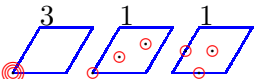
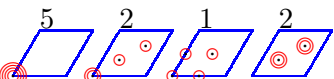
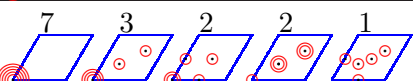
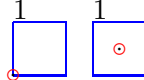
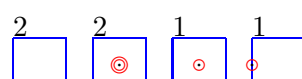
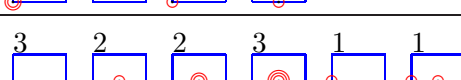
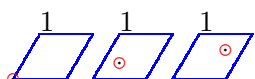
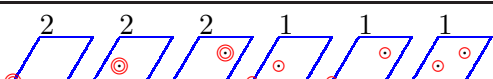
$v = \frac{\pi}{3}$	$n = 1$		$N_s = 1$
	$n = 2$		$N_s = 3$
	$n = 3$		$N_s = 5$
	$n = 4$		$N_s = 10$
	$n = 5$		$N_s = 15$
$v = \frac{\pi}{2}$	$n = 1$		$N_s = 2$
	$n = 2$		$N_s = 6$
	$n = 3$		$N_s = 12$
$v = \frac{2\pi}{3}$	$n = 1$		$N_s = 3$
	$n = 2$		$N_s = 9$

Figure 2. The complete basis of ground states. A ground state in this basis comprises of one or more single-particle states from figure 1. As in the previous figure, each ground state is depicted by the intersection of the strings with the fiber at $x_3 = 0$. Several different states could have the same depiction (if they decompose as $n = n_1 + \dots + n_p$ in different ways), and the numbers on top of each cell indicate the multiplicity. N_s is the total number of states.

Action of \mathcal{U} on single-particle states for $v = \frac{\pi}{2}$ ($\tau = i$ and $\mathbf{r} = 4$). In this case $k = 2$. For $\tilde{n} = 1$, \mathcal{U} acts as

$$\mathcal{U}|\left[\begin{smallmatrix} 1 \\ 2 + \frac{1}{2}i \end{smallmatrix}\right]\rangle = \left|\left[\begin{smallmatrix} 1 \\ 2 + \frac{1}{2}i \end{smallmatrix}\right]\right\rangle, \quad \mathcal{U}\left|\left[\begin{smallmatrix} 1 \\ 2 + \frac{1}{2}i \end{smallmatrix}\right]\right\rangle = |[0]\rangle,$$

or in pictorial notation,

$$\mathcal{U}|\square_{\bullet}\rangle = |\square_{\circ}\rangle, \quad \mathcal{U}|\square_{\circ}\rangle = |\square_{\bullet}\rangle.$$

For $\tilde{n} = 2$, \mathcal{U} acts as:

$$\mathcal{U}|[0, 0]\rangle = \left| \left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i \right] \right\rangle, \quad \mathcal{U} \left| \left[\frac{1}{2}, \frac{1}{2}i \right] \right\rangle = \left| \left[\frac{1}{2}, \frac{1}{2}i \right] \right\rangle, \quad \mathcal{U} \left| \left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i \right] \right\rangle = |[0, 0]\rangle.$$

$$\mathcal{U} \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle = \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle, \quad \mathcal{U} \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle = \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle, \quad \mathcal{U} \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle = \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle.$$

For $\tilde{n} = 3$, \mathcal{U} acts as:

$$\mathcal{U}|[0, 0, 0]\rangle = \left| \left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i \right] \right\rangle, \quad \mathcal{U} \left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i \right] = |[0, 0, 0]\rangle.$$

$$\mathcal{U} \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle = \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle, \quad \mathcal{U} \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle = \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle.$$

Action of \mathcal{U} on single-particle states for $v = \frac{\pi}{3}$ ($\tau = e^{\pi i/3}$ and $\mathbf{r} = \mathbf{6}$). In this case $k = 1$ and \mathcal{U} is the identity.

Action of \mathcal{U} on single-particle states for $v = \frac{2\pi}{3}$ ($\tau = e^{\pi i/3}$ and $\mathbf{r} = \mathbf{3}$). In this case $k = 3$. For $\tilde{n} = 1$, \mathcal{U} acts as:

$$\mathcal{U}|[0]\rangle = \left| \left[\frac{1}{3} + \frac{1}{3}\tau \right] \right\rangle, \quad \mathcal{U} \left| \left[\frac{1}{3} + \frac{1}{3}\tau \right] \right\rangle = \left| \left[\frac{2}{3} + \frac{2}{3}\tau \right] \right\rangle, \quad \mathcal{U} \left| \left[\frac{2}{3} + \frac{2}{3}\tau \right] \right\rangle = |[0]\rangle.$$

$$\mathcal{U} \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle = \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle, \quad \mathcal{U} \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle = \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle, \quad \mathcal{U} \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle = \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle.$$

For $\tilde{n} = 2$, \mathcal{U} acts as:

$$\left. \begin{aligned} \mathcal{U}|[0, 0]\rangle &= \left| \left[\frac{1}{3} + \frac{1}{3}\tau, \frac{1}{3} + \frac{1}{3}\tau \right] \right\rangle \\ \mathcal{U} \left| \left[\frac{1}{3} + \frac{1}{3}\tau, \frac{1}{3} + \frac{1}{3}\tau \right] \right\rangle &= \left| \left[\frac{2}{3} + \frac{2}{3}\tau, \frac{2}{3} + \frac{2}{3}\tau \right] \right\rangle \\ \mathcal{U} \left| \left[\frac{2}{3} + \frac{2}{3}\tau, \frac{2}{3} + \frac{2}{3}\tau \right] \right\rangle &= |[0, 0]\rangle \end{aligned} \right\}$$

$$\mathcal{U} \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle = \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle, \quad \mathcal{U} \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle = \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle, \quad \mathcal{U} \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle = \left| \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\rangle.$$

3.4 \mathbb{Z}_k winding number

Our problem has a second conserved \mathbb{Z}_k quantum number. This one is defined by the winding number of the string in the fiber direction. The winding number takes values in the first homology group of the space, which in our case is homotopically equivalent to the space W defined in (3.4). As we will check below, the first homology group is $H_1(W, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}_k$.

To see this, let us pick the origin ($z = 0, x_3 = 0$) as a marked point, and let us define three elements of the fundamental group $\pi_1(W)$ as the equivalence classes of the following three loops:

$$\left. \begin{aligned} \eta &= [t \mapsto (z = 0, x_3 = 2\pi R t)] \\ \alpha_a &= [t \mapsto (z = t, x_3 = 0)] \\ \alpha_b &= [t \mapsto (z = t\tau, x_3 = 0)] \end{aligned} \right\} \quad 0 \leq t < 1. \quad (3.11)$$

The loops that define α_a, α_b run along the T^2 fiber, while η is defined by a loop that runs along the S^1 base. Note that the fundamental group $\pi_1(W)$ is generated by η, α_a, α_b with the relations

$$\alpha_a \alpha_b = \alpha_b \alpha_a, \quad \eta^{-1} \alpha_a \eta = \alpha_a^{\mathbf{a}} \alpha_b^{\mathbf{b}}, \quad \eta^{-1} \alpha_b \eta = \alpha_a^{\mathbf{c}} \alpha_b^{\mathbf{d}},$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are the elements of the $\text{SL}(2, \mathbb{Z})$ matrix \mathbf{g} defined in section 2.2. The homology group $H_1(W, \mathbb{Z})$ is homeomorphic to the abelianization of $\pi_1(W)$. The abelianization of $\pi_1(W)$ ignores the order of operators in the relations, and so we get an abelian group generated by $\bar{\eta}$ and $\bar{\alpha}_a, \bar{\alpha}_b$ with the relations (over the ring \mathbb{Z})

$$\bar{\alpha}_a = \bar{\alpha}_a \mathbf{a} \bar{\alpha}_b \mathbf{b}, \quad \bar{\alpha}_b = \bar{\alpha}_a \mathbf{c} \bar{\alpha}_b \mathbf{d}.$$

Since it is more appropriate to denote the group operation in $H_1(W, \mathbb{Z})$ by a sum instead of a product, we switch notation from $\bar{\eta}, \bar{\alpha}_a, \bar{\alpha}_b$ to $\varrho, \beta_a, \beta_b$. The result is that $H_1(W, \mathbb{Z})$ is generated by $\varrho, \beta_a, \beta_b$, with the relations

$$\beta_a = \mathbf{a}\beta_a + \mathbf{b}\beta_b, \quad \beta_b = \mathbf{c}\beta_a + \mathbf{d}\beta_b.$$

It is useful at this point to separate the cases:

- For $\tau = i$, $\nu = \frac{\pi}{2}$, $\mathbf{g} = \mathbf{g}' \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we have $k = 2$ and $\beta_a = -\beta_b = -\beta_a$, so $H_1(W) = \mathbb{Z} \oplus \mathbb{Z}_2$, generated by $\varrho \in \mathbb{Z}$ and $\beta_a \in \mathbb{Z}_2$.
- For $\tau = e^{\pi i/3}$, $\nu = \frac{\pi}{3}$, $\mathbf{g} = \mathbf{g}'' \equiv \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, we have $k = 1$ and $\beta_b = \beta_a$ and $\beta_a = \beta_a - \beta_b$ so $\beta_a = \beta_b = 0$, and $H_1(W) = \mathbb{Z}$, generated by ϱ .
- For $\tau = e^{\pi i/3}$, $\nu = \frac{2\pi}{3}$, $\mathbf{g} = -\mathbf{g}''^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, we have $k = 3$ and $\beta_b = -\beta_a$ and $\beta_a = \beta_b - \beta_a$. So $H_1(W) = \mathbb{Z} \oplus \mathbb{Z}_3$, generated by $\varrho \in \mathbb{Z}$ and $\beta_a \in \mathbb{Z}_3$.

To a string configuration that winds n times around the base and has the homology class $n\varrho + g\beta_a$ (with $g \in \mathbb{Z}_k$), we assign a \mathbb{Z}_k charge of g . We now define the quantum operator \mathcal{V} to take the eigenvalue $e^{2\pi i g/k}$ on such a state.

There is some arbitrariness in the definition of g because of our arbitrary choice of the origin $z = 0$ of the T^2 fiber of W . Consider, for example, the case $\tau = i$. The loop $[t \mapsto (z = 0, x_3 = 2\pi t R)]$ was defined to have homology class $n\varrho$ (with $g = 0$) and the loop $[t \mapsto (z = \frac{1}{2} + \frac{1}{2}i, x_3 = 2\pi t R)]$ then has homology class $n\varrho + \beta_a$. But we could have just as well chosen the origin at $z = \frac{1}{2} + \frac{1}{2}i$, thereby switching the eigenvalues of \mathcal{V} . In general, replacing

$$\mathcal{V} \rightarrow e^{i\phi} \mathcal{V} \tag{3.12}$$

for some arbitrary (constant) phase ϕ results in an equally reasonable definition of \mathcal{V} . We will, nevertheless, stick to the definition of \mathcal{V} with the origin set at $z = 0$.

At this point we have found two operators \mathcal{V}, \mathcal{U} , acting on the Hilbert space of ground states (and, in fact, on the full Hilbert space). Each operator defines a conserved \mathbb{Z}_k quantum number, and by definition they satisfy

$$\mathcal{V}^k = \mathcal{U}^k = 1. \tag{3.13}$$

They obey the commutation relation

$$\mathcal{V}\mathcal{U}\mathcal{V}^{-1}\mathcal{U}^{-1} = e^{\frac{2\pi i n}{k}}, \tag{3.14}$$

which can be verified as follows. Consider a (classical) string configuration with $n = 1$ given by a section $z = f(x_3)$ of the fiber bundle W , where f is some continuous function on the interval $0 \leq x_3 < 2\pi R$ with $f(2\pi R) = e^{iv}f(0)$. Now, translate the section by a small c to $z = f(x_3) + c$. This no longer satisfies the boundary conditions, and so to close the string we need to add a piece of string that connects $(z = f(0) + c, x_3 = 0)$ to $(z = f(0) + e^{iv}c, x_3 = 0)$. As c is increased from 0 to $\frac{1}{k}(1 + \tau)$, the extra piece of string increases until it becomes a piece of string that stretches from start-point to end-point by the complex vector $\frac{1}{k}(1 + \tau)(1 - e^{iv})$. It is easy to check that for the cases $k = 2, 3$ this vector is in the homology class β_a . We have thus arrived at the following conclusion: acting with \mathcal{U} on a string state with homology class $n\rho + g\beta_a$ produces a string state with homology class $n\rho + (g+n)\beta_a$. The commutation relation (3.14) follows immediately from that observation.

For future reference, we list the action of \mathcal{V} on single-particle states explicitly below.

Action of \mathcal{V} on single-particle states for $v = \frac{\pi}{2}$ ($\tau = i$ and $r = 4$). In this case $k = 2$. For $\tilde{n} = 1$, \mathcal{V} acts as:

$$\begin{aligned} \mathcal{V}|[0]\rangle &= |[0]\rangle, & \mathcal{V}\left|\left[\frac{1}{2} + \frac{1}{2}i\right]\right\rangle &= -\left|\left[\frac{1}{2} + \frac{1}{2}i\right]\right\rangle. \\ \mathcal{V}|\square\rangle &= |\square\rangle, & \mathcal{V}|\square\odot\rangle &= -|\square\odot\rangle. \end{aligned} \quad (3.15)$$

For $\tilde{n} = 2$, \mathcal{V} acts as:

$$\begin{aligned} \mathcal{V}|[0, 0]\rangle &= |[0, 0]\rangle, & \mathcal{V}\left|\left[\frac{1}{2}, \frac{1}{2}i\right]\right\rangle &= -\left|\left[\frac{1}{2}, \frac{1}{2}i\right]\right\rangle, & \mathcal{V}\left|\left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\right]\right\rangle &= \left|\left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\right]\right\rangle. \\ \mathcal{V}|\square\rangle &= |\square\rangle, & \mathcal{V}|\square\odot\rangle &= |\square\odot\rangle, & \mathcal{V}|\square\ominus\rangle &= -|\square\ominus\rangle. \end{aligned}$$

For $\tilde{n} = 3$, \mathcal{V} acts as:

$$\begin{aligned} \mathcal{V}|[0, 0, 0]\rangle &= |[0, 0, 0]\rangle, & \mathcal{V}\left|\left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\right]\right\rangle &= -\left|\left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\right]\right\rangle. \\ \mathcal{V}|\square\rangle &= |\square\rangle, & \mathcal{V}|\square\odot\rangle &= -|\square\odot\rangle. \end{aligned}$$

Action on single-particle states for $v = \frac{\pi}{3}$ ($\tau = e^{\pi i/3}$ and $r = 6$). In this case $k = 1$ and \mathcal{V} is the identity.

Action on single-particle states for $v = \frac{2\pi}{3}$ ($\tau = e^{\pi i/3}$ and $r = 3$). In this case $k = 3$. For $\tilde{n} = 1$, \mathcal{V} acts as:

$$\begin{aligned} \mathcal{V}|[0]\rangle &= |[0]\rangle, & \mathcal{V}\left|\left[\frac{1}{3} + \frac{1}{3}\tau\right]\right\rangle &= e^{\frac{2\pi i}{3}}\left|\left[\frac{1}{3} + \frac{1}{3}\tau\right]\right\rangle, & \mathcal{V}\left|\left[\frac{2}{3} + \frac{2}{3}\tau\right]\right\rangle &= e^{-\frac{2\pi i}{3}}\left|\left[\frac{2}{3} + \frac{2}{3}\tau\right]\right\rangle. \\ \mathcal{V}|\square\rangle &= |\square\rangle, & \mathcal{V}|\square\odot\rangle &= e^{\frac{2\pi i}{3}}|\square\odot\rangle, & \mathcal{V}|\square\ominus\rangle &= e^{-\frac{2\pi i}{3}}|\square\ominus\rangle. \end{aligned}$$

For $\tilde{n} = 2$, \mathcal{V} acts as:

$$\mathcal{V}|[0, 0]\rangle = |[0, 0]\rangle, \quad \mathcal{V}\left|\left[\frac{1}{3} + \frac{1}{3}\tau, \frac{1}{3} + \frac{1}{3}\tau\right]\right\rangle = e^{-\frac{2\pi i}{3}}\left|\left[\frac{1}{3} + \frac{1}{3}\tau, \frac{1}{3} + \frac{1}{3}\tau\right]\right\rangle$$

$$\mathcal{V} \left| \left[\frac{2}{3} + \frac{2}{3}\tau, \frac{2}{3} + \frac{2}{3}\tau \right] \right\rangle = e^{\frac{2\pi i}{3}} \left| \left[\frac{2}{3} + \frac{2}{3}\tau, \frac{2}{3} + \frac{2}{3}\tau \right] \right\rangle.$$

$$\mathcal{V} \left| \begin{array}{c} \text{parallelogram} \\ \text{with red dot at bottom-left} \end{array} \right\rangle = \left| \begin{array}{c} \text{parallelogram} \\ \text{with red dot at bottom-left} \end{array} \right\rangle, \quad \mathcal{V} \left| \begin{array}{c} \text{parallelogram} \\ \text{with red dot at top-left} \end{array} \right\rangle = e^{-\frac{2\pi i}{3}} \left| \begin{array}{c} \text{parallelogram} \\ \text{with red dot at top-left} \end{array} \right\rangle, \quad \mathcal{V} \left| \begin{array}{c} \text{parallelogram} \\ \text{with red dot at center} \end{array} \right\rangle = e^{\frac{2\pi i}{3}} \left| \begin{array}{c} \text{parallelogram} \\ \text{with red dot at center} \end{array} \right\rangle.$$

3.5 Worldsheet symmetries

We will now discuss additional symmetries of the worldsheet CFT that do not directly correspond to symmetries of the Hilbert space of ground states, but will later be used as building blocks to construct operators that do act on the Hilbert space of ground states. Furthermore, these extra worldsheet symmetries will be useful in section 3.7 when we study T-duality.

The worldsheet theory can be regarded as a $\mathbb{Z}_{\mathbf{r}}$ orbifold of a compactification of type-IIA theory on $S^1 \times T^2$, where S^1 has radius $2\pi\mathbf{r}R$, and $\mathbb{Z}_{\mathbf{r}}$ is generated by an isometry as in (3.3). The sector of the Hilbert space that corresponds to strings of winding number \tilde{n} is a twisted sector of this orbifold theory. The rules of orbifolds [20, 21] dictate that the Hilbert space \mathcal{H} of such one-particle states is the $\mathbb{Z}_{\mathbf{r}}$ -invariant subspace of the Hilbert space $\mathcal{H}' \supseteq \mathcal{H}$ of the CFT on a circle with twisted boundary conditions given by the identification

$$(z, x_3, \zeta_1, \zeta_2, \zeta_3) \sim (e^{i\tilde{n}v}z, x_3 + 2\pi\tilde{n}R, e^{i\tilde{n}v}\zeta_1, e^{i\tilde{n}v}\zeta_2, e^{i\tilde{n}v}\zeta_3), \quad v \equiv \frac{2\pi}{\mathbf{r}}. \quad (3.16)$$

If \tilde{n} and \mathbf{r} are not relatively prime (i.e., $\tilde{n} = 2, 3, 4$ for $\mathbf{r} = 6$, and $\tilde{n} = 2$ for $\mathbf{r} = 4$), the Hilbert space \mathcal{H}' possesses discrete symmetries in addition to \mathcal{U}, \mathcal{V} . This is because the identification (3.16) generates a $\mathbb{Z}_{\mathbf{r}/\text{gcd}(\mathbf{r}, \tilde{n})} \subset \mathbb{Z}_{\mathbf{r}}$ subgroup of the orbifold group, and thus, as far as \mathcal{H}' is concerned, the effective geometry is a $\mathbb{Z}_{\mathbf{r}/\text{gcd}(\mathbf{r}, \tilde{n})}$ -orbifold, which can have a larger group of symmetries than the $\mathbb{Z}_{\mathbf{r}}$ -orbifold.

To explain this in more detail, we need a worldsheet realization of the type-IIA background, but it will be sufficient to consider only the bosons and only in the directions of W . We represent W as a $\mathbb{Z}_{\mathbf{r}}$ orbifold of $T^2 \times S^1$, where T^2 is parameterized by $z \sim z+1 \sim z+\tau$ as above, and S^1 is parameterized by $0 \leq y < 2\pi\mathbf{r}R$, and the orbifold group is generated by

$$(z, y) \mapsto (e^{iv}z, y + 2\pi R).$$

We define worldsheet coordinates (σ, η) and worldsheet bosons $Z(\sigma, \eta), Y(\sigma, \eta)$ corresponding to the coordinates z and y , so that Y is real and Z is complex. We work in a twisted sector for which

$$Z(\sigma + 2\pi, \eta) = e^{i\tilde{n}v}Z(\sigma, \eta) + M_a + M_b\tau, \quad Y(\sigma + 2\pi, \eta) = Y(\sigma, \eta) + 2\pi\tilde{n}R.$$

In this sector, the worldsheet fields have an expansion

$$Y = y_0 + P_y\eta + \tilde{n}\sigma R + \sum_{n' \neq 0} \frac{i}{n'} \gamma_{-n'} e^{in'(\eta-\sigma)} + \sum_{n' \neq 0} \frac{i}{n'} \tilde{\gamma}_{-n'} e^{in'(\eta+\sigma)}, \quad (3.17)$$

$$Z = \zeta_{M_a, M_b} + \sum_{n' \in \mathbb{Z}} \frac{i}{n' - \frac{\tilde{n}}{\mathbf{r}}} \alpha_{-n' + \frac{\tilde{n}}{\mathbf{r}}} e^{i(n' - \frac{\tilde{n}}{\mathbf{r}})(\eta-\sigma)} + \sum_{n' \in \mathbb{Z}} \frac{i}{n' + \frac{\tilde{n}}{\mathbf{r}}} \tilde{\alpha}_{-n' - \frac{\tilde{n}}{\mathbf{r}}} e^{i(n' + \frac{\tilde{n}}{\mathbf{r}})(\eta+\sigma)}, \quad (3.18)$$

where P_y is the Y -momentum, $\gamma_{-n'}$ and $\tilde{\gamma}_{-n'}$ are the integer-moded right and left moving oscillators for Y , $\alpha_{-n'-\frac{\tilde{n}}{r}}$ and $\tilde{\alpha}_{-n'-\frac{\tilde{n}}{r}}$ are the fractionally moded right and left moving oscillators for Z , and ζ_{M_a, M_b} is the solution of (3.7) and is a fixed point of rotation of the T^2 fiber by an angle $\tilde{n}v$. We define two solutions ζ_{M_a, M_b} and $\zeta_{M_{a'}, M_{b'}}$ as equivalent if they differ by a lattice vector, i.e., $\zeta_{M_a, M_b} - \zeta_{M_{a'}, M_{b'}} \in \mathbb{Z} + \mathbb{Z}\tau$. In the ground states of the CFT the oscillators are not excited and $P_y = 0$. The states fall into a finite number of sectors labeled by the inequivalent solutions ζ_{M_a, M_b} . We will denote these states by $|\zeta_{M_a, M_b}\rangle$.

Note that if ζ_{M_a, M_b} is a solution, then so is

$$e^{iv}\zeta_{M_a, M_b} = \zeta_{dM_a + bM_b, cM_a + aM_b},$$

where we have used (2.3)–(2.4). But if $e^{iv}\zeta_{M_a, M_b} - \zeta_{M_a, M_b}$ is not in $\mathbb{Z} + \mathbb{Z}\tau$, then as far as the worldsheet CFT is concerned, $|\zeta_{M_a, M_b}\rangle$ and $|e^{iv}\zeta_{M_a, M_b}\rangle$ are different states. However, to get the *string* ground states, we need to impose (i) invariance under translations in σ , and (ii) invariance under the orbifold group \mathbb{Z}_r . Now, define the operator \mathcal{R} by

$$\mathcal{R}|\zeta_{M_a, M_b}\rangle = |e^{iv}\zeta_{M_a, M_b}\rangle.$$

On ground states, it is equivalent to a combination of the \mathbb{Z}_r orbifold generator and worldsheet translation in σ . The CFT ground states that correspond to string ground states must therefore be invariant under \mathcal{R} and thus are linear combinations of states of the form

$$\sum_{j=0}^{\tilde{n}-1} |e^{ijv}\zeta_{M_a, M_b}\rangle. \tag{3.19}$$

(Note that $|e^{i\tilde{n}v}\zeta_{M_a, M_b}\rangle = |\zeta_{M_a, M_b}\rangle$, by virtue of (3.7).) Since we are not concerned with excited states, we will take \mathcal{H}' as the Hilbert space spanned by the states $|\zeta_{M_a, M_b}\rangle$, and $\mathcal{H} \subseteq \mathcal{H}'$ as the subspace spanned by the states (3.19).

Now, let us assume that $\text{gcd}(\tilde{n}, r) > 1$. We can then find additional symmetries of \mathcal{H}' that do not commute with \mathcal{R} as follows. For any $\zeta \in \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ that satisfies

$$e^{i\tilde{n}v}\zeta - \zeta \in \mathbb{Z} + \mathbb{Z}\tau, \tag{3.20}$$

we define two operators $\tilde{\mathcal{U}}(\zeta), \tilde{\mathcal{V}}(\zeta)$ on \mathcal{H}' by

$$\tilde{\mathcal{U}}(\zeta)|\zeta_{M_a, M_b}\rangle = |\zeta_{M_a, M_b} + \zeta\rangle, \quad \tilde{\mathcal{V}}(\zeta)|\zeta_{M_a, M_b}\rangle = e^{4\pi i \text{Re}(\zeta^* \zeta_{M_a, M_b})} |\zeta_{M_a, M_b}\rangle. \tag{3.21}$$

We have the commutation relations

$$\begin{aligned} \tilde{\mathcal{U}}(\zeta)\tilde{\mathcal{U}}(\zeta') &= \tilde{\mathcal{U}}(\zeta')\tilde{\mathcal{U}}(\zeta), & \tilde{\mathcal{V}}(\zeta)\tilde{\mathcal{V}}(\zeta') &= \tilde{\mathcal{V}}(\zeta')\tilde{\mathcal{V}}(\zeta), \\ \tilde{\mathcal{U}}(\zeta)\tilde{\mathcal{V}}(\zeta') &= e^{-4\pi i \text{Re}(\zeta^* \zeta')} \tilde{\mathcal{V}}(\zeta')\tilde{\mathcal{U}}(\zeta). \end{aligned}$$

For example, the symmetry operators \mathcal{U}, \mathcal{V} defined in section 3.3–3.4 can be written as

$$\mathcal{U} = \tilde{\mathcal{U}}\left(\frac{1}{k} + \frac{1}{k}\tau\right), \quad \mathcal{V} = \tilde{\mathcal{V}}\left(\frac{1}{k} + \frac{1}{k}\tau\right). \tag{3.22}$$

But in general, the operators $\tilde{\mathcal{U}}(\zeta), \tilde{\mathcal{V}}(\zeta)$ do not preserve the subspace of physical string states $\mathcal{H} \subset \mathcal{H}'$, because in general $\tilde{\mathcal{U}}(\zeta), \tilde{\mathcal{V}}(\zeta)$ do not commute with \mathcal{R} . However, we can form \mathcal{R} -invariant combinations such as

$$\sum_{j=0}^{\tilde{n}-1} \tilde{\mathcal{U}}(e^{ijv}\zeta), \quad \sum_{j=0}^{\tilde{n}-1} \tilde{\mathcal{V}}(e^{ijv}\zeta), \quad \prod_{j=0}^{\tilde{n}-1} \tilde{\mathcal{U}}(e^{ijv}\zeta)\tilde{\mathcal{V}}(e^{ijv}\zeta), \dots$$

that do preserve \mathcal{H} and therefore define operators that act on the Hilbert space of string ground states. We will return to these constructions in section 6.6.

Let us proceed to examples. For the first example, consider the case $k = 2$ ($\tau = i$ and $v = \frac{\pi}{2}$, $\mathbf{r} = 4$) and $\tilde{n} = 2$. We have four inequivalent solutions to (3.7):

$$\zeta_{0,0} = 0, \quad \zeta_{1,0} = \frac{1}{2}, \quad \zeta_{0,1} = \frac{1}{2}i, \quad \zeta_{1,1} = \frac{1}{2}(1+i) \pmod{\mathbb{Z} + \mathbb{Z}\tau}. \quad (3.23)$$

If we define

$$\tilde{\mathcal{V}}_a \equiv \tilde{\mathcal{V}}(\zeta_{1,0}), \quad \tilde{\mathcal{U}}_a \equiv \tilde{\mathcal{U}}(\zeta_{1,0}), \quad \tilde{\mathcal{V}}_b \equiv \tilde{\mathcal{V}}(\zeta_{0,1}), \quad \tilde{\mathcal{U}}_b \equiv \tilde{\mathcal{U}}(\zeta_{0,1}), \quad (3.24)$$

then they act on the four-dimensional Hilbert space \mathcal{H}' as

$$\left. \begin{aligned} \tilde{\mathcal{V}}_a |\zeta_{M_a, M_b}\rangle &= (-1)^{M_a} |\zeta_{M_a, M_b}\rangle, & \tilde{\mathcal{V}}_b |\zeta_{M_a, M_b}\rangle &= (-1)^{M_b} |\zeta_{M_a, M_b}\rangle, \\ \tilde{\mathcal{U}}_a |\zeta_{M_a, M_b}\rangle &= |\zeta_{M_a+1, M_b}\rangle, & \tilde{\mathcal{U}}_b |\zeta_{M_a, M_b}\rangle &= |\zeta_{M_a, M_b+1}\rangle, \end{aligned} \right\} \quad (3.25)$$

where $(M_a + 1)$ and $(M_b + 1)$ are understood to be additions in \mathbb{Z}_2 . The states $|\zeta_{M_a, M_b}\rangle$ are eigenstates of $\tilde{\mathcal{V}}_a, \tilde{\mathcal{V}}_b$, while

$$|K_a, K_b\rangle \equiv \frac{1}{2} \sum_{M_a, M_b \in \mathbb{Z}_2} (-1)^{K_a M_a + K_b M_b} |\zeta_{M_a, M_b}\rangle \quad (3.26)$$

are eigenstates of $\tilde{\mathcal{U}}_a, \tilde{\mathcal{U}}_b$. The operators \mathcal{U} and \mathcal{V} are related to $\tilde{\mathcal{U}}_a, \tilde{\mathcal{U}}_b$ and $\tilde{\mathcal{V}}_a, \tilde{\mathcal{V}}_b$ by

$$\mathcal{U} \equiv \tilde{\mathcal{U}}(\zeta_{1,1}) = \tilde{\mathcal{U}}_a \tilde{\mathcal{U}}_b, \quad \mathcal{V} \equiv \tilde{\mathcal{V}}(\zeta_{1,1}) = \tilde{\mathcal{V}}_a \tilde{\mathcal{V}}_b.$$

We can regard $\tilde{\mathcal{V}}_a, \tilde{\mathcal{V}}_b$ as associated with two independent winding numbers $\overline{M}_a, \overline{M}_b \in \mathbb{Z}_2$, which characterize the topology of the map $Z = Z(\sigma, \eta)$ from the worldsheet to T^2 . We define them by

$$\overline{M}_a \equiv M_a \pmod{2}, \quad \overline{M}_b \equiv M_b \pmod{2}.$$

The winding number \overline{M}_a is associated with the β_a cycle (a loop along a straight path $Z \rightarrow Z + 1$) and \overline{M}_b is associated with the β_b cycle ($Z \rightarrow Z + i$). Beyond the CFT, in the full theory, the cycles β_a, β_b were identified in homology, and only one \mathbb{Z}_2 winding number remained as an independent quantum number. But in the worldsheet CFT sector with $\tilde{n} = 2$, we have a larger symmetry. The only identification is $Z \sim -Z$ (together with $Y \sim Y + 4\pi R$). Thus, for each of the two cycles of T^2 we end up with a separate \mathbb{Z}_2 winding number in the worldsheet theory.

Now, let us recover the Hilbert subspace $\mathcal{H} \subset \mathcal{H}'$ of string ground states. In the string theory, we need to keep only the states that are invariant under the entire $\mathbb{Z}_r = \mathbb{Z}_4$ orbifold group, generated by $Z \rightarrow iZ$ together with $Y \rightarrow Y + 2\pi R$. The resulting \mathbb{Z}_r -invariant states span a 3-dimensional subspace \mathcal{H} of the 4-dimensional \mathcal{H}' , and the \mathbb{Z}_r -invariant combinations that correspond to the states in section 3.2 are:

$$|\zeta_{0,0}\rangle \rightarrow |\square_{\circlearrowleft}\rangle, \quad |\zeta_{1,1}\rangle \rightarrow |\square_{\circlearrowright}\rangle, \quad \frac{1}{\sqrt{2}}(|\zeta_{0,1}\rangle + |\zeta_{1,0}\rangle) \rightarrow |\square_{\circlearrowright}\rangle. \quad (3.27)$$

The operators

$$\tilde{\mathcal{U}}_a + \tilde{\mathcal{U}}_b, \quad \tilde{\mathcal{V}}_a + \tilde{\mathcal{V}}_b, \quad \tilde{\mathcal{V}}_a \tilde{\mathcal{U}}_a + \tilde{\mathcal{V}}_b \tilde{\mathcal{U}}_b, \dots$$

preserve the 3-dimensional Hilbert space spanned by $|\square_{\circlearrowleft}\rangle, |\square_{\circlearrowright}\rangle, |\square_{\circlearrowright}\rangle$. They act as

$$\left. \begin{aligned} (\tilde{\mathcal{V}}_a + \tilde{\mathcal{V}}_b) |\square_{\circlearrowleft}\rangle &= 2 |\square_{\circlearrowleft}\rangle, & (\tilde{\mathcal{U}}_a + \tilde{\mathcal{U}}_b) |\square_{\circlearrowleft}\rangle &= \sqrt{2} |\square_{\circlearrowright}\rangle, \\ (\tilde{\mathcal{V}}_a + \tilde{\mathcal{V}}_b) |\square_{\circlearrowright}\rangle &= 0, & (\tilde{\mathcal{U}}_a + \tilde{\mathcal{U}}_b) |\square_{\circlearrowright}\rangle &= \sqrt{2} (|\square_{\circlearrowleft}\rangle + |\square_{\circlearrowright}\rangle), \\ (\tilde{\mathcal{V}}_a + \tilde{\mathcal{V}}_b) |\square_{\circlearrowright}\rangle &= -2 |\square_{\circlearrowright}\rangle, & (\tilde{\mathcal{U}}_a + \tilde{\mathcal{U}}_b) |\square_{\circlearrowright}\rangle &= \sqrt{2} |\square_{\circlearrowleft}\rangle, \end{aligned} \right\} \quad (3.28)$$

$$\left. \begin{aligned} (\tilde{\mathcal{V}}_a \tilde{\mathcal{U}}_a + \tilde{\mathcal{V}}_b \tilde{\mathcal{U}}_b) |\square_{\circlearrowleft}\rangle &= -\sqrt{2} |\square_{\circlearrowright}\rangle, \\ (\tilde{\mathcal{V}}_a \tilde{\mathcal{U}}_a + \tilde{\mathcal{V}}_b \tilde{\mathcal{U}}_b) |\square_{\circlearrowright}\rangle &= \sqrt{2} (|\square_{\circlearrowleft}\rangle - |\square_{\circlearrowright}\rangle), \\ (\tilde{\mathcal{V}}_a \tilde{\mathcal{U}}_a + \tilde{\mathcal{V}}_b \tilde{\mathcal{U}}_b) |\square_{\circlearrowright}\rangle &= \sqrt{2} |\square_{\circlearrowleft}\rangle, \end{aligned} \right\} \quad (3.29)$$

and so on.

As another example of this technique, consider the case $k = 1$ ($\tau = e^{\pi i/3}$ and $v = \frac{\pi}{3}$, $\mathbf{r} = 6$) and $\tilde{n} = 2$. The relevant solutions to (3.7) are:

$$\zeta_{0,0} = 0, \quad \zeta_{1,0} = \frac{i}{\sqrt{3}}, \quad \zeta_{0,1} = -\frac{i}{\sqrt{3}} \pmod{\mathbb{Z} + \mathbb{Z}\tau}.$$

(But also note the equivalent solutions $\zeta_{0,0} \simeq \zeta_{1,1} \simeq \zeta_{2,2}$, $\zeta_{1,0} \simeq \zeta_{2,1} \simeq \zeta_{0,2}$, and $\zeta_{0,1} \simeq \zeta_{2,0} \simeq \zeta_{1,2}$.) The orbifold generator acts on these fixed points ζ_{M_a, M_b} as multiplication by $e^{\pi i/3}$, and the invariant combinations are

$$|\zeta_{0,0}\rangle, \quad \frac{1}{\sqrt{2}} (|\zeta_{1,0}\rangle + |\zeta_{0,1}\rangle).$$

They correspond to the string ground states

$$|\zeta_{0,0}\rangle \rightarrow |[0,0]\rangle = |\square_{\circlearrowleft}\rangle, \quad \frac{1}{\sqrt{2}} (|\zeta_{1,0}\rangle + |\zeta_{0,1}\rangle) \rightarrow \left| \left[\frac{1}{3} + \frac{1}{3}\tau, \frac{2}{3} + \frac{2}{3}\tau \right] \right\rangle = |\square_{\circlearrowright}\rangle.$$

The group of additional worldsheet symmetries is generated by

$$\tilde{\mathcal{V}}_a \equiv \tilde{\mathcal{V}}(\zeta_{1,0}), \quad \tilde{\mathcal{U}}_a \equiv \tilde{\mathcal{U}}(\zeta_{1,0}). \quad (3.30)$$

They satisfy

$$\tilde{\mathcal{V}}_a^3 = \tilde{\mathcal{U}}_a^3 = 1,$$

and can be regarded as related to \mathbb{Z}_3 winding number and momentum. They act on states as

$$\tilde{\mathcal{V}}_a |\zeta_{M_a, M_b}\rangle \equiv e^{\frac{2\pi i}{3}(M_b - M_a)} |\zeta_{M_a, M_b}\rangle, \quad \tilde{\mathcal{U}}_a |\zeta_{M_a, M_b}\rangle = |\zeta_{M_a+1, M_b}\rangle. \quad (3.31)$$

The operators

$$\tilde{\mathcal{U}}_a + \tilde{\mathcal{U}}_a^{-1}, \quad \tilde{\mathcal{V}}_a + \tilde{\mathcal{V}}_a^{-1}, \quad \tilde{\mathcal{V}}_a \tilde{\mathcal{U}}_a + \tilde{\mathcal{V}}_a^{-1} \tilde{\mathcal{U}}_a^{-1}, \dots$$

preserve the 2-dimensional Hilbert space spanned by $\left| \begin{array}{c} \square \\ \circ \end{array} \right\rangle, \left| \begin{array}{c} \square \\ \circ \circ \end{array} \right\rangle$, and act as

$$\left. \begin{aligned} (\tilde{\mathcal{V}}_a + \tilde{\mathcal{V}}_a^{-1}) \left| \begin{array}{c} \square \\ \circ \circ \end{array} \right\rangle &= - \left| \begin{array}{c} \square \\ \circ \circ \end{array} \right\rangle, & (\tilde{\mathcal{U}}_a + \tilde{\mathcal{U}}_a^{-1}) \left| \begin{array}{c} \square \\ \circ \circ \end{array} \right\rangle &= \left| \begin{array}{c} \square \\ \circ \circ \end{array} \right\rangle + \sqrt{2} \left| \begin{array}{c} \square \\ \circ \end{array} \right\rangle, \\ (\tilde{\mathcal{V}}_a + \tilde{\mathcal{V}}_a^{-1}) \left| \begin{array}{c} \square \\ \circ \end{array} \right\rangle &= 2 \left| \begin{array}{c} \square \\ \circ \end{array} \right\rangle, & (\tilde{\mathcal{U}}_a + \tilde{\mathcal{U}}_a^{-1}) \left| \begin{array}{c} \square \\ \circ \end{array} \right\rangle &= \sqrt{2} \left| \begin{array}{c} \square \\ \circ \circ \end{array} \right\rangle, \end{aligned} \right\} \quad (3.32)$$

$$\left. \begin{aligned} (\tilde{\mathcal{V}}_a \tilde{\mathcal{U}}_a + \tilde{\mathcal{V}}_a^{-1} \tilde{\mathcal{U}}_a^{-1}) \left| \begin{array}{c} \square \\ \circ \end{array} \right\rangle &= \sqrt{2} e^{-\frac{2\pi i}{3}} \left| \begin{array}{c} \square \\ \circ \circ \end{array} \right\rangle, \\ (\tilde{\mathcal{V}}_a \tilde{\mathcal{U}}_a + \tilde{\mathcal{V}}_a^{-1} \tilde{\mathcal{U}}_a^{-1}) \left| \begin{array}{c} \square \\ \circ \circ \end{array} \right\rangle &= e^{\frac{2\pi i}{3}} \left| \begin{array}{c} \square \\ \circ \circ \end{array} \right\rangle + \sqrt{2} \left| \begin{array}{c} \square \\ \circ \end{array} \right\rangle, \end{aligned} \right\} \quad (3.33)$$

and so on.

3.6 Dependence on complex structure

At the beginning of this section, we compactified the field theory on T^2 (on the type-IIB side) with periodic coordinates $0 \leq x_1 < 2\pi L_1$ and $0 \leq x_2 < 2\pi L_2$. For simplicity we took the metric to be $ds^2 = dx_1^2 + dx_2^2$, which sets the complex structure of T^2 to be $\rho = iL_1/L_2$. For this metric ρ is purely imaginary, but we can easily allow a more general flat metric with a complex structure that has a nonzero real part. We can then define the action of a group $\text{SL}(2, \mathbb{Z})$ of large diffeomorphisms on T^2 by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \mathcal{G} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathcal{G} \equiv \begin{pmatrix} \tilde{\mathbf{a}} & \tilde{\mathbf{b}} \\ \tilde{\mathbf{c}} & \tilde{\mathbf{d}} \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \quad (3.34)$$

which acts on the complex structure ρ as

$$\rho \rightarrow \frac{\tilde{\mathbf{a}}\rho + \tilde{\mathbf{b}}}{\tilde{\mathbf{c}}\rho + \tilde{\mathbf{d}}}. \quad (3.35)$$

(This $\text{SL}(2, \mathbb{Z})$ is, of course, not related to the S-duality group of section 2.2. In the following, we hope that the context makes it clear which $\text{SL}(2, \mathbb{Z})$ we are referring to.)

The full Hilbert space is fibered over the moduli space of ρ 's, which is

$$\text{SL}(2, \mathbb{Z}) \backslash \text{SL}(2, \mathbb{R}) / \text{SO}(2),$$

and two subgroups of $\text{SL}(2, \mathbb{Z})$ become symmetries at two special values of ρ : $\mathbb{Z}_4 \subset \text{SL}(2, \mathbb{Z})$ generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a symmetry at $\rho = i$, and $\mathbb{Z}_6 \subset \text{SL}(2, \mathbb{Z})$ generated by $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ is a symmetry at $\rho = e^{\pi i/3}$.

If we are only interested in the finite-dimensional Hilbert space of supersymmetric ground states, as is the case here, we can say more. This finite-dimensional Hilbert space is the fiber of a flat vector bundle over the moduli space of ρ . Thus, the fibers at different complex structures ρ can be naturally identified, and the action of $\text{SL}(2, \mathbb{Z})$, which is the

holonomy group of the vector bundle, can be naturally defined on the fiber. In this way we get a full $SL(2, \mathbb{Z})$ symmetry acting on the Hilbert space of ground states. Unlike the operators \mathcal{U}, \mathcal{V} , this $SL(2, \mathbb{Z})$ group is not a symmetry of the full theory, but only a low-energy symmetry. Let us now identify the action of this $SL(2, \mathbb{Z})$ on the type-IIA side.

3.7 T-duality

Following the sequence of dualities in table 1, we find that on the type-IIA side (the last row in the table) we can identify ρ (defined in section 3.6) as the complexified area modulus of the T^2 fiber of W :

$$\rho = \frac{i}{\alpha'_{\text{IIA}}} \text{Area}(T^2) + \frac{1}{2\pi} \int_{T^2} B. \tag{3.36}$$

Here, B is the NS-NS two-form potential. The $SL(2, \mathbb{Z})$ group from section 3.6 becomes T-duality, and is generated by

$$\mathcal{S} \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z}), \quad \mathcal{S} : \quad \rho \rightarrow -\frac{1}{\rho},$$

and

$$\mathcal{T} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z}), \quad \mathcal{T} : \quad \rho \rightarrow \rho + 1.$$

At $\rho = i$, \mathcal{S} generates a \mathbb{Z}_4 symmetry of the full theory. However, \mathcal{T} can never be extended to a symmetry of the full spectrum, while \mathcal{TS} has order 6 and is a symmetry of the full spectrum for $\rho = e^{\pi i/3}$. Let us now determine the action of \mathcal{T} and \mathcal{S} on the ground states.

Partial information can be gleaned from the commutation relations of \mathcal{S}, \mathcal{T} with \mathcal{U}, \mathcal{V} defined in section 3.3–3.4. Since we associated \mathcal{U} with \mathbb{Z}_k momentum, and \mathcal{V} with \mathbb{Z}_k winding number, and since T-duality exchanges these two quantum numbers, we set:

$$\mathcal{S}^{-1} \mathcal{V} \mathcal{S} = \mathcal{U}, \quad \mathcal{S}^{-1} \mathcal{U} \mathcal{S} = \mathcal{V}^{-1}. \tag{3.37}$$

We also expect that a general T-duality element $\mathcal{G} \in SL(2, \mathbb{Z})$ [defined in (3.34)] acts as

$$\mathcal{G}^{-1} \mathcal{V} \mathcal{G} = e^{i\phi_1} \mathcal{V}^{\tilde{\mathbf{d}}} \mathcal{U}^{-\tilde{\mathbf{c}}}, \quad \mathcal{G}^{-1} \mathcal{U} \mathcal{G} = e^{i\phi_2} \mathcal{V}^{-\tilde{\mathbf{b}}} \mathcal{U}^{\tilde{\mathbf{a}}}. \tag{3.38}$$

We included undetermined phases ϕ_1, ϕ_2 in the commutation relations, because \mathcal{U}, \mathcal{V} do not commute and their order in the expressions on the right-hand side of the equations in (3.38) is important. Part of this phase ambiguity can be absorbed by a redefinition

$$\mathcal{G} \rightarrow \mathcal{U}^{\tilde{\mathbf{p}}} \mathcal{V}^{\tilde{\mathbf{q}}} \mathcal{G}, \tag{3.39}$$

under which

$$\phi_1 \rightarrow \phi_1 + \frac{2\pi n}{k} \tilde{\mathbf{p}}, \quad \phi_2 \rightarrow \phi_2 - \frac{2\pi n}{k} \tilde{\mathbf{q}}.$$

But in general ϕ_1, ϕ_2 need to be nonzero, so that the eigenvalues of the left- and right-hand sides of the equations in (3.38) will agree.

Now, let us specialize to \mathcal{T} . The generator \mathcal{T} of $\text{SL}(2, \mathbb{Z})$ commutes with the winding number operator \mathcal{V} , so we can choose $\phi_1 = 0$ in (3.38). We can also take $\phi_2 = \pm\pi n(k-1)/k$, so that the eigenvalues of the left- and right-hand sides of the rightmost equation of (3.38) will agree. (We take $-$ sign for $k = 2$ and $+$ sign for $k = 3$.) Thus we get

$$\mathcal{T}^{-1}\mathcal{V}\mathcal{T} = \mathcal{V}, \quad \mathcal{T}^{-1}\mathcal{U}\mathcal{T} = e^{\frac{i\pi n(-1)^{k+1}(k-1)}{k}}\mathcal{U}\mathcal{V}^{-1}. \quad (3.40)$$

For single-particle ground states of winding number $\tilde{n} = n$ that is relatively prime to \mathbf{r} , equations (3.37) and (3.40) are sufficient to determine \mathcal{S} and \mathcal{T} , up to multiplication by an overall phase and the freedom (3.39). In principle, these ambiguities can be further restricted by requiring the $\text{SL}(2, \mathbb{Z})$ relations $\mathcal{S}^2 = (\mathcal{S}\mathcal{T})^3 = -1$, but this will not be required for our present purposes. The results for \mathcal{S}, \mathcal{T} are listed in appendix A.

If $\text{gcd}(n, \mathbf{r}) > 1$, (3.37) and (3.40) are insufficient to completely determine \mathcal{S} and \mathcal{T} , and we need to study the worldsheet theory more carefully. In this case, the worldsheet theory, as we saw in section 3.5, possesses additional discrete symmetries that can be regarded as additional components of \mathbb{Z}_2 or \mathbb{Z}_3 winding and momentum. These symmetries do not commute with the $\mathbb{Z}_{\mathbf{r}}$ -orbifold symmetry generator \mathcal{R} and therefore do not lead to symmetries of the Hilbert space of string ground states. However, since T-duality is a duality at the level of CFT, we can use the additional discrete symmetries to glean additional information about the action of \mathcal{S}, \mathcal{T} . We will demonstrate how this works below.

As a first example, consider the case $v = \frac{\pi}{2}$ ($\tau = i$). We will start with the $n = 1$, for which of course $\text{gcd}(n, \mathbf{r}) = 1$ and we do not get additional worldsheet symmetries; but it is still instructive to start with this case. Referring to the notation of section 3.5, we have only two inequivalent solutions to (3.7):

$$\zeta_{1,1} \simeq \zeta_{0,0} = 0, \quad \zeta_{1,0} \simeq \zeta_{0,1} \simeq \frac{1}{2}(1+i) \pmod{\mathbb{Z} + \mathbb{Z}\tau}.$$

The states $|\zeta_{0,0}\rangle$ and $|\zeta_{1,1}\rangle$ are eigenstates of winding, while $\frac{1}{\sqrt{2}}(|\zeta_{0,0}\rangle \pm |\zeta_{1,1}\rangle)$ are eigenstates of the translation $Z \rightarrow Z + \frac{1}{2}(1+i)$. Hence, \mathcal{S} maps $|\zeta_{0,0}\rangle$ to $\frac{1}{\sqrt{2}}(|\zeta_{0,0}\rangle + |\zeta_{1,1}\rangle)$, and maps $|\zeta_{1,1}\rangle$ to $\frac{1}{\sqrt{2}}(|\zeta_{0,0}\rangle - |\zeta_{1,1}\rangle)$.

Now consider the case $v = \frac{\pi}{2}$ and $n = 2$, for which $\text{gcd}(n, \mathbf{r}) = 2$, and we do get additional worldsheet symmetries, $\tilde{\mathcal{V}}_a, \tilde{\mathcal{V}}_b, \tilde{\mathcal{U}}_a, \tilde{\mathcal{U}}_b$, as explained in section 3.5. The T-duality generators \mathcal{S}, \mathcal{T} are required to satisfy commutation relations similar to (3.37)–(3.40):

$$\mathcal{S}^{-1}\tilde{\mathcal{V}}_a\mathcal{S} = \tilde{\mathcal{U}}_b, \quad \mathcal{S}^{-1}\tilde{\mathcal{V}}_b\mathcal{S} = \tilde{\mathcal{U}}_a^{-1}, \quad \mathcal{S}^{-1}\tilde{\mathcal{U}}_a\mathcal{S} = \tilde{\mathcal{V}}_b, \quad \mathcal{S}^{-1}\tilde{\mathcal{U}}_b\mathcal{S} = \tilde{\mathcal{V}}_a^{-1}, \quad (3.41)$$

$$\mathcal{T}^{-1}\tilde{\mathcal{V}}_a\mathcal{T} = \tilde{\mathcal{V}}_a, \quad \mathcal{T}^{-1}\tilde{\mathcal{V}}_b\mathcal{T} = \tilde{\mathcal{V}}_b, \quad \mathcal{T}^{-1}\tilde{\mathcal{U}}_a\mathcal{T} = \tilde{\mathcal{U}}_a\tilde{\mathcal{V}}_b^{-1}, \quad \mathcal{T}^{-1}\tilde{\mathcal{U}}_b\mathcal{T} = \tilde{\mathcal{U}}_b\tilde{\mathcal{V}}_a. \quad (3.42)$$

Solving (3.41)–(3.42), we find the explicit expressions:

$$\mathcal{S}|\zeta_{K_a, K_b}\rangle = \frac{1}{2} \sum_{M_a, M_b \in \mathbb{Z}_2} (-1)^{K_a M_a + K_b M_b} |\zeta_{M_a, M_b}\rangle \equiv |K_a, K_b\rangle, \quad (3.43)$$

and

$$\mathcal{T}|\zeta_{K_a, K_b}\rangle = (-1)^{K_a K_b} |\zeta_{K_a, K_b}\rangle. \quad (3.44)$$

The action of \mathcal{S}, \mathcal{T} on the subspace of string ground states can be deduced from (3.27) and (3.43)–(3.44). The complete expressions are listed in appendix A.

As another example of this technique, consider the case $v = \frac{\pi}{3}$ ($\tau = e^{\pi i/3}$) and $n = 2$. The commutation relations of \mathcal{S}, \mathcal{T} with the extra symmetry generators (3.30) are:

$$\mathcal{S}^{-1} \tilde{\mathcal{V}}_a \mathcal{S} = \tilde{\mathcal{U}}_a, \quad \mathcal{S}^{-1} \tilde{\mathcal{U}}_a \mathcal{S} = \tilde{\mathcal{V}}_a^{-1}, \quad \mathcal{T}^{-1} \tilde{\mathcal{V}}_a \mathcal{T} = \tilde{\mathcal{V}}_a, \quad \mathcal{T}^{-1} \tilde{\mathcal{U}}_a \mathcal{T} = e^{\frac{2\pi i}{3}} \tilde{\mathcal{U}}_a \tilde{\mathcal{V}}_a, \quad (3.45)$$

where we have chosen the phase in the rightmost equation so that (i) the eigenvalues of the left- and right-hand sides will agree, and (ii) so that the subspace of string states will be invariant under \mathcal{T} . The solutions for \mathcal{S}, \mathcal{T} are listed in appendix A.

Worldsheet derivation of the action of \mathcal{T} . We will end this section by checking the formulas for \mathcal{T} directly from the worldsheet description. Denote

$$\omega_F \equiv \frac{1}{2i \operatorname{Im} \tau} dz \wedge d\bar{z}.$$

The integral of ω_F on any T^2 fiber of W is 1. The operator \mathcal{T} acts by shifting the NS-NS 2-form B -field of type-IIA by

$$B \rightarrow B + \omega_F.$$

We will now check how this shift affects the phase of scattering amplitudes of the string ground states. A string ground state, as discussed in section 3.2, corresponds to a curve γ in target space, which we can take for the present discussion to be W . Consider a worldsheet configuration that contributes to a scattering amplitude taking string ground states that correspond to the curves $\gamma_1, \gamma_2, \dots, \gamma_p$ into another ground states corresponding to curves $\gamma'_1, \gamma'_2, \dots, \gamma'_q$. The image of this worldsheet configuration in target space is a surface Σ whose boundary is

$$\partial\Sigma = \left(\bigcup_{i=1}^p \gamma_i^{-1} \right) \cup \left(\bigcup_{j=1}^q \gamma'_j \right), \quad (3.46)$$

where γ_i^{-1} is the curve with opposite orientation of γ_i . Define the phase factor

$$e^{i\Phi(\gamma_1^{-1}, \gamma_2^{-1}, \dots, \gamma_p^{-1}, \gamma'_1, \gamma'_2, \dots, \gamma'_q)} \equiv \exp \left(i \int_{\Sigma} \omega_F \right).$$

This phase is clearly independent of which Σ we choose, as long as it satisfies (3.46), because $\omega_F/2\pi$ is an integral cohomology class, whose integral over any closed surface is an integer. Thus, the phase $\exp(i\Phi)$ only depends on the curves $\gamma_1^{-1}, \dots, \gamma'_q$.

Consider, for example, the case $v = \frac{\pi}{2}$ and $n = 1$. We show in appendix A that

$$\mathcal{T} | \square \rangle = | \square \rangle, \quad \mathcal{T} | \circ \rangle = e^{\frac{i\pi}{2}} | \circ \rangle.$$

We would like to verify this phase difference of $e^{\pi i/2} = i$ using the explicit worldsheet considerations as above. So we study the action of \mathcal{T} on a scattering amplitude with initial state $| \square \rangle$ and final state $| \circ \rangle$. But because of \mathcal{V} -conservation (see (3.15)), we have to

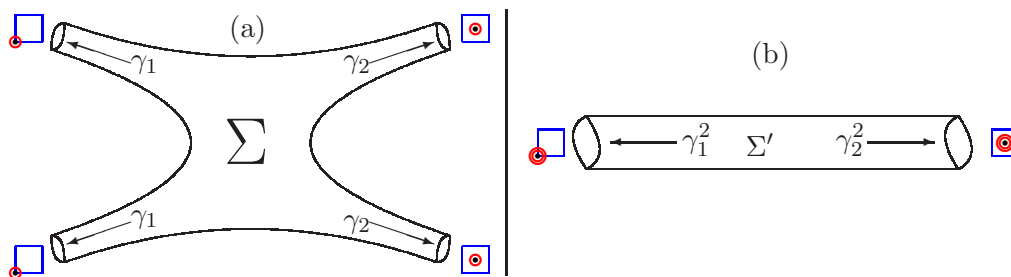


Figure 3. The image in target space of string worldsheets representing a scattering amplitude of string ground states. (a) Scattering of two identical string ground states into two other identical string ground states. The boundary of the image of the worldsheet is the union of four loops, corresponding to the four string states; the phase acquired under $B \rightarrow B + \frac{1}{2i} dz \wedge d\bar{z}$ is equal to the phase that the same transformation induces in (b) a worldsheet diagram for a 2-point function of string ground states of winding number 2 .

have an even number of $|\square\rangle$ in the final state. So, we consider the scattering amplitude of two $|\square\rangle$ states into two $|\square\rangle$ states. (See figure 3.) \mathcal{T} acts as multiplication by $i^2 = -1$ on this 4-point scattering amplitude, and this is what we wish to verify.

With the parameterization

$$0 \leq t < 1,$$

define the loops

$$\gamma_1 = [t \mapsto (z = 0, x_3 = 2\pi Rt)], \quad \gamma_2 = \left[t \mapsto \left(z = \frac{1}{2}(1 + \tau), x_3 = 2\pi Rt \right) \right].$$

We also use the standard loop-space product to define the double-wound loops:

$$\gamma_1^2 = [t \mapsto (z = 0, x_3 = 4\pi Rt)], \quad \gamma_2^2 = \left[t \mapsto \left(z = \frac{1}{2}(1 + \tau), x_3 = 4\pi Rt \right) \right].$$

In addition define the loops

$$\begin{aligned} \alpha_a &= [t \mapsto (z = t, x_3 = 0)], & \alpha_b &= [t \mapsto (z = t\tau, x_3 = 0)], \\ \alpha_{a+b} &= [t \mapsto (z = t(\tau + 1), x_3 = 0)]. \end{aligned}$$

(See (3.11) for similar definitions.) The phase Φ is clearly additive, so

$$\Phi(\gamma_1^{-1}, \gamma_1^{-1}, \gamma_2, \gamma_2) \equiv \Phi(\gamma_1^{-2}, \gamma_2^2) \pmod{2\pi}.$$

We calculate the latter as follows. First note that the following 3-point phase vanishes:

$$\Phi(\gamma_1^{-2}, \gamma_2^2, \alpha_{a+b}) \equiv 0 \pmod{2\pi}.$$

To quickly see this, take Σ to be the following surface:

$$\Sigma = \left[(\sigma, \eta) \mapsto \left(z = \frac{1}{2}\sigma(1 + \tau), x_3 = 4\pi R\eta \right) \right], \quad 0 \leq \sigma, \eta < 1,$$

for which $\partial\Sigma = \gamma_1^{-2} \cup \gamma_2^2 \cup \alpha_{a+b}$ and $\int_\Sigma \omega_F = 0$. We also have

$$\Phi(\alpha_a^{-1}, \alpha_b^{-1}) \equiv 0 \pmod{2\pi},$$

which can be verified by taking

$$\Sigma = [(\sigma, \eta) \mapsto (z = \sigma, x_3 = 2\pi R\eta)], \quad 0 \leq \sigma, \eta < 1.$$

Finally, note that

$$\Phi(\alpha_a^{-1}, \alpha_b^{-1}, \alpha_{a+b}) \equiv \pi \pmod{2\pi}.$$

To see this, consider Σ that is confined to one fiber at $x_3 = 0$ and is bounded by the three cycles $\alpha_a^{-1}, \alpha_b^{-1}, \alpha_{a+b}$. This Σ is a triangle and integrating ω_F on it gives π . The above results imply that

$$\Phi(\gamma_1^{-2}, \gamma_2^2) \equiv \Phi(\alpha_{a+b}) \equiv \pi \pmod{2\pi},$$

as claimed.

As another example, consider a 3-string scattering amplitude $|\begin{smallmatrix} \square \\ \bullet \end{smallmatrix}\rangle \rightarrow |\begin{smallmatrix} \square & \square \\ \bullet & \bullet \end{smallmatrix}\rangle$ (which preserves both \mathbb{Z}_2 momentum and winding). The action of \mathcal{T} on this amplitude will tell us the phase difference between the \mathcal{T} -eigenvalue of $|\begin{smallmatrix} \square \\ \bullet \end{smallmatrix}\rangle$ and the \mathcal{T} -eigenvalue of $|\begin{smallmatrix} \square & \square \\ \bullet & \bullet \end{smallmatrix}\rangle$. (This nontrivial phase will have an important consequence in section 6.5.) From (A.3) and (A.5), we know that this phase difference is $e^{-\pi i/2}$. To verify it, define the loop,

$$\gamma_3 = \left[t \mapsto \left(z = \frac{1}{2}, x_3 = 4\pi R t \right) \right].$$

The loop γ_3 corresponds to the state $|\begin{smallmatrix} \square \\ \bullet \end{smallmatrix}\rangle$, since at $t = \frac{1}{2}$ we have $(z = \frac{1}{2}, x_3 = 2\pi R) \simeq (z = \frac{1}{2}\tau, x_3 = 0)$ by (3.4). What we need then is the phase $\Phi(\gamma_3, \gamma_1^{-1}, \gamma_2^{-1})$. To calculate it, consider the following two surfaces (here $\tau = i$):

$$\Sigma_1 = \left[(\sigma, \eta) \mapsto \left(z = \frac{1}{2}\sigma, x_3 = 2\pi R\eta \right) \right], \quad 0 \leq \sigma, \eta < 1,$$

and

$$\Sigma_2 = \left[(\sigma, \eta) \mapsto \left(z = \frac{1}{2}\sigma\tau, x_3 = 2\pi R\eta \right) \right], \quad 0 \leq \sigma, \eta < 1.$$

Also, define the curve:

$$\delta = \left[t \mapsto \left\{ \begin{array}{ll} (z = 2t, x_3 = 0) & 0 \leq t \leq \frac{1}{4} \\ (z = \frac{1}{2} + 2(t - \frac{1}{4})\tau, x_3 = 0) & \frac{1}{4} \leq t \leq \frac{1}{2} \\ (z = 2(\frac{3}{4} - t) + \frac{1}{2}\tau, x_3 = 0) & \frac{1}{2} \leq t \leq \frac{3}{4} \\ (z = 2(1 - t)\tau, x_3 = 0) & \frac{3}{4} \leq t \leq 1 \end{array} \right\} \right].$$

Note that δ traces a square with vertices $z = 0, \frac{1}{2}, \frac{1}{2}(1 + \tau), \frac{1}{2}\tau$ inside the fiber over $x_3 = 0$, and the area bounded by it is $\frac{1}{4}$. To complete the calculation of the phase, we note that

$$\partial(\Sigma_1 \cup \Sigma_2) = \gamma_1^{-1} \cup \gamma_2^{-1} \cup \gamma_3 \cup \delta, \quad \text{and} \quad \int_{\Sigma_1 \cup \Sigma_2} \omega_F = 0.$$

Thus,

$$\Phi(\gamma_1^{-1}, \gamma_2^{-1}, \gamma_3) \equiv -\Phi(\delta) \equiv -\frac{\pi}{2} \pmod{2\pi}.$$

4 Warm-up: C-twist

In section 2.2 we were interested only in $\mathbf{g} \in \text{SL}(2, \mathbb{Z})$ that act nonperturbatively and fix a strongly-coupled value of τ . But there is another element \mathbf{g} that we can consider: $\mathbf{g} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. It preserves every τ , and acts on the theory as charge conjugation. It corresponds to $v = \pi$ and has order $\mathbf{r} = 2$. We will refer to this twist as a *C-twist*.

Compactification with C-twist actually preserves the full $\mathcal{N} = 8$ supersymmetry in three dimensions. In addition, we can keep $\text{Im } \tau \gg 1$, so as to have a weakly-coupled theory. We will now study the C-twist and demonstrate some of the ideas in the previous section explicitly in this setting. We will study only the cases of U(1) and U(2) gauge group. The case of U(n) with $n \geq 3$ is more involved and will not be addressed here.

4.1 Group theory

Combining the C-twist with the appropriate R-twist, and adjusting (2.8)–(2.10) to include a charge conjugation, we get the following boundary conditions

$$\left[\overline{\psi}_a^\alpha(x_0, x_1, x_2, x_3 + 2\pi R) \right]^* = i\Lambda^{-1} \psi_a^\alpha(x_0, x_1, x_2, x_3) \Lambda, \quad a = 1, \dots, 4, \quad (4.1)$$

$$\left[\Phi^I(x_0, x_1, x_2, x_3 + 2\pi R) \right]^* = -\Lambda^{-1} \Phi^I(x_0, x_1, x_2, x_3) \Lambda, \quad I = 1 \dots 6, \quad (4.2)$$

$$-A_\mu^*(x_0, x_1, x_2, x_3 + 2\pi R) = \Lambda^{-1} A_\mu(x_0, x_1, x_2, x_3) \Lambda - i\Lambda^{-1} \partial_\mu \Lambda, \quad (4.3)$$

where $[\dots]^*$ is the complex conjugate $n \times n$ matrix (not the adjoint matrix), and Λ is an arbitrary gauge transformation.

Now consider a closed path C at a constant $x_3 = 0$ that starts and ends at the origin, and consider the U(n)-holonomy $g = P \exp(i \oint_C A)$. Set $\Omega = \Lambda(0, 0, 0, 0)$. The combined charge conjugation and gauge transformation act on g as

$$g \mapsto [\Omega^{-1} g \Omega]^*.$$

We will need the invariant subgroup of U(n), which is the subgroup of solutions to

$$g = [\Omega^{-1} g \Omega]^*.$$

We denote it by $G_\Omega^{(\text{inv})} \subset \text{U}(n)$, since it generally depends on Ω . We now proceed to study the U(1) and U(2) cases in more detail.

4.2 U(1) gauge group

In this case $G_\Omega^{(\text{inv})} = O(1) \simeq \mathbb{Z}_2$. At low-energy, no propagating degrees of freedom survive the twist (4.1)–(4.3). The low-energy gauge group is $O(1) \simeq \mathbb{Z}_2$, which means that when we compactify the 2D space on T^2 we can have nontrivial \mathbb{Z}_2 Wilson lines around the two independent 1-cycles of T^2 . Let $w_a \in \mathbb{Z}_2 \simeq \{1, -1\}$ be the \mathbb{Z}_2 Wilson line along 1-cycle a ($a = 1, 2$). (For convenience, we take the \mathbb{Z}_2 group to be multiplicative instead of additive.) The four vacua are then labeled by $|w_1, w_2\rangle$, and we have a mass gap of $1/(2R)$.

Now consider the type-IIA dual description of the vacua, as in section 3.2. The effect of charge conjugation here is that it rotates the (dual) T^2 by $v = \pi$. The four vacua are therefore

$$|\square\rangle = |\{0\}\rangle, \quad |\square\rangle = \left| \left\{ \frac{1}{2} \right\} \right\rangle, \quad |\square\rangle = \left| \left\{ \frac{1}{2}\tau \right\} \right\rangle, \quad |\square\rangle = \left| \left\{ \frac{1}{2}(1+\tau) \right\} \right\rangle.$$

In order to match these type-IIA states with the field theory vacua $|w_1, w_2\rangle$, we define, as in section 3.3, the \mathbb{Z}_2 momentum operators

$$\left. \begin{aligned} \mathcal{U}_1 \left| \left[\frac{1}{2}M + \frac{1}{2}N\tau \right] \right\rangle &= \left| \left[\frac{1}{2}(1-M) + \frac{1}{2}N\tau \right] \right\rangle, \\ \mathcal{U}_2 \left| \left[\frac{1}{2}M + \frac{1}{2}N\tau \right] \right\rangle &= \left| \left[\frac{1}{2}M + \frac{1}{2}(1-N)\tau \right] \right\rangle, \end{aligned} \right\} \quad (4.4)$$

and the \mathbb{Z}_2 winding number operators, as in section 3.4,

$$\left. \begin{aligned} \mathcal{V}_1 \left| \left[\frac{1}{2}M + \frac{1}{2}N\tau \right] \right\rangle &= (-1)^M \left| \left[\frac{1}{2}M + \frac{1}{2}N\tau \right] \right\rangle, \\ \mathcal{V}_2 \left| \left[\frac{1}{2}M + \frac{1}{2}N\tau \right] \right\rangle &= (-1)^N \left| \left[\frac{1}{2}M + \frac{1}{2}N\tau \right] \right\rangle. \end{aligned} \right\} \quad (4.5)$$

We will now argue that

$$\mathcal{U}_1 = (-1)^{\mathbf{m}_1}, \quad \mathcal{U}_2 = (-1)^{\mathbf{e}_1}, \quad \mathcal{V}_1 = (-1)^{\mathbf{e}_2}, \quad \mathcal{V}_2 = (-1)^{\mathbf{m}_2}, \quad (4.6)$$

where $\mathbf{e}_1, \mathbf{e}_2$ are the electric flux operators in directions 1, 2 respectively, and $\mathbf{m}_1, \mathbf{m}_2$ are the magnetic flux operators.

Equations (4.6) can be derived by following the chain of dualities of table 1 backwards. Starting on the type-IIA side (the last row of table 1), take a state with Kaluza-Klein momenta $p_1, p_{10} \in \mathbb{Z}$ in directions x_1, x_{10} . (We can assume that the state is localized in the x_3 direction.) The unitary operator \mathcal{U}_1 acts as a translation in the direction of x_{10} and therefore multiplies the state by the phase $e^{\pi i p_{10}}$. Similarly, \mathcal{U}_2 multiplies the state by $e^{\pi i p_1}$. Following the chain of dualities backwards in table 1, we find that on the type-IIB side p_1 becomes fundamental string (F1) winding number in direction x_1 , while p_{10} becomes D1 winding number in direction x_1 . (See table 2.) The Kaluza-Klein state on the type-IIA side therefore becomes a (p, q) -string, with $p = p_1$ and $q = p_{10}$. Bound to n D3-branes, these quantum numbers become [12, 22] $\mathbf{e}_1 = p_1$ units of electric flux in direction 1 and $\mathbf{m}_1 = p_{10}$ units of magnetic flux in the same direction. Similarly, \mathcal{V}_1 corresponds to the exponential of string winding number in direction 10, and \mathcal{V}_2 to the exponential of string winding number in direction 1. On the type-IIB side, these become fundamental string winding number and D1-brane winding number in direction 2.

Now, let's relate the $|w_1, w_2\rangle$ basis (on the field theory/type-IIB side) to the $\left| \left[\frac{1}{2}M + \frac{1}{2}N\tau \right] \right\rangle$ basis on the type-IIA side. On the field theory side, $\mathcal{V}_1 = (-1)^{\mathbf{e}_2}$ can be interpreted as the operator of a large gauge transformation acting on the components of the gauge field as

$$A_1 \rightarrow A_1, \quad A_2 \rightarrow A_2 + \frac{1}{2L_2}.$$

Similarly, $\mathcal{U}_2 = (-1)^{\mathbf{e}_1}$ can be interpreted as the operator of a large gauge transformation

$$A_1 \rightarrow A_1 + \frac{1}{2L_1}, \quad A_2 \rightarrow A_2.$$

We can therefore identify the action on eigenstates of Wilson lines as

$$\mathcal{V}_1|w_1, w_2\rangle = |w_1, -w_2\rangle, \quad \mathcal{U}_2|w_1, w_2\rangle = |-w_1, w_2\rangle. \quad (4.7)$$

Comparing (4.7) to (4.4)–(4.5) we find

$$\left| \left[\frac{1}{2}M + \frac{1}{2}N\tau \right] \right\rangle_{\text{IIA}} = \frac{1}{\sqrt{2}} \sum_{M'=0,1} (-1)^{M'M} |(-1)^N, (-1)^{M'}\rangle_{\text{IIB}}. \quad (4.8)$$

Now consider the operators $\mathcal{U}_1, \mathcal{V}_2$, which according to (4.6) are related to magnetic flux. Using (4.4)–(4.5) and (4.8), we find

$$\mathcal{U}_1|w_1, w_2\rangle = w_2|w_1, w_2\rangle, \quad \mathcal{V}_2|w_1, w_2\rangle = w_1|w_1, w_2\rangle. \quad (4.9)$$

So w_1 is the eigenvalue of magnetic flux $(-1)^{\mathbf{m}_2}$, and w_2 is the eigenvalue of magnetic flux $(-1)^{\mathbf{m}_1}$.

The connection between the discrete \mathbb{Z}_2 Wilson line w_1 and the magnetic flux \mathbf{m}_2 can be understood as follows. Let us pick a uniform gauge field with Wilson line $w_1 = -1$: $A = \frac{1}{2L_1}dx_1$. The charge conjugate field is $-A$, so we have to pick a nonzero Λ in (4.3). Specifically, the gauge transformation that converts A to $-A$ is $\Lambda = \exp(-ix_1/L_1)$. This gauge transformation accompanies the coordinate transformation $x_3 \rightarrow x_3 + 2\pi R$, and for an ordinary T^3 compactification it would be interpreted [23, 24] as one unit of magnetic flux in direction 2, i.e., $\mathbf{m}_2 = 1$. The connection between w_2 and \mathbf{m}_1 is similar.

We can now understand the action of \mathcal{V}_1 and \mathcal{U}_2 as follows. According to (4.6), $\mathcal{U}_2 = (-1)^{\mathbf{e}_1}$ and therefore acts as a discontinuous gauge transformation with gauge parameter $\tilde{\Lambda}(x_1, x_2) = \exp(-ix_1/2L_1)$. Such a gauge transformation does not preserve the boundary conditions (4.3), because charge conjugation converts $\tilde{\Lambda}$ to $\tilde{\Lambda}^{-1} = \exp(ix_1/2L_1)$, but this can be fixed by modifying the gauge transformation Λ that appears in (4.3) to

$$\Lambda \rightarrow \Lambda e^{\frac{ix_1}{L_1}}.$$

This implies that \mathcal{U}_2 changes the magnetic flux \mathbf{m}_2 by one unit (modulo 2). Similarly, $\mathcal{V}_1 = (-1)^{\mathbf{e}_2}$ changes the magnetic flux \mathbf{m}_1 by one unit. Since, as we have seen in (4.9) [combined with (4.6)], w_1, w_2 can be identified with the magnetic fluxes $(-1)^{\mathbf{m}_2}, (-1)^{\mathbf{m}_1}$, we recover the expressions (4.7) for the action of $\mathcal{V}_1, \mathcal{U}_2$ on states. We have therefore completely mapped the field theory ground states to the type-IIA ground states.

Let us conclude this subsection with a few additional comments. First we note that the magnetic flux \mathbf{m}_3 has to vanish, because charge conjugation acts on it as $\mathbf{m}_3 \rightarrow -\mathbf{m}_3$, and this cannot be fixed by any gauge transformation Λ in (4.3). The electric flux \mathbf{e}_3 therefore also vanishes by S-duality. Finally, let us also write down the T-duality transformations \mathcal{S}, \mathcal{T} . On the type-IIB (field theory) side they act geometrically, so we have

$$\mathcal{S}|w_1, w_2\rangle = |w_2, w_1\rangle, \quad \mathcal{T}|w_1, w_2\rangle = |w_1, w_1 w_2\rangle.$$

4.3 U(2) gauge group

Let's now study the case of gauge group U(2). On the type-IIA side, a basis state is of one of two types: (i) a single string with winding number 2; or (ii) two strings with winding number 1.

The single-particle string states of winding number 2 are built from one of the four types of states

$$| [z, -z] \rangle, \quad \left| \left[z, \frac{1}{2} - z \right] \right\rangle, \quad \left| \left[z, \frac{1}{2}\tau - z \right] \right\rangle, \quad \left| \left[z, \frac{1}{2} + \frac{1}{2}\tau - z \right] \right\rangle,$$

where z is a free parameter on T^2/\mathbb{Z}_2 , which needs to be quantized. In addition, the location of the strings in the \mathbb{R}^6 transverse directions is free and needs to be quantized, too. This results in a continuous spectrum.

The two-particle states are given by combining two strings of winding number 1. Each of these strings can be at any of the four locations studied in section 4.2, and since they are identical bosons, the order is not important. We denote the states by $| \{ [z], [z'] \} \rangle$, where $z, z' \in \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ are any one of $0, \frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2} + \frac{1}{2}\tau$. Altogether we get 10 states (two identical bosons with 4 single-particle states). Similarly to (4.4), we define the symmetry operators \bar{U}_1, \bar{U}_2 by

$$\left. \begin{aligned} \bar{U}_1 | \{ [\frac{1}{2}M + \frac{1}{2}N\tau], [\frac{1}{2}M' + \frac{1}{2}N'\tau] \} \rangle &= | [\frac{1}{2}(1-M) + \frac{1}{2}N\tau], [\frac{1}{2}(1-M') + \frac{1}{2}N'\tau] \rangle \\ \bar{U}_2 | \{ [\frac{1}{2}M + \frac{1}{2}N\tau], [\frac{1}{2}M' + \frac{1}{2}N'\tau] \} \rangle &= | [\frac{1}{2}M + \frac{1}{2}(1-N)\tau], [\frac{1}{2}M' + \frac{1}{2}(1-N')\tau] \rangle \end{aligned} \right\}, \quad (4.10)$$

and similarly to (4.5), we define \bar{V}_1, \bar{V}_2 by

$$\left. \begin{aligned} \bar{V}_1 | \{ [\frac{1}{2}M + \frac{1}{2}N\tau], [\frac{1}{2}M' + \frac{1}{2}N'\tau] \} \rangle &= (-1)^{M+M'} | \{ [\frac{1}{2}M + \frac{1}{2}N\tau], [\frac{1}{2}M' + \frac{1}{2}N'\tau] \} \rangle \\ \bar{V}_2 | \{ [\frac{1}{2}M + \frac{1}{2}N\tau], [\frac{1}{2}M' + \frac{1}{2}N'\tau] \} \rangle &= (-1)^{N+N'} | \{ [\frac{1}{2}M + \frac{1}{2}N\tau], [\frac{1}{2}M' + \frac{1}{2}N'\tau] \} \rangle \end{aligned} \right\}. \quad (4.11)$$

Note that the 4 operators $\bar{U}_1, \bar{U}_2, \bar{V}_1, \bar{V}_2$ are mutually commuting.

Next, let us see how to get this spectrum from the field theory (type-IIB) side. Since $U(2) = [SU(2) \times U(1)]/\mathbb{Z}_2$, we can start by separately discussing the SU(2) and U(1) degrees of freedom, and then consider how they combine to form states of the full U(2) theory.

We begin with the SU(2) degrees of freedom. Since SU(2) is pseudo-real, charge conjugation is equivalent to a gauge transformation. Explicitly, the gauge transformation is realized by the matrix $i\sigma_2 \in SU(2)$ (we denote the Pauli matrices by $\sigma_1, \sigma_2, \sigma_3$), and for an adjoint-valued field ϕ we have

$$-\phi^* = (i\sigma_2)^{-1} \phi (i\sigma_2).$$

Thus, combining the extra gauge parameter $i\sigma_2$ with Λ in (4.1)–(4.3), we find that the C-twist has no effect on the SU(2) degrees of freedom. As far as the SU(2) degrees of freedom are concerned, we therefore have a standard compactification of $\mathcal{N} = 4$ SU(2) SYM on T^3 , preserving 16 supersymmetries, and we are interested in the normalizable ground states.

Let $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ be the \mathbb{Z}_2 't Hooft electric fluxes of a state of the SU(2) theory, and $\mathbf{m}'_1, \mathbf{m}'_2, \mathbf{m}'_3$ the \mathbb{Z}_2 't Hooft magnetic fluxes.⁴ It turns out [25] that there is one supersymmetric ground state for every combination of 't Hooft fluxes that satisfies

$$\mathbf{e}'_1 \mathbf{m}'_2 - \mathbf{e}'_2 \mathbf{m}'_1 = \mathbf{e}'_1 \mathbf{m}'_3 - \mathbf{e}'_3 \mathbf{m}'_1 = \mathbf{e}'_2 \mathbf{m}'_3 - \mathbf{e}'_3 \mathbf{m}'_2 = 0. \quad (4.12)$$

We denote the corresponding ground state by $|\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3, \mathbf{m}'_1, \mathbf{m}'_2, \mathbf{m}'_3\rangle_{\text{SU}(2)}$, and by convention this state is identically zero if (4.12) is not satisfied. We will soon require $\mathbf{e}'_3 = \mathbf{m}'_3 = 0$, and then the only nontrivial condition in (4.12) is

$$\mathbf{e}'_1 \mathbf{m}'_2 - \mathbf{e}'_2 \mathbf{m}'_1 = 0. \quad (4.13)$$

There are exactly 10 combinations of the \mathbb{Z}_2 fluxes $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{m}'_1, \mathbf{m}'_2$ that satisfy (4.13).

Let us comment that the result on the number of SU(2) ground states can be obtained in several ways. One way is to count the supersymmetric bound states of 2 D3-branes on T^3 . This system is described at low-energy by U(2) super Yang-Mills theory, and its Hilbert space is a tensor product of sectors of U(1) and SU(2) Hilbert spaces with the SU(2) electric and magnetic fluxes determined by the modulo 2 residue of the U(1) electric and magnetic fluxes. The U(1) electric and magnetic fluxes correspond to fundamental string (F1) and D1-charge. The result, which can be established by T-duality on the three directions of T^3 , is that there is one supersymmetric bound state for each combination of the electric and magnetic fluxes. It is a “bound state at threshold” if all magnetic fluxes vanish, and not at threshold otherwise. The condition (4.12) ensures that the total momentum carried by the U(1) flux is an integer. Alternatively, the result can be established entirely in field theory (with the assumption that the Witten index is identical to the number of ground states) [25], using results on the number of normalizable ground states in theories with 16 supersymmetries [12, 26–28].

Next, let us discuss the U(1) degrees of freedom. In section 4.2 we showed that the U(1) theory has 4 ground states, $|w_1, w_2\rangle$ (with $w_1, w_2 = \pm 1$). However, the discussion of section 4.2 needs to be modified in order to be applicable to the U(2) theory, as we shall now explain. Generally speaking, the problem is that the pure U(1) theory is invariant under certain large gauge transformations that can no longer be considered good gauge transformations in the U(2) theory. To explain this in detail, we need to discuss the electric and magnetic fluxes more thoroughly.

Consider a U(2) gauge configuration $A^{\text{U}(2)}$ which we regard locally as a 2×2 matrix of 1-forms. From this matrix we construct a U(1) gauge field by taking the trace, $A^{\text{U}(1)} = \text{tr} A^{\text{U}(2)}$. This normalization is actually a matter of convention. For example, for a standard toroidal compactification of U(n) gauge theory on T^3 , we can choose to define $A^{\text{U}(1)} = \text{tr} A^{\text{U}(n)}$, which corresponds to a surjective map $\text{U}(n) \xrightarrow{\det} \text{U}(1)$, or we can choose to define $A^{\text{U}(1)} = \frac{1}{n} \text{tr} A^{\text{U}(n)}$, which corresponds to an injective map $\text{U}(1) \xrightarrow{\cdot I} \text{U}(n)$. Neither choice

⁴Here we do not restrict the magnetic or electric fluxes, since we need to combine the SU(2) degrees of freedom with the U(1) later on. Of course, if we had just the SU(2) degrees of freedom, we would have had to set all magnetic fluxes to zero, and if we had just $\text{SO}(3) \simeq \text{SU}(2)/\mathbb{Z}_2$ we would have had to set all electric fluxes to zero.

is optimal, however, because with the second choice we are forced to include sectors with fractional magnetic flux (like $\frac{1}{n}$), and with the first choice we are forced to include sectors with fractional electric flux. We will see a manifestation of this below (4.16) where we will have to include sectors for which a proper gauge transformation ($\det \tilde{\Lambda}$) does not act as the identity operator. In section section 6.1 we will choose to work with the second convention but for the present section we proceed with $A^{U(1)} = \text{tr } A^{U(2)}$.

We now introduce U(1) magnetic fluxes $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$. We have already seen in section 4.2 that the C-twist requires $\mathbf{m}_3 = 0$. States with integer U(1) magnetic fluxes $\mathbf{m}_1, \mathbf{m}_2$ can be realized by the following classical solution to (4.3):

$$A^{U(2)} = \begin{pmatrix} \frac{\mathbf{m}_2}{2L_1} dx_1 + \frac{\mathbf{m}_1}{2L_2} dx_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} e^{-\frac{i\mathbf{m}_2 x_1}{L_1} - \frac{i\mathbf{m}_1 x_2}{L_2}} & 0 \\ 0 & 1 \end{pmatrix}.$$

Following (4.6) we define

$$\mathcal{U}_1 \equiv (-1)^{\mathbf{m}_1}, \quad \mathcal{V}_2 \equiv (-1)^{\mathbf{m}_2}. \tag{4.14}$$

Now, let's find the SU(2) magnetic fluxes $\mathbf{m}'_1, \mathbf{m}'_2 \in \mathbb{Z}_2$ of this configuration. Locally, we can split Λ into U(1) and SU(2) parts as

$$\Lambda = e^{-\frac{i\mathbf{m}_2 x_1}{2L_1} - \frac{i\mathbf{m}_1 x_2}{2L_2}} \begin{pmatrix} e^{-\frac{i\mathbf{m}_2 x_1}{2L_1} - \frac{i\mathbf{m}_1 x_2}{2L_2}} & 0 \\ 0 & e^{\frac{i\mathbf{m}_2 x_1}{2L_1} + \frac{i\mathbf{m}_1 x_2}{2L_2}} \end{pmatrix}.$$

We can read off the SU(2) 't Hooft magnetic fluxes $\mathbf{m}'_1, \mathbf{m}'_2$ from the SU(2) matrix on the right-hand side. This shows that, as in ordinary toroidal compactifications, the SU(2) 't Hooft magnetic flux is determined by the U(1) magnetic flux according to

$$0 = \mathbf{m}_1 + \mathbf{m}'_1 = \mathbf{m}_2 + \mathbf{m}'_2 \pmod{2}. \tag{4.15}$$

Similarly, $\mathbf{m}'_3 = \mathbf{m}_3 \pmod{2}$, and since $\mathbf{m}_3 = 0$ we get $\mathbf{m}'_3 = 0$.

Let us now turn to electric fluxes. Consider the large gauge transformation

$$\tilde{\Lambda} = \begin{pmatrix} e^{\frac{ix_1}{L_1}} & 0 \\ 0 & 1 \end{pmatrix} = e^{\frac{ix_1}{2L_1}} \begin{pmatrix} e^{\frac{ix_1}{2L_1}} & 0 \\ 0 & e^{-\frac{ix_1}{2L_1}} \end{pmatrix}. \tag{4.16}$$

All states of the U(2) theory must be invariant under $\tilde{\Lambda}$. On the right-hand side of (4.16) we decomposed $\tilde{\Lambda}$ locally into a U(1) gauge transformation and an SU(2) gauge transformation. However, note that the latter actually generates a discontinuous gauge transformation of SU(2)—applying this gauge transformation locally is equivalent to acting with the operator $(-1)^{e'_1}$, according to 't Hooft's definition [23, 24]. On the U(1) degrees of freedom, with our normalization, $\tilde{\Lambda}$ acts as $\det \tilde{\Lambda} = \exp(ix_1/L_1)$, which is a proper gauge transformation.

In section 4.2 we defined the operators $\mathcal{U}_2, \mathcal{V}_1$ which correspond to gauge transformations by discontinuous gauge parameters $\exp(ix_1/2L_1)$ and $\exp(ix_2/2L_2)$, respectively. The U(1) part of the gauge transformation $\tilde{\Lambda}$ can therefore be identified with \mathcal{U}_2^2 , and we conclude that in the U(2) theory all states must satisfy

$$\mathcal{U}_2^2 (-1)^{e'_1} = \mathcal{V}_1^2 (-1)^{e'_2} = 1. \tag{4.17}$$

In the realization (4.4)–(4.5) of $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1, \mathcal{V}_2$, we had $\mathcal{V}_1^2 = \mathcal{U}_2^2 = 1$ identically, but in the present context of the U(2) theory, this is too restrictive. For example, \mathcal{U}_2^2 corresponds to the gauge parameter

$$\begin{pmatrix} e^{\frac{ix_1}{2L_1}} & 0 \\ 0 & e^{\frac{ix_1}{2L_1}} \end{pmatrix},$$

which is discontinuous in U(2) and therefore not required to be the identity on physical states.

Thus, the Hilbert space of the U(1) theory has to be a representation of the algebra generated by $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1, \mathcal{V}_2$ with the relations

$$\begin{aligned} \mathcal{U}_1^2 = \mathcal{V}_2^2 = 1, \quad \mathcal{U}_1 \mathcal{U}_2 = \mathcal{U}_2 \mathcal{U}_1, \quad \mathcal{V}_1 \mathcal{V}_2 = \mathcal{V}_2 \mathcal{V}_1, \\ \mathcal{U}_i \mathcal{V}_j = (-1)^{\delta_{ij}} \mathcal{V}_j \mathcal{U}_i, \quad i, j = 1, 2, \end{aligned} \tag{4.18}$$

and we can add the conditions

$$\mathcal{V}_1^4 = \mathcal{U}_2^4 = 1, \tag{4.19}$$

since \mathcal{U}_2^4 and \mathcal{V}_1^4 are generated by the continuous large gauge transformations $e^{\frac{ix_1}{L_1}}$ and $e^{\frac{ix_2}{L_2}}$, and those do have to be the identity on physical states.

Note that \mathcal{V}_1^2 and \mathcal{U}_2^2 are central elements of this algebra, and all irreducible representations with $\mathcal{V}_1^2 = \mathcal{U}_2^2 = 1$ are equivalent to the one we studied in section 4.2. But we can find other irreducible representations by allowing one or both of \mathcal{V}_1^2 and \mathcal{U}_2^2 to be (-1) . In light of (4.17), we can identify

$$\mathcal{U}_2^2 = (-1)^{\mathbf{e}'_1}, \quad \mathcal{V}_1^2 = (-1)^{\mathbf{e}'_2}.$$

The algebra (4.18)–(4.19) then has the following 4-dimensional irreducible representation with states $|w_1, w_2, \mathbf{e}_1, \mathbf{e}_2\rangle$, where $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{Z}_2$ are fixed (and we dropped the primes now), and $w_1, w_2 \in \{-1, 1\}$ take all possible values:

$$\begin{cases} \mathcal{U}_1 |w_1, w_2, \mathbf{e}_1, \mathbf{e}_2\rangle = w_2 |w_1, w_2, \mathbf{e}_1, \mathbf{e}_2\rangle, \\ \mathcal{U}_2 |w_1, w_2, \mathbf{e}_1, \mathbf{e}_2\rangle = i^{\mathbf{e}_1} |w_1, w_2, \mathbf{e}_1, \mathbf{e}_2\rangle, \\ \mathcal{V}_1 |w_1, w_2, \mathbf{e}_1, \mathbf{e}_2\rangle = i^{\mathbf{e}_2} |w_1, -w_2, \mathbf{e}_1, \mathbf{e}_2\rangle, \\ \mathcal{V}_2 |w_1, w_2, \mathbf{e}_1, \mathbf{e}_2\rangle = w_1 |w_1, w_2, \mathbf{e}_1, \mathbf{e}_2\rangle. \end{cases} \tag{4.20}$$

(Note that replacing $i^{\mathbf{e}_j}$ by $(-i)^{\mathbf{e}_j}$ can be absorbed by a change of basis, so we picked one choice of square-root of $(-1)^{\mathbf{e}_j}$ at random.)

We can now combine the U(1) and SU(2) parts to form physical U(2) states. Incorporating the conditions (4.15) and (4.17) with the identifications (4.14) and (4.20), we can construct a basis of physical states of the form

$$\begin{aligned} |\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{m}'_1, \mathbf{m}'_2\rangle_{\text{U}(2)} \equiv \\ |w_1 = (-1)^{\mathbf{m}'_2}, w_2 = (-1)^{\mathbf{m}'_1}, \mathbf{e}'_1, \mathbf{e}'_2\rangle_{\text{U}(1)} \otimes |\mathbf{e}'_1, \mathbf{e}'_2, 0, \mathbf{m}'_1, \mathbf{m}'_2, 0\rangle_{\text{SU}(2)}, \end{aligned} \tag{4.21}$$

with

$$\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{m}'_1, \mathbf{m}'_2 \in \mathbb{Z}_2, \quad \mathbf{e}'_3 = \mathbf{m}'_3 = 0.$$

These are a total of $2^4 = 16$ states, but they are reduced to 10 states when we implement the condition (4.13).

Let us now map the basis of states (4.21) to the basis of states $\left\{ \left[\frac{1}{2}M + \frac{1}{2}N\tau \right], \left[\frac{1}{2}M' + \frac{1}{2}N'\tau \right] \right\}$ discussed at the beginning of this subsection. The discussion above motivates us to postulate the following relations between the type-IIA and field-theory symmetry operators:

$$\bar{\mathcal{U}}_1 = \mathcal{U}_1, \quad \bar{\mathcal{U}}_2 = \mathcal{U}_2^2, \quad \bar{\mathcal{V}}_1 = \mathcal{V}_1^2, \quad \bar{\mathcal{V}}_2 = \mathcal{V}_2.$$

With these identifications, the commutation relations agree. Comparing (4.10)–(4.11) with (4.20), using (4.21), we find the relation between the type-IIA and U(2) field-theory states:

$$\begin{aligned} & \left\{ \left[\frac{1}{2}M + \frac{1}{2}N\tau \right], \left[\frac{1}{2}M' + \frac{1}{2}N'\tau \right] \right\}_{\text{IIA}} = \\ & = \frac{1}{2} \sum_{K=0}^1 \sum_{L=0}^1 (-1)^{MK+NL} |e'_1 = L, e'_2 = M - M', \mathbf{m}'_1 = K, \mathbf{m}'_2 = N - N'\rangle_{\text{U}(2)}, \end{aligned} \quad (4.22)$$

where $M - M'$, and $N - N'$ are understood to be mod 2, and of course, we have used $M - M' \equiv M + M' \pmod{2}$ and $N - N' \equiv N + N' \pmod{2}$. We have also fixed an arbitrary phase in the definition of the states $|e'_1, e'_2, \mathbf{m}'_1, \mathbf{m}'_2\rangle_{\text{U}(2)}$.

Where does the condition (4.13) come from? It comes from the fact that the type-IIA strings are identical bosons. To see this, note that the expression (4.22) is a priori not symmetric under the interchange $(M, N) \leftrightarrow (M', N')$. This exchange does not affect $(M - M')$ and $(N - N')$, since they are \mathbb{Z}_2 -valued, but it replaces $(-1)^{MK+NL}$ by $(-1)^{M'K+N'L}$. The operator that exchanges $(M, N) \leftrightarrow (M', N')$ therefore acts as multiplication by $(-1)^{(M-M')K+(N-N')L}$ and can be identified with $(-1)^{e'_1 \mathbf{m}'_2 - e'_2 \mathbf{m}'_1}$ acting on the states $|e'_1, e'_2, \mathbf{m}'_1, \mathbf{m}'_2\rangle_{\text{U}(2)}$. Requiring the states to be invariant under the exchange $(M, N) \leftrightarrow (M', N')$ is therefore equivalent to (4.13).

We conclude the discussion of the U(2) C-twist by writing down the reverse transformation from the type-IIA states to the field theory states:

$$\begin{aligned} & |e'_1, e'_2, \mathbf{m}'_1, \mathbf{m}'_2\rangle_{\text{U}(2)} = \\ & = \frac{1}{2} \sum_{M=0}^1 \sum_{N=0}^1 (-1)^{M\mathbf{m}'_1 + Ne'_1} \left\{ \left[\frac{1}{2}M + \frac{1}{2}N\tau \right], \left[\frac{1}{2}(M + e'_2) + \frac{1}{2}(N + \mathbf{m}'_2)\tau \right] \right\}_{\text{IIA}}. \end{aligned} \quad (4.23)$$

5 Solution for U(1) gauge theory

We will now present in detail the solution of the problem presented in section 2 for U(1) gauge group, in which case the action of the three-dimensional field theory can be written down exactly.

5.1 The field theory side

The scalars, fermions, and gauge fields decouple from each other, and the scalars and fermions are described by a free field theory with R-twisted boundary conditions as in section 2.3:

$$\begin{aligned} \psi_\alpha^a(x_0, x_1, x_2, x_3 + 2\pi R) &= e^{i\varphi_a} \psi_\alpha^a(x_0, x_1, x_2, x_3), & a = 1, \dots, 4. \\ Z^j(x_0, x_1, x_2, x_3 + 2\pi R) &= e^{i(\varphi_j + \varphi_4)} Z^j(x_0, x_1, x_2, x_3), & j = 1, 2, 3. \end{aligned}$$

Here we take the $\mathcal{N} = 6$ twist (2.11), for which $\varphi_1 = \varphi_2 = \varphi_3 = \frac{1}{2}\nu$ and $\varphi_4 = -\frac{3}{2}\nu$ and there are no zero modes. The scalars and fermions therefore do not give rise to any 2+1D low-energy fields.

The vector field is a bit more involved. The action for the vector field contains two terms: a 3+1D bulk term in the coordinate range $0 < x_3 < 2\pi R$, and a 2+1D “boundary” term at $x_3 = 0$ (or $x_3 = 2\pi R$) associated with the S-twist. The bulk term is a standard U(1) Yang-Mills action on the interval $0 < x_3 < 2\pi R$, but instead of identifying the two endpoints, we allow the gauge fields at $x_3 = 0$ and $x_3 = 2\pi R$ to be independent, and define the 2+1D fields

$$A(0) \equiv \sum_{\mu=0}^2 A_\mu(x_0, x_1, x_2, x_3 = 0) dx^\mu, \quad A(2\pi R) \equiv \sum_{\mu=0}^2 A_\mu(x_0, x_1, x_2, x_3 = 2\pi R) dx^\mu.$$

The gauge transformations are also not required to be periodic in x_3 .

The additional boundary term depends on the specific element $\mathbf{g} \in \text{SL}(2, \mathbb{Z})$ used in the twist. For $\mathbf{g} = \mathbf{g}' \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the S-twist is incorporated by adding the following 2+1D term to the action [1, 29, 30]:

$$I_S(\mathbf{g}') = \frac{1}{2\pi} \int A(0) \wedge dA(2\pi R). \tag{5.1}$$

Here, the exterior derivative d in $dA(2\pi R)$ is a 2+1D derivative. One way to see that this term realizes the S-twist is to switch to Euclidean signature and think of x_3 as a Euclidean time coordinate (instead of x_0). Then, in the Hamiltonian formalism, e^{iI_S} represents the kernel of the S-duality operator which acts on wavefunctions (in the $A_3 = 0$ gauge) as a kind of “Fourier transform”:

$$\Psi(A) \rightarrow \tilde{\Psi}(\tilde{A}) = \int [DA] \exp \left\{ \frac{i}{2\pi} \int A \wedge d\tilde{A} \right\} \Psi(A), \tag{5.2}$$

and the last expression can be seen by requiring the \mathbf{g}' -duality to act on operators as

$$E_i \rightarrow B_i, \quad B_i \rightarrow -E_i. \tag{5.3}$$

If instead of $\mathbf{g} = \mathbf{g}'$ we had picked $\mathbf{g} = -\mathbf{g}' = (\mathbf{g}')^{-1}$, we would have ended up with the boundary action

$$I_S(-\mathbf{g}') = -\frac{1}{2\pi} \int A(0) \wedge dA(2\pi R). \tag{5.4}$$

For $\mathbf{g} = \mathbf{g}'' \equiv \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{g}'$ we get

$$I_S(\mathbf{g}'') = \frac{1}{4\pi} \int \{-A(2\pi R) \wedge dA(2\pi R) + 2A(0) \wedge dA(2\pi R)\}, \quad (5.5)$$

since the effect of the $SL(2, \mathbb{Z})$ transformation $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on wavefunctions is multiplication by a Chern-Simons term at level $k = -1$:

$$\Psi(A) \rightarrow \exp \left\{ -\frac{i}{4\pi} \int A \wedge dA \right\} \Psi(A).$$

Similarly, the action of $\mathbf{g} = -\mathbf{g}'' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (\mathbf{g}')^{-1}$ is realized by the boundary term

$$I_S(-\mathbf{g}'') = \frac{1}{4\pi} \int \{-A(2\pi R) \wedge dA(2\pi R) - 2A(0) \wedge dA(2\pi R)\}. \quad (5.6)$$

Now, given the various expressions for the boundary terms I_S in (5.1),(5.5),(5.6), it is easy to take the low-energy limit. We simply set $A(0) = A(2\pi R)$ (up to a gauge transformation), and find that $I_S(\pm\mathbf{g}')$ reduces to a Chern-Simons action at level $k = \pm 2$, and $I_S(\pm\mathbf{g}'')$ reduces to a Chern-Simons action at level $k = -1 \pm 2$. This can be summarized in the formula

$$I_S \rightarrow \frac{2 - \mathbf{a} - \mathbf{d}}{4\pi\mathbf{c}} \int A \wedge dA,$$

which is a $U(1)$ Chern-Simons theory at level

$$k \equiv (2 - \mathbf{a} - \mathbf{d})/\mathbf{c}. \quad (5.7)$$

Note that the gauge transformation parameter Λ , appearing in the gauge transformation $A \rightarrow A + d\Lambda$, is not required to be periodic in x_3 . Therefore, unlike an ordinary S^1 compactification, the Wilson line $\int_0^{2\pi R} A_3 dx_3$ (at fixed x_0, x_1, x_2) can be gauged away, and there is no additional massless mode arising from the dimensional reduction of A_3 .

Let us summarize the results in the following list:

- for $\tau = i, v = \frac{\pi}{2}, \mathbf{g} = \mathbf{g}' \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we have $k = 2$;
- for $\tau = e^{\pi i/3}, v = \frac{\pi}{3}, \mathbf{g} = \mathbf{g}'' \equiv \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, we have $k = 1$;
- for $\tau = e^{\pi i/3}, v = \frac{2\pi}{3}, \mathbf{g} = -\mathbf{g}''^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, we have $k = 3$.

Note that the value of the Chern-Simons level k is the same as the number of ground states with winding number $n = 1$ in type-IIA theory, given in (3.9). This is consistent with the fact that $U(1)$ Chern-Simons theory at level k has k ground states, as we will see in section 5.3.

5.2 Toroidal compactification

The expressions (5.1), (5.4), and (5.5) that we found in section 5.1 assume that the theory is formulated on $\mathbb{R}^{2,1}$ with certain boundary conditions that will become evident shortly. When these boundary conditions are relaxed, or when the theory is compactified on T^2 , additional terms need to be added due to the possibility of electric or magnetic fluxes, as we will now explain. For the sake of the discussion, let us assume Euclidean signature, and let us compactify all directions 0, 1, 2 on T^3 , so that

$$0 \leq x_0 < 2\pi L_0, \quad 0 \leq x_1 < 2\pi L_1, \quad 0 \leq x_2 < 2\pi L_2.$$

In (5.2) we wrote down the transformation from a wavefunction $\Psi(A)$ to the dual wavefunction $\tilde{\Psi}(\tilde{A})$. The expression contained the integral $\int A \wedge d\tilde{A}$. On T^3 , this expression is not well-defined, because A is not globally well-defined in sectors with nonzero magnetic flux. To obtain correct expression, consider a sector with magnetic flux $(\bar{\mathbf{m}}_0, \bar{\mathbf{m}}_1, \bar{\mathbf{m}}_2)$, where $\bar{\mathbf{m}}_0, \bar{\mathbf{m}}_1, \bar{\mathbf{m}}_2$ are integers. We define the associated gauge field

$$A = \frac{\bar{\mathbf{m}}_0 x_1}{2\pi L_1 L_2} dx_2 + \frac{\bar{\mathbf{m}}_1 x_2}{2\pi L_2 L_0} dx_0 + \frac{\bar{\mathbf{m}}_2 x_0}{2\pi L_0 L_1} dx_1 + A',$$

where A' is a globally defined 1-form. Similarly, set

$$\tilde{A} = \frac{\tilde{\mathbf{m}}_0 x_1}{2\pi L_1 L_2} dx_2 + \frac{\tilde{\mathbf{m}}_1 x_2}{2\pi L_2 L_0} dx_0 + \frac{\tilde{\mathbf{m}}_2 x_0}{2\pi L_0 L_1} dx_1 + \tilde{A}'.$$

Then (5.2) should read

$$\begin{aligned} & \Psi(A'; \bar{\mathbf{m}}_0, \bar{\mathbf{m}}_1, \bar{\mathbf{m}}_2) \rightarrow \\ & \tilde{\Psi}(\tilde{A}; \tilde{\mathbf{m}}_0, \tilde{\mathbf{m}}_1, \tilde{\mathbf{m}}_2) = \sum_{\bar{\mathbf{m}}_0 \in \mathbb{Z}} \sum_{\bar{\mathbf{m}}_1 \in \mathbb{Z}} \sum_{\bar{\mathbf{m}}_2 \in \mathbb{Z}} \int [DA'] \exp \left\{ \right. \\ & i \int A' \wedge \left(\frac{\tilde{\mathbf{m}}_0}{4\pi^2 L_1 L_2} dx_1 \wedge dx_2 + \frac{\tilde{\mathbf{m}}_1}{4\pi^2 L_2 L_0} dx_2 \wedge dx_0 + \frac{\tilde{\mathbf{m}}_2}{4\pi^2 L_0 L_1} dx_0 \wedge dx_1 \right) \\ & - i \int \tilde{A}' \wedge \left(\frac{\bar{\mathbf{m}}_0}{4\pi^2 L_1 L_2} dx_1 \wedge dx_2 + \frac{\bar{\mathbf{m}}_1}{4\pi^2 L_2 L_0} dx_2 \wedge dx_0 + \frac{\bar{\mathbf{m}}_2}{4\pi^2 L_0 L_1} dx_0 \wedge dx_1 \right) \\ & \left. + \frac{i}{2\pi} \int A' \wedge d\tilde{A}' \right\} \Psi(A'; \bar{\mathbf{m}}_0, \bar{\mathbf{m}}_1, \bar{\mathbf{m}}_2). \end{aligned} \quad (5.8)$$

The expression within the curly brackets $\{\dots\}$ can be written in terms of the discontinuous A and \tilde{A} fields as

$$\begin{aligned} & \frac{i}{2\pi} \int A \wedge d\tilde{A} + i\pi(\bar{\mathbf{m}}_0 \tilde{\mathbf{m}}_1 + \bar{\mathbf{m}}_1 \tilde{\mathbf{m}}_2 + \bar{\mathbf{m}}_2 \tilde{\mathbf{m}}_0) + \frac{i\bar{\mathbf{m}}_0}{L_2} \int \tilde{A}_0(x_0, 0, x_2) dx_0 dx_2 \\ & + \frac{i\bar{\mathbf{m}}_1}{L_0} \int \tilde{A}_1(x_0, x_1, 0) dx_0 dx_1 + \frac{i\bar{\mathbf{m}}_2}{L_1} \int \tilde{A}_2(0, x_1, x_2) dx_1 dx_2. \end{aligned} \quad (5.9)$$

The last three terms may seem a little odd, especially since they are evaluated at arbitrary locations ($x_1 = 0$, $x_2 = 0$, and $x_0 = 0$), but they are required because of the discontinuity in A at those locations.

We can now Wick-rotate the expression (5.9) by setting

$$\overline{\mathbf{m}}_0 = \int dA \wedge \delta(x_0) dx_0, \tag{5.10}$$

and

$$\begin{aligned} \overline{\mathbf{m}}_1 &= \int dA \wedge \delta(x_1) dx_1 = \frac{1}{2\pi L_1} \int dA \wedge dx_1, \\ \overline{\mathbf{m}}_2 &= \int dA \wedge \delta(x_2) dx_2 = \frac{1}{2\pi L_2} \int dA \wedge dx_2, \end{aligned} \tag{5.11}$$

and similarly,

$$\widetilde{\mathbf{m}}_0 = \int d\tilde{A} \wedge \delta(x_0) dx_0, \tag{5.12}$$

and

$$\begin{aligned} \widetilde{\mathbf{m}}_1 &= \int d\tilde{A} \wedge \delta(x_1) dx_1 = \frac{1}{2\pi L_1} \int d\tilde{A} \wedge dx_1, \\ \widetilde{\mathbf{m}}_2 &= \int d\tilde{A} \wedge \delta(x_2) dx_2 = \frac{1}{2\pi L_2} \int d\tilde{A} \wedge dx_2. \end{aligned} \tag{5.13}$$

We now find the correction to (5.1) by combining (5.9) with (5.10)–(5.13), and setting $A \equiv A(0)$ and $\tilde{A} \equiv A(2\pi R)$.

For most purposes, (5.1) will be sufficient. In particular, the low-energy limit is simply the compactification on T^2 of the Chern-Simons theory found in section 5.1. This can be seen by setting $A = \tilde{A}$ (i.e., $A' = \tilde{A}'$ and $\overline{\mathbf{m}}_j = \widetilde{\mathbf{m}}_j$) in (5.9). However, one place where we should be careful is when we consider electric fluxes. For example, let us discuss the electric flux \mathbf{e}_1 in the direction of x_1 . First, note that \mathbf{e}_1 by itself is not S-duality invariant and so is ill-defined in our setting. However, if we add the magnetic flux \mathbf{m}_1 we find that the combination $\mathbf{e}_1 + \mathbf{m}_1 \pmod{2}$ is S-duality invariant. Thus,

$$(-1)^{\mathbf{e}_1 + \mathbf{m}_1}$$

is a well-defined \mathbb{Z}_2 quantum number. More generally, for other $\text{SL}(2, \mathbb{Z})$ elements, $\mathbf{e}_j + \mathbf{m}_j \pmod{k}$, where k is the Chern-Simons level defined in (5.7), is invariant under the S-duality twist, and

$$e^{\frac{2\pi i}{k}(\mathbf{e}_j + \mathbf{m}_j)}$$

is a well-defined \mathbb{Z}_k quantum number. Let us see how to interpret this statement from the action.

The operator $(-1)^{\mathbf{e}_1}$ acts as the discontinuous gauge transformation

$$A \rightarrow A + \frac{dx_1}{2L_1}.$$

In the action (5.8) this translates to

$$A' \rightarrow A' + \frac{dx_1}{2L_1}, \quad \tilde{A}' \rightarrow \tilde{A}' + \frac{dx_1}{2L_1}, \tag{5.14}$$

while keeping $\bar{\mathbf{m}}_j$ and $\tilde{\bar{\mathbf{m}}}_j$ ($j = 0, 1, 2$) unchanged. Under (5.14), the action (5.9) then picks up an extra term $i\pi(\tilde{\bar{\mathbf{m}}}_1 - \bar{\mathbf{m}}_1)$, which, using (5.10)–(5.13), can be written as

$$\frac{i}{2} \int [F_{02}(x_3 = 0) - F_{02}(x_3 = 2\pi R)] dx_0 dx_2.$$

Assuming that A_0 is periodic in x_2 , this becomes the difference of Wilson lines:

$$i\pi \int [A_2(x_3 = 2\pi R, x_0 = \infty) - A_2(x_3 = 0, x_0 = \infty)] dx^2 - i\pi \int [A_2(x_3 = 2\pi R, x_0 = -\infty) - A_2(x_3 = 0, x_0 = -\infty)] dx^2. \quad (5.15)$$

Now, what is the operator $(-1)^{\mathbf{m}_1}$? Acting on quantum states, it would multiply the wavefunction by

$$\exp \left\{ i\pi \int F_{23} dx_2 dx_3 \right\} \xrightarrow{A_3=0} \exp \left\{ i\pi \left(\int [A_2(x_3 = 2\pi R) - A_2(x_3 = 0)] dx^2 \right) \right\}.$$

If a symmetry multiplies quantum states by $e^{i\phi(x_0)}$, where $\phi(x_0)$ is a time-dependent phase, then it multiplies the path-integral by $\exp\{i(\phi(x_0 = \infty) - \phi(x_0 = -\infty))\}$. Thus, altogether $(-1)^{\mathbf{m}_1 + \mathbf{e}_1}$ keeps the action invariant. We also see that this operator reduces to $(-1)^{\mathbf{e}_1}$ in the low-energy Chern-Simons theory, because the low-energy Chern-Simons theory action depends only on $A' = \tilde{A}'$ on which (5.14) acts as $(-1)^{\mathbf{e}_1}$.

5.3 U(1) Chern-Simons theory on T^2

Now we describe in detail the theory upon compactification of the two spatial directions on T^2 (parameterized by $0 \leq x_j \leq 2\pi L_j$ for $j = 1, 2$). As we have seen in section 5.1, the low-energy theory is a U(1) Chern-Simons theory at level k with action

$$I = \frac{k}{4\pi} \int A \wedge dA.$$

We are interested in the Hilbert space of states of this theory on T^2 (all are ground states since the theory is topological), and we will now take a few paragraphs to review it (see [33] for more details).

We denote the k^{th} root of unity by

$$\omega \equiv e^{\frac{2\pi i}{k}}.$$

We also define two independent Wilson loop operators in terms of the integrals of the gauge fields on two independent 1-cycles of T^2 :

$$\mathcal{W}_1 = \exp \left\{ \int_0^{2\pi L_1} A_1(t, 0) dt \right\}, \quad \mathcal{W}_2 = \exp \left\{ \int_0^{2\pi L_2} A_2(0, t) dt \right\}. \quad (5.16)$$

The Hilbert space of Chern-Simons theory on T^2 at level k has k states, and we can pick a basis where $\mathcal{W}_1, \mathcal{W}_2$ are represented by

$$\mathcal{W}_1 = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \ddots & & \\ & & & \omega^{k-2} & \\ & & & & \omega^{k-1} \end{pmatrix}, \quad \mathcal{W}_2 = \begin{pmatrix} & & & & 1 \\ & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & & 1 \end{pmatrix}. \quad (5.17)$$

These operators satisfy the relations

$$\mathcal{W}_1 \mathcal{W}_2 = \omega \mathcal{W}_2 \mathcal{W}_1, \quad \mathcal{W}_1^k = \mathcal{W}_2^k = 1. \quad (5.18)$$

We denote the states in the basis in which (5.17) holds by

$$|p\rangle, \quad p = 0, \dots, k-1 \in \mathbb{Z}/k\mathbb{Z}, \quad (5.19)$$

so that

$$\mathcal{W}_1 |p\rangle = \omega^p |p\rangle, \quad \mathcal{W}_2 |p\rangle = |p+1\rangle.$$

The Chern-Simons theory is topological and therefore independent of the metric on T^2 . There is an $\text{SL}(2, \mathbb{Z})$ group of large diffeomorphisms, introduced in section 3.6, that acts on T^2 as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \mathcal{G} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathcal{G} \equiv \begin{pmatrix} \tilde{\mathbf{a}} & \tilde{\mathbf{b}} \\ \tilde{\mathbf{c}} & \tilde{\mathbf{d}} \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \quad (5.20)$$

(We stress again that this $\text{SL}(2, \mathbb{Z})$ should not be confused with the S-duality group.) For the same reason as in section 3.7, it is represented projectively on the Hilbert space (i.e., commutation relations close up to a phase). That is, we can require an element $\mathcal{G} \in \text{SL}(2, \mathbb{Z})$ to satisfy

$$\mathcal{G}^{-1} \mathcal{W}_1 \mathcal{G} = \mathcal{W}_2^{-\tilde{\mathbf{b}}} \mathcal{W}_1^{\tilde{\mathbf{a}}}, \quad \mathcal{G}^{-1} \mathcal{W}_2 \mathcal{G} = \mathcal{W}_2^{\tilde{\mathbf{d}}} \mathcal{W}_1^{-\tilde{\mathbf{c}}}, \quad (5.21)$$

but the order of the operators on the right-hand side of each equation is arbitrary, since the Wilson operators \mathcal{W}_1 and \mathcal{W}_2 do not commute, and this is why we get only a projective representation. Another ordering would correspond to replacing \mathcal{G} by $\mathcal{W}_1^l \mathcal{W}_2^m \mathcal{G}$ for some integers l, m . For our purposes we will only need to realize two elements of $\text{SL}(2, \mathbb{Z})$, and the projective nature of the representation will not be important to us. The elements that we need are listed below.

The element $\mathcal{S} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ acts as

$$\mathcal{S}|p\rangle = \frac{1}{\sqrt{k}} \sum_q \omega^{pq} |q\rangle, \quad (5.22)$$

and is a special case of the Verlinde matrix [31, 32]. It is easy to verify that

$$\mathcal{S}^{-1} \mathcal{W}_1 \mathcal{S} = \mathcal{W}_2, \quad \mathcal{S}^{-1} \mathcal{W}_2 \mathcal{S} = \mathcal{W}_1^{-1}.$$

The other generator of $\text{SL}(2, \mathbb{Z})$ is $\mathcal{T} \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For even k it acts as

$$\mathcal{T}|p\rangle = e^{\frac{\pi i}{k} p^2} |p\rangle, \quad k \equiv 0 \pmod{2}. \quad (5.23)$$

For odd k , (5.23) is ill-defined since the phase $\exp(\frac{\pi i}{k} p^2)$ depends on p and not just on $p \pmod{k}$. In fact, as we will discuss in section 5.4, things get a little more complicated for odd k , but with the proper modification of the definition of \mathcal{T} , it turns out that we can use the expression

$$\mathcal{T}|p\rangle = e^{\frac{\pi i}{k} p(p+k)} |p\rangle. \quad (5.24)$$

In general, there is some freedom in the expressions that we have given above for \mathcal{T} . For odd as well as even k , we can replace $\mathcal{T} \rightarrow e^{i\phi} \mathcal{W}_1^{N'} \mathcal{T}$ for some arbitrary integer N' and phase ϕ , and this will only introduce an inconsequential phase in the commutation relation $\mathcal{T}^{-1} \mathcal{W}_2 \mathcal{T}$. In principle, ϕ can be determined if we wish to preserve the relation $(\mathcal{T}\mathcal{S})^3 = -1$, not just up to a phase.

Now, let us discuss electric flux. Consider large discontinuous $U(1)$ gauge transformations of the form

$$\Lambda_1(x_1, x_2) = e^{\frac{i\nu x_1}{L_1}}, \quad \Lambda_2(x_1, x_2) = e^{\frac{i\nu x_2}{L_2}}, \quad (5.25)$$

where $0 < \nu < 1$ is arbitrary, for the time being. Let Ω_1, Ω_2 be the corresponding operators on the Hilbert space, which we can identify with exponentials of the electric fluxes $\mathbf{e}_1, \mathbf{e}_2$:

$$\Omega_1 = e^{2\pi i \nu \mathbf{e}_1}, \quad \Omega_2 = e^{2\pi i \nu \mathbf{e}_2}. \quad (5.26)$$

They act by conjugation on the Wilson operators:

$$\Omega_1^{-1} \mathcal{W}_1 \Omega_1 = e^{2\pi i \nu} \mathcal{W}_1, \quad \Omega_2^{-1} \mathcal{W}_1 \Omega_2 = \mathcal{W}_1, \quad \Omega_1^{-1} \mathcal{W}_2 \Omega_1 = \mathcal{W}_2, \quad \Omega_2^{-1} \mathcal{W}_2 \Omega_2 = e^{2\pi i \nu} \mathcal{W}_2.$$

These equations are solvable only if ν is an integer multiple of $1/k$. Setting

$$\nu = \frac{1}{k}, \quad \Lambda_1(x_1, x_2) = e^{\frac{i x_1}{k L_1}}, \quad \Lambda_2(x_1, x_2) = e^{\frac{i x_2}{k L_2}}, \quad (5.27)$$

we can identify

$$\Omega_1 \equiv \mathcal{W}_2, \quad \Omega_2 \equiv \mathcal{W}_1^{-1}, \quad (5.28)$$

and they act on states as:

$$\Omega_1|p\rangle = |p+1\rangle, \quad \Omega_2|p\rangle = \omega^{-p}|p\rangle. \quad (5.29)$$

Thus, electric flux is defined only modulo k . The state $|p\rangle$ is an eigenstate of Ω_2 with eigenvalue ω^{-p} , and hence has $\mathbf{e}_2 = -p$, and the state

$$\frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} \omega^{-qj} |j\rangle$$

is an eigenstate of Ω_1 with eigenvalue ω^q , and hence has $\mathbf{e}_1 = q$.

5.4 Wavefunctions

An explicit description of the Hilbert space of Chern-Simons theory on T^2 can be given in terms of θ -functions [33, 34, 36, 37].

In general, the states of Chern-Simons theory on a Riemann surface can be obtained by quantization of the space of flat connections on the Riemann surface [32]. For $U(1)$ gauge group on T^2 of complex structure $\rho = \rho_1 + i\rho_2$, a flat connection corresponds to a constant gauge field $A \equiv A_1 dx_1 + A_2 dx_2$. We parameterize it by the complex combination

$$\mathbf{a} \equiv -\frac{i\rho_2}{\pi} A_{\bar{z}} = \frac{1}{2\pi}(-\rho A_1 + A_2),$$

where we used the complex coordinate $z \equiv x_1 + \rho x_2$. Because gauge equivalent configurations are identified, we find that \mathbf{a} lives on a T^2 of complex structure ρ , with $\mathbf{a} \simeq \mathbf{a}+1 \simeq \mathbf{a}+\rho$.

The Chern-Simons action implies a nonzero commutation relation between the operator $\hat{\mathbf{a}}$ that represents \mathbf{a} and its conjugate $\hat{\mathbf{a}}^\dagger$. These commutation relations can be represented by the following operators,

$$\hat{\mathbf{a}}^\dagger = \frac{\rho_2}{\pi k} \frac{\partial}{\partial \mathbf{a}}, \quad \hat{\mathbf{a}} = \mathbf{a}, \tag{5.30}$$

acting on analytic functions $\psi(\mathbf{a})$. The formulas above for $\hat{\mathbf{a}}$ and its complex conjugate $\hat{\mathbf{a}}^\dagger$ are compatible with an inner product [34] given by

$$\langle \psi | \psi \rangle = \int e^{-\frac{\pi k}{\rho_2} |\mathbf{a}|^2} |\psi|^2 d^2 \mathbf{a}. \tag{5.31}$$

Imposing the periodicity conditions $\mathbf{a} \simeq \mathbf{a}+1 \simeq \mathbf{a}+\rho$, we get a k -dimensional Hilbert space with a basis

$$\psi_p(\mathbf{a}) = \theta(k\mathbf{a} + p\rho; k\rho) e^{\frac{\pi k}{2\rho_2} \mathbf{a}^2 + \frac{1}{k} \pi i \rho p^2 + 2\pi i p \mathbf{a}}, \tag{5.32}$$

where the θ -function is given by

$$\theta(\mathbf{a}; \rho) \equiv \sum_{n=-\infty}^{\infty} e^{\pi i \rho n^2 + 2\pi i n \mathbf{a}}.$$

The operators $\mathcal{W}_1, \mathcal{W}_2$ act on a generic wavefunction $\psi(\mathbf{a})$ [which is understood to be a linear combination of the $\psi_p(\mathbf{a})$'s] as

$$\mathcal{W}_1 \psi(\mathbf{a}) = e^{-\frac{\pi}{\rho_2} \mathbf{a} - \frac{\pi}{2k\rho_2}} \psi\left(\mathbf{a} + \frac{1}{k}\right), \quad \mathcal{W}_2 \psi(\mathbf{a}) = e^{-\frac{\pi \bar{\rho}}{\rho_2} \mathbf{a} - \frac{\pi |\rho|^2}{2k\rho_2}} \psi\left(\mathbf{a} + \frac{\rho}{k}\right). \tag{5.33}$$

The factors $e^{-\frac{\pi}{\rho_2} \mathbf{a} - \frac{\pi}{2k\rho_2}}$ and $e^{-\frac{\pi \bar{\rho}}{\rho_2} \mathbf{a} - \frac{\pi |\rho|^2}{2k\rho_2}}$ are required in order to preserve unitarity, which can be understood as follows: the transformation $\mathbf{a} \rightarrow \mathbf{a} + \frac{1}{k}$, for example, needs to be accompanied by $\mathbf{a}^\dagger \rightarrow \mathbf{a}^\dagger + \frac{1}{k}$, and the latter is generated by $e^{-\frac{\pi}{\rho_2} \mathbf{a}}$.

In this representation it is easy to check the \mathcal{T} -transformation

$$\mathcal{T}|p\rangle = e^{\frac{\pi i}{k} p^2} |p\rangle \tag{5.34}$$

for even k (up to an unimportant overall phase), and odd k will be discussed below.

Odd k . For odd k , the definition of Chern-Simons theory requires a spin structure on the three-manifold [38]. For that reason, theories with odd k are sometimes referred to as *spin Chern-Simons* theories [39].

In our case, we can see a manifestation of this in the behavior of the wavefunctions (5.32) under $\text{SL}(2, \mathbb{Z})$. There are 4 distinct spin structures on T^2 , and the wavefunction (5.32) corresponds to one particular spin structure. The transformation $\rho \rightarrow \rho + 1$ does not preserve this spin structure, and indeed, for odd k the Hilbert space is not closed under it. (It is only closed under its square $\rho \rightarrow \rho + 2$.) We can see this by a direct calculation:

$$\psi_p(\mathbf{a}, \rho + 1) = e^{\frac{1}{k}\pi ip^2 + \pi ip} e^{-\frac{\pi k}{2\rho_2}\mathbf{a}} e^{-\frac{\pi k}{8\rho_2}} \psi_p\left(\mathbf{a} + \frac{1}{2}, \rho\right).$$

However, if we define

$$\mathcal{T}\psi(\mathbf{a}, \rho) \equiv e^{-\frac{\pi k}{2\rho_2}\mathbf{a} - \frac{\pi k}{8\rho_2}} \psi\left(\mathbf{a} + \frac{1}{2}, \rho + 1\right), \tag{5.35}$$

then we find closure:

$$\mathcal{T}\psi_p = e^{\frac{1}{k}\pi ip^2 + \pi ip} \psi_p.$$

Note that the factor $e^{-\frac{\pi k}{2\rho_2}\mathbf{a}}$ in (5.35) can be understood as realizing $\mathbf{a}^\dagger \rightarrow \mathbf{a}^\dagger + \frac{1}{2}$, in agreement with (5.34). Thus, (5.35) represents the large diffeomorphism $\rho \rightarrow \rho + 1$ augmented by a change of coordinates that represents translation by $1/2$ of the T^2 .

The dependence of the theory on the spin structure of T^2 is related to the dependence of the partition function of a 5+1D (anti-)self-dual free 2-form on spin structure [40]. The connection arises because our setting is related to a compactification of the (2,0)-theory on $W \times T^2$ (see section 7).

5.5 Connecting to the type-IIA picture

We would like to match the states $|p\rangle$ ($p = 0, \dots, k-1$) of Chern-Simons theory with linear combinations of the k ground states of the type-IIA theory that we found in section 3.2. Our strategy is to identify the symmetry operators \mathcal{U}, \mathcal{V} that we defined in section 3.3–3.4 with operators on the Chern-Simons Hilbert space.

We believe that the correct identification is

$$\mathcal{V} = \mathcal{W}_1, \quad \mathcal{U} = \mathcal{W}_2. \tag{5.36}$$

As a check, note that with this identification the commutation relations (3.13)–(3.14) agree with (5.18).

To motivate (5.36) further, we can compare the connection between the operators above and electric flux. Combining (5.26) (with $\nu = 1/k$) and (5.28) we get

$$\mathcal{W}_1 = \omega^{-e_2}, \quad \mathcal{W}_2 = \omega^{e_1}. \tag{5.37}$$

Now, similarly to what we did in section 4.2, we can follow the chain of dualities of section 3 backwards, starting with \mathcal{U}, \mathcal{V} on the type-IIA side, to find out what they do on the type-IIB

side. On the type-IIA side (the last row of table 1), take an eigenstate with Kaluza-Klein momenta $p_1, p_{10} \in \mathbb{Z}$ in directions x_1, x_{10} . (We can assume that the eigenstate is localized in the x_3 direction.) The unitary operator \mathcal{U} acts as a translation, and therefore multiplies its eigenstate by the phase $e^{2\pi i(p_1+p_{10})/k}$. Following the chain of dualities backwards in table 1, we saw in section 4.2 that the Kaluza-Klein state on the type-IIA side becomes a (p, q) -string with $p = p_1$ and $q = p_{10}$ on the type-IIB side, and that bound to n D3-branes, these quantum numbers become [12, 22] $\mathbf{e}_1 = p_1$ units of electric flux and $\mathbf{m}_1 = p_{10}$ units of magnetic flux in direction 1. We conclude that on the gauge theory side \mathcal{U} acts as

$$\mathcal{U} = e^{\frac{2\pi i}{k}(\mathbf{e}_1 + \mathbf{m}_1)}. \tag{5.38}$$

Similarly, \mathcal{V} is related to fundamental string winding number, and we find

$$\mathcal{V} = e^{-\frac{2\pi i}{k}(\mathbf{e}_2 + \mathbf{m}_2)}. \tag{5.39}$$

To see this, we note, for example, that the operator that has the eigenvalue $e^{2\pi i(p_1+p_{10})/k}$ under the adjoint action of \mathcal{V} is simply \mathcal{W}_1 .

To interpret (5.38)–(5.39) correctly, we need to discuss how to define the electric and magnetic fluxes in the presence of the S-duality twist. First, note that $\mathbf{e}_j + \mathbf{m}_j$ ($j = 1, 2$) is generally not invariant under S-duality, but it is not hard to check that if k is determined by \mathbf{g} as in (3.9), then $(\mathbf{e}_j + \mathbf{m}_j)$ is invariant mod k . As we have argued in section 5.2, the operator $\exp[\frac{2\pi i}{k}(\mathbf{e}_j + \mathbf{m}_j)]$ in the full 3+1D theory reduces to $\exp[\frac{2\pi i}{k}\mathbf{e}_j]$ in the low-energy Chern-Simons theory. We can therefore identify \mathcal{V} and \mathcal{U} as

$$\mathcal{U} = \omega^{\mathbf{e}_1}, \quad \mathcal{V} = \omega^{-\mathbf{e}_2}, \tag{5.40}$$

which together with (5.37) leads to (5.36).

The basis states defined at the end of section 3.2 are eigenstates of \mathcal{V} with eigenvalues ω^p . Up to an unimportant phase, they can be identified as eigenstates $|p\rangle$ of \mathcal{W}_1 defined in (5.19). We conclude with a list of identifications of these states.

Single-particle states for $\nu = \frac{\pi}{2}$ ($\tau = i$ and $k = 2$).

$$|\square\rangle = |0\rangle, \quad |\circledsquare\rangle = |1\rangle.$$

Single-particle states for $\nu = \frac{\pi}{3}$ ($\tau = e^{\pi i/3}$ and $k = 1$).

$$|\square\rangle = |0\rangle.$$

Single-particle states for $\nu = \frac{2\pi}{3}$ ($\tau = e^{\pi i/3}$ and $k = 3$).

$$|\square\rangle = |0\rangle, \quad |\circledsquare\rangle = |1\rangle, \quad |\circledcircledsquare\rangle = |2\rangle. \tag{5.41}$$

6 U(n) gauge group on T²

In section 5 we have seen that for U(1) gauge group, the solution to our problem (as posed in section 2) is a Chern-Simons theory at one of the levels $k = 1, 2, 3$. The level is determined by the choice of coupling constant τ and the $SL(2, \mathbb{Z})$ element \mathbf{g} . We now turn to the nonabelian case of U(n) gauge group. A natural question, then, is whether the solution to the U(n) problem (with the restrictions on n as given in section 2.4) is also a Chern-Simons theory. And if not, what is it? To explore this question we will use the dual type-IIA description of the Hilbert space of ground states of the U(n) theory on T^2 that we found in section 3, and compare it to the Hilbert space of Chern-Simons theory at the appropriate level.

As we saw in (3.6), the type-IIA Hilbert space $\mathcal{H}(n, v)$ can be decomposed into subspaces $\mathcal{H}_{(n_1, n_2, \dots, n_p)}(v)$ by specifying the winding numbers n_1, \dots, n_p of the individual strings. We will analyze these subspaces separately, using the following three tools:

1. The T-duality group $SL(2, \mathbb{Z})$ generated by \mathcal{T}, \mathcal{S} ;
2. The $\mathbb{Z}_k \times \mathbb{Z}_k$ symmetry generated by \mathcal{U}, \mathcal{V} (which is useful for $k > 1$);
3. The decomposition of the gauge group $U(n) = [U(1) \times SU(n)]/\mathbb{Z}_n$.

Together with the known solution for U(1), the last point allows us to construct from each of the $\mathcal{H}_{(n_1, \dots, n_p)}$'s another Hilbert space $\tilde{\mathcal{H}}_{(n_1, \dots, n_p)}$ of states that we can associate with the $SU(n)$ degrees of freedom only. This will be done in section 6.1. The coupling to the U(1) degrees of freedom is encoded in the action of large gauge transformations related to the $\mathbb{Z}_n \subset SU(n)$ center. They form a $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetry group that is generated by a pair of large gauge transformations. In this way, we will end up with Hilbert subspaces $\tilde{\mathcal{H}}_{(n_1, \dots, n_p)}(v)$ on which an action of the semidirect product of $SL(2, \mathbb{Z})$ and $\mathbb{Z}_n \times \mathbb{Z}_n$ is given. This will also be described in detail in section 6.1 [see (6.5)].

In general the analysis depends on the total winding number n and the angle v , but there is one observation that we can make independently of them. For every n and v , there is unique sector $\tilde{\mathcal{H}}_{(1, 1, \dots, 1)}(v)$ with n particles of winding number 1. In section 6.4 we will argue that this sector corresponds to the Hilbert space of $SU(n)$ Chern-Simons theory at level k . The interpretation of other sectors is more mysterious and we will defer the discussion of them to section 6.5.

We take the T^2 to be in directions x_1, x_2 , and as in section 2, the S^1 is of radius R in direction x_3 .

6.1 The center $U(1) \subset U(n)$

Except for global issues related to electric and magnetic fluxes, the U(1) center of the gauge group U(n) decouples. Arguments similar to those in section 5 lead to the conclusion that at low-energy (below the compactification scale $1/R$) it gives rise to a decoupled sector of U(1) Chern-Simons theory. (The global issues will be discussed below.) The level is $k' = kn$, where k is that of the U(1) problem given in section 5. This can be seen as follows.

A gauge field A of the $U(1)$ center is diagonally embedded in $U(n)$ as

$$A \rightarrow \begin{pmatrix} A & & \\ & A & \\ & & \ddots \end{pmatrix}.$$

Inserted into the $U(n)$ Yang-Mills action, this normalization of A gives rise to a $U(1)$ action with coupling constant $n\tau$. Then the S-dual $U(1)$ action, expressed in terms of a dual $U(1)$ gauge field \tilde{A} , has coupling constant $-\frac{1}{n\tau}$. Now consider the S-twist by \mathfrak{g}' (the case with $v = \frac{\pi}{2}$ and $k = 2$), which requires the theory to be self-dual. To achieve self-duality, we need to rescale the dual gauge field by defining $\tilde{A} = n\tilde{A}'$. We can then set the S-twisted boundary conditions to be, roughly speaking, $\tilde{A}'(2\pi R) = A(0)$. Inspecting (5.2), and repeating the arguments of section 5, then shows that the proper Chern-Simons level is $k' = 2n$. Similarly, it can be checked that the effective $U(1)$ Chern-Simons level is $k' = kn$ for the other values of k , where k is the function of v defined in (3.9). (When checking this, note that the shift of the $U(n)$ θ -angle that corresponds to $\tau \rightarrow \tau + 1$ induces $\tau \rightarrow \tau + n$ for the $U(1)$ variables.)

Now compactify the remaining two spatial directions of the theory on T^2 (parameterized by $0 \leq x_j \leq 2\pi L_j$ for $j = 1, 2$). The Hilbert space of states of $U(1)$ Chern-Simons theory at level $k' = kn$ on T^2 has k' states, which we denote by

$$|p\rangle_{U(1)}, \quad p = 0, \dots, k' - 1.$$

We pick a basis of the Hilbert space so that these states are eigenstates of the $U(1)$ Wilson line operator \mathcal{W}_1 corresponding to the 1-cycle around the x_1 -axis,

$$\mathcal{W}_1 = \exp \left\{ \int_0^{2\pi L_1} A_1(x_1, 0) dx_1 \right\}, \tag{6.1}$$

so that

$$\mathcal{W}_1 |p\rangle_{U(1)} = e^{\frac{2\pi i p}{kn}} |p\rangle_{U(1)}.$$

(For a quick review of $U(1)$ Chern-Simons theory on T^2 , see section 5.3, replacing k that appeared there with k' .) A general state $|\psi\rangle_{U(n)}$ of the Hilbert space of the $U(n)$ theory can then be decomposed as

$$|\psi\rangle_{U(n)} = \sum_{p=0}^{kn-1} |\psi; p\rangle_{SU(n)} \otimes |p\rangle_{U(1)}, \tag{6.2}$$

where $|\psi; p\rangle_{SU(n)}$ are the ‘‘coefficients’’ which can be interpreted as wavefunctions of the $SU(n)$ degrees of freedom only.

Let us now discuss the global issues that arise because $U(n)$ is not $U(1) \times SU(n)$ but rather $[U(1) \times SU(n)]/\mathbb{Z}_n$. When compactifying a $U(n)$ gauge theory on T^2 , we require the Hilbert space of states to be invariant under large $U(n)$ gauge transformations. In particular, we need to consider the two gauge transformations Ω_j ($j = 1, 2$) with gauge parameters

$$\Omega_j(x_1, x_2) = \text{diag} \left(e^{\frac{ix_j}{L_j}}, 1, \dots, 1 \right),$$

which are continuous in $U(n)$, but cannot be lifted to continuous gauge transformations in $U(1) \times SU(n)$. Indeed, they can be written as $\Omega_j = \Omega'_j \Omega''_j$ with

$$\Omega'_j(x_1, x_2) = \text{diag} \left(e^{\frac{ix_j}{nL_j}}, e^{\frac{ix_j}{nL_j}}, \dots, e^{\frac{ix_j}{nL_j}} \right) \in U(1), \quad (6.3)$$

$$\Omega''_j(x_1, x_2) = \text{diag} \left(e^{\frac{(n-1)ix_j}{nL_j}}, e^{-\frac{ix_j}{nL_j}}, \dots, e^{-\frac{ix_j}{nL_j}} \right) \in SU(n), \quad (6.4)$$

but Ω'_j and Ω''_j have a discontinuity at $x_j = 0 \equiv 2\pi L_j$.

Nevertheless, Ω'_1 and Ω'_2 define unitary operators on the Hilbert space of $U(1)$ Chern-Simons theory on T^2 which act on the states $|p\rangle_{U(1)}$ as (see section 5.3)

$$\Omega'_1 |p\rangle_{U(1)} = |p+k\rangle_{U(1)}, \quad \Omega'_2 |p\rangle_{U(1)} = e^{-\frac{2\pi ip}{n}} |p\rangle_{U(1)}.$$

The decomposition (6.2) then implies that

$$\Omega''_1 |\psi; p\rangle_{SU(n)} = |\psi; p+k\rangle_{SU(n)}, \quad \Omega''_2 |\psi; p\rangle_{SU(n)} = e^{\frac{2\pi ip}{n}} |\psi; p\rangle_{SU(n)}. \quad (6.5)$$

We conclude that a state of the form (6.2) is in the Hilbert space of the $U(n)$ theory provided that the $SU(n)$ states in the decomposition satisfy (6.5).

Identifying the \mathbb{Z}_k momentum and winding number operators. In section 3.3–3.4 we defined the symmetry operators \mathcal{U}, \mathcal{V} on the type-IIA dual. Let us identify these operators on the gauge theory side. From the definition it is clear that \mathcal{U}, \mathcal{V} act only on the $U(1) \subset U(n)$ degrees of freedom, since on the type-IIA side they are defined in terms of the “center-of-mass” of the strings. We therefore turn to the analysis of large gauge transformations that act only on the $U(1)$ degrees of freedom.

Define the discontinuous $U(1)$ gauge transformations

$$\Upsilon_j^{(\alpha)}(x_1, x_2) = \text{diag} \left(e^{\frac{i\alpha x_j}{k'L_j}}, \dots, e^{\frac{i\alpha x_j}{k'L_j}} \right) \in U(1), \quad j = 1, 2,$$

where α is a real parameter. We will see momentarily that it has to be an integer. Let \mathcal{W}_1 be the $U(1)$ Wilson line as in (6.1), and define the Wilson line \mathcal{W}_2 in direction 2 similarly. Then, by definition, $\Upsilon_1^{(\alpha)}$ has the following commutation relations with the Wilson lines:

$$(\Upsilon_1^{(\alpha)})^{-1} \mathcal{W}_1 \Upsilon_1^{(\alpha)} = e^{\frac{2\pi i\alpha}{k'}} \mathcal{W}_1, \quad (\Upsilon_1^{(\alpha)})^{-1} \mathcal{W}_2 \Upsilon_1^{(\alpha)} = \mathcal{W}_2, \quad (6.6)$$

and similarly for $\Upsilon_2^{(\alpha)}$. Given the explicit k' -dimensional representation of $\mathcal{W}_1, \mathcal{W}_2$ (see section 5.3), it is not hard to check that a solution to (6.6) exists only for integer α , in which case we can take

$$\Upsilon_1^{(\alpha)} = \mathcal{W}_2^\alpha, \quad \Upsilon_2^{(\alpha)} = \mathcal{W}_1^{-\alpha}.$$

But even with $\alpha \in \mathbb{Z}$, the operators $\Upsilon_j^{(\alpha)}$ might not preserve the Hilbert space of $U(n)$ Chern-Simons theory. For example, acting on (6.2) we get

$$\Upsilon_2^{(\alpha)} |\psi\rangle_{U(n)} = \sum_{p=0}^{kn-1} e^{-\frac{2\pi i\alpha p}{kn}} |\psi; p\rangle_{SU(n)} \otimes |p\rangle_{U(1)},$$

and assuming the “coefficients” $|\psi; p\rangle_{\text{SU}(n)}$ satisfy the condition (6.5), we find that the new “coefficients” $e^{\frac{2\pi i \alpha p}{kn}} |\psi; p\rangle_{\text{SU}(n)}$ do not satisfy the rightmost equation of (6.5) unless $\alpha \in n\mathbb{Z}$. Similarly,

$$\Upsilon_1^{(\alpha)} |\psi\rangle_{\text{U}(n)} = \sum_{p=0}^{kn-1} |\psi; p - \alpha\rangle_{\text{SU}(n)} \otimes |p\rangle_{\text{U}(1)},$$

and the leftmost condition of (6.5) is not satisfied unless $\alpha \in n\mathbb{Z}$. In order to preserve the Hilbert space, we therefore require α to be an integer multiple of n . We therefore define

$$\mathcal{K}_1 \equiv \Upsilon_1^{(n)} = \mathcal{W}_2^n, \quad \mathcal{K}_2 \equiv \Upsilon_2^{(n)} = \mathcal{W}_1^{-n}. \quad (6.7)$$

They generate a k^2 -dimensional group that preserves the $\text{U}(n)$ Hilbert space, and they act as

$$\begin{aligned} \mathcal{K}_1 |\psi\rangle_{\text{U}(n)} &= \sum_{p=0}^{kn-1} |\psi; p - n\rangle_{\text{SU}(n)} \otimes |p\rangle_{\text{U}(1)}, \\ \mathcal{K}_2 |\psi\rangle_{\text{U}(n)} &= \sum_{p=0}^{kn-1} e^{-\frac{2\pi i p}{k}} |\psi; p\rangle_{\text{SU}(n)} \otimes |p\rangle_{\text{U}(1)}. \end{aligned} \quad (6.8)$$

The operators $\mathcal{K}_1, \mathcal{K}_2$ satisfy the clock-and-shift relations

$$\mathcal{K}_1 \mathcal{K}_2 = e^{\frac{2\pi i n}{k}} \mathcal{K}_2 \mathcal{K}_1, \quad (\mathcal{K}_1)^k = (\mathcal{K}_2)^k = 1.$$

We can now connect the type-IIA operators \mathcal{U}, \mathcal{V} to the gauge theory by identifying

$$\mathcal{U} = \mathcal{K}_1, \quad \mathcal{V} = \mathcal{K}_2^{-1}, \quad (6.9)$$

in analogy with (5.36). We therefore get

$$\begin{aligned} \mathcal{V} |\psi\rangle_{\text{U}(n)} &= \sum_{p=0}^{kn-1} e^{\frac{2\pi i p}{k}} |\psi; p\rangle_{\text{SU}(n)} \otimes |p\rangle_{\text{U}(1)}, \\ \mathcal{U} |\psi\rangle_{\text{U}(n)} &= \sum_{p=0}^{kn-1} |\psi; p - n\rangle_{\text{SU}(n)} \otimes |p\rangle_{\text{U}(1)}. \end{aligned} \quad (6.10)$$

Action of $\text{SL}(2, \mathbb{Z})$. There are two more operators that we find useful to define on the Hilbert space of the $\text{U}(n)$ theory on T^2 . In section 3.6 we discussed the $\text{SL}(2, \mathbb{Z})$ action of large diffeomorphisms of T^2 , and we mentioned that it induces an action on the Hilbert space of ground states. The action of $\text{SL}(2, \mathbb{Z})$ on the $\text{U}(1)$ states is well-known. Setting the generators of $\text{SL}(2, \mathbb{Z})$ to be

$$\mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we have (see section 5.3)

$$\begin{aligned} \mathcal{S}|p\rangle_{U(1)} &= \frac{1}{\sqrt{kn}} \sum_{q=0}^{kn-1} e^{\frac{2\pi i}{kn}pq} |q\rangle_{U(1)}, \\ \mathcal{T}|p\rangle_{U(1)} &= \begin{cases} e^{\frac{i\pi p^2}{kn}} |p\rangle_{U(1)} & \text{for even } kn, \\ e^{\frac{i\pi p(p+kn)}{kn}} |p\rangle_{U(1)} & \text{for odd } kn. \end{cases} \end{aligned} \tag{6.11}$$

There are restrictions on the action of \mathcal{T}, \mathcal{S} on the $SU(n)$ states since they have to preserve the $U(n)$ Hilbert space (6.2) with the conditions (6.5). The restrictions have certain implications for the commutation relations among \mathcal{T}, \mathcal{S} and Ω''_1, Ω''_2 . We have

$$\mathcal{T}|\psi\rangle_{U(n)} = \begin{cases} \sum_{p=0}^{kn-1} e^{\frac{\pi ip^2}{kn}} \mathcal{T}|\psi; p\rangle_{SU(n)} \otimes |p\rangle_{U(1)} & \text{for even } kn, \\ \sum_{p=0}^{kn-1} e^{\frac{\pi ip(p+kn)}{kn}} \mathcal{T}|\psi; p\rangle_{SU(n)} \otimes |p\rangle_{U(1)} & \text{for odd } kn, \end{cases}$$

and therefore (6.5) implies

$$\mathcal{T}^{-1}\Omega''_2\mathcal{T}|\psi; p\rangle_{SU(n)} = e^{\frac{2\pi ip}{n}} |\psi; p\rangle_{SU(n)}, \tag{6.12}$$

and

$$\mathcal{T}^{-1}\Omega''_1\mathcal{T}|\psi; p\rangle_{SU(n)} = \begin{cases} e^{\frac{2\pi ip}{n} + \frac{\pi ik}{n}} |\psi; p+k\rangle_{SU(n)} & \text{for even } kn, \\ e^{\frac{2\pi ip}{n} + \frac{\pi ik(n+1)}{n}} |\psi; p+k\rangle_{SU(n)} & \text{for odd } kn. \end{cases} \tag{6.13}$$

Using (6.5) again, we can rewrite these relations as

$$\mathcal{T}^{-1}\Omega''_2\mathcal{T} = \Omega''_2, \quad \mathcal{T}^{-1}\Omega''_1\mathcal{T} = \begin{cases} e^{-\frac{\pi ik}{n}} \Omega''_2\Omega''_1 & \text{for even } kn \\ e^{\frac{\pi ik(n-1)}{n}} \Omega''_2\Omega''_1 & \text{for odd } kn \end{cases} \quad (\text{on ground states}). \tag{6.14}$$

Similarly, we find

$$\mathcal{S}^{-1}\Omega''_1\mathcal{S} = \Omega''_1, \quad \mathcal{S}^{-1}\Omega''_2\mathcal{S} = (\Omega''_1)^{-1} \quad (\text{on ground states}). \tag{6.15}$$

So, in order for \mathcal{T} and \mathcal{S} to preserve the Hilbert space with the conditions (6.5), they must obey the commutation relations (6.14)–(6.15).

6.2 $U(n)$ Chern-Simons Hilbert space as a symmetric product

The states of $SU(n)$ Chern-Simons theory on T^2 at level k are in one-to-one correspondence with irreducible representations of $SU(n)$ that correspond to Young diagrams with at most k columns [32]. For example, for $SU(2)$ the states are labeled by an irreducible representation of $SU(2)$ with spin j at most $k/2$, and so we can label the states by an integer $m = 2j = 0, \dots, k$.

We are interested in how the $SL(2, \mathbb{Z})$ generators \mathcal{T}, \mathcal{S} act on the states, as well as in the large gauge transformations Ω''_1, Ω''_2 , which are defined similarly to (6.4) and generate two \mathbb{Z}_n symmetries of the Hilbert space. For this purpose, we work with a particularly convenient representation of the Hilbert space that is derived from the Hilbert space of $U(n)$ Chern-Simons theory, as we shall now explain.

To describe Chern-Simons theory with $U(n)$ gauge group requires two levels — a level k' for $U(1)$ and a level k for $SU(n)$. The theory is then denoted by

$$[U(1)_{k'} \times SU(n)_k] / \mathbb{Z}_n,$$

where the \mathbb{Z}_n quotient refers to modding out by large gauge transformations $\Omega_j = \Omega'_j \Omega''_j$ defined similarly to (6.3)–(6.4). The level k' can in principle be any integer multiple of n , and can be changed by adding to the Chern-Simons action a $U(1)$ Chern-Simons term for the trace of the gauge field. One customary choice is $k' = n(k + n)$, for which the (bare) Lagrangian doesn't have a separate Chern-Simons term for the trace of the gauge field. Another choice is $k' = kn$. This choice is particularly convenient, because we have the equivalence of Hilbert spaces [33, 34]:

$$\mathcal{H}([U(1)_{kn} \times SU(n)_k] / \mathbb{Z}_n) \simeq \mathcal{H}(U(1)_k)^{\otimes n} / S_n \tag{6.16}$$

where $[\dots] / S_n$ denotes the symmetric part of the tensor product.

Equation (6.16) is to be understood as follows: both sides are equivalent representations of \mathcal{T}, \mathcal{S} , as well as Ω_1, Ω_2 . In particular, the dimensions of both sides are equal:

$$\begin{aligned} \dim \mathcal{H}([U(1)_{kn} \times SU(n)_k] / \mathbb{Z}_n) &= kn \binom{n+k-1}{k} \frac{1}{n^2} = \binom{n+k-1}{k-1} \\ &= \dim [\mathcal{H}(U(1)_k)^{\otimes n}]_S. \end{aligned} \tag{6.17}$$

In fact, (6.16) can be understood in terms of wavefunctions as well. To explain this, we need to first discuss the wavefunctions of $U(n)$ Chern-Simons theory on T^2 . The states of Chern-Simons theory on a Riemann surface can be obtained by quantization of the space of flat connections on the Riemann surface [32]. For T^2 , the flat connections can be encoded in the conjugacy class of the two commuting holonomies of the gauge field around two independent cycles of T^2 . The resulting wavefunctions have been explicitly described in [33–37]. For $U(1)$ gauge group, the two holonomies can be combined into a complex variable that takes values on a dual T^2 . This dual T^2 also has complex structure ρ , and the wavefunctions are related to θ -functions, as we reviewed in section 5.4. For $U(n)$ gauge group, with the help of a gauge transformation, the two commuting holonomies can be reduced to a maximal torus $U(1)^n \subset U(n)$. The two holonomies associated with the i^{th} $U(1)$ factor ($i = 1, \dots, n$) can be combined into a complex variable \mathbf{a}_i which takes values in T^2 , and so is subject to the identifications $\mathbf{a}_i \sim \mathbf{a}_i + 1 \sim \mathbf{a}_i + \rho$. The wavefunctions $\psi(\mathbf{a}_1, \dots, \mathbf{a}_n)$ are required to be symmetric in the n variables (because of the Weyl group S_n) and can be expressed in terms of partition functions of $U(n)$ WZW models at level k (which are characters of the corresponding affine Lie algebra [41, 42]). Explicit expressions can be found in [33, 34, 36, 37].

On the other hand, the wavefunctions of $\mathcal{H}(U(1)_k)^{\otimes n}$ are proportional to symmetrized products of the wavefunctions $\psi_{p_i}(\mathbf{a}_i)$ described in section 5.4:

$$\Psi_{p_1, \dots, p_n}(\mathbf{a}_1, \dots, \mathbf{a}_n) \equiv \sum_{\sigma \in S_n} \prod_{i=1}^n \psi_{p_i}(\mathbf{a}_{\sigma(i)}). \tag{6.18}$$

Here, again, the U(1) wavefunctions can be expressed in terms of θ -functions as reviewed in section 5.4. Now, the main point is that these symmetrized products of θ -functions (6.18) span the same Hilbert space as the characters of U(n) WZW at level k , as explained in [33, 34]!

Using (6.16) it is easy to calculate the action of $\mathcal{T}, \mathcal{S}, \Omega_1'', \Omega_2''$ on $\mathcal{H}(\text{SU}(n)_k)$ by calculating the action of $\mathcal{T}, \mathcal{S}, \Omega_1, \Omega_2$ on the right-hand side using the formulas of section 5, and then extracting the SU(n) degrees of freedom from the left-hand side using the U(1) results in section 6.1. In doing so, it is very useful, using (6.9), to compare the action of \mathcal{U}, \mathcal{V} on type-IIA states and on the expansion (6.2) [as given in (6.10)], and derive restrictions on the the SU(n) states that appear in the expansion (6.2) as coefficients. Once we have these restrictions, we can derive the action of the SL(2, \mathbb{Z}) operators \mathcal{T} and \mathcal{S} on the SU(n) states. We illustrate this procedure with an example in section 6.3. The general case is described in appendix B.

6.3 Example: U(2)

We will demonstrate the decomposition into U(1) and SU(n) degrees of freedom in the case $k = 2$ and $n = 2$.

Consider the sector of 2-particle states on the type-IIA side where each string has winding number 1. Since we identified the single-particle states with those of U(1) Chern-Simons theory in section 5, (6.16) implies that states in this sector can be identified with those of U(2) Chern-Simons theory. We expand the basis states given in section 3.2 using (6.10):

$$\begin{aligned}
 |\square \square\rangle &= \sum_{p=0}^3 \left(|\square \square; p\rangle_{\text{SU}(n)} \otimes |p\rangle_{\text{U}(1)} \right), \\
 |\square \circ\rangle &= \sum_{p=0}^3 \left(|\square \circ; p\rangle_{\text{SU}(n)} \otimes |p\rangle_{\text{U}(1)} \right), \\
 |\circ \circ\rangle &= \sum_{p=0}^3 \left(|\circ \circ; p\rangle_{\text{SU}(n)} \otimes |p\rangle_{\text{U}(1)} \right).
 \end{aligned}$$

Now let's compare the eigenvalues of \mathcal{V} on both sides. Since $|\circ \circ\rangle$ and $|\square \square\rangle$ have \mathcal{V} -eigenvalue +1, only even p 's can appear in their expansions [see (6.10)]. Similarly, $|\square \square\rangle$ has \mathcal{V} -eigenvalue -1, and therefore only odd p 's can appear in its expansion. Next, we note that $|\square \square\rangle + |\circ \circ\rangle$ and $|\square \circ\rangle$ have \mathcal{U} -eigenvalue +1 while $|\square \square\rangle - |\circ \circ\rangle$ has \mathcal{U} -eigenvalue -1. We conclude that the expansion of U(2) states in terms of SU(2) and U(1) states must take the form

$$|\square \square\rangle = |a\rangle_{\text{SU}(2)} \otimes |0\rangle_{\text{U}(1)} + |c\rangle_{\text{SU}(2)} \otimes |2\rangle_{\text{U}(1)} \quad (6.19)$$

$$|\square \circ\rangle = |b\rangle_{\text{SU}(2)} \otimes \left(|1\rangle_{\text{U}(1)} + |3\rangle_{\text{U}(1)} \right), \quad (6.20)$$

$$|\circ \circ\rangle = |c\rangle_{\text{SU}(2)} \otimes |0\rangle_{\text{U}(1)} + |a\rangle_{\text{SU}(2)} \otimes |2\rangle_{\text{U}(1)}, \quad (6.21)$$

where $|a\rangle, |b\rangle, |c\rangle$ are 3 states of the SU(2) degrees of freedom.

We now note that the $SL(2, \mathbb{Z})$ action of large diffeomorphisms becomes T-duality on the type-IIA side. Using the action of \mathcal{T}, \mathcal{S} on the single-particle states as listed in appendix A, and the action of \mathcal{T}, \mathcal{S} on the $U(1)$ variables as given in (6.11), we can find the action of \mathcal{T}, \mathcal{S} on the $SU(2)$ basis of states $|a\rangle, |b\rangle, |c\rangle$:

$$\mathcal{S} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{\pi i}{4}} & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (6.22)$$

We also find, using (6.5), the action of large gauge transformations:

$$\Omega_1'' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Omega_2'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6.23)$$

We will now present explicit expressions for the wavefunctions that realize the decomposition (6.19)–(6.21). The single-particle states $|\square\rangle$ and $|\circ\rangle$ can be identified with wavefunctions of $U(1)$ Chern-Simons theory. The latter can be explicitly represented in terms of θ -functions, as we recalled in (5.32). In order to realize (6.19)–(6.21), we need to recast the product of two such wavefunctions in a way that separates the “center of mass” $U(1)$ variable.

We denote the wavefunctions of Chern-Simons theory for any k as⁵ (see section 5.4)

$$\psi_{p,k}(\mathbf{a}) = \theta(k\mathbf{a} + p\rho; k\rho) e^{\frac{\pi k}{2\rho^2} \mathbf{a}^2 + \frac{1}{k} \pi i \rho p^2 + 2\pi i p \mathbf{a}}, \quad p = 0, \dots, k-1. \quad (6.24)$$

The correspondence between the single-particle states and their wavefunctions can be found using (5.33) and (5.36); for example, for $k = 2$, we get

$$|\square\rangle \rightarrow \psi_{0,2}, \quad |\circ\rangle \rightarrow \psi_{1,2}.$$

Next, we use the identity

$$\theta(z_1; \tau)\theta(z_2; \tau) = \theta(z_1 + z_2; 2\tau)\theta(z_1 - z_2; 2\tau) + e^{\pi i(\tau + 2z_2)}\theta(z_1 + z_2 + \tau; 2\tau)\theta(z_1 - z_2 - \tau; 2\tau)$$

to rewrite the 2-particle wavefunctions as

$$\begin{aligned} & \psi_{p_1,k}(\mathbf{a}_1)\psi_{p_2,k}(\mathbf{a}_2) + \psi_{p_1,k}(\mathbf{a}_2)\psi_{p_2,k}(\mathbf{a}_1) \\ &= \psi_{p_1+p_2,2k} \left(\frac{\mathbf{a}_1 + \mathbf{a}_2}{2} \right) \left[\psi_{p_1-p_2,2k} \left(\frac{\mathbf{a}_1 - \mathbf{a}_2}{2} \right) + \psi_{p_2-p_1,2k} \left(\frac{\mathbf{a}_1 - \mathbf{a}_2}{2} \right) \right] \\ & \quad + \psi_{p_1+p_2+k,2k} \left(\frac{\mathbf{a}_1 + \mathbf{a}_2}{2} \right) \left[\psi_{p_1-p_2-k,2k} \left(\frac{\mathbf{a}_1 - \mathbf{a}_2}{2} \right) + \psi_{p_2-p_1+k,2k} \left(\frac{\mathbf{a}_1 - \mathbf{a}_2}{2} \right) \right]. \end{aligned} \quad (6.25)$$

We interpret the functions of $(\mathbf{a}_1 + \mathbf{a}_2)$ as the $U(1)$ parts,

$$|p\rangle_{U(1)} \rightarrow \psi_{p,2k} \left(\frac{\mathbf{a}_1 + \mathbf{a}_2}{2} \right),$$

⁵For the purposes of this discussion, we can actually be more general, and do not need to restrict ourselves to $k = 2$.

and the factors in the square brackets $[\dots]$ in (6.25) as the $SU(2)$ parts. Specializing to $k = 2$ again, we see that the decomposition (6.19)–(6.21) is consistent with:

$$\begin{aligned} |a\rangle_{SU(2)} &\rightarrow \psi_{0,4} \left(\frac{\mathbf{a}_1 - \mathbf{a}_2}{2} \right), \\ |b\rangle_{SU(2)} &\rightarrow \frac{1}{\sqrt{2}} \left[\psi_{1,4} \left(\frac{\mathbf{a}_1 - \mathbf{a}_2}{2} \right) + \psi_{3,4} \left(\frac{\mathbf{a}_1 - \mathbf{a}_2}{2} \right) \right], \\ |c\rangle_{SU(2)} &\rightarrow \psi_{2,4} \left(\frac{\mathbf{a}_1 - \mathbf{a}_2}{2} \right). \end{aligned} \tag{6.26}$$

In section 6.4 we will interpret these as wavefunctions of $SU(2)$ Chern-Simons theory at level 2.

SL(2, Z) action on SU(2) Chern-Simons theory. For future reference, we list here the action of $\mathcal{T}, \mathcal{S}, \Omega''_1, \Omega''_2$ on the Hilbert space of $SU(2)$ Chern-Simons theory at level $k = 1, 2, 3$.

For $k = 1$, the basis states $|a\rangle, |b\rangle$ are defined, using (6.16), by

$$\left| \begin{array}{c} \text{[diagram: two blue parallelograms with red dots]} \end{array} \right\rangle = |a\rangle_{SU(2)} \otimes |0\rangle_{U(1)} + |b\rangle_{SU(2)} \otimes |1\rangle_{U(1)}.$$

In this basis, we have

$$\mathcal{T} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\pi i/2} \end{pmatrix}, \quad \mathcal{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{6.27}$$

$$\Omega''_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Omega''_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.28}$$

For $k = 2$, the results are in (6.22)–(6.23).

For $k = 3$, the basis states $|a\rangle, |b\rangle, |c\rangle, |d\rangle$ are defined by the decomposition

$$\begin{aligned} \left| \begin{array}{c} \text{[diagram: two blue parallelograms with red dots]} \end{array} \right\rangle &= |a\rangle_{SU(2)} \otimes |0\rangle_{U(1)} + |b\rangle_{SU(2)} \otimes |3\rangle_{U(1)}, \\ \left| \begin{array}{c} \text{[diagram: two blue parallelograms with red dots]} \end{array} \right\rangle &= |a\rangle_{SU(2)} \otimes |2\rangle_{U(1)} + |b\rangle_{SU(2)} \otimes |5\rangle_{U(1)}, \\ \left| \begin{array}{c} \text{[diagram: two blue parallelograms with red dots]} \end{array} \right\rangle &= |a\rangle_{SU(2)} \otimes |4\rangle_{U(1)} + |b\rangle_{SU(2)} \otimes |1\rangle_{U(1)}, \\ \left| \begin{array}{c} \text{[diagram: two blue parallelograms with red dots]} \end{array} \right\rangle &= |c\rangle_{SU(2)} \otimes |1\rangle_{U(1)} + |d\rangle_{SU(2)} \otimes |4\rangle_{U(1)}, \\ \left| \begin{array}{c} \text{[diagram: two blue parallelograms with red dots]} \end{array} \right\rangle &= |c\rangle_{SU(2)} \otimes |3\rangle_{U(1)} + |d\rangle_{SU(2)} \otimes |0\rangle_{U(1)}, \\ \left| \begin{array}{c} \text{[diagram: two blue parallelograms with red dots]} \end{array} \right\rangle &= |c\rangle_{SU(2)} \otimes |5\rangle_{U(1)} + |d\rangle_{SU(2)} \otimes |2\rangle_{U(1)}, \end{aligned}$$

where we have used the same argument as in the paragraph preceding (6.19)–(6.21) to simplify the decomposition into $U(1)$ and $SU(n)$ degrees of freedom. We get

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\pi i/2} & 0 & 0 \\ 0 & 0 & e^{-5\pi i/6} & 0 \\ 0 & 0 & 0 & e^{2\pi i/3} \end{pmatrix}, \quad \mathcal{S} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & \sqrt{2} & \sqrt{2} \\ 1 & -1 & -\sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 1 & -1 \\ \sqrt{2} & \sqrt{2} & -1 & -1 \end{pmatrix}, \tag{6.29}$$

$$\Omega_1'' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \Omega_2'' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (6.30)$$

Note that \mathcal{S}, \mathcal{T} , as given above, are not the same as the realization of $SL(2, \mathbb{Z})$ on Chern-Simons states as given by [31, 32]. In [31, 32], \mathcal{S} is realized by the Verlinde matrix:

$$\mathcal{S}_V |m\rangle = \sqrt{\frac{2}{k+2}} \sum_{m'} \sin \frac{\pi(m+1)(m'+1)}{k+2} |m'\rangle, \quad m = 0, \dots, k, \quad (6.31)$$

where $|m\rangle$ is the state corresponding to spin $m/2$, and \mathcal{T} is realized diagonally by

$$\mathcal{T}_V |m\rangle = e^{\frac{\pi i m(m+2)}{2(k+2)}} |m\rangle \quad m = 0, \dots, k. \quad (6.32)$$

Formulas (6.31)–(6.32) were derived from the equivalence between states of Chern-Simons theory on T^2 and characters of the WZW model, while formulas (6.22) and (6.27),(6.29) were derived by realizing the Hilbert space of $SU(2)$ Chern-Simons theory as a subspace of $\mathcal{H}(U(1)_k)^2/S_2$. The discrepancy between (6.31)–(6.32) and (6.22), (6.27),(6.29) is because they are written in difference bases, and the transformation from one basis to the other involves nontrivial coefficients that are functions of the complex structure ρ . These coefficients transform nontrivially themselves under \mathcal{S} and \mathcal{T} , and hence the resulting formulas are different. This point will be demonstrated explicitly in an example in section 6.4.

6.4 Chern-Simons theory and the $[\sigma]$ -untwisted sector $\mathcal{H}_{[1]}(v)$

Each of the three cases $k = 1, 2, 3$ considered in section 3.2 has a special sector $\mathcal{H}_{[1]}(v) = \mathcal{H}_{(1,1,\dots,1)}(v)$ comprising of n string states, all of which have winding number 1. The location of these strings can be any one of k choices, so altogether the Hilbert space is the symmetric product of single-particle Hilbert spaces:

$$\mathcal{H}_{(1,1,\dots,1)}(v) \simeq \mathcal{H}(U(1)_k)^{\otimes n} / S_n. \quad (6.33)$$

In fact, for the present discussion the restrictions on n from section 2.4 can be relaxed, and we can allow any $n \geq 1$, because even the cases $n \geq \mathbf{r}$ still have a finite-dimensional subspace of normalizable ground states (even though there is no mass gap now). The finite-dimensional Hilbert space $\mathcal{H}_{[1]}(v)$ is therefore well-defined for all n . This is the sector we referred to in section 3.2 as the $[\sigma]$ -untwisted sector.

We can now state our main observation: $\mathcal{H}_{[1]}$ is equivalent to the Hilbert space of $[U(1)_{kn} \times SU(n)_k] / \mathbb{Z}_n$ Chern-Simons theory at level k . This follows immediately from (6.16).

As an example, take the case $n = 2$. In section 6.3 we studied the basis of symmetric 2-particle states of $\mathcal{H}(U(1)_k)^{\otimes 2}$ with wavefunctions of the form $\psi_{p_1}(\mathbf{a}_1)\psi_{p_2}(\mathbf{a}_2) + \psi_{p_1}(\mathbf{a}_2)\psi_{p_2}(\mathbf{a}_1)$. In (6.25) we expressed these products as a linear combination of products of wavefunctions of $(\mathbf{a}_1 + \mathbf{a}_2)$ and wavefunctions of $(\mathbf{a}_1 - \mathbf{a}_2)$. According to (6.16), the symmetric part of the space $\mathcal{H}(U(1)_k)^{\otimes 2}$ is the Hilbert space of

U(2) Chern-Simons theory where the U(1) center is at level $2k$ and the SU(2) is at level k . Indeed, the functions of $(\mathbf{a}_1 + \mathbf{a}_2)$ correspond to the U(1) factor and are wavefunctions of U(1) theory at level $2k$, while the functions of $(\mathbf{a}_1 - \mathbf{a}_2)$ can be recast as wavefunctions of SU(2) theory at level k using the connection [31–34] between the latter and characters of affine Lie algebras. Let us demonstrate how this is done.

The characters of the SU(2) affine Lie algebra at level k are related to the wavefunctions we found in (6.25) as follows. Consider the Weyl-Kac characters

$$\text{ch}_k^\lambda(\mathbf{a}) = \sum_{m=1-k}^k C_m^\lambda(\rho) \Theta_{m,k}, \quad \Theta_{m,k} \equiv \sum_{n \in \mathbb{Z} + m/2k} e^{2\pi i(n^2 \rho - n\mathbf{a})}, \quad (6.34)$$

where $C_m^\lambda(\rho)$ are the “string functions,” which satisfy the following relations [43]:

$$C_m^\lambda = 0 \quad \forall \lambda \neq m \pmod{2}, \quad C_m^\lambda = C_{-m}^\lambda, \quad C_m^\lambda = C_{k+m}^{k-\lambda}. \quad (6.35)$$

Here $\lambda = 0, \dots, k$ corresponds to twice the “spin” of the highest weight of the representation. For $k = 2$, for example, these constraints yield 3 independent C_m^λ ’s whose exact forms are

$$C_+ \equiv C_0^0 + C_2^0 = \frac{\eta(q)}{\eta(\sqrt{q})\eta(q^2)}, \quad C_- \equiv C_0^0 - C_2^0 = \frac{\eta(\sqrt{q})}{[\eta(q)]^2}, \quad C_1^1 = \frac{\eta(q^2)}{[\eta(q)]^2}, \quad (6.36)$$

where $\eta(q)$ is the Dedekind function of $q \equiv e^{2\pi i\rho}$. Relating them back to our wavefunctions in (6.26), we find the relations:

$$|a\rangle_{\text{SU}(2)} = \frac{e^{\pi k \mathbf{a}_-^2 / 2\rho_2}}{C_+ C_-} (C_0^0 \text{ch}_2^0(\mathbf{a}_-) - C_2^0 \text{ch}_2^2(\mathbf{a}_-)), \quad (6.37)$$

$$|c\rangle_{\text{SU}(2)} = \frac{e^{\pi k \mathbf{a}_-^2 / 2\rho_2}}{C_+ C_-} (-C_2^0 \text{ch}_2^0(\mathbf{a}_-) + C_0^0 \text{ch}_2^2(\mathbf{a}_-)), \quad (6.38)$$

$$|b\rangle_{\text{SU}(2)} = \frac{e^{\pi k \mathbf{a}_-^2 / 2\rho_2}}{C_1^1} \text{ch}_2^1(\mathbf{a}_-), \quad (6.39)$$

where $\mathbf{a}_- \equiv (\mathbf{a}_2 - \mathbf{a}_1)/2$.

Thus, we see explicitly that the wavefunctions of $\mathcal{H}_{(1,1)}(v)$ correspond to a basis of the wavefunctions of U(2) Chern-Simons theory at level k . In particular, the states $|a\rangle, |b\rangle, |c\rangle$ of (6.26) correspond to a linear combination of the states of SU(2) at level $k = 2$ with highest weight $j = 0, 1, 2$, respectively. The subtle point about the linear coefficients being functions of ρ is that in the language of holomorphic quantization, the basis furnished by the string theory is not yet normalized. A straightforward computation reveals that the modular transformation properties of the string functions explain the discrepancy between our formulae for \mathcal{T}, \mathcal{S} and those found in standard literature for Chern-Simons theory. For example, under $\rho \rightarrow \rho + 1$ we find

$$C_1^1 \rightarrow C_1^1, \quad C_- \rightarrow e^{-\frac{\pi i}{8}} C_+, \quad C_+ \rightarrow e^{-\frac{\pi i}{8}} C_-.$$

Thus, if one further orthonormalizes the string theory states, then, as shown above, the states are those of nonabelian Chern-Simons theory.

6.5 $[\sigma]$ -twisted sectors

We now turn to the sectors of the Hilbert space where some strings have winding number greater than 1. These are the sectors $\mathcal{H}_{(n_1, \dots, n_p)}(v)$ that we called $[\sigma]$ -twisted in section 3.2. We distinguish two kinds of sectors:

1. *Irreducible sectors*—those sectors for which all individual strings have the same winding number, i.e., $n_1 = \dots = n_p$. Those Hilbert spaces cannot be written as a product of two Hilbert spaces with a smaller number of strings and same value of v . In particular, all single-particle sectors ($p = 1$ and $n_1 = n$) are irreducible.
2. *Reducible sectors*—those sectors for which at least two individual strings have different winding numbers, i.e., $n_1 > n_p$. Those Hilbert spaces can always be written as a product of at least two Hilbert spaces with a smaller number of strings and same value of v .

The Hilbert spaces of reducible sectors can always be written as tensor products of Hilbert spaces of irreducible sectors, and are therefore equivalent to Hilbert spaces of a sum of decoupled Chern-Simons theories with gauge groups of lower rank. For example, for $v = \frac{\pi}{2}$ and $n = 3$ we have

$$\mathcal{H}_{(2,1)}\left(\frac{\pi}{2}\right) \simeq \mathcal{H}_{(1)}\left(\frac{\pi}{2}\right) \otimes \mathcal{H}_{(2)}\left(\frac{\pi}{2}\right).$$

We have already identified $\mathcal{H}_{(1)}(\frac{\pi}{2})$ as equivalent to the Hilbert space of $U(1)_2$ Chern-Simons theory, and below we will identify $\mathcal{H}_{(2)}(\frac{\pi}{2})$ as the Hilbert space of $[U(1)_4 \times SU(2)_{-2}]/\mathbb{Z}_2$ Chern-Simons theory (the gauge group here is $U(2) \simeq [U(1) \times SU(2)]/\mathbb{Z}_2$), so altogether we can identify $\mathcal{H}_{(2,1)}(\frac{\pi}{2})$ as equivalent to the Hilbert space of a $U(1) \times U(2)$ Chern-Simons theory. It therefore suffices to study the irreducible sectors, which we shall undertake below.

Let us here begin by outlining the plan. In each case we will decompose an irreducible space $\mathcal{H}_{(n_1, \dots, n_p)}(v)$ into irreducible representations of $SL(2, \mathbb{Z})$ and present the action of \mathcal{T}, \mathcal{S} . We then extract the $SU(n)$ degrees of freedom as in (6.2) and calculate the action of \mathcal{T}, \mathcal{S} on the resulting states $|\psi; p\rangle_{SU(n)}$. Next, we will attempt to map the states $|\psi; p\rangle_{SU(n)}$ to ground states of $SU(n)$ Chern-Simons theory on T^2 at some level k'' . We will find that this is possible in all single-particle cases, and we will identify the level. A useful tool is the action of the $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetry group generated by the large gauge transformations Ω'_1, Ω'_2 as in (6.5).

We should make it clear that at this point we are not claiming that the theory *is* Chern-Simons theory at level k'' (although it is very likely), but only that the single-particle Hilbert space $\mathcal{H}_{(n)}(v)$ is equivalent as a representation of $SL(2, \mathbb{Z})$ and \mathcal{U}, \mathcal{V} to the Hilbert space of $[U(1)_{kn} \times SU(n)_{k''}]/\mathbb{Z}_n$. However, whether the low-energy theory really is Chern-Simons theory or not, the $SU(n)$ states $|\psi; p\rangle_{SU(n)}$ have to form a representation of the $\mathbb{Z}_n \times \mathbb{Z}_n$ group, and in this sense we can say that at the very least the low-energy theory is a \mathbb{Z}_n gauge theory (but of course \mathbb{Z}_n might be a subgroup of a bigger gauge group).

Accepting the equivalence between the single-particle sectors and their corresponding $U(n)$ Chern-Simons Hilbert spaces, and given the equivalence between the untwisted sector $\mathcal{H}_{(1,1, \dots, 1)}(v)$ and the corresponding $U(n)$ Chern-Simons Hilbert space that we

established in section 6.4, it is straightforward to construct the Chern-Simons Hilbert space equivalent of reducible sectors $\mathcal{H}_{(n_1, \dots, n_p)}(v)$, as long as no $n_j \geq 2$ appears more than once in the sequence (n_1, \dots, n_p) . Thus, assuming

$$n_1 > \dots > n_{p-q} > n_{p-q+1} = \dots = n_p = 1, \quad (6.40)$$

and denoting, for brevity,

$$U(n')_{kn', k''} \equiv \mathcal{H}([U(1)_{kn'} \times SU(n')_{k''}] / \mathbb{Z}_{n'}),$$

we can identify

$$\mathcal{H}_{(n_1, \dots, n_p)}(v) \simeq U(n_1)_{kn_1, k_1} \otimes \dots \otimes U(n_{p-q})_{kn_{p-q}, k_{p-q}} \otimes U(q)_{kq, k}, \quad (6.41)$$

where $U(n_j)_{kn_j, k_j}$ is our proposal, to be developed below, for the single-particle sector $\mathcal{H}_{(n_j)}(v)$, and k_j ($j = 1, \dots, p - q$) depends on n_j and k .

Note that (6.41) is not explicitly in the form $[U(1)_{kn} \times (\dots)] / \mathbb{Z}_n$. To reconcile (6.41) with our discussion on the level kn of the $U(1)$ center in section 6.1, we can consider a wavefunction in the right-hand side of (6.41). It is a product of wavefunctions of the component $U(n')_{kn', k''}$ Hilbert spaces. As we have explained in section 6.3, these wavefunctions are products of θ -functions in variables $\mathbf{a}_1, \mathbf{a}_2, \dots$, which take values on T^2 . To address the question of the $U(1)$ center, we fix $\mathbf{a}_1, \mathbf{a}_2, \dots$, and translate all variables by ζ , which we take to be some holomorphic coordinate on T^2 :

$$\mathbf{a}_1 \rightarrow \mathbf{a}_1 + \zeta, \quad \mathbf{a}_2 \rightarrow \mathbf{a}_2 + \zeta, \quad \dots$$

The main point is that a wavefunction in $U(n')_{kn', k''}$ is a linear combination of level- kn' θ -functions in ζ . In other words, it is a section of a holomorphic line bundle over T^2 with first Chern class $c_1 = kn'$. As a function of ζ , the product of the wavefunctions in all the component Hilbert spaces on the right-hand side of (6.41) is a linear combination of θ -functions of level $k(q + \sum_{j=1}^{p-q} n_j) = kn$, as it should be.

The condition (6.40) is satisfied by all irreducible sectors except $\mathcal{H}_{(2,2)}(\frac{\pi}{3})$. There are therefore two sectors that are not covered by our results, both for $k = 1$. The first is $\mathcal{H}_{(2,2)}(\frac{\pi}{3})$ itself for $n = 4$, and the second is the reducible sector $\mathcal{H}_{(2,2,1)}(\frac{\pi}{3})$ for $n = 5$, which decomposes as $U(1)_1 \times \mathcal{H}_{(2,2)}(\frac{\pi}{3})$. We discuss the sector $\mathcal{H}_{(2,2)}(\frac{\pi}{3})$ in some detail in appendix C, but it generally remains a mystery to us.

We now turn to a case-by-case analysis of the single-particle irreducible sectors.

6.5.1 $v = \frac{\pi}{2}$ ($k = 2$)

For $k = 2$ and $n = 2$ we have, on the type-IIA side, 3 single-particle states

$$|\square\rangle, \quad |\square\rangle, \quad |\square\rangle,$$

which are a basis for a subspace we denote by $\mathcal{H}_{(2)}(\frac{\pi}{2})$. Using the same argument as in the paragraph preceding (6.19)–(6.21), we can separate the $SU(2)$ degrees of freedom as follows:

$$|\square\rangle = |a\rangle_{SU(2)} \otimes |0\rangle_{U(1)} + |c\rangle_{SU(2)} \otimes |2\rangle_{U(1)} \quad (6.42)$$

$$|\square_{\circlearrowleft}\rangle = |b\rangle_{\text{SU}(2)} \otimes (|1\rangle_{\text{U}(1)} + |3\rangle_{\text{U}(1)}), \quad (6.43)$$

$$|\square_{\circlearrowright}\rangle = |c\rangle_{\text{SU}(2)} \otimes |0\rangle_{\text{U}(1)} + |a\rangle_{\text{SU}(2)} \otimes |2\rangle_{\text{U}(1)}. \quad (6.44)$$

[We used the same notation $|a\rangle, |b\rangle, |c\rangle$ as in (6.19)–(6.21), but these states are, of course, unrelated to the states $|a\rangle, |b\rangle, |c\rangle$ of (6.42)–(6.44).] Then, we can read off the action of the $\text{SL}(2, \mathbb{Z})$ generators \mathcal{T}, \mathcal{S} , and the large \mathbb{Z}_2 gauge transformations Ω_1'', Ω_2'' on $|a\rangle, |b\rangle, |c\rangle$:

$$\mathcal{T}|a\rangle_{\text{SU}(2)} = |a\rangle_{\text{SU}(2)}, \quad \mathcal{T}|b\rangle_{\text{SU}(2)} = e^{-\frac{\pi i}{4}}|b\rangle_{\text{SU}(2)}, \quad \mathcal{T}|c\rangle_{\text{SU}(2)} = -|c\rangle_{\text{SU}(2)}, \quad (6.45)$$

$$\left. \begin{aligned} \mathcal{S}|a\rangle_{\text{SU}(2)} &= \frac{1}{2}|a\rangle_{\text{SU}(2)} + \frac{1}{\sqrt{2}}|b\rangle_{\text{SU}(2)} + \frac{1}{2}|c\rangle_{\text{SU}(2)}, \\ \mathcal{S}|b\rangle_{\text{SU}(2)} &= \frac{1}{\sqrt{2}}|a\rangle_{\text{SU}(2)} - \frac{1}{\sqrt{2}}|c\rangle_{\text{SU}(2)}, \\ \mathcal{S}|c\rangle_{\text{SU}(2)} &= \frac{1}{2}|a\rangle_{\text{SU}(2)} - \frac{1}{\sqrt{2}}|b\rangle_{\text{SU}(2)} + \frac{1}{2}|c\rangle_{\text{SU}(2)}, \end{aligned} \right\} \quad (6.46)$$

and

$$\Omega_1''|a\rangle_{\text{SU}(2)} = |c\rangle_{\text{SU}(2)}, \quad \Omega_1''|b\rangle_{\text{SU}(2)} = |b\rangle_{\text{SU}(2)}, \quad \Omega_1''|c\rangle_{\text{SU}(2)} = |a\rangle_{\text{SU}(2)}, \quad (6.47)$$

$$\Omega_2''|a\rangle_{\text{SU}(2)} = |a\rangle_{\text{SU}(2)}, \quad \Omega_2''|b\rangle_{\text{SU}(2)} = -|b\rangle_{\text{SU}(2)}, \quad \Omega_2''|c\rangle_{\text{SU}(2)} = |c\rangle_{\text{SU}(2)}. \quad (6.48)$$

Comparing the above with (6.22)–(6.23), we see that the action of $\mathcal{T}, \mathcal{S}, \Omega_1'', \Omega_2''$ agrees with that on the Hilbert space of $\text{SU}(2)$ Chern-Simons theory at level $k = -2$. [To see this, note that the eigenvalues of \mathcal{T} above are, up to an overall phase, conjugates of those in (6.22).] Chern-Simons theories with negative levels $k < 0$ are equivalent to the theories with positive levels $(-k)$ but with the opposite orientation of spacetime. Thus, if we wish to keep the same spacetime orientation for all the sectors of the theory, we have to include negative Chern-Simons levels. We conclude that $\mathcal{H}_{(2)}(\frac{\pi}{2})$ is equivalent to the Hilbert space of $[\text{U}(1)_4 \times \text{SU}(2)_{-2}]/\mathbb{Z}_2$ Chern-Simons theory.

For $n = 3$ we get two $[\sigma]$ -twisted sectors. The first, corresponding to $[\sigma] = (3)$, is 2-dimensional and spanned by

$$|\square_{\circlearrowleft}\rangle, \quad |\square_{\circlearrowright}\rangle.$$

We denote it by $\mathcal{H}_{(3)}(\frac{\pi}{2})$. Let us first separate the $\text{U}(1)_6$ center, as in (6.2):

$$\begin{aligned} |\square_{\circlearrowleft}\rangle &= |a\rangle_{\text{SU}(3)} \otimes |0\rangle_{\text{U}(1)} + |b\rangle_{\text{SU}(3)} \otimes |2\rangle_{\text{U}(1)} + |c\rangle_{\text{SU}(3)} \otimes |4\rangle_{\text{U}(1)}, \\ |\square_{\circlearrowright}\rangle &= |a\rangle_{\text{SU}(3)} \otimes |3\rangle_{\text{U}(1)} + |b\rangle_{\text{SU}(3)} \otimes |5\rangle_{\text{U}(1)} + |c\rangle_{\text{SU}(3)} \otimes |1\rangle_{\text{U}(1)}, \end{aligned}$$

where $|p\rangle_{\text{U}(1)}$ ($p = 0, \dots, 5$) are states of $\text{U}(1)$ Chern-Simons theory at level $kn = 6$, and we have used the known action of \mathcal{U}, \mathcal{V} to simplify the decomposition. $|a\rangle_{\text{SU}(3)}, |b\rangle_{\text{SU}(3)}, |c\rangle_{\text{SU}(3)}$ are unspecified states associated with the $\text{SU}(3)$ degrees of freedom only. Using (A.7) and (6.11) we calculate (up to an overall phase):

$$\mathcal{T}|a\rangle_{\text{SU}(3)} = |a\rangle_{\text{SU}(3)}, \quad \mathcal{T}|b\rangle_{\text{SU}(3)} = e^{-\frac{2\pi i}{3}}|b\rangle_{\text{SU}(3)}, \quad \mathcal{T}|c\rangle_{\text{SU}(3)} = e^{-\frac{2\pi i}{3}}|c\rangle_{\text{SU}(3)},$$

and

$$\left. \begin{aligned} \mathcal{S}|a\rangle_{\text{SU}(3)} &= \frac{1}{\sqrt{3}} \left(|a\rangle_{\text{SU}(3)} + |b\rangle_{\text{SU}(3)} + |c\rangle_{\text{SU}(3)} \right) \\ \mathcal{S}|b\rangle_{\text{SU}(3)} &= \frac{1}{\sqrt{3}} \left(|a\rangle_{\text{SU}(3)} + e^{\frac{2\pi i}{3}} |b\rangle_{\text{SU}(3)} + e^{-\frac{2\pi i}{3}} |c\rangle_{\text{SU}(3)} \right) \\ \mathcal{S}|c\rangle_{\text{SU}(3)} &= \frac{1}{\sqrt{3}} \left(|a\rangle_{\text{SU}(3)} + e^{-\frac{2\pi i}{3}} |b\rangle_{\text{SU}(3)} + e^{\frac{2\pi i}{3}} |c\rangle_{\text{SU}(3)} \right) \end{aligned} \right\}. \quad (6.49)$$

The $\mathbb{Z}_3 \subset \text{SU}(3)$ center acts, according to (6.5), as

$$\Omega_2''|a\rangle_{\text{SU}(3)} = |a\rangle_{\text{SU}(3)}, \quad \Omega_2''|b\rangle_{\text{SU}(3)} = e^{-\frac{2\pi i}{3}}|b\rangle_{\text{SU}(3)}, \quad \Omega_2''|c\rangle_{\text{SU}(3)} = e^{\frac{2\pi i}{3}}|c\rangle_{\text{SU}(3)},$$

and

$$\Omega_1''|a\rangle_{\text{SU}(3)} = |b\rangle_{\text{SU}(3)}, \quad \Omega_1''|b\rangle_{\text{SU}(3)} = |c\rangle_{\text{SU}(3)}, \quad \Omega_1''|c\rangle_{\text{SU}(3)} = |a\rangle_{\text{SU}(3)}.$$

These formulas for $\mathcal{T}, \mathcal{S}, \Omega_1'', \Omega_2''$, are consistent with the Hilbert space of $\text{SU}(3)$ Chern-Simons theory at level $k = -1$. To check that $|a\rangle_{\text{SU}(3)}, |b\rangle_{\text{SU}(3)}, |c\rangle_{\text{SU}(3)}$ agree with the states of $\text{SU}(3)_{-1}$ Chern-Simons theory, we note that $\text{U}(3)_{-1} = [\text{U}(1)_{-3} \times \text{SU}(3)_{-1}] / \mathbb{Z}_3$ has a one-dimensional Hilbert space, spanned by a state of the form

$$|0\rangle_{\text{U}(1)_{-3}} \otimes |a'\rangle_{\text{SU}(3)_{-1}} + |1\rangle_{\text{U}(1)_{-3}} \otimes |b'\rangle_{\text{SU}(3)_{-1}} + |2\rangle_{\text{U}(1)_{-3}} \otimes |c'\rangle_{\text{SU}(3)_{-1}}$$

where, as the notation suggests, $|p\rangle_{\text{U}(1)_{-3}}$ ($p = 0, 1, 2$) are the states of $\text{U}(1)_{-3}$ Chern-Simons theory. The \mathcal{T}, \mathcal{S} transformations of $|a'\rangle_{\text{SU}(3)_{-1}}, |b'\rangle_{\text{SU}(3)_{-1}}, |c'\rangle_{\text{SU}(3)_{-1}}$ can then be recovered from (5.22) and (5.24).

The second $[\sigma]$ -twisted sector for $n = 3$ corresponds to $[\sigma] = (2, 1)$ and is spanned by

$$|\square\square\rangle, \quad |\square\circ\square\rangle, \quad |\square\circ\circ\rangle, \quad |\circ\square\rangle, \quad |\circ\circ\rangle, \quad |\circ\circ\rangle. \quad (6.50)$$

We denote it by $\mathcal{H}_{(2,1)}(\frac{\pi}{2})$. As explained at the top of section 6.5, this sector is reducible, and equivalent to the Hilbert space of $\text{U}(1)_2 \times [\text{U}(1)_4 \times \text{SU}(2)_{-2}] / \mathbb{Z}_2$ Chern-Simons theory.

So, altogether, in the case $n = 3$ we found that the Hilbert space is a direct sum of three Hilbert spaces:

$$\text{U}(3)_{6,2} \oplus \text{U}(3)_{6,-1} \oplus [\text{U}(1)_2 \otimes \text{U}(2)_{4,-2}].$$

The first two have gauge group $\text{U}(3)$, and the third has gauge group $\text{U}(1) \times \text{U}(2)$.

6.5.2 $v = \frac{\pi}{3}$ ($k = 1$) and $v = \frac{2\pi}{3}$ ($k = 3$)

Except for the mysterious $\mathcal{H}_{(2,2)}(\frac{\pi}{3})$ sector mentioned above, we again find that each single-particle sector $\mathcal{H}_{(n)}(v)$ is equivalent to a Chern-Simons Hilbert space. The derivations are presented in appendix C, and the results are summarized in table 3 in the concluding section.

6.6 Wilson loop operators

So far we have found a correspondence between the Hilbert space of ground states of the S-duality twisted compactification of $\mathcal{N} = 4$ SYM of section 2 and the Hilbert space of ground states of the type-IIA background of section 3. The next step is to extend this correspondence to operators. The natural operators to start with on the type-IIB (gauge theory) side are Wilson loops at a constant x_3 and along a curve $C \subset \mathbb{R}^{2,1}$ where $\mathbb{R}^{2,1}$ corresponds to directions 0, 1, 2. Let us denote a Wilson loop operator in the fundamental representation of $U(n)$ by $\mathcal{W}(C, x_3)$. We assume that the curvature of C is much smaller than the compactification scale $1/R$. We can also consider supersymmetric extensions of Wilson loops, as constructed in [44, 45]. These include additional terms that depend on the scalar fields of $\mathcal{N} = 4$ SYM, but since we have eliminated all the zero modes of scalar fields in section 2.3 we can expect that at low-energy the scalar fields are effectively zero, and it is likely that the difference between ordinary and supersymmetric Wilson loops disappears. In any case, we ask to what operator $\mathcal{W}(C, x_3)$ flows to at low-energy. This question will be addressed in more detail in an upcoming paper [46], but we will make a few preliminary remarks in the present subsection. For simplicity, we restrict to the case $k = 2$.

Note that because of the S-duality twist, the operator $\mathcal{W}(C, x_3)$ satisfies the boundary conditions

$$\mathcal{W}(C, x_3) = \mathcal{M}(C, x_3 + 2\pi R) = \mathcal{W}(C, x_3 + 4\pi R)^\dagger = \mathcal{M}(C, x_3 + 6\pi R)^\dagger = \mathcal{W}(C, x_3 + 8\pi R), \quad (6.51)$$

where \mathcal{M} is the magnetic dual 't Hooft loop operator, and \mathcal{W}^\dagger is the charge-conjugate Wilson loop operator in the anti-fundamental representation of $U(n)$. We now define linear combinations which diagonalize the boundary conditions (6.51):

$$\mathcal{V}^{(p)}(C, x_3) \equiv \mathcal{W}(C, x_3) + i^p \mathcal{M}(C, x_3) + (-1)^p \mathcal{W}(C, x_3)^\dagger + (-i)^p \mathcal{M}(C, x_3)^\dagger, \quad p = 0, 1, 2, 3. \quad (6.52)$$

Their Fourier transforms along x_3 are

$$\mathcal{V}^{(p)}(C, x_3) = \sum_{m \in \mathbb{Z}} \widehat{\mathcal{V}}_{m+\frac{p}{4}}^{(p)}(C) e^{(m+\frac{p}{4})\frac{ix_3}{R}}. \quad (6.53)$$

For $p \neq 0$, when acting on the ground states all the modes $\widehat{\mathcal{V}}_{m+\frac{p}{4}}^{(p)}(C)$ create linear combinations of states with nonzero fractional Kaluza-Klein momentum, and therefore have energy at least $\frac{1}{4R}$. Thus, when we project these operators to the Hilbert space of ground states they all vanish except $\mathcal{V}^{(0)}$. We can therefore surmise that the operators $\mathcal{W}(C, x_3)$, $\mathcal{M}(C, x_3)$, $\mathcal{W}(C, x_3)^\dagger$, and $\mathcal{M}(C, x_3)^\dagger$, all flow at low-energy to the same operator:

$$\mathcal{W}(C, x_3), \mathcal{M}(C, x_3), \mathcal{W}(C, x_3)^\dagger, \mathcal{M}(C, x_3)^\dagger \xrightarrow{\text{IR}} \frac{1}{4} \widehat{\mathcal{V}}_0^{(0)}. \quad (6.54)$$

In other words, at low-energy only the S-duality invariant combination

$$\mathcal{V}^{(0)}(C, x_3) \equiv \mathcal{W}(C, x_3) + \mathcal{M}(C, x_3) + \mathcal{W}(C, x_3)^\dagger + \mathcal{M}(C, x_3)^\dagger$$

is relevant. And in particular we note that even though the gauge group is complex, $\mathcal{V}^{(0)}(C, x_3)$ is real and gives rise to a self-adjoint operator on the Hilbert space of ground

states. For example, for $U(1)$ gauge group we saw that the Wilson loops $\mathcal{W}_1, \mathcal{W}_2$ defined in (5.17) are self-adjoint for $k = 2$, even though they are not self-adjoint in Chern-Simons theory at level $k > 2$. A similar phenomenon occurs for the C-twist that we studied in section 4. This time the combinations that survive the low-energy limit are $\mathcal{W} + \mathcal{W}^\dagger$. In section 4.2 we saw that starting with $U(2)$ gauge group, with the help of a C-twist, we get a low-energy $SU(2)$ gauge theory. So, while Wilson loop operators in $U(2)$ gauge theory are not self-adjoint, they are in $SU(2)$ since its fundamental and anti-fundamental representations are equivalent!

The action of $\mathcal{V}^{(0)}(C, x_3)$ on ground states can be studied using the type-IIA dual by introducing probe strings, but this is beyond the scope of the present paper and will be discussed in detail in [46]. We will only mention that the operators defined in (3.28) play a role in the construction.

7 Realization via the (2, 0)-theory

S-duality is geometrically realized in terms of the six-dimensional (2, 0)-theory. In this section we will discuss a geometrical construction in terms of the (2, 0)-theory of a setting similar to that of section 2. The (2, 0)-theory that was proposed by Witten in [48] is still poorly understood, but there are at least two proposals for a definition: one as a M(atr ix)-model [49, 50] and another in terms of deconstruction [51]. (For some attempts in other directions see [52]–[55].) In this section we will actually not have to use any of the fundamental definitions, however, because known results about the low-energy description of the theory will suffice. The (2, 0)-theory has an $SO(5)$ R-symmetry, so we cannot reproduce the identical setting of section 2, because the full $SO(6)$ R-symmetry twist cannot be realized in terms of the (2, 0)-theory. Instead, we will produce a closely related setting as follows.

As Witten proposed [48], $\mathcal{N} = 4$ $U(n)$ super Yang-Mills theory with coupling constant τ is the low-energy limit of a six-dimensional theory compactified on T^2 , with τ being the complex structure parameter of the torus, so that S-duality $\tau \rightarrow (\mathbf{a}\tau + \mathbf{b})/(\mathbf{c}\tau + \mathbf{d})$ is realized as an element of the mapping class group of the T^2 . This immediately leads to a realization of the S-duality twisted compactification defined in section 2.2: we simply take the (2, 0)-theory (for the appropriate n) and compactify it on the space W defined in section 3.1. Recall that $W \simeq (T^2 \times S^1)/\mathbb{Z}_r$, where S^1 has radius $2\pi Rr$. The torus T^2 has complex structure τ , and we denote its area by \mathcal{A} , so that in the limit

$$\mathcal{A} \ll R^2, \tag{7.1}$$

we recover the S-duality twisted compactification of section 2.2.

The R-symmetry twist of section 2.3, however, is more difficult to realize because the (2, 0)-theory only has an $SO(5)$ global R-symmetry, not $SO(6)$. The enhanced $SO(6)$ R-symmetry of $\mathcal{N} = 4$ SYM only arises as an effective low-energy symmetry. To get around this obstacle, we note that the R-symmetry twist we used in section 2.3 can be continuously

deformed while preserving $\mathcal{N} = 4$ SUSY in 2+1D by replacing (2.11) with

$$\gamma' = \begin{pmatrix} e^{\frac{i}{2}v} & & & \\ & e^{\frac{i}{2}v} & & \\ & & e^{i\epsilon - \frac{i}{2}v} & \\ & & & e^{-i\epsilon - \frac{i}{2}v} \end{pmatrix} \in \text{SU}(4)_R. \tag{7.2}$$

For $\epsilon = v$ we recover (2.11), while for $\epsilon = 0$ we get an R-symmetry twist in a subgroup $\text{SO}(2) \subset \text{SO}(5) \subset \text{SO}(6)$, and thus it can be realized inside the $\text{SO}(5)$ R-symmetry of the (2,0)-theory. For $\epsilon = 0$ we also get additional bosonic and fermionic zero-modes from the scalars and gluinos, but for $0 < \epsilon \leq v$ they are absent. Presumably, the low-energy description for $0 < \epsilon \leq v$ is independent of ϵ (by supersymmetry, or if the theory is indeed topological), and so we can study the theory at $\epsilon = 0$ first, and then deform by a small ϵ , provided we can understand that deformation in the low-energy description of the $\epsilon = 0$ setting. In fact, for the specific purpose of understanding some of the $[\sigma]$ -twisted sectors in section 7.3, it will suffice to study the $\epsilon = 0$ case.

To better understand the low-energy limit of the $\epsilon = 0$ theory, which is a 2+1D theory with $\mathcal{N} = 4$ supersymmetry, we will make the plausible assumption that the low-energy theory is independent of the dimensionless parameter \mathcal{A}/R^2 and take the limit opposite to (7.1), namely

$$\frac{\mathcal{A}}{R^2} \rightarrow \infty. \tag{7.3}$$

To analyze this limit, it is convenient to describe W as an S^1 fibration over a base T^2/\mathbb{Z}_r . The fibers are constructed as follows. Fix a point on T^2 that corresponds to coordinate z (with the identification $z \sim z + 1 \sim z + \tau$), and consider the set of all points with coordinates (z, x_3) , where x_3 is arbitrary. The generic fiber is an S^1 of circumference $2\pi Rr$. Because of the \mathbb{Z}_r action, the fibers that we get for z and $e^{2\pi i/r}z$ are identical, so the base is T^2/\mathbb{Z}_r , as stated above.

This fibration is not quite a circle bundle, however, because there exist special points on the T^2/\mathbb{Z}_r base where the fiber is smaller than the generic one. This happens if z is invariant (up to $\mathbb{Z} + \mathbb{Z}\tau$) under some nontrivial element of the orbifold group \mathbb{Z}_r . For $\tau = i$ this is the case for three inequivalent z 's: $z = 0, \frac{1}{2},$ and $\frac{1}{2}(1 + i)$. The T^2 points 0 and $\frac{1}{2}(1 + i)$ are fixed by the entire \mathbb{Z}_4 , and the fiber over those points is of size $2\pi R$, i.e., $\frac{1}{4}$ of the generic fiber. The point $\frac{1}{2}$ is fixed by a $\mathbb{Z}_2 \subset \mathbb{Z}_4$ subgroup and the fiber over it is of size $4\pi R$, i.e., $\frac{1}{2}$ of the generic fiber. We can choose the fundamental domain of the \mathbb{Z}_4 action on T^2 to be a triangle with vertices $z = 0, \frac{1}{2}(1 + i), 1$ and with extra identifications on the boundary of the triangle which are induced by the identification $z \simeq 1 + iz$ and $z \simeq 1 - z$. The result is depicted in the $k = 2$ portion of figure 4. The situation is similar for $\tau = e^{\pi i/3}$. Here again there are three special points of T^2/\mathbb{Z}_r which are invariant under subgroups of \mathbb{Z}_r , and the fibers there are smaller than the generic fiber.

For $i = 1, 2, 3$, we denote the i^{th} special point by $Q_i \in T^2/\mathbb{Z}_r$. We denote the order of the subgroup of \mathbb{Z}_r that fixes the special point Q_i by p_i . The generic fiber has circumference

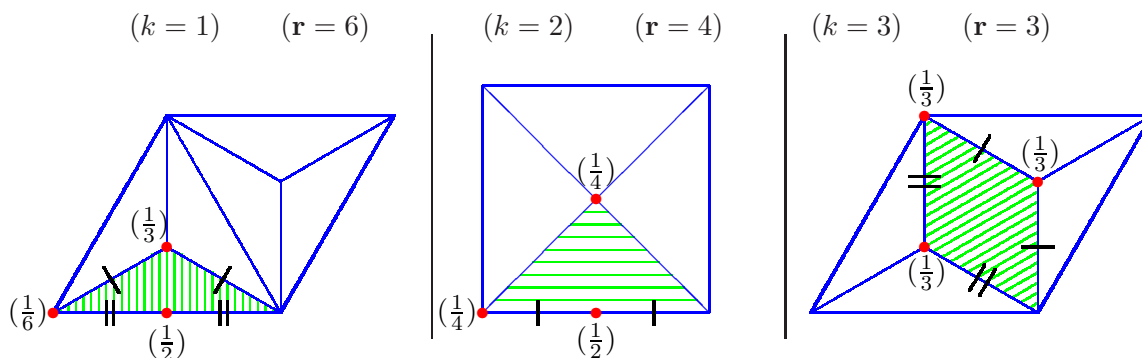


Figure 4. Fundamental domains of the \mathbb{Z}_r action on T^2 . Fundamental domains are triangles for $k = 1, 2$ or a rhombus for $k = 3$ with edges identified as indicated by the markings. The special points are fixed points of \mathbb{Z}_r or a proper subgroup of it, and the fractions indicate the size of the S^1 fiber. These fractions are the inverses of the orders of the fixed-point subgroup.

$2\pi rR$, and so the fiber at Q_i has circumference $2\pi rR/p_i$. We find the following values of p_i :

$$(p_1, p_2, p_3) = \begin{cases} (3, 3, 3) & \text{for } r = 3 \ (\tau = e^{\pi i/3}), \\ (4, 4, 2) & \text{for } r = 4 \ (\tau = i), \\ (6, 3, 2) & \text{for } r = 6 \ (\tau = e^{\pi i/3}). \end{cases} \quad (7.4)$$

Figure 4 shows convenient representations of T^2/\mathbb{Z}_r with the special points marked by the fraction $1/p_i$. Note that in all three cases

$$1 = \sum_{i=1}^3 \frac{1}{p_i}. \quad (7.5)$$

7.1 Reduction to 4+1D and 2+1D

When the (2,0)-theory is compactified on W in the limit (7.3), we can “dimensionally reduce” the theory on the generic S^1 fiber to get, away from the three singular points Q_1, Q_2, Q_3 , a low-energy 4+1D $\mathcal{N} = 2$ super Yang-Mills theory (with 16 supersymmetry generators). The theory is formulated on $\mathbb{R}^{2,1} \times (T^2/\mathbb{Z}_r)$, and has a coupling constant

$$g_{\text{YM}}^{(5\text{D})} = 2\pi(2Rr)^{\frac{1}{2}}.$$

The space T^2/\mathbb{Z}_r is locally flat, except for curvature singularities at the special points Q_1, Q_2, Q_3 .

We denote the bulk 4+1D $U(n)$ gauge field by \mathbf{C}' , and for simplicity of the discussion ignore the superpartners. The resulting low-energy description is constructed by combining the bulk 4+1D action for \mathbf{C}' with additional localized interactions at the special points Q_1, Q_2, Q_3 .

What is the contribution of the special point Q_i to the action? Near Q_i the base looks like $\mathbb{R}^2/\mathbb{Z}_{p_i}$, which is a cone. The total space looks like $(S^1 \times \mathbb{R}^2)/\mathbb{Z}_{p_i}$ where \mathbb{Z}_{p_i} acts as rotation by $2\pi/p_i$ on \mathbb{R}^2 and translation by $2\pi Rr/p_i$ on S^1 . To proceed, we switch to the M-theory realization where we have n M5-branes on $(S^1 \times \mathbb{R}^2)/\mathbb{Z}_{p_i}$. We also need to realize the

R-symmetry twist. For (7.2) with $\epsilon = 0$, this twist can be expressed as a $2\pi/p_i$ rotation in an additional \mathbb{R}^2 plane transverse to the M5-branes. Altogether, combining this transverse $\mathbb{R}^2 \simeq \mathbb{C}$ with the $\mathbb{R}^2 \simeq \mathbb{C}$ that appears in $(S^1 \times \mathbb{R}^2)/\mathbb{Z}_{p_i}$, we get M-theory on $(S^1 \times \mathbb{C}^2)/\mathbb{Z}_{p_i}$ where the generator of \mathbb{Z}_{p_i} acts on a point of $(S^1 \times \mathbb{C}^2)$ with coordinates (x_3, ζ_1, ζ_2) as:

$$\mathbb{Z}_{p_i} : (x_3, \zeta_1, \zeta_2) \mapsto \left(x_3 + \frac{2\pi R\mathbf{r}}{p_i}, e^{\frac{2\pi i}{p_i}} \zeta_1, e^{-\frac{2\pi i}{p_i}} \zeta_2 \right). \quad (7.6)$$

We are now ready to describe the low-energy contribution of Q_i to the action. We can arrive at the answer by combining a thirteen-year-old result of Witten [57] with a fairly recent result of Gaiotto and Witten [1]. In [57], Witten showed that M-theory on $(S^1 \times \mathbb{C}^2)/\mathbb{Z}_{p_i}$ in the $R \rightarrow 0$ limit and in the region near the origin ($\zeta_1 = \zeta_2 = 0$) is dual to a $(1, p_i)$ 5-brane (an object with p_i units of NS5-brane charge and 1 unit of D5-brane charge) of type-IIB string theory. We will review Witten's arguments below, and see that under the duality the n M5-branes are transformed into n D3-branes that end on the $(1, p_i)$ 5-brane. Luckily, in the last section of [1], Gaiotto and Witten described the boundary interaction of n D3-branes ending on a $(1, p_i)$ 5-brane, and so we can use that interaction to describe the vicinity of our special point Q_i .

Before we proceed to the details of the interaction, let us review the part of Witten's arguments from [57] that apply to our case. Starting with $(S^1 \times \mathbb{C}^2)/\mathbb{Z}_{p_i}$, we first replace \mathbb{C}^2 with a Taub-NUT space, whose metric can be written as

$$ds^2 = \left(1 + \frac{S}{2r}\right)^{-1} (dy + \cos \theta d\phi)^2 + \left(1 + \frac{S}{2r}\right) (dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)), \quad (7.7)$$

where y is a periodic coordinate with range $0 \leq y < 2\pi S$. The origin $r = 0$ is a smooth point, and the isometry that acts as $y \rightarrow y + 2\pi S/p_i$ (keeping the other coordinates unchanged) rotates the tangent plane at the origin in exactly the same way that the \mathbb{C}^2 parameterized by (ζ_1, ζ_2) is rotated in (7.6). We then replace \mathbb{C}^2 in the space $(S^1 \times \mathbb{C}^2)/\mathbb{Z}_{p_i}$ with the Taub-NUT space (7.7) and take the limit of large S .

Next, we take $r \rightarrow \infty$ at constant θ, ϕ , and focus on the T^2 in the (x_3, y) directions. The periodicities and the \mathbb{Z}_{p_i} orbifold induce the identifications

$$(x_3, y) \simeq (x_3, y + 2\pi S) \simeq \left(x_3 + \frac{2\pi R\mathbf{r}}{p_i}, y - \frac{2\pi S}{p_i} \right).$$

Changing coordinates to a complex variable

$$w = \frac{1}{2\pi R\mathbf{r}}(x_3 + iy),$$

we find the identifications

$$w \simeq w + 1 \simeq w - \frac{1}{p_i} + i\frac{S}{p_i R\mathbf{r}}.$$

We now reduce M-theory on the T^2 that is in the (x_3, y) directions to type-IIB with complex coupling constant

$$\tau_{\text{IIB}} = -\frac{1}{p_i} + i\frac{S}{p_i R\mathbf{r}}. \quad (7.8)$$

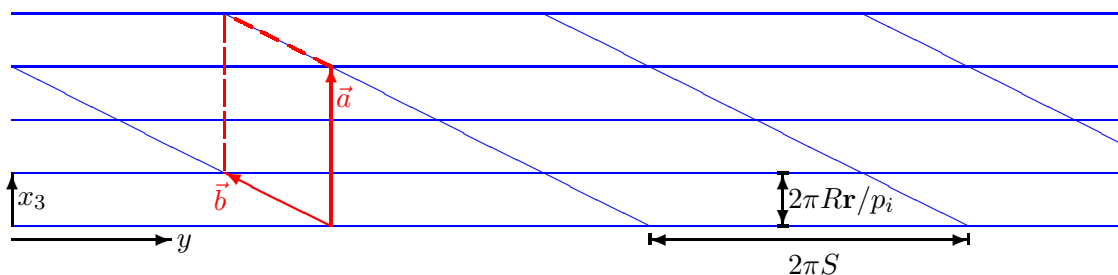


Figure 5. The fiber of the space $(S^1 \times \text{Taub-NUT})/\mathbb{Z}_{p_i}$ at $r \rightarrow \infty$ and constant θ, ϕ is a T^2 . The S^1 is in direction x_3 , and the Taub-NUT fiber is in direction y . The T^2 is represented here as a fundamental cell of a lattice, and we picked the fundamental cell generated by the vectors \vec{a}, \vec{b} . The Taub-NUT direction is then $\vec{a} - p_i \vec{b}$. In this example $p_i = 3$.

To identify which (p, q) 5-brane we get in type-IIB, we have to find the Taub-NUT charge of the metric in terms of w . More explicitly, for fixed and large r , the T^2 is fibered over the S^2 (parameterized by the θ, ϕ coordinates), and the structure group of the fibration is generated by translations in y . (See figure 5.) In terms of w , the translation $(x_3, y) \rightarrow (x_3, y + \epsilon)$ is equivalent to $w \rightarrow w + \epsilon(p_i \tau_{\text{IIB}} + 1)$. The combination $p_i \tau_{\text{IIB}} + 1$ identifies the Taub-NUT charge as the one that reduces to the $(1, p_i)$ 5-brane.

So, after reduction to type-IIB, we get n D3-branes ending on a $(1, p_i)$ 5-brane. Let \mathbf{C}''' be the 3+1D $U(n)$ gauge field on the D3-branes, and let \mathbf{C} be the 2+1D boundary value of the gauge field at the endpoint where the D3-branes meet the $(1, p_i)$ 5-brane. From the 4+1D perspective, \mathbf{C} can be identified with the restriction of the 2+1D components of the bulk gauge field \mathbf{C}' to the special point Q_i . In the discussion that follows we will suppress the superpartners for simplicity.

The description that Gaiotto and Witten provide for n D3-branes ending on a $(1, p_i)$ 5-brane was derived as the S-dual of the description of n D3-branes ending on a $(p_i, 1)$ 5-brane. The latter configuration is described simply by adding a Chern-Simons coupling for the boundary gauge field. The Chern-Simons level is p_i , and this can be derived by a standard $SL(2, \mathbb{Z})$ transformation that maps a $(p_i, 1)$ 5-brane to a $(0, 1)$ 5-brane while changing the type-IIB coupling constant as $\tau_{\text{IIB}} \rightarrow \tau_{\text{IIB}} + p_i$ (see [58]). We denote the $U(n)$ gauge field of this Chern-Simons theory by \mathbf{B}_i .

Following Gaiotto and Witten, S-duality is realized by coupling \mathbf{B}_i to \mathbf{C} through additional degrees of freedom with global $U(n) \times U(n)$ symmetry (the “ $T(U(n))$ ” theory of [1]) and gauging one $U(n)$ factor with \mathbf{B}_i and the other with \mathbf{C} . (See section 7.2 for an example of how this works for $U(1)$ gauge theory.) This description is valid if $\text{Re}\tau_{\text{IIB}} = 0$. But in our case, the type-IIB coupling constant (7.8) has a nonzero real part $\text{Re}\tau_{\text{IIB}} = -1/p_i$. This adds an additional interaction in terms of $F = d\mathbf{C}''' + \mathbf{C}''' \wedge \mathbf{C}'''$:

$$-\frac{1}{4\pi p_i} \int_{D3} \text{tr}(F \wedge F) = \frac{1}{4\pi p_i} \int \left(\mathbf{C} \wedge d\mathbf{C} + \frac{2}{3} \mathbf{C} \wedge \mathbf{C} \wedge \mathbf{C} \right), \quad (7.9)$$

where we have integrated $\text{tr}(F \wedge F)$ to obtain a Chern-Simons coupling at level $1/p_i$ at

the boundary. In principle there is an equal and opposite term at the other end of the D3-branes, wherever it may be, but this is of no concern to us since we are only interested in the interactions near the end of the $(1, p_i)$ 5-brane.

The final step is to reduce to the 2+1D low-energy theory. At low energy the 2+1D components of the bulk gauge field \mathbf{C}' can be assumed to be constant along T^2/\mathbb{Z}_r , and in particular we can identify the three \mathbf{C} gauge fields as one and the same. Adding up the 3 fractional Chern-Simons interactions (7.9) at levels $1/p_i$, and using (7.5), we find that the low-energy effective action has a Chern-Simons interaction at level 1 for \mathbf{C} :

$$\frac{1}{4\pi} \text{tr} \int \left(\mathbf{C} \wedge d\mathbf{C} + \frac{2}{3} \mathbf{C} \wedge \mathbf{C} \wedge \mathbf{C} \right).$$

In addition, the Lagrangian has 3 Chern-Simons interactions – an interaction at level p_i for \mathbf{B}_i ($i = 1, 2, 3$):

$$\sum_{i=1}^3 \frac{p_i}{4\pi} \text{tr} \int \left(\mathbf{B}_i \wedge d\mathbf{B}_i + \frac{2}{3} \mathbf{B}_i \wedge \mathbf{B}_i \wedge \mathbf{B}_i \right),$$

and three copies ($i = 1, 2, 3$) of the $T(U(n))$ theories described in [1], each coupled to \mathbf{B}_i and \mathbf{C} . At this point we point out again that superpartners of the gauge fields have been suppressed.

7.2 Recovering the U(1) result

For U(1) gauge theory, we construct the low-energy 2+1D interactions as follows. First, we have a low-energy gauge field \mathbf{C} that descends from the bulk field \mathbf{C}' . It has a Chern-Simons interaction at level 1, i.e., $\frac{1}{4\pi} \int \mathbf{C} \wedge d\mathbf{C}$. Then, we have additional degrees of freedom from the three special points. These are equivalent to the degrees of freedom of a D3-brane that ends on a $(1, p)$ 5-brane. The description of that system was given in [1] in terms of the action

$$\frac{1}{4\pi} \int (p\mathbf{B} \wedge d\mathbf{B} + 2\mathbf{B} \wedge d\mathbf{C}_b), \tag{7.10}$$

where \mathbf{C}_b is the bulk D3-brane gauge field, restricted to the boundary. We can identify it with our low-energy field \mathbf{C} .

Let us briefly comment on how the expression (7.10) was derived. It is the S-dual of the boundary interaction of a D3-brane ending on a $(p, 1)$ 5-brane, the latter being given by a level p Chern-Simons interaction of the boundary gauge field \mathbf{C}_b . As explained in [59], the $2\mathbf{B} \wedge d\mathbf{C}_b$ term realizes the S-duality [see (5.1)].

For each of the cases listed in (7.4), we have three special points, so we need to include three interactions of the type (7.10), with the appropriate values of p . We denote the 3 localized gauge fields by $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$. The various values of p are the denominators of the fractions appearing in figure 4. Thus, we have

$$I = I_{\text{sp}} + \frac{1}{2\pi} \int (\mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3) \wedge d\mathbf{C}, \tag{7.11}$$

with

$$\left. \begin{aligned} I_{\text{sp}} &= \frac{1}{4\pi} \int (6\mathbf{B}_1 \wedge d\mathbf{B}_1 + 3\mathbf{B}_2 \wedge d\mathbf{B}_2 + 2\mathbf{B}_3 \wedge d\mathbf{B}_3), & (k=1) \\ I_{\text{sp}} &= \frac{1}{4\pi} \int (4\mathbf{B}_1 \wedge d\mathbf{B}_1 + 4\mathbf{B}_2 \wedge d\mathbf{B}_2 + 2\mathbf{B}_3 \wedge d\mathbf{B}_3), & (k=2) \\ I_{\text{sp}} &= \frac{1}{4\pi} \int (3\mathbf{B}_1 \wedge d\mathbf{B}_1 + 3\mathbf{B}_2 \wedge d\mathbf{B}_2 + 3\mathbf{B}_3 \wedge d\mathbf{B}_3), & (k=3) \end{aligned} \right\}. \tag{7.12}$$

The general form of the interaction (7.11)–(7.12) is therefore

$$I = \frac{1}{4\pi} \int \left(\sum_{i=1}^3 p_i \mathbf{B}_i \wedge d\mathbf{B}_i + 2d\mathbf{C} \wedge \sum_{i=1}^3 \mathbf{B}_i \right), \quad (7.13)$$

where p_1, p_2, p_3 are integers determined by the level k .

Now consider an abelian Chern-Simons theory with action

$$\frac{1}{4\pi} \int \sum_{i,j} h_{ij} \mathbf{B}_i \wedge d\mathbf{B}_j, \quad (7.14)$$

where h_{ij} are integer elements of a nonsingular symmetric matrix. Compactified on T^2 , the number of states that we get is the determinant

$$N_{\text{states}} = \det\{h_{ij}\}.$$

However, the 4×4 matrix corresponding to (7.13) is singular:

$$\det \begin{pmatrix} p_1 & 0 & 0 & 1 \\ 0 & p_2 & 0 & 1 \\ 0 & 0 & p_3 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = p_1 p_2 p_3 \left(1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3} \right) = 0.$$

Nevertheless, the zero mode can easily be extracted by changing variables:

$$\mathbf{B}_1 \equiv \mathbf{B}'_1, \quad \mathbf{B}_2 \equiv \mathbf{B}'_2 + \frac{p_1}{p_2} \mathbf{B}'_1, \quad \mathbf{B}_3 \equiv \mathbf{B}'_3 + \frac{p_1}{p_3} \mathbf{B}'_1, \quad \mathbf{C} \equiv \mathbf{C}' - p_1 \mathbf{B}'_1.$$

Note that this transformation is always in $\text{SL}(4, \mathbb{Z})$, since we have arranged the p_1, p_2, p_3 in (7.12) so that $\frac{p_1}{p_2}$ and $\frac{p_1}{p_3}$ are integers.

The action (7.13) can now be written as

$$I = \frac{1}{4\pi} \int \left(\sum_{i=2}^3 p_i \mathbf{B}'_i \wedge d\mathbf{B}'_i + 2d\mathbf{C}' \wedge \sum_{i=2}^3 \mathbf{B}'_i \right), \quad (7.15)$$

and \mathbf{B}'_1 does not appear in the action. This means that when we look for ground states on T^2 , we should include a canonical kinetic term proportional to $\int d\mathbf{B}'_1 \wedge *d\mathbf{B}'_1$ (originating from the gauge kinetic term for \mathbf{C} and other terms). Such a term will ensure that states coming from excitations with nonzero $d\mathbf{B}'_1$ are not ground states. If we are only interested in the ground states of the system it is therefore sufficient to concentrate on the abelian Chern-Simons theory (7.15). The reduced matrices h_{ij} corresponding to (7.15) for the cases $k = 1, 2, 3$ are:

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

and their determinants are 1, 2, 3, respectively! Thus, we have recovered the correct number of ground states. We conclude this subsection with a few comments.

1. We can trace the zero mode \mathbf{B}'_1 back to the scalar field coming from the component of the $(2, 0)$ anti-self-dual 2-form $B^{(-)}$ along the z, \bar{z} directions, i.e., $B_{z\bar{z}}^{(-)}$. This term corresponds to the 6th scalar field of $\mathcal{N} = 4$ SYM, and in the construction of section 2 this zero mode gets lifted by an R-symmetry twist. In our $(2, 0)$ -realization this scalar is special, and while we could not add an R-symmetry twist to lift the zero mode at the outset, we can add it to the low-energy theory at the end. In any case, we do not get any additional multiplicity of ground states.
2. Naïvely, we can attempt to integrate out $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$ in (7.13). The result is obtained by setting $\mathbf{B}_i = -\frac{1}{p_i}\mathbf{C}$, but plugging this back into (7.13) we get a vanishing action. The problem with this prescription is that it ignores the integral periodicity of the gauge fields. In fact, this is precisely what Gaiotto and Witten warned us not to do when dealing with a D3-brane ending on a $(p_i, 1)$ 5-brane (see §8.3 of [1]). Here, we see an explicit manifestation of what can go wrong if we disregard their advice!
3. There is a connection between the relations among the generators of the homology group $H_2(T^2 \times W)$ and operator relations in the Hilbert space of ground states. Consider an abelian Chern-Simons theory of the form (7.14), with $i, j = 1, \dots, d$, compactified on T^2 . Let $0 \leq x_1, x_2 \leq 2\pi$ be coordinates on this T^2 , α_a', α_b' the 1-cycles along directions 1 and 2 respectively, and define Wilson lines along the α_a', α_b' cycles:

$$\mathcal{W}_{1i} = e^{i \oint_{\alpha_a'} \mathbf{B}_i}, \quad \mathcal{W}_{2i} = e^{i \oint_{\alpha_b'} \mathbf{B}_i}.$$

If $\{h_{ij}\}$ is invertible with inverse h^{ij} , the commutation relations are

$$\mathcal{W}_{1i}\mathcal{W}_{2j} = e^{2\pi i h^{ij}} \mathcal{W}_{2j}\mathcal{W}_{1i}.$$

The Hilbert space is a representation of this algebra. We then find that for every i ,

$$X_i \equiv \prod_{j=1}^d \mathcal{W}_{1j}^{h_{ij}} \quad \text{and} \quad Y_i \equiv \prod_{j=1}^d \mathcal{W}_{2j}^{h_{ij}}$$

commute with all $\mathcal{W}_{1i}, \mathcal{W}_{2j}$ and so are central elements of the algebra. Without loss of generality, we can set their value to 1. The Hilbert space can now be constructed by diagonalizing all \mathcal{W}_{1i} ($i = 1, \dots, d$) simultaneously. Let $|\psi\rangle$ be any common eigenstate of all \mathcal{W}_{1i} . Then, the full Hilbert space can be constructed by acting with the \mathcal{W}_{2i} on $|\psi\rangle$ and obtaining states of the form $\prod_{i=1}^d \mathcal{W}_{2i}^{N_i} |\psi\rangle$, where $(N_1, \dots, N_d) \in \mathbb{Z}^d$ is a vector of integers. The states of the Hilbert space thus correspond to lattice points in \mathbb{Z}^d , but not all lattice points give distinct states. Since we have identified $Y_i = 1$, we find that the lattice points $(h_{i1}, h_{i2}, \dots, h_{id})$ correspond to the same state as $(0, 0, \dots, 0)$. Let $\Gamma \subset \mathbb{Z}^d$ be the sublattice generated by the d vectors $(h_{i1}, h_{i2}, \dots, h_{id})$ ($i = 1, \dots, d$). Then, the basis states of the Hilbert space thus constructed can be identified with the finite-dimensional set \mathbb{Z}^d/Γ . It is not hard to see that X_i, Y_i are central elements even when $\{h_{ij}\}$ is not invertible.

The point of this note is that in our case, the relations $X_i = Y_i = 1$ have a natural interpretation in terms of the $(2, 0)$ theory. The Wilson loop operators $\mathcal{W}_{1i}, \mathcal{W}_{2i}$ descend from surface operators of the $(2, 0)$ theory. The surface operators $\mathcal{W}(\mathcal{S})$ are associated with closed surfaces $\mathcal{S} \subset T^2 \times W$. In our case, since we are working with the abelian theory and since we are only interested in the ground states, the surface operators only depend on the homology class of \mathcal{S} . This can be argued by noting that in the ground state the anti-self-dual 3-form flux of the $(2, 0)$ -theory vanishes, and that when \mathcal{S} and \mathcal{S}' are in the same homology class, we can write $\mathcal{S} - \mathcal{S}' = \partial\Sigma_3$ so that the difference between the integral of the 2-form of the $(2, 0)$ -theory on \mathcal{S} and on \mathcal{S}' is the integral of the anti-self-dual 3-form field-strength on Σ_3 . We can therefore denote the surface operators as $\mathcal{W}([\mathcal{S}])$ where $[\mathcal{S}]$ is the homology class of \mathcal{S} .

Now, we can match the Wilson lines of $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{C}$ with surface operators as follows. Let γ_i be the homology class of the exceptional fiber at the i^{th} special point ($i = 1, 2, 3$), and let γ_0 be the homology class of the generic fiber. Then, we match

$$\mathcal{W}(\alpha_a' \times \gamma_i) \rightarrow e^{i \oint_{\alpha_a'} \mathbf{B}_i}, \quad \mathcal{W}(\alpha_a' \times \gamma_0) \rightarrow e^{i \oint_{\alpha_a'} \mathbf{C}},$$

and similarly,

$$\mathcal{W}(\alpha_b' \times \gamma_i) \rightarrow e^{i \oint_{\alpha_b'} \mathbf{B}_i}, \quad \mathcal{W}(\alpha_b' \times \gamma_0) \rightarrow e^{i \oint_{\alpha_b'} \mathbf{C}}.$$

The relations $X_i = Y_i = 1$ are then seen to be a consequence of similar relations in homology. (See [60] for a discussion of the commutation relations for the nonabelian $(2, 0)$ -theory.)

7.3 The U(2) theory

We now turn to the nonabelian gauge group U(2). Schematically, the action is of the form

$$I = \frac{1}{4\pi} \int \left\{ \sum_{i=1}^3 p_i \text{tr} \left(\mathbf{B}_i \wedge d\mathbf{B}_i + \frac{2}{3} \mathbf{B}_i \wedge \mathbf{B}_i \wedge \mathbf{B}_i \right) + \text{tr} \left(\mathbf{C} \wedge d\mathbf{C} + \frac{2}{3} \mathbf{C} \wedge \mathbf{C} \wedge \mathbf{C} \right) \right\} + \sum_{i=1}^3 I_i^{[T(U(2))]}(\mathbf{B}_i, \mathbf{C}), \tag{7.16}$$

where $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$ are the 2+1D U(2) gauge fields coming from the singular points Q_1, Q_2, Q_3 , \mathbf{C} is also a 2+1D U(2) gauge field, and $I_i^{[T(U(2))]}$ is the coupling between \mathbf{B}_i and \mathbf{C} through the additional $T(U(2))$ degrees of freedom. Roughly speaking, this coupling realizes the nonabelian S-duality [1], whereby \mathbf{B}_i and \mathbf{C} are regarded as S-dual variables. (As noted above, we are ignoring the superpartners in this discussion.)

Although Gaiotto and Witten have provided an explicit realization of $T(U(2))$ as the low-energy limit of a certain $\mathcal{N} = 4$ 2+1D gauge theory, the full $U(2) \times U(2)$ symmetry, and hence the coupling to \mathbf{B}_i and \mathbf{C} , relies on an enhanced symmetry of $T(U(2))$ that is not explicit. We therefore do not know how to proceed at this moment. However, we can make some comments about the $[\sigma]$ -twisted sector.

What is the interpretation of the $[\sigma]$ -twisted sectors in terms of the $(2, 0)$ -theory construction? For $U(2)$ gauge group, there is only one $[\sigma]$ -twisted sector. In the type-IIA description, the nontrivial $\sigma \in S_2$ exchanges the two strings as we go in a loop from $x_3 = 0$ to $x_3 = 2\pi R$, and so it is reasonable to expect that in our present M-theory description, we need to exchange the two M5-branes as we go from $x_3 = 0$ to $x_3 = 2\pi R$. This exchange of the two branes accompanies the identification $(z, x_3) \sim (e^{iv}z, x_3 + 2\pi R)$ of (3.4).

At this point we need to distinguish between two cases – even \mathbf{r} and odd \mathbf{r} . If \mathbf{r} is odd, then the identification $(z, x_3) \sim (z, x_3 + 2\pi R\mathbf{r})$ is accompanied by an exchange of the two branes. If \mathbf{r} is even this identification is not accompanied by an exchange of branes. The two even cases are $\mathbf{r} = 4, 6$. Since $(z, x_3) \sim (z, x_3 + 2\pi R\mathbf{r})$ is not accompanied by exchange of the branes, the reduction to the 4+1D theory proceeds as in section 7.1.

One of the effects of the nontrivial σ on this low-energy 4+1D $U(2)$ gauge theory is that as we go around a special point Q_i with odd p_i , we have to also exchange the branches of the D4-branes (that we formally get from the M5-branes). This can be interpreted as a holonomy for \mathbf{C}' , which after a suitable conjugation can be written as

$$P \exp \oint_{Q_i} \mathbf{C}' = \begin{pmatrix} 1 & \\ & (-1)^{\mathbf{r}/p_i} \end{pmatrix}. \tag{7.17}$$

For odd \mathbf{r}/p_i , this breaks the gauge group $U(2) \rightarrow U(1) \times U(1)$. In addition, the boundary interaction at Q_i also needs to be modified. Altogether, the action appears quite complicated and we will not attempt to develop it further in this paper. It will be interesting to explore this in a future work.

8 Discussion

We have analyzed the Hilbert space of ground states of the S-duality twisted compactification of $\mathcal{N} = 4$ $U(n)$ SYM on T^2 , and have seen that in almost all cases, at least as a representation of $SL(2, \mathbb{Z})$ and the \mathbb{Z}_k symmetry operators \mathcal{U}, \mathcal{V} , it breaks up into a direct sum of Hilbert spaces of Chern-Simons theories with gauge groups of the form $U(n_1) \times U(n_2) \times \dots \times U(n_s)$ (with $n = \sum_{j=1}^s n_j$). Chern-Simons theory with $U(n_j)$ gauge group is described by specifying the level of $SU(n_j)$ and the level of the $U(1)$ center, so we use the notation

$$U(n_j)_{k'_j, k''_j} \simeq [U(1)_{k'_j} \times SU(n_j)_{k''_j}] / \mathbb{Z}_{n_j}.$$

(There were also two exceptional cases, which involved the Hilbert space $\mathcal{H}_{(2,2)}(\frac{\pi}{3})$.) The various decompositions that we get are listed in table 3. In particular, we saw in section 6.4 that in all cases there is a distinguished sector — the $[\sigma]$ -untwisted sector — which is described by the Hilbert space of $U(n)_{kn, k}$.

In this paper we only studied compactification on T^2 . What do our results suggest for the theory formulated on $\mathbb{R}^{2,1}$? Is Chern-Simons theory the low-energy theory, and if so what is the role of the various sectors with their different Chern-Simons levels and gauge groups (as listed in table 3)? To make this question more precise, we need to connect the operators of Chern-Simons theory to physical operators in our theory. But the low-energy

$v = \frac{\pi}{3}$ ($k = 1$)	$n = 1$	$U(1)_1$
	$n = 2$	$U(2)_{2,1} \oplus U(2)_{2,-3}$
	$n = 3$	$U(3)_{3,1} \oplus [U(1)_1 \times U(2)_{2,-3}] \oplus U(3)_{3,-2}$
	$n = 4$	$U(4)_{4,1} \oplus 2[U(2)_{2,1} \times U(2)_{2,-3}] \oplus [U(1)_1 \times U(3)_{3,-2}] \oplus \mathcal{H}_{(2,2)}$
	$n = 5$	$U(5)_{5,1} \oplus U(5)_{5,1} \oplus 2[U(3)_{3,1} \times U(2)_{2,-3}] \oplus [U(1)_1 \times \mathcal{H}_{(2,2)}] \oplus [U(2)_{2,1} \times U(3)_{3,-2}] \oplus [U(2)_{2,-3} \times U(3)_{3,-2}]$
$v = \frac{\pi}{2}$ ($k = 2$)	$n = 1$	$U(1)_2$
	$n = 2$	$U(2)_{4,2} \oplus U(2)_{4,-2}$
	$n = 3$	$U(3)_{6,2} \oplus [U(1)_2 \times U(2)_{4,-2}] \oplus U(3)_{6,-1}$
$v = \frac{2\pi}{3}$ ($k = 3$)	$n = 1$	$U(1)_3$
	$n = 2$	$U(2)_{6,3} \oplus U(2)_{6,-1}$

Table 3. The decomposition of the Hilbert spaces $\mathcal{H}(n, v)$ into direct sums of Hilbert spaces of Chern-Simons theories. The data in the table is collected from results in appendix C. Trivial $U(n')_{n',1}$ factors were added to conform to the form (6.41). Note that for $n = 4$ we have two copies of $U(2)_{2,1} \times U(2)_{2,-3}$. They come from the sectors $\mathcal{H}_{(2,1,1)}$ and $\mathcal{H}_{(4)}$. Also, note that the sector $\mathcal{H}_{(2,2)}$ is unresolved.

limit of Wilson loops in the $\mathcal{N} = 4$ theory cannot in general be simply a Wilson loop in Chern-Simons theory, because for $k = 2$ for example, the latter is not generally self-adjoint while the former is, as we have argued in section 6.6. If there is a connection between Chern-Simons theory and the low-energy limit of the S-duality twisted compactification on $\mathbb{R}^{2,1}$ it would certainly have to be more complicated than the “crude” conjectures presented in [11], which at best only captured the $[\sigma]$ -untwisted sector. The answer to most of these questions may lie in the proper description of the low-energy limit of Wilson loops. The tools we have developed in this paper in principle allow the analysis of this problem as well, by probing the type-IIB D3-branes with open strings. We hope to report on this matter soon [46].

In this paper we have concentrated on gauge groups of low rank, as we were restricted by the condition $n < \mathbf{r}$. In these cases, as explained in section 2.4, we expect a mass gap. It would be interesting to extend the analysis to $n \geq \mathbf{r}$. Here there are several questions that we can ask. First, we can still look for a low-energy description on $\mathbb{R}^{2,1}$. Since we are dealing with a 2+1D theory with $\mathcal{N} = 6$ supersymmetry that we also expect to be conformally invariant in the low-energy limit, the ABJM theories [61] spring as a natural candidate. Indeed, we expect the low-energy limit for $n \geq \mathbf{r}$ to be an ABJM theory with an appropriate gauge group that can be determined from the moduli space. (Note that for $n < \mathbf{r}$ the low-energy theories that we have proposed are supersymmetric in a trivial way — as topological theories, all their SUSY generators are zero.) Second, we can explore the subspace of normalizable ground states on T^2 . Thus, for example, the $\mathcal{H}_{(n_1, n_2, \dots, n_p)}(v)$ sector makes sense as long as $n_j < \mathbf{r}$ (for $j = 1, \dots, p$), even if $n = \sum_1^p n_j \geq \mathbf{r}$. The states in this sector define the normalizable ground states of the T^2 compactification of the theory, even though the full theory has a continuum of states that start at zero energy. As a simple

case-study, we recall the analysis of the C-twist with $U(2)$ gauge group in section 4.3. There, we found that even though there is no mass gap, there is a finite-dimensional Hilbert space of *normalizable* ground states. Moreover, we matched the ground states with the states of the type-IIA strings, and the bosonic nature of the strings had an interesting interpretation in terms of a restriction on the electric and magnetic fluxes [see (4.13)]. In the more complicated case of an S-duality twist, we can also ask whether the subset of normalizable ground states has a description in terms of a topological field theory. For example, in section section 6.4 we saw that the subsectors $\mathcal{H}_{(1,1,\dots,1)}(v)$ correspond to ground states of $U(n)_{kn,k}$ Chern-Simons theory, even for $n \geq \mathbf{r}$. Other sectors might also have extensions for $n \geq \mathbf{r}$. This avenue of investigation might be connected to ideas developed in [62] about isolating the ground states of supersymmetric theories and finding a simpler description for them separately. We conclude with a summary of a few of the open problems:

1. What is the underlying principle that determines the levels and gauge groups of the $[\sigma]$ -twisted sectors? Does the decomposition into sectors survive when the theory is formulated on $\mathbb{R}^{2,1}$, and if so are they to be regarded as “superselection sectors”, or are there any physical operators that connect different sectors? And can the permutation σ be interpreted as a discrete $S_n \subset U(n)$ Wilson line? If so, why is it restricted to S_n , i.e., how do fluctuations away from S_n receive a potential? Can the unresolved sector $\mathcal{H}_{(2,2)}(\frac{\pi}{3})$ be interpreted as a Chern-Simons Hilbert space? We note that under certain circumstances flavor symmetry twists can induce Chern-Simons interactions [63], but in our case the R-symmetry twist alone cannot induce a low-energy Chern-Simons term because the R-symmetry is nonabelian. The Chern-Simons couplings are intimately related to the S-duality twist.
2. How can these results be recovered directly from the (2,0)-theory? In particular, what is the low-energy description of the action (7.16)? What do the Chern-Simons theories that we found teach us about the S-duality generating $T(SU(n))$ theories that Gaiotto and Witten have found in [1]?
3. The description of the low-energy limit of Wilson loops as operators on the Hilbert space of ground states is currently under investigation [46].
4. How can the results be extended to $n \geq \mathbf{r}$? In this case we can also explore the large n limit in the context of the AdS/CFT correspondence [64]–[66]. If the $[\sigma]$ -untwisted sector $\mathcal{H}_{(1,1,\dots,1)}$ survives the large n limit, it would be interesting to find its holographic dual. Perhaps the holographic dual of Chern-Simons theory [67] will somehow make an appearance.
5. The analysis of the S-duality and R-symmetry twist can also be performed on the topologically twisted $\mathcal{N} = 4$ SYM theories [68]. (Some preliminary results were discussed in [11].) For $n \geq \mathbf{r}$, it might be interesting to look for connections with the topologically twisted supersymmetric Chern-Simons theories [69, 70].
6. Recently, new ideas about surprising mathematical aspects of Chern-Simons theory have emerged (see for instance [71, 72]). In this paper, we have suggested that

Chern-Simons theory is related to S-duality. The latter is also intimately connected to the Langlands correspondence [15]. So, perhaps it is worthwhile to search for a connection between Chern-Simons theory and the Langlands program.

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A Explicit action of $SL(2, \mathbb{Z})$ on ground states

Below we present the operators \mathcal{S} and \mathcal{T} that generate $SL(2, \mathbb{Z})$ in the various single-particle Hilbert spaces. The action of $SL(2, \mathbb{Z})$ on the multi-particle Hilbert spaces can, of course, be calculated from the tensor products of the expressions below. We recall the ambiguity (3.39) in the definition of \mathcal{S} and \mathcal{T} , to which we can also add a phase ambiguity

$$\mathcal{T} \rightarrow e^{i\phi} \mathcal{U}^{\tilde{\mathbf{p}}} \mathcal{V}^{\tilde{\mathbf{q}}} \mathcal{T}, \quad \mathcal{S} \rightarrow e^{i\phi'} \mathcal{U}^{\tilde{\mathbf{p}}} \mathcal{V}^{\tilde{\mathbf{q}}} \mathcal{S}, \quad (\text{A.1})$$

as long as we preserve the group relations $\mathcal{S}^2 = (\mathcal{ST})^3 = -1$. Below, we pick arbitrary $\tilde{\mathbf{p}}, \tilde{\mathbf{q}}, \phi, \phi'$. Thus, the expressions below satisfy the group relations $\mathcal{S}^2 = (\mathcal{ST})^3 = -1$ only up to a phase. This can easily be fixed by choosing appropriate phases ϕ, ϕ' in (A.1), but we find the formulas easier to read without these phases, so we have not included them. We note that only matrix elements of operators (constructed from \mathcal{S} and \mathcal{T}) between initial and final states that preserve \mathcal{U} and \mathcal{V} are physically meaningful, and the ambiguity (A.1) does not affect those matrix elements.

A.1 Action of \mathcal{S} on single-particle states for $v = \frac{\pi}{2}$ ($\tau = i$ and $\mathbf{r} = 4$)

In this case $k = 2$. For $n = 1$, \mathcal{S} acts as:

$$\mathcal{S}|\square_{\bullet}\rangle = \frac{1}{\sqrt{2}}(|\square_{\bullet}\rangle + |\square_{\circ}\rangle), \quad \mathcal{S}|\square_{\circ}\rangle = \frac{1}{\sqrt{2}}(|\square_{\bullet}\rangle - |\square_{\circ}\rangle), \quad (\text{A.2})$$

or, equivalently,

$$\mathcal{S}|[0]\rangle = \frac{1}{\sqrt{2}}\left(|[0]\rangle + \left|\left[\frac{1}{2} + \frac{1}{2}i\right]\right\rangle\right), \quad \mathcal{S}\left|\left[\frac{1}{2} + \frac{1}{2}i\right]\right\rangle = \frac{1}{\sqrt{2}}\left(|[0]\rangle - \left|\left[\frac{1}{2} + \frac{1}{2}i\right]\right\rangle\right).$$

And for \mathcal{T} we have:

$$\mathcal{T}|\square_{\bullet}\rangle = |\square_{\bullet}\rangle, \quad \mathcal{T}|\square_{\circ}\rangle = e^{\frac{i\pi}{2}}|\square_{\circ}\rangle, \quad (\text{A.3})$$

or, equivalently,

$$\mathcal{T}|[0]\rangle = |[0]\rangle, \quad \mathcal{T}\left|\left[\frac{1}{2} + \frac{1}{2}i\right]\right\rangle = e^{\frac{i\pi}{2}}\left|\left[\frac{1}{2} + \frac{1}{2}i\right]\right\rangle.$$

For $n = 2$, \mathcal{S} acts as:

$$\begin{cases} \mathcal{S}|\square_{\bullet}\rangle = \frac{1}{2}|\square_{\bullet}\rangle + \frac{1}{\sqrt{2}}|\square_{\circ}\rangle + \frac{1}{2}|\square_{\circ}\rangle, \\ \mathcal{S}|\square_{\circ}\rangle = \frac{1}{\sqrt{2}}|\square_{\bullet}\rangle - \frac{1}{2}|\square_{\circ}\rangle, \\ \mathcal{S}|\square_{\circ}\rangle = \frac{1}{2}|\square_{\bullet}\rangle - \frac{1}{\sqrt{2}}|\square_{\circ}\rangle + \frac{1}{2}|\square_{\circ}\rangle, \end{cases} \quad (\text{A.4})$$

or, equivalently,

$$\begin{cases} \mathcal{S}|[0, 0]\rangle &= \frac{1}{2}|[0, 0]\rangle + \frac{1}{\sqrt{2}}\left|\left[\frac{1}{2}, \frac{1}{2}i\right]\right\rangle + \frac{1}{2}\left|\left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\right]\right\rangle, \\ \mathcal{S}\left|\left[\frac{1}{2}, \frac{1}{2}i\right]\right\rangle &= \frac{1}{\sqrt{2}}|[0, 0]\rangle - \frac{1}{\sqrt{2}}\left|\left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\right]\right\rangle, \\ \mathcal{S}\left|\left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\right]\right\rangle &= \frac{1}{2}|[0, 0]\rangle - \frac{1}{\sqrt{2}}\left|\left[\frac{1}{2}, \frac{1}{2}i\right]\right\rangle + \frac{1}{2}\left|\left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\right]\right\rangle. \end{cases}$$

And for \mathcal{T} we have:

$$\mathcal{T}|\square_{\bullet}\rangle = |\square_{\bullet}\rangle, \quad \mathcal{T}|\square_{\circ}\rangle = |\square_{\circ}\rangle, \quad \mathcal{T}|\square_{\circ}\rangle = -|\square_{\circ}\rangle, \quad (\text{A.5})$$

or, equivalently,

$$\mathcal{T}|[0, 0]\rangle = |[0, 0]\rangle, \quad \mathcal{T}\left|\left[\frac{1}{2}, \frac{1}{2}i\right]\right\rangle = \left|\left[\frac{1}{2}, \frac{1}{2}i\right]\right\rangle, \quad \mathcal{T}\left|\left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\right]\right\rangle = -\left|\left[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\right]\right\rangle.$$

For $n = 3$, \mathcal{S} acts as:

$$\mathcal{S}|\square_{\bullet}\rangle = \frac{1}{\sqrt{2}}(|\square_{\bullet}\rangle + |\square_{\circ}\rangle), \quad \mathcal{S}|\square_{\circ}\rangle = \frac{1}{\sqrt{2}}(|\square_{\bullet}\rangle - |\square_{\circ}\rangle), \quad (\text{A.6})$$

or, equivalently,

$$\begin{cases} \mathcal{S} |[0, 0, 0]\rangle &= \frac{1}{\sqrt{2}} (|[0, 0, 0]\rangle + |[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i]\rangle), \\ \mathcal{S} |[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i]\rangle &= \frac{1}{\sqrt{2}} (|[0, 0, 0]\rangle - |[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i]\rangle). \end{cases}$$

And for \mathcal{T} we have:

$$\mathcal{T} \left| \begin{array}{c} \square \\ \bullet \end{array} \right\rangle = \left| \begin{array}{c} \square \\ \bullet \end{array} \right\rangle, \quad \mathcal{T} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle = e^{-\frac{i\pi}{2}} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle, \quad (\text{A.7})$$

or, equivalently,

$$\begin{cases} \mathcal{T} |[0, 0, 0]\rangle &= |[0, 0, 0]\rangle, \\ \mathcal{T} |[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i]\rangle &= e^{-\frac{i\pi}{2}} |[\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i]\rangle. \end{cases}$$

A.2 Action of \mathcal{S} on single-particle states for $v = \frac{\pi}{3}$ ($\tau = e^{\pi i/3}$ and $\mathbf{r} = 6$)

In this case $k = 1$. For $n = 1$, \mathcal{S} and \mathcal{T} act as:

$$\mathcal{S} \left| \begin{array}{c} \square \\ \bullet \end{array} \right\rangle = \left| \begin{array}{c} \square \\ \bullet \end{array} \right\rangle, \quad \mathcal{T} \left| \begin{array}{c} \square \\ \bullet \end{array} \right\rangle = \left| \begin{array}{c} \square \\ \bullet \end{array} \right\rangle, \quad (\text{A.8})$$

or, equivalently,

$$\mathcal{S} |[0]\rangle = |[0]\rangle, \quad \mathcal{T} |[0]\rangle = |[0]\rangle.$$

For $n = 2$, \mathcal{S} acts as:

$$\mathcal{S} \left| \begin{array}{c} \square \\ \bullet \end{array} \right\rangle = \frac{1}{\sqrt{3}} \left| \begin{array}{c} \square \\ \bullet \end{array} \right\rangle + \sqrt{\frac{2}{3}} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle, \quad \mathcal{S} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle = \sqrt{\frac{2}{3}} \left| \begin{array}{c} \square \\ \bullet \end{array} \right\rangle - \frac{1}{\sqrt{3}} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle, \quad (\text{A.9})$$

or, equivalently,

$$\begin{cases} \mathcal{S} |[0, 0]\rangle &= \frac{1}{\sqrt{3}} |[0, 0]\rangle + \sqrt{\frac{2}{3}} \left[\frac{1}{\sqrt{3}} e^{\frac{i\pi}{6}}, \frac{2}{\sqrt{3}} e^{\frac{i\pi}{6}} \right], \\ \mathcal{S} \left[\frac{1}{\sqrt{3}} e^{\frac{i\pi}{6}}, \frac{2}{\sqrt{3}} e^{\frac{i\pi}{6}} \right] &= \sqrt{\frac{2}{3}} |[0, 0]\rangle - \frac{1}{\sqrt{3}} \left[\frac{1}{\sqrt{3}} e^{\frac{i\pi}{6}}, \frac{2}{\sqrt{3}} e^{\frac{i\pi}{6}} \right]. \end{cases}$$

\mathcal{T} acts as:

$$\mathcal{T} \left| \begin{array}{c} \square \\ \bullet \end{array} \right\rangle = \left| \begin{array}{c} \square \\ \bullet \end{array} \right\rangle, \quad \mathcal{T} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle = e^{-\frac{2\pi i}{3}} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle, \quad (\text{A.10})$$

or, equivalently,

$$\mathcal{T} |[0, 0]\rangle = |[0, 0]\rangle, \quad \mathcal{T} \left[\frac{1}{\sqrt{3}} e^{\frac{i\pi}{6}}, \frac{2}{\sqrt{3}} e^{\frac{i\pi}{6}} \right] = e^{-\frac{2\pi i}{3}} \left[\frac{1}{\sqrt{3}} e^{\frac{i\pi}{6}}, \frac{2}{\sqrt{3}} e^{\frac{i\pi}{6}} \right].$$

For $n = 3$, we have two \mathbb{Z}_2 worldsheet momentum operators $\tilde{\mathcal{U}}_a, \tilde{\mathcal{U}}_b$ and two \mathbb{Z}_2 worldsheet winding number operators $\tilde{\mathcal{V}}_a, \tilde{\mathcal{V}}_b$. For the commutation relations we choose

$$\mathcal{S}^{-1} \tilde{\mathcal{V}}_a \mathcal{S} = \tilde{\mathcal{U}}_b, \quad \mathcal{S}^{-1} \tilde{\mathcal{V}}_b \mathcal{S} = \tilde{\mathcal{U}}_a^{-1}, \quad \mathcal{S}^{-1} \tilde{\mathcal{U}}_a \mathcal{S} = \tilde{\mathcal{V}}_b, \quad \mathcal{S}^{-1} \tilde{\mathcal{U}}_b \mathcal{S} = \tilde{\mathcal{V}}_a^{-1}, \quad (\text{A.11})$$

$$\mathcal{T}^{-1} \tilde{\mathcal{V}}_a \mathcal{T} = \tilde{\mathcal{V}}_a, \quad \mathcal{T}^{-1} \tilde{\mathcal{V}}_b \mathcal{T} = -\tilde{\mathcal{V}}_b, \quad \mathcal{T}^{-1} \tilde{\mathcal{U}}_a \mathcal{T} = -\tilde{\mathcal{U}}_a \mathcal{V}_b^{-1}, \quad \mathcal{T}^{-1} \tilde{\mathcal{U}}_b \mathcal{T} = \tilde{\mathcal{U}}_b \mathcal{V}_a. \quad (\text{A.12})$$

Note the $(-)$ sign on the second and third equations of (A.12). We found that this phase assignment is necessary so that \mathcal{T} will commute with the orbifold action and keep invariant the subspace spanned by

$$\left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle = |\zeta_{0,0}\rangle, \quad \left| \begin{array}{c} \square \\ \odot \odot \end{array} \right\rangle = \frac{1}{\sqrt{3}}(|\zeta_{0,1}\rangle + |\zeta_{1,0}\rangle + |\zeta_{1,1}\rangle).$$

We then find that \mathcal{S} acts as:

$$\mathcal{S} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle = \frac{1}{2} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle + \frac{\sqrt{3}}{2} \left| \begin{array}{c} \square \\ \odot \odot \end{array} \right\rangle, \quad \mathcal{S} \left| \begin{array}{c} \square \\ \odot \odot \end{array} \right\rangle = \frac{\sqrt{3}}{2} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle - \frac{1}{2} \left| \begin{array}{c} \square \\ \odot \odot \end{array} \right\rangle, \quad (\text{A.13})$$

or, equivalently,

$$\begin{cases} \mathcal{S}|[0, 0, 0]\rangle &= \frac{1}{2}|[0, 0, 0]\rangle + \frac{\sqrt{3}}{2}|[\frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2} + \frac{1}{2}\tau]\rangle, \\ \mathcal{S}|[\frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2} + \frac{1}{2}\tau]\rangle &= \frac{\sqrt{3}}{2}|[0, 0, 0]\rangle - \frac{1}{2}|[\frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2} + \frac{1}{2}\tau]\rangle, \end{cases}$$

and \mathcal{T} acts as

$$\mathcal{T} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle = \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle, \quad \mathcal{T} \left| \begin{array}{c} \square \\ \odot \odot \end{array} \right\rangle = - \left| \begin{array}{c} \square \\ \odot \odot \end{array} \right\rangle. \quad (\text{A.14})$$

or, equivalently,

$$\mathcal{T}|[0, 0, 0]\rangle = |[0, 0, 0]\rangle, \quad \mathcal{T} \left| \left[\frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2} + \frac{1}{2}\tau \right] \right\rangle = - \left| \left[\frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2} + \frac{1}{2}\tau \right] \right\rangle.$$

For $n = 4$, \mathcal{S} acts (similarly to the $n = 2$ case) as:

$$\mathcal{S} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle = \frac{1}{\sqrt{3}} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle + \sqrt{\frac{2}{3}} \left| \begin{array}{c} \square \\ \odot \odot \end{array} \right\rangle, \quad \mathcal{S} \left| \begin{array}{c} \square \\ \odot \odot \end{array} \right\rangle = \sqrt{\frac{2}{3}} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle - \frac{1}{\sqrt{3}} \left| \begin{array}{c} \square \\ \odot \odot \end{array} \right\rangle, \quad (\text{A.15})$$

or, equivalently,

$$\begin{cases} \mathcal{S}|[0, 0, 0, 0]\rangle &= \frac{1}{\sqrt{3}}|[0, 0, 0, 0]\rangle + \sqrt{\frac{2}{3}} \left| \left[\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{2}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{2}{\sqrt{3}}e^{i\frac{\pi}{6}} \right] \right\rangle, \\ \mathcal{S} \left| \left[\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{2}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{2}{\sqrt{3}}e^{i\frac{\pi}{6}} \right] \right\rangle &= \sqrt{\frac{2}{3}}|[0, 0, 0, 0]\rangle - \frac{1}{\sqrt{3}} \left| \left[\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{2}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{2}{\sqrt{3}}e^{i\frac{\pi}{6}} \right] \right\rangle, \end{cases}$$

and \mathcal{T} acts as

$$\mathcal{T} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle = \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle, \quad \mathcal{T} \left| \begin{array}{c} \square \\ \odot \odot \end{array} \right\rangle = e^{\frac{2\pi i}{3}} \left| \begin{array}{c} \square \\ \odot \odot \end{array} \right\rangle, \quad (\text{A.16})$$

or, equivalently,

$$\begin{cases} \mathcal{T}|[0, 0, 0, 0]\rangle &= |[0, 0, 0, 0]\rangle, \\ \mathcal{T} \left| \left[\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{2}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{2}{\sqrt{3}}e^{i\frac{\pi}{6}} \right] \right\rangle &= e^{\frac{2\pi i}{3}} \left| \left[\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{2}{\sqrt{3}}e^{i\frac{\pi}{6}}, \frac{2}{\sqrt{3}}e^{i\frac{\pi}{6}} \right] \right\rangle. \end{cases}$$

For $n = 5$, \mathcal{S}, \mathcal{T} act as the identity:

$$\mathcal{S} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle = \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle, \quad \mathcal{T} \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle = \left| \begin{array}{c} \square \\ \odot \end{array} \right\rangle,$$

or, equivalently,

$$\mathcal{S}|[0, 0, 0, 0, 0]\rangle = |[0, 0, 0, 0, 0]\rangle, \quad \mathcal{T}|[0, 0, 0, 0, 0]\rangle = |[0, 0, 0, 0, 0]\rangle.$$

\mathcal{T} acts as

$$\mathcal{T} \left| \begin{array}{c} \text{blue parallelogram} \\ \text{red dot at bottom-left} \end{array} \right\rangle = \left| \begin{array}{c} \text{blue parallelogram} \\ \text{red dot at bottom-left} \end{array} \right\rangle, \quad \mathcal{T} \left| \begin{array}{c} \text{blue parallelogram} \\ \text{red dot at top-left} \end{array} \right\rangle = e^{\frac{2\pi i}{3}} \left| \begin{array}{c} \text{blue parallelogram} \\ \text{red dot at top-left} \end{array} \right\rangle, \quad \mathcal{T} \left| \begin{array}{c} \text{blue parallelogram} \\ \text{red dot at top-right} \end{array} \right\rangle = e^{\frac{2\pi i}{3}} \left| \begin{array}{c} \text{blue parallelogram} \\ \text{red dot at top-right} \end{array} \right\rangle, \quad (\text{A.20})$$

or, equivalently,

$$\left[\begin{array}{l} \mathcal{T}|[0, 0]\rangle = |[0, 0]\rangle, \\ \mathcal{T} \left| \left[\frac{1}{\sqrt{3}} e^{\frac{i\pi}{6}}, \frac{1}{\sqrt{3}} e^{\frac{i\pi}{6}} \right] \right\rangle = e^{\frac{2\pi i}{3}} \left| \left[\frac{1}{\sqrt{3}} e^{\frac{i\pi}{6}}, \frac{1}{\sqrt{3}} e^{\frac{i\pi}{6}} \right] \right\rangle, \\ \mathcal{T} \left| \left[\frac{2}{\sqrt{3}} e^{\frac{i\pi}{6}}, \frac{2}{\sqrt{3}} e^{\frac{i\pi}{6}} \right] \right\rangle = e^{\frac{2\pi i}{3}} \left| \left[\frac{2}{\sqrt{3}} e^{\frac{i\pi}{6}}, \frac{2}{\sqrt{3}} e^{\frac{i\pi}{6}} \right] \right\rangle. \end{array} \right.$$

B Action of $\text{SL}(2, \mathbb{Z})$ on Chern-Simons Hilbert spaces

The Hilbert space of $U(n) = [U(1) \times \text{SU}(n)]/\mathbb{Z}_n$ Chern-Simons theory at level k on T^2 , where $U(1)$ is at level kn and $\text{SU}(n)$ is at level k , is equivalent to the symmetric product of n copies of the Hilbert space of $U(1)_k$. We use this to extract the $\text{SL}(2, \mathbb{Z})$ representation of the $\text{SU}(n)_k$ Hilbert space.

We can write the states of $U(1)_k$ as $|p\rangle$ with $p = 0, \dots, k-1$, and the states of the product of n copies as $|p_1, \dots, p_n\rangle$ with $0 \leq p_i \leq k-1$. We then decompose

$$\sum_{\sigma \in S_n} |p_{\sigma(1)}, \dots, p_{\sigma(n)}\rangle = \sum_{p=0}^{kn-1} |p_1, \dots, p_n; p\rangle_{\text{SU}(n)} |p\rangle_{U(1)}.$$

As we will soon see, only n out of the kn terms on the right-hand side are nonzero, and the normalization is

$$\langle p_1, \dots, p_n; p | p_1, \dots, p_n; p \rangle = \frac{1}{n} N_{p_1 \dots p_n},$$

where $N_{p_1 \dots p_n}$ is calculated as follows. For $0 \leq j < k$, let m_j be the number of indices i for which $p_i = j$. Then $\sum_{j=0}^{k-1} m_j = n$ and

$$N_{p_1 \dots p_n} = \frac{n!}{\prod_j m_j!}.$$

We also need to match the action of large $U(1)$ gauge transformations that reside entirely inside the $U(1)$ factor and do not affect the $\text{SU}(n)$ degrees of freedom. They form a $\mathbb{Z}_k \times \mathbb{Z}_k$ group, and act as

$$\begin{aligned} \sum_{\sigma \in S_n} |p_{\sigma(1)} + 1, \dots, p_{\sigma(n)} + 1\rangle &= \mathcal{K}_1 \sum_{\sigma \in S_n} |p_{\sigma(1)}, \dots, p_{\sigma(n)}\rangle \\ &= \sum_{p=0}^{kn-1} |p_1, \dots, p_n; p\rangle_{\text{SU}(n)} |p+n\rangle_{U(1)}, \end{aligned}$$

and

$$\begin{aligned} e^{-\frac{2\pi i}{k} \sum_i p_i} \sum_{\sigma \in S_n} |p_{\sigma(1)}, \dots, p_{\sigma(n)}\rangle &= \mathcal{K}_2 \sum_{\sigma \in S_n} |p_{\sigma(1)}, \dots, p_{\sigma(n)}\rangle \\ &= \sum_{p=0}^{kn-1} e^{-\frac{2\pi i}{k} p} |p_1, \dots, p_n; p\rangle_{\text{SU}(n)} |p\rangle_{U(1)}. \end{aligned}$$

So $|p_1, \dots, p_n; p\rangle$ is nonzero only if $p = \sum_i p_i \pmod{k}$, and we also get

$$|p_1, \dots, p_n; p - n\rangle = |p_1 + 1, \dots, p_n + 1; p\rangle. \quad (\text{B.1})$$

B.1 Action of \mathcal{T}

For even k we have

$$\begin{aligned} e^{\frac{i\pi}{k} \sum_i p_i^2} \sum_{\sigma \in S_n} |p_{\sigma(1)}, \dots, p_{\sigma(n)}\rangle &= \sum_{\sigma \in S_n} \mathcal{T} |p_{\sigma(1)}, \dots, p_{\sigma(n)}\rangle \\ &= \sum_{p=0}^{kn-1} e^{\frac{i\pi}{kn} p^2} \mathcal{T} |p_1, \dots, p_n; p\rangle_{\text{SU}(n)} |p\rangle_{\text{U}(1)}. \end{aligned}$$

So,

$$\mathcal{T} |p_1, \dots, p_n; p\rangle_{\text{SU}(n)} = e^{\frac{i\pi}{k} \left(\sum_{i=1}^n p_i^2 - \frac{1}{n} p^2 \right)} |p_1, \dots, p_n; p\rangle_{\text{SU}(n)}.$$

For odd k and any n we have

$$\mathcal{T} |p_1, \dots, p_n; p\rangle_{\text{SU}(n)} = (-1)^{p - \sum_{i=1}^n p_i} e^{\frac{i\pi}{k} \left(\sum_{i=1}^n p_i^2 - \frac{1}{n} p^2 \right)} |p_1, \dots, p_n; p\rangle_{\text{SU}(n)}, \quad (\text{B.2})$$

where we have used a freedom similar to (3.39) to add an extra factor of $(-1)^p$ so that for even k we have $p - \sum_{i=1}^n p_i \equiv 0 \pmod{k}$ and therefore $(-1)^{p - \sum_{i=1}^n p_i} = 1$. Note also that for odd n the extra $(-1)^p$ factor is necessary for consistency with (B.1).

B.2 Action of \mathcal{S}

For \mathcal{S} we have,

$$\begin{aligned} \frac{1}{k^{n/2}} \sum_{\sigma \in S_n} \sum_{q_1=0}^{k-1} \dots \sum_{q_n=0}^{k-1} e^{\frac{2\pi i}{k} \sum_{i=1}^n q_i p_{\sigma(i)}} |q_1, \dots, q_n\rangle &= \sum_{\sigma \in S_n} \mathcal{S} |p_{\sigma(1)}, \dots, p_{\sigma(n)}\rangle \\ &= \frac{1}{\sqrt{kn}} \sum_{q=0}^{kn-1} e^{\frac{2\pi i}{kn} pq} \mathcal{S} |p_1, \dots, p_n; p\rangle_{\text{SU}(n)} |q\rangle_{\text{U}(1)}. \end{aligned}$$

We now take the (partial) inner product of that state with $\frac{1}{\sqrt{kn}} \sum_{q=0}^{kn-1} e^{\frac{2\pi i}{kn} pq} |q\rangle_{\text{U}(1)}$, and after some algebra we get

$$\begin{aligned} \mathcal{S} |p_1, \dots, p_n; p\rangle_{\text{SU}(n)} &= \\ &= \frac{1}{\sqrt{k^{n+1}n}} \sum_{q_1=0}^{k-1} \dots \sum_{q_n=0}^{k-1} e^{\frac{2\pi i}{k} \sum_i q_i (p_i - \frac{1}{n} p)} \sum_{m=0}^{n-1} e^{-\frac{2\pi i}{n} pm} \left| q_1, \dots, q_n; mk + \sum_i q_i \right\rangle_{\text{SU}(n)}. \end{aligned}$$

B.3 Action of Z_n

The $\text{U}(n)$ states are invariant, so

$$\begin{aligned} \Omega_1'' |p_1, \dots, p_n; p\rangle_{\text{SU}(n)} &= |p_1, \dots, p_n; p + k\rangle_{\text{SU}(n)}, \\ \Omega_2'' |p_1, \dots, p_n; p\rangle_{\text{SU}(n)} &= e^{-\frac{2\pi i}{n} p} |p_1, \dots, p_n; p\rangle_{\text{SU}(n)}. \end{aligned}$$

B.4 Example: $[U(1)_2 \times SU(2)_{-3}]/\mathbb{Z}_2$

We have $SU(2)_{-3}$ states of the form

$$\left| p_1, p_2; 3m + \sum p_i \right\rangle_{SU(2)}, \quad 0 \leq p_1, p_2 < 3, \quad 0 \leq m < 2.$$

We add the $U(1)_2$ states to get states of the form

$$\left| p_1, p_2; 3m + \sum p_i \right\rangle_{SU(2)} |q\rangle_{U(1)}$$

with $0 \leq q < 2$. We then mod out by $\mathbb{Z}_2 \times \mathbb{Z}_2$ as follows. First,

$$q \equiv 3m + \sum_{i=1}^2 p_i \equiv m + \sum_{i=1}^2 p_i \pmod{2},$$

so, q is completely determined by m, p_1, p_2 . We therefore do not specify q any more. Next, we need to keep only the Ω_1'' -invariant combinations:

$$|p_1, p_2\rangle_s \equiv \sum_{m=0}^1 \left| p_1, p_2; 3m + \sum p_i \right\rangle_{SU(2)} \left| 3m + \sum p_i \right\rangle_{U(1)}.$$

In order for the space spanned by $|p_1, p_2\rangle_s$ to be closed under \mathcal{T} (and not just \mathcal{T}^2), we need to augment (B.2) by an extra factor of $(-1)^p$. After some algebra, we then get

$$\mathcal{T}|p_1, p_2\rangle_s = (-1)^{\sum p_i} e^{\frac{\pi i}{3} [2(\sum p_i)^2 - \sum p_i^2]} |p_1, p_2\rangle_s. \quad (\text{B.3})$$

These phases were used for identifying $\mathcal{H}_{(2)}(\frac{\pi}{3})$ in appendix C. In the notation of (C.1) we have

$$\begin{aligned} \left| \begin{array}{c} \text{blue parallelogram} \\ \text{red dot at bottom-left} \end{array} \right\rangle &= \sqrt{2}|0, 0\rangle_s = \sqrt{2}|1, 1\rangle_s = \sqrt{2}|2, 2\rangle_s, \\ \left| \begin{array}{c} \text{blue parallelogram} \\ \text{red dots at bottom-left and bottom-right} \end{array} \right\rangle &= |0, 1\rangle_s = |1, 2\rangle_s = |2, 0\rangle_s = |1, 0\rangle_s = |2, 1\rangle_s = |0, 2\rangle_s, \end{aligned}$$

and up to an overall phase (see the explanation at the beginning of appendix A), we find that (B.3) agrees with (A.10).

C Decomposition of $\mathcal{H}_{(n_1, \dots, n_p)}$ into Chern-Simons Hilbert spaces

C.1 $v = \frac{\pi}{3}$ ($k = 1$)

For $k = 1$, $n = 2$, we have

$$\mathcal{H}\left(2, \frac{\pi}{3}\right) = \mathcal{H}_{(1,1)}\left(\frac{\pi}{3}\right) \oplus \mathcal{H}_{(2)}\left(\frac{\pi}{3}\right).$$

The factor $\mathcal{H}_{(1,1)}$ was discussed in section 6.4, so it only remains to discuss $\mathcal{H}_{(2)}$. There are two states which we decompose according to (6.2):

$$\begin{aligned} \left| \begin{array}{c} \text{blue parallelogram} \\ \text{red dot at bottom-left} \end{array} \right\rangle &= |a\rangle_{SU(2)} \otimes |0\rangle_{U(1)} + |b\rangle_{SU(2)} \otimes |1\rangle_{U(1)}, \\ \left| \begin{array}{c} \text{blue parallelogram} \\ \text{red dots at bottom-left and bottom-right} \end{array} \right\rangle &= |d\rangle_{SU(2)} \otimes |0\rangle_{U(1)} + |c\rangle_{SU(2)} \otimes |1\rangle_{U(1)}, \end{aligned}$$

where the U(1) is at level $kn = 2$.

Up to an unimportant overall phase, we find

$$\mathcal{T}|a\rangle = |a\rangle, \quad \mathcal{T}|b\rangle = e^{-\frac{\pi i}{2}}|b\rangle, \quad \mathcal{T}|c\rangle = e^{\frac{5\pi i}{6}}|c\rangle, \quad \mathcal{T}|d\rangle = e^{-\frac{2\pi i}{3}}|d\rangle. \quad (\text{C.1})$$

The overall phase can be fixed (up to a cubic root of unity) by calculating, with the above assignments,

$$(\mathcal{T}\mathcal{S})^3 = e^{-\frac{\pi i}{4}}.$$

So multiplying \mathcal{T} by $e^{\frac{\pi i}{12}}$, for example, would fix the phase. We also have

$$\begin{aligned} \mathcal{S}|a\rangle &= \frac{1}{\sqrt{6}}|a\rangle + \frac{1}{\sqrt{6}}|b\rangle + \sqrt{\frac{1}{3}}|c\rangle + \sqrt{\frac{1}{3}}|d\rangle, \\ \mathcal{S}|b\rangle &= \frac{1}{\sqrt{6}}|a\rangle - \frac{1}{\sqrt{6}}|b\rangle - \sqrt{\frac{1}{3}}|c\rangle + \sqrt{\frac{1}{3}}|d\rangle, \\ \mathcal{S}|c\rangle &= \sqrt{\frac{1}{3}}|a\rangle - \sqrt{\frac{1}{3}}|b\rangle + \frac{1}{\sqrt{6}}|c\rangle - \frac{1}{\sqrt{6}}|d\rangle, \\ \mathcal{S}|d\rangle &= \sqrt{\frac{1}{3}}|a\rangle + \sqrt{\frac{1}{3}}|b\rangle - \frac{1}{\sqrt{6}}|c\rangle - \frac{1}{\sqrt{6}}|d\rangle, \end{aligned}$$

and

$$\begin{aligned} \Omega_1''|a\rangle &= |b\rangle, & \Omega_1''|b\rangle &= |a\rangle, & \Omega_1''|c\rangle &= |d\rangle, & \Omega_1''|d\rangle &= |c\rangle, \\ \Omega_2''|a\rangle &= |a\rangle, & \Omega_2''|b\rangle &= -|b\rangle, & \Omega_2''|c\rangle &= -|c\rangle, & \Omega_2''|d\rangle &= |d\rangle. \end{aligned}$$

These results agree with those of the SU(2) Chern-Simons theory at level $k = -3$ [see (6.29)]. So we get

$$\mathcal{H}_{(2)}\left(\frac{\pi}{3}\right) = \mathcal{H}([\text{U}(1)_2 \times \text{SU}(2)_{-3}]/\mathbb{Z}_2).$$

For $n = 3$, we have sectors corresponding to $[\sigma] = (1, 1, 1)$, $(2, 1)$, and (3) . The first was discussed in section 6.4, and the second is a reducible sector. We now discuss the third case.

We decompose the basis states into the U(1) and SU(3) degrees of freedom as follows:

$$\begin{aligned} \left| \begin{array}{c} \text{[diagram: blue parallelogram with one red dot]}\end{array} \right\rangle &= |a\rangle_{\text{SU}(3)} \otimes |0\rangle_{\text{U}(1)} + |b\rangle_{\text{SU}(3)} \otimes |1\rangle_{\text{U}(1)} + |c\rangle_{\text{SU}(3)} \otimes |2\rangle_{\text{U}(1)}, \\ \left| \begin{array}{c} \text{[diagram: blue parallelogram with two red dots]}\end{array} \right\rangle &= |d\rangle_{\text{SU}(3)} \otimes |0\rangle_{\text{U}(1)} + |e\rangle_{\text{SU}(3)} \otimes |1\rangle_{\text{U}(1)} + |f\rangle_{\text{SU}(3)} \otimes |2\rangle_{\text{U}(1)}. \end{aligned}$$

As usual, we extract the action of $\mathcal{T}, \mathcal{S}, \Omega_1'', \Omega_2''$ on SU(3) degrees of freedom by using the known results for U(1) theory at level $kn = 3$. We get

$$\begin{aligned} \mathcal{T}|a\rangle_{\text{SU}(3)} &= -|a\rangle_{\text{SU}(3)}, & \mathcal{T}|b\rangle_{\text{SU}(3)} &= -e^{\frac{2\pi i}{3}}|b\rangle_{\text{SU}(3)}, & \mathcal{T}|c\rangle_{\text{SU}(3)} &= -e^{\frac{2\pi i}{3}}|c\rangle_{\text{SU}(3)}, \\ \mathcal{T}|d\rangle_{\text{SU}(3)} &= |d\rangle_{\text{SU}(3)}, & \mathcal{T}|e\rangle_{\text{SU}(3)} &= e^{\frac{2\pi i}{3}}|e\rangle_{\text{SU}(3)}, & \mathcal{T}|f\rangle_{\text{SU}(3)} &= e^{\frac{2\pi i}{3}}|f\rangle_{\text{SU}(3)}, \end{aligned}$$

up to an overall phase, and

$$\mathcal{S} = \frac{1}{2} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 1 & 1 & 1 \\ \frac{1}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} & \frac{\omega^2}{\sqrt{3}} & 1 & \omega & \omega^2 \\ \frac{1}{\sqrt{3}} & \frac{\omega^2}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} & 1 & \omega^2 & \omega \\ 1 & 1 & 1 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 1 & \omega & \omega^2 & -\frac{1}{\sqrt{3}} & -\frac{\omega}{\sqrt{3}} & -\frac{\omega^2}{\sqrt{3}} \\ 1 & \omega^2 & \omega & -\frac{1}{\sqrt{3}} & -\frac{\omega^2}{\sqrt{3}} & -\frac{\omega}{\sqrt{3}} \end{pmatrix},$$

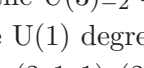
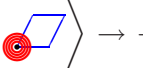
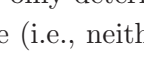

where $\omega = e^{\frac{2\pi i}{3}}$, in the basis $|a\rangle_{\text{SU}(3)}, \dots, |f\rangle_{\text{SU}(3)}$. We also have

$$\begin{aligned} \Omega_1'' |a\rangle_{\text{SU}(3)} &= |b\rangle_{\text{SU}(3)}, & \Omega_1'' |b\rangle_{\text{SU}(3)} &= |c\rangle_{\text{SU}(3)}, & \Omega_1'' |c\rangle_{\text{SU}(3)} &= |a\rangle_{\text{SU}(3)}, \\ \Omega_1'' |d\rangle_{\text{SU}(3)} &= |e\rangle_{\text{SU}(3)}, & \Omega_1'' |e\rangle_{\text{SU}(3)} &= |f\rangle_{\text{SU}(3)}, & \Omega_1'' |f\rangle_{\text{SU}(3)} &= |d\rangle_{\text{SU}(3)}, \end{aligned}$$

and

$$\begin{aligned} \Omega_2'' |a\rangle_{\text{SU}(3)} &= |a\rangle_{\text{SU}(3)}, & \Omega_2'' |b\rangle_{\text{SU}(3)} &= e^{\frac{2\pi i}{3}} |b\rangle_{\text{SU}(3)}, & \Omega_2'' |c\rangle_{\text{SU}(3)} &= e^{-\frac{2\pi i}{3}} |c\rangle_{\text{SU}(3)}, \\ \Omega_2'' |d\rangle_{\text{SU}(3)} &= |d\rangle_{\text{SU}(3)}, & \Omega_2'' |e\rangle_{\text{SU}(3)} &= e^{\frac{2\pi i}{3}} |e\rangle_{\text{SU}(3)}, & \Omega_2'' |f\rangle_{\text{SU}(3)} &= e^{-\frac{2\pi i}{3}} |f\rangle_{\text{SU}(3)}. \end{aligned}$$

The results agree with those of $\text{SU}(3)$ Chern-Simons theory at $k = -2$. The latter can be checked, for example, by studying the $\text{U}(3)_{-2} = [\text{U}(1)_{-6} \times \text{SU}(3)_{-2}]/\mathbb{Z}_3$ Chern-Simons theory, using the known results for the $\text{U}(1)$ degrees of freedom and (6.16).

For $n = 4$, we have $[\sigma] = (1, 1, 1, 1), (2, 1, 1), (2, 2), (3, 1)$, and (4) sectors. The last case is equivalent as representation of $\text{SL}(2, \mathbb{Z})$ to the $n = 2, [\sigma] = (2)$ case [see (A.15)–(A.16)]. To see this, one has to change basis ( \rightarrow ,  \rightarrow , and recall that \mathcal{T}, \mathcal{S} as appear in appendix A are only determined up to an overall phase.)

The only remaining nontrivial case (i.e., neither untwisted nor reducible) is the $[\sigma] = (2, 2)$ sector. We can write $\mathcal{H}_{(2,2)}(\frac{\pi}{3})$ as a symmetric product $\mathcal{H}_{(2,2)}(\frac{\pi}{3}) \simeq \mathcal{H}_{(2)}(\frac{\pi}{3})^{\otimes 2}/S_2$, and using the result $\mathcal{H}_{(2)}(\frac{\pi}{3}) \simeq \text{U}(2)_{2,-3}$ from above, we can write $\mathcal{H}_{(2,2)}(\frac{\pi}{3})$ as the symmetric product of Chern-Simons Hilbert spaces:

$$\mathcal{H}_{(2,2)}\left(\frac{\pi}{3}\right) \simeq \text{U}(2)_{2,-3}^{\otimes 2}/S_2.$$

This, however, is not good enough for our purposes, because we would like to present each sector as the Hilbert space of a gauge theory, and $\text{U}(2)^{\otimes 2}/S_2$ is not a group.

The dimension of $\mathcal{H}_{(2,2)}(\frac{\pi}{3})$ is 3, so if we attempt to write it as $[\text{U}(1)_4 \times \tilde{\mathcal{H}}_{(2,2)}(\frac{\pi}{3})]/\mathbb{Z}_4$ we find that we need $\dim \tilde{\mathcal{H}}_{(2,2)}(\frac{\pi}{3}) = 12$. The dimension of the Hilbert space of $\text{SU}(n')_{k'}$ is $(n' + k' - 1)!/k'!(n' - 1)!$, so if we assume $n' \leq n$ we find only $(n' = 2, k' = 11)$ and $(n' = 12, k' = 1)$, but these are easily ruled out. We have also explored product gauge groups such as $\text{U}(2)_{2k',k'} \times \text{U}(2)_{2k'',k''}$ with $k' + k'' = 2$, to no avail.

Perhaps we can obtain a clue to the solution by noting that the symmetric product of n' copies of $\text{SU}(2)_{k'}$ is equivalent to the Hilbert space of a symplectic group,

$$\mathcal{H}[\text{SU}(2)_{k'}]^{\otimes n'}/S_{n'} \simeq \mathcal{H}[\text{Sp}(n')_{k'}], \quad (\text{C.2})$$

as can easily be verified by writing out the explicit wavefunctions, using the character formulas for affine Lie algebras [42] (see also [73] for a historical review and references). Another observation is that $\mathcal{H}_{(2,2)}(\frac{\pi}{3})$ is equivalent as a representation of only \mathcal{T}^2 and \mathcal{S} to the Hilbert space of $\text{Sp}(2)/\mathbb{Z}_2 \simeq \text{SO}(5)$ Chern-Simons theory at level 3. To see this, take $n' = 2$ and $k' = 3$ in (C.2). The Hilbert space for $\text{SU}(2)$ Chern-Simons theory at level 3 was discussed at the end of section 6.3 — in particular, its dimension is 4, and the \mathbb{Z}_2 large gauge transformation acts as (6.30). We can therefore obtain the Hilbert space of $\text{Sp}(2)/\mathbb{Z}_2$ Chern-Simons theory by first taking the symmetric product of two copies of $\mathcal{H}[\text{SU}(2)_3]$, and then requiring invariance under (6.30).

The result is that it is a three-dimensional space spanned by

$$|a, a\rangle + |b, b\rangle, \quad |a, d\rangle + |b, c\rangle, \quad |c, c\rangle + |d, d\rangle,$$

where, for example,

$$\begin{aligned} |a, a\rangle &\equiv |a\rangle_{\text{SU}(2)} \otimes |a\rangle_{\text{SU}(2)}, \\ |a, d\rangle &\equiv \frac{1}{\sqrt{2}}(|a\rangle_{\text{SU}(2)} \otimes |d\rangle_{\text{SU}(2)} + |d\rangle_{\text{SU}(2)} \otimes |a\rangle_{\text{SU}(2)}), \end{aligned}$$

with $|a\rangle, |b\rangle, |c\rangle, |d\rangle$ as defined in section 6.3. Other states like $|b, b\rangle$ and $|b, c\rangle$ are defined similarly. We can also read off the action of \mathcal{T}^2 and \mathcal{S} in this basis:

$$\mathcal{T}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{4\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}, \quad \mathcal{S} = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}.$$

This matches exactly the action of \mathcal{T}^2 and \mathcal{S} on $\mathcal{H}_{(2,2)}(\frac{\pi}{3})$, which can be found from (A.9) and (A.10).

There is a caveat in this discussion, however, in that we only checked the action of \mathcal{T}^2 , not \mathcal{T} . The latter is actually not well-defined in the $\text{SO}(5)$ Chern-Simons theory Hilbert space, because it does not commute with Ω_2'' of (6.30). In other words, $\mathcal{H}_{(2,2)}(\frac{\pi}{3})$ is equivalent to the Hilbert space of $\text{SO}(5)$ Chern-Simons theory, not as a representation of the full $\text{SL}(2, \mathbb{Z})$, but as a representation of its subgroup $\Gamma(2)$. The situation is reminiscent of $\text{U}(1)$ Chern-Simons theory at an odd level k , discussed in section 5.4, where the theory depends on the choice of spin structure of T^2 . Another problem with identifying the $[\sigma] = (2, 2)$ sector with $\text{Sp}(2)/\mathbb{Z}_2$ Chern-Simons theory is that it is not a subgroup of our gauge group $\text{U}(4)$, only its double-cover $\text{Sp}(2)$ is. At this point, therefore, we are not making any claims about the sector $\mathcal{H}_{(2,2)}(\frac{\pi}{3})$.

For $n = 5$, we have $[\sigma] = (1, 1, 1, 1, 1), (2, 1, 1, 1), (2, 2, 1), (3, 1, 1), (3, 2), (4, 1)$, and (5). All sectors are either untwisted or reducible, except for the last one. But the (5) sector is a trivial one-dimensional Hilbert space, so we may set

$$\mathcal{H}_{(5)}\left(\frac{\pi}{3}\right) = \mathcal{H}[\text{U}(5)_{5,1}].$$

C.2 $v = \frac{2\pi}{3}$ ($k = 3$)

For $n = 2$, we have

$$\mathcal{H}\left(2, \frac{2\pi}{3}\right) = \mathcal{H}_{(1,1)}\left(\frac{2\pi}{3}\right) \oplus \mathcal{H}_{(2)}\left(\frac{2\pi}{3}\right).$$

The factor $\mathcal{H}_{(1,1)}$ was discussed in section 6.4, so it only remains to discuss $\mathcal{H}_{(2)}$. There are three states which we decompose according to (6.2):

$$\begin{aligned} \left| \begin{array}{c} \text{[diagram: blue parallelogram with red dot at top-left]} \\ \text{[diagram: blue parallelogram with red dot at top-right]} \\ \text{[diagram: blue parallelogram with red dot at center]} \end{array} \right\rangle &= |a\rangle_{\text{SU}(2)} \otimes |0\rangle_{\text{U}(1)} + |b\rangle_{\text{SU}(2)} \otimes |3\rangle_{\text{U}(1)}, \\ \left| \begin{array}{c} \text{[diagram: blue parallelogram with red dot at top-left]} \\ \text{[diagram: blue parallelogram with red dot at top-right]} \\ \text{[diagram: blue parallelogram with red dot at center]} \end{array} \right\rangle &= |a\rangle_{\text{SU}(2)} \otimes |2\rangle_{\text{U}(1)} + |b\rangle_{\text{SU}(2)} \otimes |5\rangle_{\text{U}(1)}, \\ \left| \begin{array}{c} \text{[diagram: blue parallelogram with red dot at top-left]} \\ \text{[diagram: blue parallelogram with red dot at top-right]} \\ \text{[diagram: blue parallelogram with red dot at center]} \end{array} \right\rangle &= |a\rangle_{\text{SU}(2)} \otimes |4\rangle_{\text{U}(1)} + |b\rangle_{\text{SU}(2)} \otimes |1\rangle_{\text{U}(1)}, \end{aligned}$$

where the U(1) is at level $kn = 6$. Following the usual procedure, we get

$$\mathcal{S}|a\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle), \quad \mathcal{S}|b\rangle = \frac{1}{\sqrt{2}}(|a\rangle - |b\rangle),$$

and, up to a phase,

$$\mathcal{T}|a\rangle = |a\rangle, \quad \mathcal{T}|b\rangle = e^{\frac{\pi i}{2}}|b\rangle.$$

We also have

$$\Omega_1''|a\rangle = |a\rangle, \quad \Omega_1''|b\rangle = -|b\rangle, \quad \Omega_2''|a\rangle = |b\rangle, \quad \Omega_2''|b\rangle = |a\rangle.$$

These relations agree with the states of $\text{SU}(2)_{-1}$, so

$$\mathcal{H}_{(2)}\left(\frac{2\pi}{3}\right) = \mathcal{H}([\text{U}(1)_6 \times \text{SU}(2)_{-1}]/\mathbb{Z}_2).$$

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References

- [1] D. Gaiotto and E. Witten, *S-duality of boundary conditions in $N = 4$ super Yang-Mills theory*, [arXiv:0807.3720](#) [[SPIRES](#)].
- [2] D. Gaiotto and E. Witten, *Supersymmetric boundary conditions in $N = 4$ super Yang-Mills theory*, [arXiv:0804.2902](#) [[SPIRES](#)].
- [3] C. Vafa, *Evidence for F-theory*, *Nucl. Phys. B* **469** (1996) 403 [[hep-th/9602022](#)] [[SPIRES](#)].
- [4] A. Dabholkar and J.A. Harvey, *String islands*, *JHEP* **02** (1999) 006 [[hep-th/9809122](#)] [[SPIRES](#)].
- [5] O.J. Ganor, *U-duality twists and possible phase transitions in $(2 + 1)D$ supergravity*, *Nucl. Phys. B* **549** (1999) 145 [[hep-th/9812024](#)] [[SPIRES](#)].

- [6] A. Dabholkar and C. Hull, *Duality twists, orbifolds and fluxes*, *JHEP* **09** (2003) 054 [[hep-th/0210209](#)] [[SPIRES](#)].
- [7] J. Shelton, W. Taylor and B. Wecht, *Nongeometric flux compactifications*, *JHEP* **10** (2005) 085 [[hep-th/0508133](#)] [[SPIRES](#)].
- [8] D. Vegh and J. McGreevy, *Semi-flatland*, *JHEP* **10** (2008) 068 [[arXiv:0808.1569](#)] [[SPIRES](#)].
- [9] S. Hellerman and J. Walcher, *Worldsheet CFTs for flat monodrofolds*, [hep-th/0604191](#) [[SPIRES](#)].
- [10] E. Witten, *On S duality in Abelian gauge theory*, *Selecta Math.* **1** (1995) 383 [[hep-th/9505186](#)] [[SPIRES](#)].
- [11] O.J. Ganor and Y.P. Hong, *Selfduality and Chern-Simons theory*, [arXiv:0812.1213](#) [[SPIRES](#)].
- [12] E. Witten, *Bound states of strings and p-branes*, *Nucl. Phys. B* **460** (1996) 335 [[hep-th/9510135](#)] [[SPIRES](#)].
- [13] C.M. Hull and P.K. Townsend, *Unity of superstring dualities*, *Nucl. Phys. B* **438** (1995) 109 [[hep-th/9410167](#)] [[SPIRES](#)].
- [14] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys. B* **443** (1995) 85 [[hep-th/9503124](#)] [[SPIRES](#)].
- [15] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, [hep-th/0604151](#) [[SPIRES](#)].
- [16] E. Witten, *On the Landau-Ginzburg description of $N = 2$ minimal models*, *Int. J. Mod. Phys. A* **9** (1994) 4783 [[hep-th/9304026](#)] [[SPIRES](#)].
- [17] Y.-K.E. Cheung, O.J. Ganor and M. Krogh, *On the twisted (2,0) and little-string theories*, *Nucl. Phys. B* **536** (1998) 175 [[hep-th/9805045](#)] [[SPIRES](#)].
- [18] Y.-K.E. Cheung, O.J. Ganor, M. Krogh and A.Y. Mikhailov, *Instantons on a non-commutative T^4 from twisted (2,0) and little-string theories*, *Nucl. Phys. B* **564** (2000) 259 [[hep-th/9812172](#)] [[SPIRES](#)].
- [19] J.G. Russo and A.A. Tseytlin, *Magnetic flux tube models in superstring theory*, *Nucl. Phys. B* **461** (1996) 131 [[hep-th/9508068](#)] [[SPIRES](#)].
- [20] L.J. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds*, *Nucl. Phys. B* **261** (1985) 678 [[SPIRES](#)].
- [21] L.J. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds. 2*, *Nucl. Phys. B* **274** (1986) 285 [[SPIRES](#)].
- [22] M.R. Douglas, *Branes within branes*, [hep-th/9512077](#) [[SPIRES](#)].
- [23] G. 't Hooft, *On the phase transition towards permanent quark confinement*, *Nucl. Phys. B* **138** (1978) 1 [[SPIRES](#)].
- [24] G. 't Hooft, *A property of electric and magnetic flux in nonabelian gauge theories*, *Nucl. Phys. B* **153** (1979) 141 [[SPIRES](#)].
- [25] M. Henningson and N. Wyllard, *Low-energy spectrum of $N = 4$ super-Yang-Mills on T^3 : flat connections, bound states at threshold and S-duality*, *JHEP* **06** (2007) 001 [[hep-th/0703172](#)] [[SPIRES](#)].

- [26] M. Claudson and M.B. Halpern, *Supersymmetric ground state wave functions*, *Nucl. Phys. B* **250** (1985) 689 [SPIRES].
- [27] P. Yi, *Witten index and threshold bound states of D-branes*, *Nucl. Phys. B* **505** (1997) 307 [[hep-th/9704098](#)] [SPIRES].
- [28] S. Sethi and M. Stern, *D-brane bound states redux*, *Commun. Math. Phys.* **194** (1998) 675 [[hep-th/9705046](#)] [SPIRES].
- [29] Y. Lozano, *S duality in gauge theories as a canonical transformation*, *Phys. Lett. B* **364** (1995) 19 [[hep-th/9508021](#)] [SPIRES].
- [30] O.J. Ganor, *A note on zeroes of superpotentials in F-theory*, *Nucl. Phys. B* **499** (1997) 55 [[hep-th/9612077](#)] [SPIRES].
- [31] E.P. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, *Nucl. Phys. B* **300** (1988) 360 [SPIRES].
- [32] E. Witten, *Quantum field theory and the Jones polynomial*, *Commun. Math. Phys.* **121** (1989) 351 [SPIRES].
- [33] S. Elitzur, G.W. Moore, A. Schwimmer and N. Seiberg, *Remarks on the canonical quantization of the Chern-Simons-Witten theory*, *Nucl. Phys. B* **326** (1989) 108 [SPIRES].
- [34] M. Bos and V.P. Nair, *Coherent state quantization of Chern-Simons theory*, *Int. J. Mod. Phys. A* **5** (1990) 959 [SPIRES].
- [35] H. Murayama, *Explicit quantization of the Chern-Simons action*, *Z. Phys. C* **48** (1990) 79 [SPIRES].
- [36] J.M.F. Labastida and A.V. Ramallo, *Chern-Simons and conformal field theories*, *Nucl. Phys. Proc. Suppl.* **16** (1990) 594 [SPIRES].
- [37] J.M. Isidro, J.M.F. Labastida and A.V. Ramallo, *Coset constructions in Chern-Simons gauge theory*, *Phys. Lett. B* **282** (1992) 63 [[hep-th/9201027](#)] [SPIRES].
- [38] E. Witten, *SL(2, Z) action on three-dimensional conformal field theories with Abelian symmetry*, [[hep-th/0307041](#)] [SPIRES].
- [39] D. Belov and G.W. Moore, *Classification of Abelian spin Chern-Simons theories*, [[hep-th/0505235](#)] [SPIRES].
- [40] E. Witten, *Five-brane effective action in M-theory*, *J. Geom. Phys.* **22** (1997) 103 [[hep-th/9610234](#)] [SPIRES].
- [41] K. Bardakci and M.B. Halpern, *New dual quark models*, *Phys. Rev. D* **3** (1971) 2493 [SPIRES].
- [42] V.G. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press, Cambridge U.K. (1990) [SPIRES].
- [43] V.G. Kac and D.H. Peterson, *Infinite dimensional Lie algebras, theta functions and modular forms*, *Adv. Math.* **53** (1984) 125 [SPIRES].
- [44] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large-N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J. C* **22** (2001) 379 [[hep-th/9803001](#)] [SPIRES].
- [45] J.M. Maldacena, *Wilson loops in large-N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [[hep-th/9803002](#)] [SPIRES].

- [46] O.J. Ganor, Y.P. Hong and H.-S. Tan, in preparation.
- [47] J.M.F. Labastida, P.M. Llatas and A.V. Ramallo, *Knot operators in Chern-Simons gauge theory*, *Nucl. Phys. B* **348** (1991) 651 [SPIRES].
- [48] E. Witten, *Some comments on string dynamics*, [hep-th/9507121](#) [SPIRES].
- [49] O. Aharony, M. Berkooz, S. Kachru, N. Seiberg and E. Silverstein, *Matrix description of interacting theories in six dimensions*, *Adv. Theor. Math. Phys.* **1** (1998) 148 [[hep-th/9707079](#)] [SPIRES].
- [50] O. Aharony, M. Berkooz and N. Seiberg, *Light-cone description of $(2,0)$ superconformal theories in six dimensions*, *Adv. Theor. Math. Phys.* **2** (1998) 119 [[hep-th/9712117](#)] [SPIRES].
- [51] N. Arkani-Hamed, A.G. Cohen, D.B. Kaplan, A. Karch and L. Motl, *Deconstructing $(2,0)$ and little string theories*, *JHEP* **01** (2003) 083 [[hep-th/0110146](#)] [SPIRES].
- [52] O.J. Ganor, *Six-dimensional tensionless strings in the large- N limit*, *Nucl. Phys. B* **489** (1997) 95 [[hep-th/9605201](#)] [SPIRES].
- [53] O. Ganor and L. Motl, *Equations of the $(2,0)$ theory and knitted fivebranes*, *JHEP* **05** (1998) 009 [[hep-th/9803108](#)] [SPIRES].
- [54] P.-M. Ho, Y. Imamura, Y. Matsuo and S. Shiba, *M5-brane in three-form flux and multiple M2-branes*, *JHEP* **08** (2008) 014 [[arXiv:0805.2898](#)] [SPIRES].
- [55] S.-J. Rey and F. Sugino, *A nonperturbative proposal for nonabelian tensor gauge theory and dynamical quantum Yang-Baxter maps*, [arXiv:1002.4636](#) [SPIRES].
- [56] N. Lambert and C. Papageorgakis, *Nonabelian $(2,0)$ tensor multiplets and 3-algebras*, *JHEP* **08** (2010) 083 [[arXiv:1007.2982](#)] [SPIRES].
- [57] E. Witten, *New ‘gauge’ theories in six dimensions*, *JHEP* **01** (1998) 001 [*Adv. Theor. Math. Phys.* **2** (1998) 61] [[hep-th/9710065](#)] [SPIRES].
- [58] T. Kitao, K. Ohta and N. Ohta, *Three-dimensional gauge dynamics from brane configurations with (p,q) -fivebrane*, *Nucl. Phys. B* **539** (1999) 79 [[hep-th/9808111](#)] [SPIRES].
- [59] D. Gaiotto, *$N = 2$ dualities*, [arXiv:0904.2715](#) [SPIRES].
- [60] M. Henningson, *Commutation relations for surface operators in six-dimensional $(2,0)$ theory*, *JHEP* **03** (2001) 011 [[hep-th/0012070](#)] [SPIRES].
- [61] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *$N = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [[arXiv:0806.1218](#)] [SPIRES].
- [62] N.A. Nekrasov and S.L. Shatashvili, *Quantum integrability and supersymmetric vacua*, *Prog. Theor. Phys. Suppl.* **177** (2009) 105 [[arXiv:0901.4748](#)] [SPIRES].
- [63] E. Poppitz and M. Ünsal, *Index theorem for topological excitations on $R^3 \times S^1$ and Chern-Simons theory*, *JHEP* **03** (2009) 027 [[arXiv:0812.2085](#)] [SPIRES].
- [64] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [*Adv. Theor. Math. Phys.* **2** (1998) 231] [[hep-th/9711200](#)] [SPIRES].
- [65] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)] [SPIRES].

- [66] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[SPIRES](#)].
- [67] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415 [[hep-th/9811131](#)] [[SPIRES](#)].
- [68] C. Vafa and E. Witten, *A strong coupling test of S duality*, *Nucl. Phys. B* **431** (1994) 3 [[hep-th/9408074](#)] [[SPIRES](#)].
- [69] K. Lee, S. Lee and J.-H. Park, *Topological twisting of multiple M2-brane theory*, *JHEP* **11** (2008) 014 [[arXiv:0809.2924](#)] [[SPIRES](#)].
- [70] A. Kapustin and N. Saulina, *Chern-Simons-Rozansky-Witten topological field theory*, *Nucl. Phys. B* **823** (2009) 403 [[arXiv:0904.1447](#)] [[SPIRES](#)].
- [71] S. Gukov and J. Walcher, *Matrix factorizations and Kauffman homology*, [hep-th/0512298](#) [[SPIRES](#)].
- [72] D.S. Freed, *Remarks on Chern-Simons theory*, [arXiv:0808.2507](#) [[SPIRES](#)].
- [73] K. Bardakci and M.B. Halpern, *The dual quark models*, [arXiv:0907.2705](#) [[SPIRES](#)].