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Equivariant $\mathrm{U}(N)$ Verlinde algebra from Bethe/gauge correspondence

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Abstract: We compute the topological partition function (twisted index) of $\mathcal{N}=2$ $\mathrm{U}(N)$ Chern-Simons theory with an adjoint chiral multiplet on $\Sigma_g \times S^1$. The localization technique shows that the underlying Frobenius algebra is the equivariant Verlinde algebra which is obtained from the canonical quantization of the complex Chern-Simons theory regularized by U(1) equivariant parameter t. Our computation relies on a Bethe/Gauge correspondence which allows us to represent the equivariant Verlinde algebra in terms of the Hall-Littlewood polynomials $P_\lambda(x_B, t)$ with a specialization by Bethe roots x_B of the q-boson model. We confirm a proposed duality to the Coulomb branch limit of the lens space superconformal index of four dimensional $\mathcal{N}=2$ theories for SU(2) and SU(3) with lower levels. In SU(2) case we also present more direct computation based on Jeffrey-Kirwan residue operation.

Keywords: Topological Field Theories, Bethe Ansatz, Chern-Simons Theories

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Contents

1 Introduction

Localization method in the computation of supersymmetric (SUSY) gauge theories ([1] and references therein) gives exact results which are useful for confirming non-perturbative dualities and correspondence among the partition functions and correlation functions. It also reveals a relation to the integrable system such as Bethe/Gauge correspondence, which was first observed in [2] and later developed in [3–6]. A kind of "mother" theory of non-trivial dualities involving low dimensional SUSY theories is the six dimensional $\mathcal{N}=(2,0)$ superconformal field theory that arises as a low energy effective world-volume theory of $M5$ branes. Namely we can make use of the fact that twisted[1] compactification of the

[1]To keep supersymmetry in lower dimensions, it is often necessary to make a twist along the curved compactification manifold. This twist should be distinguished from another twist which is used to obtain topological theory from the resulting supersymmetric theory.

six dimensional theory on a manifold M_n of dimension n gives a SUSY gauge theory in $(6-n)$ dimensions, which is a source of otherwise unexpected correspondence between the field theory on M_n and the SUSY gauge theory. One of the most intriguing examples is AGT(W) correspondence [7, 8], where twisted compactification on a punctured Riemann surface $\Sigma_{g,n}$ gives four dimensional $\mathcal{N}=2$ superconformal theories of class $\mathcal{S}$ and the instanton partition function of the class $\mathcal{S}$ theory computes conformal blocks on $\Sigma_{g,n}$. The theory of our interest in this paper is the compactification on a three manifold M_3 denoted as $T[M_3]$. With an appropriate twisting we can keep $\mathcal{N}=2$ supersymmetry and the theory $T[M_3]$ gives supersymmetric theory on complementary 3-manifold $\widetilde{M}_3$. In spite of interesting proposal of so-called 3d-3d correspondence [9–12], the theory $T[M_3]$ for general three manifold is only partly explored.

However, if M_3 is a Seifert manifold which is an S^1 bundle over a Riemann surface Σ, we have good chances for getting detailed information about $T[M_3]$, since S^1-compactification of $\mathcal{N}=(2,0)$ theory gives 5 dimensional super Yang-Mills theory which is relatively tractable. For example, it is known that complex Chern-Simons theory (Chern-Simons theory with a complexified gauge group $G_{\mathbb{C}}$) is obtained by a compactification of 6d theory on the (squashed) lens space $L(\kappa,1)_b$ [13–17]. The Chern-Simons theory with a compact gauge group G is a renowned example of topological quantum field theory (TQFT) [18]. After the canonical quantization on $\widetilde{M}_3=\Sigma\times S^1$ with the periodic time along S^1 which gives the trace, the partition function gives a two dimensional TQFT on Σ that counts the dimensions of physical Hilbert space due to the vanishing TQFT Hamiltonian. If one introduces the Wilson loop operators along the time direction, they create punctures on Σ and the Hilbert space is mathematically identified with the space of conformal blocks of WZNW model. Hence, the Verlinde algebra or the fusion ring of the current algebra [19–22] underlies the two dimensional TQFT from the Chern-Simons theory. Quite similarly complex Chern-Simons theory is also 3 dimensional TQFT and the quantization on $\Sigma\times S^1$ gives a 2d TQFT [23, 24]. Furthermore, in [25] by considering six dimensional $\mathcal{N}=(2,0)$ superconformal field theory on $L(\kappa,1)\times\Sigma\times S^1$, it is proposed that 2d TQFT obtained from complex Chern-Simons theory or $\mathcal{N}=2$ Chern-Simons theory with an adjoint chiral multiplet on $\Sigma\times S^1$ has a dual description in terms of the Coulomb branch limit of the superconformal index on the lens space $L(\kappa,1)$. When $G_{\mathbb{C}}=\mathrm{SL}(N,\mathbb{C})$, the superconformal index of our concern is associated with the class $\mathcal{S}$ theory of A_{N-1} type obtained by the compactification on Σ. The family of these indices defines 2d TQFT [26, 27] and according to the general principles of 2d TQFT, the Coulomb branch limit of the lens space index can be evaluated by gluing those associated with three punctured sphere[2] $\Sigma_{0,3}$. In this paper we check this proposal by explicitly computing the genus g partition functions on the Chern-Simons theory side.

The powerful localization technique for $\mathcal{N}=2$ SUSY theory on Seifert manifold was first worked out in [28] and further elaborated by [29–31]. When the manifold is $\Sigma\times S^1$, we can make the partition function localized on discrete SUSY vacua (critical points of superpotential) which coincide with the solutions (Bethe roots) to the Bethe ansatz equation

[2]The A_{N-1} type class $\mathcal{S}$ theory associated with $\Sigma_{0,3}$ is called T_N theory. Except for $N=2$ it does not allow Lagrangian description in general.

of integrable lattice models called phase model and q-boson model [32–34]. By supersymmetric localization the path integral of topologically twisted Chern-Simons-matter theory reduces to infinite magnetic sum and multi-contour residue integrals called Jeffrey-Kirwan (JK) residues [35]. One might hope that one can check the duality to the superconformal index of the class $\mathcal{S}$ theory by summing up infinitely many JK residues in the localization formula. Unfortunately, there is a crucial subtlety in evaluating the partition function (topologically twisted index)[3] on higher genus Riemann surfaces. When the genus is larger than one and the gauge group is non-abelian, the projective condition of the JK residues is violated due to the one-loop determinant of vector multiplet. Hence, JK residues are ill-defined and naive residue operation does not reproduce lens space index. Therefore we need an alternative method to evaluate the topological partition function on higher genus surfaces. In this paper, we employ 2d TQFT viewpoint and quantum integrable structure behind Chern-Simons-matter theory a.k.a Bethe/Gauge correspondence. Since the Chern-Simons-matter theories are topologically twisted along Riemann surface, we expect that the Chern-Simons-matter theories possess the structure of 2d TQFT. As we summarized in appendix A, in 2d TQFT the partition function on higher genus Riemann surface is reconstructed by genus zero correlation functions. Although JK residues for genus zero case are well-defined, it is technically difficult to evaluate infinitely many JK residues in practice, for example SU(3) theories.

If the magnetic sum is performed before the JK residue operation and integration contours are deformed to enclose saddle points of effective twisted superpotential, the topological partition function and correlation functions are given by finite summations over solutions of the saddle point equations. But it is still hard to evaluate them, because it is usually impossible to solve the saddle point equations explicitly. To overcome such a difficulty, in this paper we use a combinatorial algorithm to evaluate correlation functions without knowing explicit form of solutions. This algorithm features the Hall-Littlewood polynomials $P_\lambda(x,t)$ that arise naturally from the algebraic Bethe ansatz of the q-boson model [36]. In $\mathrm{U}(N)$ theories, the saddle point equations agree with Bethe ansatz for N particle sector of q-boson model, while the number of sites corresponds to the level κ of the Chern-Simons theory. Thus, the Bethe/Gauge correspondence helps us to compute the partition function of 2d TQFT on Chern-Simons theory side, whose algebraic structure (the deformed Verlinde algebra) is related to the algebra of Hall-Littlewood polynomials on the space of Bethe roots. We emphasize that it is $\mathrm{U}(N)$ Chern-Simons theories that are related to the q-boson model, or the Hall-Littlewood polynomials. However, once the topological partition function and correlation functions of $\mathrm{U}(N) = (\mathrm{U}(1) \times \mathrm{SU}(N))/\mathbb{Z}_N$ theories are given, those of $\mathrm{SU}(N)$ theories are obtained by decomposing $\mathrm{U}(N)$ theory to $\mathrm{U}(1)$ part and $\mathrm{SU}(N)$ part. We show that the twisted indices of $\mathrm{SU}(N)$ theories reproduce Coulomb branch limit of lens space index for SU(2) and SU(3) with lower levels, which confirms the proposal in [25]. We also provide a result for level 2 U(4) theory, but there is no corresponding computation on the superconformal index side at the moment.

[3]In this paper we call topologically twisted index of Chern-Simons-matter theory simply (topological) partition function.

JHEP02(2019)097

1.1 Complex Chern-Simons theory and $\mathcal{N} = 2$ Chern-Simons theory with adjoint matter

Compactification of 6d theory on the (squashed) lens space

$$L(\kappa,1)_b := \{(z,w) \in \mathbb{C}^2; b^2|z|^2 + b^{-2}|w|^2 = 1\}/\mathbb{Z}_\kappa, \tag{1.1}$$

has been shown to give a complex Chern-Simons theory[4] [14–17]. The orbifold action in (1.1) is defined by $(z,w) \mapsto (e^{2\pi i/\kappa}z, e^{-2\pi i/\kappa}w)$. The action of complex Chern-Simons theory is

$$\mathcal{S} = \frac{q}{8\pi}\int_{M_3} \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) + \frac{\bar{q}}{8\pi}\int_{M_3} \mathrm{Tr}\left(\bar{\mathcal{A}}\wedge d\bar{\mathcal{A}} + \frac{2}{3}\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\right), \tag{1.2}$$

where $\mathcal{A} = \mathbf{A} + i\Phi$ is a complex gauge field and $q = \kappa + i\sigma$ is the complex coupling constant. For the invariance under the large gauge transformations the real part of the coupling κ has to be integer. Under the complex gauge transformation both $\mathbf{A}$ and Φ transform as a gauge field. But if the gauge transformation is restricted to be real, Φ transforms as a matter in the adjoint representation. Or the pair $(\mathbf{A}, \Phi)$ is regarded as coordinates on the cotangent bundle of the space of connections for the compact gauge group $\mathrm{U}(N)$ or $\mathrm{SU}(N)$. When we consider the compactification on $L(\kappa,1)_b$, the imaginary part of the Chern-Simons coupling is related to the squashing parameter by $\sigma = \kappa\frac{1-b^2}{1+b^2}$ [14, 15].

In this paper we only consider the case $b = 1$,[5] namely $\sigma = 0$. The action on $\Sigma \times S^1$ becomes

$$S^{\sigma=0} = \frac{\kappa}{4\pi}\int_{\Sigma\times S^1} \mathrm{Tr}\left(A\wedge D_0 A + 2A_0\wedge(dA + A\wedge A) - 2\phi_0\wedge d_A\phi - \phi\wedge D_0\phi\right), \tag{1.3}$$

where we have made a decomposition $\mathbf{A} = A + A_0 dx_0$, $\Phi = \phi + \phi_0 dx_0$ and x_0 is a coordinate along S^1. Since there are no time derivatives of A_0 and ϕ_0 we obtain

$$F_A - \phi\wedge\phi = 0, \qquad d_A\phi = 0 \tag{1.4}$$

as constraints for the Hilbert space of the canonical quantization of complex Chern-Simons theory. In fact (1.4) is the flatness condition of the total curvature $\mathcal{F} := d\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$ restricted on Σ. A crucial fact which connects the complex Chern-Simons theory and the twist of $\mathcal{N} = 2$ super Chern-Simons theory with an adjoint matter multiplet is the fact that

$$\mathcal{M}_H := \{F_A - \phi\wedge\phi = 0,\ \ d_A\phi = 0\}/\mathcal{G}_{\mathbb{C}} = \{F_A - \phi\wedge\phi = 0,\ \ d_A\phi = d_A^\dagger\phi = 0\}/\mathcal{G}, \tag{1.5}$$

which gives an equivalence of a holomorphic and a Hermitian description of the Hitchin moduli space.[6] In (1.5) $\mathcal{G}_{\mathbb{C}}$ is the group of complex gauge transformations, while $\mathcal{G}$ is that of real gauge transformations. It is the relations in the second description of the Hitchin

[4] For $G = \mathrm{U}(N), \mathrm{SU}(N)$ the complexified gauge group is $G_{\mathbb{C}} = \mathrm{GL}(N,\mathbb{C}), \mathrm{SL}(N,\mathbb{C})$. In this paper we only consider these cases.

[5] When $b = 1$ we denote the lens space simply by $L(\kappa, 1)$.

[6] Precisely speaking we have to impose some stability condition in the holomorphic description.

JHEP02(2019)097

moduli space that arise naturally as the equations of motion (topological gauge fixing conditions) in the topological twist of $\mathcal{N}=2$ Chern-Simons theory with an adjoint matter multiplet [23, 34]. In the second description we impose the additional condition $d_A^\dagger \phi = 0$ in compensation for the reduced gauge symmetry $\mathcal{G}$. An important role of this additional condition is that this allows us to introduce SO(2) rotation acting on the space components (ϕ_1, ϕ_2) of the one form ϕ [23]. In fact, the equation $d_A \phi = 0$ is invariant under the SO(2) rotation only when it is combined with the condition $d_A^\dagger \phi = 0$.

On the other hand the adjoint chiral multiplet $\phi = \phi_1 + i\phi_2$ in the $\mathcal{N}=2$ Chern-Simons theory is originally a complex scalar field with U(1) flavor symmetry. But the R-symmetry of the 3 dimensional $\mathcal{N}=2$ SUSY algebra is $\mathrm{U}(1)_R$ and there is a freedom of $\mathrm{U}(1)_R$ charge assignment r for ϕ. Since the topological twist of 3d theory on $\Sigma \times S^1$ is a redefinition of 2d local Lorentz symmetry $\mathrm{SO}(2)_\Sigma$ on Σ as the diagonal part of $\mathrm{U}(1)_R \times \mathrm{SO}(2)_\Sigma$, the adjoint matter ϕ has spin $r/2$ after the topological twist. In particular the R-charge has to be $r=2$ for matching with the complex Chern-Simons theory where ϕ is a one form. Thus the twisted $\mathcal{N}=2$ Chern-Simons theory with an adjoint matter ϕ with $\mathrm{U}(1)_R$ charge $r=2$ gives another description of the 2d TQFT that comes from the complex Chern-Simons theory. As remarked above, the complex Chern-Simons theory in the Hermitian description has SO(2) symmetry which rotates the one form components of ϕ. In $\mathcal{N}=2$ Chern-Simons theory this symmetry is nothing but the U(1) flavor symmetry of the adjoint chiral multiplet. As was proposed in [23], this U(1) symmetry can be used to regularize the problem of divergence due to the fact that the Hilbert space of complex Chern-Simons theory is infinite dimensional. Let us introduce the equivariant parameter $t := e^{-m}$ for the U(1) rotation, where the parameter m can be regarded as the mass for the adjoint matter.[7] The parameter t is physically regarded as a Wilson loop of a background gauge field of U(1) flavor symmetry. Then the corresponding 2d TQFT computes

$$Z(\Sigma_g) = \mathrm{Tr}_{\mathcal{H}} e^{-\beta H - mF} = \sum_{n=0}^{\infty} t^n \dim \mathcal{H}^{(n)}, \tag{1.6}$$

where F is the charge of flavor symmetry and $\mathcal{H}^{(n)}$ is the charge n sector of the physical Hilbert space. Since we have a smooth $t \to 0$ limit, which is the decoupling limit of the adjoint matter that gives the pure Chern-Simons theory, no negative powers of t appear. The underlying algebra of this 2d TQFT is called equivariant Verlinde algebra in [23]. As we will see in our computation based on Bethe/Gauge correspondence, the U(1) equivariant parameter t corresponds to the parameter of the Hall-Littlewood polynomial[8] $P_\lambda(x,t)$ where $t \to 0$ limit gives the Schur function $s_\lambda(x)$.

1.2 Coulomb branch limit of superconformal index

Now let us see the other side of 6d theory on $L(\kappa, 1) \times \Sigma \times S^1$. The superconformal index of four dimensional $\mathcal{N}=2$ theory is defined as the partition function on $S^3 \times S^1$, where we take

[7] In the Nekrasov partition function of five dimensional SUSY Yang-Mills theory with the adjoint hypermultiplet a similar equivariant parameter appears.

[8] See appendix B for a definition and basic properties of $P_\lambda(x,t)$.

the trace over S^1 direction regarded as time coordinate. When the $\mathcal{N}=2$ superconformal theory is of class $\mathcal{S}$, the superconformal indices give a 2d TQFT on the punctured Riemann surface $\Sigma_{g,n}$ associated with the class $\mathcal{S}$ theory [26, 27]. This is regarded as a TQFT version of AGT correspondence, where conformal blocks are replaced by topological correlation functions. As a 2d TQFT the basic ingredients are the indices for the superconformal theories coming from the genus zero surface with three punctures $\Sigma_{0,3}$, which are identified with the topological three point functions $C_{\mu\nu\lambda}$. The associativity condition for $C_{\mu\nu\lambda}$ is equivalent to the S-duality of the class $\mathcal{S}$ theories. One can also consider the index on the lens space by introducing the orbifold action on S^3 [37, 38]. In general the superconformal index has three fugacities $\mathfrak{p}, \mathfrak{q}$ and $\mathfrak{t}$ [39, 40]. There is a special limit called Coulomb branch limit which is defined by $\mathfrak{p}, \mathfrak{q}, \mathfrak{t} \to 0$ while $t := \mathfrak{p}\mathfrak{q}/\mathfrak{t}$ fixed [41]. According to the proposal in [25] the U(1) equivariant parameter t is identified with the equivariant U(1) parameter on the Chern-Simons side[9] The significant feature of the Coulomb branch limit is that the hypermultiplet does not contribute in the limit except the zero mode contributions. The superconformal theory obtained by twisted compactifications on $\Sigma_{0,3}$ of 6d $\mathcal{N}=(2,0)$ theory of type A_{N-1} is called T_N theory. When $N>2$ the theory does not allow the Lagrangian description and there is no weak coupling region. In [25] the computation of the superconformal indices for T_3 theory has been made by invoking the Argyres-Seiberg duality that allows a weak coupling region. Unfortunately this approach cannot be generalized to T_N theory for $3<N$. In this paper we compute the partition function of U(4) theory which is expected to match with the superconformal indices of T_4 theory.

1.3 Organization of the paper

This paper is organized as follows; in the next section we review localization formula for $\mathcal{N}=2$ Chern-Simons matter theories in general. The final result involves an infinite magnetic sum of the multi-contour integrals which are called Jeffrey-Kirwan (JK) residues. In section 3, we evaluate the integral of localization formula by the direct JK residue computation. For technical reason, the computation is possible for rank one case, namely SU(2) theory. We find a complete agreement with the result in [24] based on the geometry of SU(2) Hitchin system. Section 4 is the main part of the paper; we use Bethe/Gauge correspondence to compute the structure constants of the equivariant U(N) Verlinde algebra. The localization formula shows that the equivariant Verlinde algebra is realized by the algebra of Hall-Littlewood polynomials $P_\lambda(x,t)$ with the specialization by the Bethe roots of the q-boson model, where N corresponds to the number of excitations. Namely we substitute the solutions to the Bethe ansatz equation to the symmetric polynomial $P_\lambda(x,t)$. After the specialization there arise relations among $P_\lambda(x,t)$ which are related by the affine Weyl group of A_{N-1} acting on the partition λ [36]. We can understand these relations as a result of the quotient by the ideal $\mathcal{I}_{N,\kappa}$ determined by the space of Bethe roots. The characterization of the space by an ideal of the polynomial algebra is one of the basic ideas in algebraic geometry. In fact this is a generalization of what Gepner showed for the Verlinde algebra

[9] In the theory of the superconformal index there is a so-called Hall-Littlewood slice [41]. Though the Hall-Littlewood polynomials are featured in the present paper, this has nothing to do with the Hall-Littlewood slice.

(fusion ring) [20], where the ideal is generated by derivatives of a potential $W(x)$ [21, 22]. After taking the relation of $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ theories into account, we can confirm the agreement of our result with the superconformal indices of T_2 and T_3 theories. In section 5, we discuss several aspects of the equivariant $\mathrm{U}(N)$ Verlinde algebra, such as the recurrence relation among genus g partition functions and the level-rank duality. Finally backgrounds of 2d TQFT and the Hall-Littlewood polynomials are collected in appendices.

2 Localization of topologically twisted Chern-Simons-matter theories

In this section we consider topologically twisted Chern-Simons-matter (CS-matter) theories on $\Sigma_g \times S^1$. The R-symmetry of $\mathcal{N}=2$ supersymmetric theory in 3 dimensions is $\mathrm{U}(1)_R$ and the topological twist is made along Σ_g with local Lorentz symmetry $\mathrm{U}(1)_{\mathrm{spin}}$.[10] Namely we redefine the local Lorentz symmetry on Σ_g as the diagonal subgroup of $\mathrm{U}(1)_R \times \mathrm{U}(1)_{\mathrm{spin}}$. The observables of the CS-matter theories are supersymmetric Wilson loops W_λ. Here λ expresses a representation of the gauge group G and operators $\mathcal{O}_\lambda$'s in the correlation function are either supersymmetric Wilson loops or background flavor Wilson loops. Supersymmetric localization can be applied to correlation functions of W_λ wrapping on S^1 and located at a point of Σ_g. When the rank of G is N, the path integral reduces to N-dimensional contour integral (more precisely Jeffrey-Kirwan residue) by the localization formula [29–31];

$$\langle\prod_{i=1}^{n}\mathcal{O}_{\mu_i}\rangle_g = \frac{1}{|W(G)|}\oint_{\mathrm{JK}(\eta)}\prod_{a=1}^{N}\frac{dx_a}{2\pi i x_a}\sum_{\mathbf{k}\in\Gamma(G^\vee)}\left(\prod_{i=1}^{n}\mathcal{O}_{\mu_i}(x,t)\right)e^{-S_{cl}^{(\mathbf{k})}} \\ \times Z_{\mathrm{vec}}^{(\mathbf{k})}(x,g)Z_{\mathrm{chi}}^{(\mathbf{k})}(x,t,g,r)H(x,\kappa,t)^g, \tag{2.1}$$

where $|W(G)|$ is the order of the Weyl group of G. The integration variables x_a's are saddle point values of the Wilson loops associated to the a-th $\mathrm{U}(1)$ Cartan of G and the choice of the contour $\mathrm{JK}(\eta)$ is determined by an N-dimensional vector η. The set of vectors $\mathbf{k}\in\Gamma(G^\vee)$ represents an element of the magnetic lattice of G and the integrand comes from several multiplets in this susy model: $Z_{\mathrm{vec}}^{(\mathbf{k})}$ is the one-loop determinant of the super Yang-Mills fields with the magnetic charge $\mathbf{k}=(k_1,\cdots,k_N)$:

$$Z_{\mathrm{vec}}^{(\mathbf{k})}(x,g) = (-1)^{\sum_{\alpha>0}\alpha(\mathbf{k})}\prod_{\alpha\neq 0}(1-x^\alpha)^{1-g}\,, \tag{2.2}$$

where $\sum_{\alpha>0}$ and $\prod_{\alpha\neq0}$ express summation over the positive root vectors and product over the root vectors, respectively. x^α stands for a paring of the Cartan part of Wilson loop x and a root α. $Z_{\mathrm{chi}}^{(\mathbf{k})}(x,t,g,r)$ is the one-loop determinant of a chiral multiplet in a representation $\mathbf{R}$ of the Lie algebra of G:

$$Z_{\mathrm{chi}}^{(\mathbf{k})}(x,t,g,r) = \prod_{\rho\in\Delta(\mathbf{R})}\left(\frac{x^{\frac{\rho}{2}}t^{\frac{1}{2}}}{1-x^\rho t}\right)^{\rho(\mathbf{k})+(1-g)(1-r)}\,, \tag{2.3}$$

[10] Topological twist on general three manifold is more non-trivial, since the local Lorentz symmetry is $\mathrm{SU}(2)_{\mathrm{spin}}$.

where $\Delta(\mathbf{R})$ expresses the set of weight vectors of the representation $\mathbf{R}$ and r is R-charge for the lowest component scalar in the chiral multiplet. x^ρ stands for the paring of the Cartan part of Wilson loop x and a weight ρ. The parameter t originates in the background flavor Wilson loop. In general, one can introduce background U(1) flavor Wilson loops for the Cartan part of the flavor symmetry. Later we consider the adjoint representation, which has only U(1) flavor symmetry. The Q-closed action is a sum of the (mixed) Chern-Simons terms in three dimensions. Here Q is a generator of supersymmetric transformation used in the localization computation. Their saddle point values are written by x_a and t

$$e^{-S_{cl}^{(\mathbf{k})}} = \left(\prod_{a,b=1}^{N} x_a^{\kappa^{ab}k_b}\right) t^{\kappa_{(\mathrm{rf})}(g-1)} \dots \tag{2.4}$$

where $\kappa^{ab} := \kappa \mathrm{Tr}(H_a H_b)$ and $\{H_a\}_{a=1}^{N}$ represents the Cartan part of the Lie algebra of G in the Chevalley basis. κ and $\kappa_{(\mathrm{rf})}$ are respectively gauge CS level and mixed CS level between flavor symmetry and R-symmetry ((rf)-mixed CS level). The symbol "$\cdots$" stands for other mixed CS terms which are not included in the model we will treat in the following sections. We will also choose $\kappa_{(\mathrm{rf})} = \frac{N^2(1-r)}{2}$ for $G = \mathrm{U}(N)$ and $\kappa_{(\mathrm{rf})} = \frac{(N^2-1)(1-r)}{2}$ for $G = \mathrm{SU}(N)$ when we look at the relation with the Coulomb branch limit of lens space index. But it is easy to recover genus g partition function with generic value of $\kappa_{(\mathrm{rf})}$, because the (rf)-mixed CS term is independent of integration variables and magnetic charges. Finally $H(x,\kappa,t)$ is the Hessian of the effective twisted superpotential $W_{\mathrm{eff}}(x)$ which comes from integration over gaugino zero modes;

$$H(x,\kappa,t) := \det_{a,b}\left(\frac{(2\pi i)^2 \partial^2 W_{\mathrm{eff}}}{\partial \log x_a \partial \log x_b}\right) = \det_{a,b}\left(\kappa^{ab} + \sum_{\alpha\in\Delta(\mathbf{R})} \rho^a \rho^b \frac{1}{2}\left(\frac{1+tx^\rho}{1-tx^\rho}\right)\right), \tag{2.5}$$

with

$$\begin{aligned}(2\pi i)^2 W_{\mathrm{eff}}(x,\kappa,t) = &\frac{1}{2}\sum_{a,b=1}^{N} \kappa^{ab}(\log x_a)(\log x_b) - 2\pi^2 \sum_{\alpha>0}\alpha \\ &+ \sum_{\rho\in\Delta(\mathbf{R})}\left(\mathrm{Li}_2(x^\rho t) + \frac{1}{4}(\rho(\log x) + \log t)^2\right) + \cdots .\end{aligned} \tag{2.6}$$

In (2.6), ellipse "$\cdots$" stands for the gauge flavor mixed CS-term which is taken as zero in our calculation.

If the magnetic sum is performed before the evaluation of the integral and the contour is deformed to enclose saddle point configurations of the effective twisted superpotential

$e^{2\pi i \partial_{\log x_a} W_{\text{eff}}} = 1$, the correlation function is expressed as

$$
\begin{aligned}
\langle \prod_{i=1}^{n} \mathcal{O}_{\mu_i} \rangle_g &= \frac{1}{|W(G)|} \sum_{x_* \in \text{Sol}} \oint_{x=x_*} \prod_{a=1}^{N} \frac{dx_a}{2\pi i x_a} \left(\prod_{i=1}^{n} \mathcal{O}_{\mu_i}(x,t) \right) \\
&\quad \times \left(\prod_{a=1}^{N} \frac{1}{1 - e^{2\pi i \partial_{\log x_a} W_{\text{eff}}}} \right) e^{-S_{cl}^{(0)}} Z_{\text{vec}}^{(0)} Z_{\text{chi}}^{(0)} H^g && (2.7) \\
&= \sum_{x \in \text{Sol}} \left(\prod_{i=1}^{n} \mathcal{O}_{\mu_i} \right) e^{-S_{cl}^{(0)}} Z_{\text{vec}}^{(0)} Z_{\text{chi}}^{(0)} H^{g-1} \,. && (2.8)
\end{aligned}
$$

When the gauge group is non-Abelian, the summation $\sum_{x_* \in \text{Sol}}$ is taken over the roots of the saddle point equation of the twisted superpotential $e^{2\pi i \partial_{\log x_a} W_{\text{eff}}} = 1$ except for $x^\alpha = 1$ for any root α. If the roots x with $x^\alpha = 1$ are included in the residue operation, we find that the genus one partition function $\langle 1 \rangle_{g=1}$ from the expressions (2.7) and (2.8) does not reproduce the correct Witten index and also the higher genus partition functions $\langle 1 \rangle_{g \geq 2}$ do not agree with the results predicted from the Coulomb branch limit of lens space indices in our models. Thus we have to remove the roots satisfying $x^\alpha = 1$ and Sol is given by

$$
\text{Sol} := \left\{ x = (x_1, \cdots, x_N) \middle| e^{2\pi i \partial_{\log x_a} W_{\text{eff}}} = 1,\, a = 1, \cdots, N,\, x^\alpha \neq 1 \text{ for all the root } \alpha \right\} / \sim \tag{2.9}
$$

Here "$\sim$" means that we identify solutions which are equal up to the Weyl permutation. Since the theory is topologically twisted and does not depend on the metric on Riemann surfaces, we expect the correlation functions satisfy the axiom of 2d TQFT or equivalently the set of observables $\mathcal{O}_\lambda$'s gives a finite dimensional commutative Frobenius algebra. In appendix A we summarize properties of 2d TQFT used in this paper. Especially, the definitions of the structure constant $C_{\mu\nu}^{\lambda}$, the metric $\eta_{\mu\nu}$ and the handle operator $(H \cdot C)_\mu^\nu$ are given by (A.2), (A.3) and (A.9), respectively. Note that we assume that the reduction of the Chern-Simons-matter theory to 2 dimensions gives 2d TQFT and compute the partition functions and correlation functions in higher genus from genus zero two point and three point functions, for which we employ the localization formula. It is an interesting problem to check that the predictions based on 2d TQFT agree with the result of the direct computations of the localization formula.[11]

3 Direct (residue) computations in SU(2) case

Let us apply the localization formula in the last section to SU(2) CS-matter theory with an adjoint chiral multiplet. Since SU(2) is rank one, the residue evaluation and the saddle point are relatively simple. Unfortunately the direct computation in this section gets technically involved for higher rank gauge group. We can evaluate the genus g partition function by two methods: one is gluing the genus zero three point functions and the other is the direct residue evaluation of the higher genus partiton function in the summed form (2.7). Each

[11] A 2d TQFT which reproduces the localization computation of twisted CS-matter theory with an adjoint matter of R charge $r = 2$ is constructed in [24] based on the moduli space of the Higgs bundle.

method has its advantages and disadvantages. In the first method we make use of the properties of 2d TQFT and once we obtain the genus zero three point functions it is rather easy to compute the partition functions for any higher genus. However, the computation becomes quite involved for higher level κ, since the dimensions of the Frobenius algebra $\mathcal{A}$ increase with κ. On the other hand, in the second method we do not have to rely on 2d TQFT structure and there is no complication with higher κ mentioned above. But the higher genus computations are difficult in this case. Thus we can obtain the result for arbitrary level κ but only for lower genera.

The saddle point equation of the twisted superpotential is given in the SU(2) model

$$\exp\left(2\pi i \frac{\partial W_{\text{eff}}}{\partial \log x}\right) = x^{2\kappa+4}\left(\frac{1-tx^{-2}}{1-tx^{2}}\right)^{2} = 1. \tag{3.1}$$

First, we shall directly evaluate the residue in the resumed form (2.7). We can write down the genus g partition function of this $\mathrm{SU}(2)_\kappa$ theory with R-charge $r=2$

$$\langle\prod_{i=1}^{n}\mathcal{O}_{\mu_i}\rangle_g = \sum_{x_*\in\mathrm{Sol}}\oint_{x=x_*}\frac{dx}{2\pi i}\left(\prod_{i=1}^{n}\mathcal{O}_{\mu_i}(x,t)\right)\omega_g(x,t,\kappa)\,, \tag{3.2}$$

where

$$\begin{aligned}\omega_g(x,t,\kappa) := {} & \frac{1}{2x}\left(1-x^{2\kappa+4}\left(\frac{1-tx^{-2}}{1-tx^{2}}\right)^{2}\right)^{-1} \\ & \times\left[(1-t)\prod_{d=\pm1}(1-x^{2d})(1-tx^{2d})\right]^{1-g} H(x,\kappa,t)^g\,,\end{aligned} \tag{3.3}$$

with

$$H(x,\kappa,t) = 2\kappa + 2\frac{1+tx^2}{1-tx^2} + 2\frac{1+tx^{-2}}{1-tx^{-2}}. \tag{3.4}$$

The roots of the effective twisted superpotential except $x^\alpha = 1$ are collected into the set "Sol"

$$\mathrm{Sol} = \left\{x\middle| x^{2\kappa+4} - 2tx^{2\kappa+2} + t^2x^{2\kappa} - t^2x^4 + 2tx^2 - 1 = 0,\ x^2 \neq 1\right\}\,. \tag{3.5}$$

Then we find that $|\mathrm{Sol}|/|W(\mathrm{SU}(2))| = \kappa+1$ reproduces the correct Witten index for the SU(2) theory. The higher genus partition function is given by

$$Z_g^{\mathrm{SU}(2)_\kappa} = \sum_{x_*\in\mathrm{Sol}}\oint_{x=x_*}\omega_g(x,t,\kappa)\,. \tag{3.6}$$

Let us evaluate these higher genus partition functions. When $g\geq 2$, the poles of $\omega_g(x,t,\kappa)$ are located at $\{\pm1,\pm t^{1/2},\pm t^{-1/2}\}\cup\mathrm{Sol}$ on the Riemann sphere $\mathbb{C}\cup\{\infty\}\ni x$ and we have

$$Z_g^{\mathrm{SU}(2)_\kappa} = \sum_{x_*\in\mathrm{Sol}}\oint_{x=x_*}\frac{dx}{2\pi i}\,\omega_g(x,t,\kappa) = -\sum_{x_*=\pm t^{\frac12},\pm t^{-\frac12},\pm1}\oint_{x=x_*}\frac{dx}{2\pi i}\,\omega_g(x,t,\kappa), \quad (g\geq 2)\,. \tag{3.7}$$

For example, we can show the partition functions with $g=2,3$ explicitly

$$Z_{g=2}^{\mathrm{SU}(2)_\kappa}=\frac{1}{6(t-1)^6(t+1)^3}\Big[\kappa^3\left(1-t^2\right)^3+6\kappa^2\left(t^2-1\right)^2\left(t^2+1\right)-\kappa(11t^6-36t^5 \\ -9t^4+9t^2+36t-11)+6\left(-16t^{\kappa+3}+t^6-6t^5+15t^4-4t^3+15t^2-6t+1\right)\Big], \tag{3.8}$$

and

$$Z_{g=3}^{\mathrm{SU}(2)_\kappa}=\frac{1}{180(t-1)^{12}(t+1)^6}\Big[\kappa^6\left(t^2-1\right)^6-12\kappa^5(t-1)^5(t+1)^7 \\ +10\kappa^4(t-1)^4\left(7t^2-2t+7\right)(t+1)^6-240\kappa^3\left(t^2-1\right)^3\left(t^6-t^5-4t^4-10t^3-4t^2-t+1\right) \\ +\kappa^2\left(t^2-1\right)^2(31680t^{\kappa+4}+469t^8-2280t^7+44t^6-6360t^5+7614t^4 \\ -6360t^3+44t^2-2280t+469)-36\kappa\left(t^2-1\right)(4160t^{\kappa+4}+3200t^{\kappa+5}+4160t^{\kappa+6}+13t^{10} \\ -114t^9+361t^8+296t^7+2986t^6+1556t^5+2986t^4+296t^3+361t^2-114t+13) \\ +180(960t^{\kappa+4}+1536t^{\kappa+5}+2944t^{\kappa+6}+1536t^{\kappa+7}+960t^{\kappa+8}+64t^{2\kappa+6}+t^{12}-12t^{11} \\ +66t^{10}-220t^9-465t^8-2328t^7-2084t^6-2328t^5-465t^4-220t^3+66t^2-12t+1)\Big]. \tag{3.9}$$

Eqs. (3.8) and (3.9) are consistent with the results in [24], which were obtained from the geometry of the moduli space of SU(2) Hitchin system.

Next we calculate partition functions based on 2d TQFT structure, namely by gluing genus zero three point functions obtained by evaluating the Jeffrey-Kirwan residues. As we remarked at the beginning of the section, the computations are made with level by level, since the underlying Frobenius algebra $\mathcal{A}$ depends on the level. The localization formula (2.1) for the $\mathrm{SU}(2)_\kappa$ model with $r=2$ tells us that genus zero three point function is

$$\langle\mathcal{O}_\mu\mathcal{O}_\nu\mathcal{O}_\lambda\rangle_{g=0}=\frac{1-t}{2}\oint_{\mathrm{JK}(\eta)}\frac{dx}{2\pi ix}\;\mathcal{O}_\mu(x,t)\mathcal{O}_\nu(x,t)\mathcal{O}_\lambda(x,t)\cdot(1-x^2)(1-x^{-2}) \\ \times\sum_{k\in\mathbb{Z}}x^{2\kappa k}\left(\frac{x}{1-t\,x^2}\right)^{2k-1}\left(\frac{x^{-1}}{1-t\,x^{-2}}\right)^{-2k-1}. \tag{3.10}$$

When we choose a vector $\eta<0$, the Jeffrey-Kirwan residue operation is evaluated at the poles $x=\pm t^{\frac12},0$. On the other hand, when we choose a vector $\eta>0$, the residue is evaluated at $x=\pm t^{-\frac12},\infty$. Since there are no poles except for $x=\pm t^{\frac12},0,\pm t^{-\frac12},\infty$ in the genus zero case (3.10), the genus zero correlation functions evaluated at positive and negative η cause the same result up to an overall sign. In the next section we will compute U(2) case. The comparison of the following results with those in the next section gives a supporting evidence for the relation of $\mathrm{SU}(N)$ and $\mathrm{U}(N)$ partition functions derived in the next section (see (4.3)). In fact the (mutually distinct) roots y_i of the characteristic polynomial of the handle operator $(H\cdot C)$ are related by

$$\left(\frac{\kappa}{2}\right)y_i^{\mathrm{SU}(2)}=(1-t)\cdot y_i^{\mathrm{U}(2)} \tag{3.11}$$

for $\kappa=2,3,4$. Note that the dimensions of the Frobenius algebra $\mathcal{A}$ are different for SU(2) and U(2) and the multiplicity of each root y_i is also different. The multiplicity of $\mathrm{U}(N)$ theory is κ/N times that of $\mathrm{SU}(N)$ theory.

Level $\kappa = 2$. First we study $\kappa = 2$ case. The field configuration of a supersymmetric Wilson loop $W_\lambda = \mathrm{Tr}_\lambda x$ in a representation λ is symmetric under the exchange $x \leftrightarrow x^{-1}$ at the saddle points and the Wilson loop algebra consists of functions of $x + x^{-1}$. We can also include the background Wilson loop t for the U(1) flavor symmetry in the correlation function. Thus an element of the Wilson loop algebra takes value in $\mathbb{C}[[t]][x, x^{-1}]^{\mathfrak{S}_2}$. The equivalence relation $\mathcal{I}$ for this theory can be constructed by the saddle point equation (3.1)

$$\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2} - 2t\right) = 0\,. \tag{3.12}$$

Here we have removed $(x^2 - 1)$ to produce the ideal $\mathcal{I}$ correctly and the algebra of the Wilson loops is given by

$$\mathcal{A} = \mathbb{C}[[t]][x, x^{-1}]^{\mathfrak{S}_2} / \langle (x + x^{-1})(x^2 + x^{-2} - 2t) \rangle\,. \tag{3.13}$$

We can take a basis of (3.13) as $\{1, x + x^{-1}, x^2 + x^{-2}\}$. Then the number of generators is equal to the genus one partition function (Witten index) $Z_{g=1} = 3$ for $\kappa = 2$. Products among $\{1, x + x^{-1}, x^2 + x^{-2}\}$ lead to structure constants;

$$C^1_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4t \end{pmatrix}, \qquad C^2_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2t \\ 0 & 2t & 0 \end{pmatrix}, \quad C^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2(t-1) \end{pmatrix}. \tag{3.14}$$

From (3.10) with insertion of operators $\{1, x + x^{-1}, x^2 + x^{-2}\}$, we obtain the metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1-t^4 & 0 & -t^5+3t^4+2t^3-2t^2-t-1 \\ 0 & (1-t)^3(t+1)^2 & 0 \\ -t^5+3t^4+2t^3-2t^2-t-1 & 0 & 2\left(1+2t+t^2-4t^3-t^4+2t^5-t^6\right) \end{pmatrix}. \tag{3.15}$$

From (3.14) and (3.15), we can compute the characteristic polynomial $\det(yI - H \cdot C) = (y - y_1)^2 (y - y_2)$ with

$$y_1 = \frac{4}{(1-t)^3(t+1)}, \quad y_2 = \frac{2}{(1-t)(t+1)^3}\,, \tag{3.16}$$

and the partition function with genus g is given by

$$Z_g = 2y_1^{g-1} + y_2^{g-1}\,. \tag{3.17}$$

For example, we shall show partition functions in lower genera

$$Z_{g=0} = 1 - t^4\,, \qquad Z_{g=1} = 3\,, \qquad Z_{g=2} = \frac{2(5 + 6t + 5t^2)}{(1-t^2)^3}\,, \tag{3.18}$$

$$Z_{g=3} = \frac{4\left(9t^4 + 28t^3 + 54t^2 + 28t + 9\right)}{(1-t^2)^6}\,, \tag{3.19}$$

$$Z_{g=4} = \frac{8\left(17t^6 + 90t^5 + 255t^4 + 300t^3 + 255t^2 + 90t + 17\right)}{(1-t^2)^9}\,. \tag{3.20}$$

This result reproduces[12] table 1 in [25] and eqs. (3.8) (3.9).

In a similar manner, we can evaluate the genus g partition functions for the SU(2) models with $\kappa = 3, 4$. We summarize our results in these models: the algebra of Wilson loops $\mathcal{A}$, structure constants $C^\lambda_{\mu\nu}$ in a basis $\{1, x^l + x^{-l}\}_{l=1,\ldots,\kappa}$, the metric $\eta_{\mu\nu}$ and the characteristic polynomial of the handle operator.

[12] It seems there are a few typos in the table.

Level $\kappa = 3$.

- The algebra of Wilson loops

$$\mathcal{A} = \mathbb{C}[[t]][x, x^{-1}]^{\mathfrak{S}_2} / \langle (1-t)^2 + (1-2t)(x^2 + x^{-2}) + x^4 + x^{-4} \rangle \,. \tag{3.21}$$

- The structure constants $C_{\mu\nu}^{\lambda}$ in a basis $\{1, x + x^{-1}, x^2 + x^{-2}, x^3 + x^{-3}\}$

$$C^1_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -(t-1)^2 \\ 0 & 0 & -t^2+2t+1 & 0 \\ 0 & -(t-1)^2 & 0 & -2t^3+5t^2+1 \end{pmatrix}, \quad C^2_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -t^2+4t-1 \\ 0 & 0 & -t^2+4t-1 & 0 \end{pmatrix},$$
$$C^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2t \\ 1 & 0 & 2t-1 & 0 \\ 0 & 2t & 0 & (t-1)(3t+1) \end{pmatrix}, \qquad C^4_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2(t-1) \\ 1 & 0 & 2(t-1) & 0 \end{pmatrix}. \tag{3.22}$$

- The metric

$$\eta_{\mu\nu} = \begin{pmatrix} (1-t)\left(t^2+t+1\right) & 0 & (1-t)\left(t^3-t^2-2t-1\right) & 0 \\ 0 & (1-t)\left(t^3+t^2+1\right) & 0 & (1-t)\left(t^4-t^3-4t^2-t-1\right) \\ (1-t)\left(t^3-t^2-2t-1\right) & 0 & (1-t)\left(t^4-2t^3-t^2+3t+2\right) & 0 \\ 0 & (1-t)\left(t^4-t^3-4t^2-t-1\right) & 0 & (1-t)\left(t^5-2t^4-2t^3+8t^2+5t+2\right) \end{pmatrix}. \tag{3.23}$$

- The characteristic polynomial

$$\det(yI - H \cdot C) = (y - y_+)^2 (y - y_-)^2 \,, \tag{3.24}$$

with

$$y_{\pm} = \frac{4t^3 + 9t^2 + 9t + 5 \pm \left(3t^2 + 5t + 1\right) \sqrt{4t+5}}{(1-t^2)^3} \,. \tag{3.25}$$

Level $\kappa = 4$.

- The algebra of Wilson loops

$$\mathcal{A} = \mathbb{C}[[t]][x, x^{-1}]^{\mathfrak{S}_2} / \langle (1-t)^2 \left(x + x^{-1}\right) + (1-2t) \left(x^3 + x^{-3}\right) + x^5 + x^{-5} \rangle \,. \tag{3.26}$$

- The structure constants $C_{\mu\nu}^{\lambda}$ in a basis $\{1, x + x^{-1}, x^2 + x^{-2}, x^3 + x^{-3}, x^4 + x^{-4}\}$

$$C^1_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -2(t-1)^2 \\ 0 & 0 & 0 & -2(t-2)t & 0 \\ 0 & 0 & -2(t-1)^2 & 0 & -4t^3+10t^2-4t+2 \end{pmatrix}, \quad C^2_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -(t-1)^2 \\ 0 & 1 & 0 & (2-t)t & 0 \\ 0 & 0 & (2-t)t & 0 & -2(t-2)t^2 \\ 0 & -(t-1)^2 & 0 & -2(t-2)t^2 & 0 \end{pmatrix},$$
$$C^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -t^2+4t-1 \\ 0 & 1 & 0 & -t^2+4t-2 & 0 \\ 0 & 0 & -t^2+4t-1 & 0 & -2(t-3)(t-1)t \end{pmatrix}, \quad C^4_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2t \\ 0 & 1 & 0 & 2t-1 & 0 \\ 1 & 0 & 2t-1 & 0 & (t-1)(3t+1) \\ 0 & 2t & 0 & (t-1)(3t+1) & 0 \end{pmatrix},$$
$$C^5_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2(t-1) \\ 0 & 1 & 0 & 2(t-1) & 0 \\ 1 & 0 & 2(t-1) & 0 & (t-1)(3t-1) \end{pmatrix}, \tag{3.27}$$

- The metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1-t^3 & 0 & t^3+t^2-t-1 & 0 & -t^4+t^3-t^2+t \\ 0 & -t^3+t^2-t+1 & 0 & -t^4+2t^3-1 & 0 \\ t^3+t^2-t-1 & 0 & -t^4-t^3-t^2+t+2 & 0 & -t^5+3t^4+2t^3-2t^2-t-1 \\ 0 & -t^4+2t^3-1 & 0 & -t^5+3t^4-t^3-3t^2+2 & 0 \\ -t^4+t^3-t^2+t & 0 & -t^5+3t^4+2t^3-2t^2-t-1 & 0 & -t^6+3t^5-10t^3+3t^2+3t+2 \end{pmatrix}, \tag{3.28}$$

- The characteristic polynomial

$$\begin{aligned} &\det(yI - H\cdot C) = (y-y_1)^2(y-y_2)^2(y-y_3)\,, \\ &\qquad \text{with} \quad y_1 = \frac{4}{1-t^2}, \quad y_2 = \frac{4(t+3)}{(1-t)^3}, \quad y_3 = \frac{t+3}{(1-t)(t+1)^3}\,. \end{aligned} \tag{3.29}$$

4 Equivariant U(N) Verlinde algebra via Bethe ansatz

When the rank N of the gauge group G is greater than one, the evaluation of the residue integral becomes difficult. It is desirable to have alternative method to compute the correlation functions and it is here that the Bethe/Gauge correspondence saves the day. In this section, we will evaluate partition functions of $\mathrm{U}(N)_\kappa$ Chern-Simons theory with an adjoint chiral multiplet with $r = 2$ for $N = 2, 3$ and 4 with lower κ. As we explain in appendix A, the partition function of 2d TQFT is characterized by the structure constant $C^\lambda_{\mu\nu}$ and the metric $\eta_{\mu\nu}$. The Bethe/Gauge correspondence allows us to obtain these quantities from the algebra of Hall-Littlewood polynomials with the specialization on the set of explicit solutions (Bethe roots) to the Bethe ansatz equation. A crucial fact is that in this approach we do not have to solve the Bethe ansatz equation explicitly. What we need is the generating relations among Hall-Littlewood polynomials with the specialization, which mathematically define an ideal of the algebra of Hall-Littlewood polynomials. More precisely speaking the realization by the Hall-Littlewood polynomials is obtained, when the adjoint matter ϕ has the R-charge $r = 0$, while the equivariant Verlinde algebra is related to the case $r = 2$ [23]. However, we can control the dependence of the Frobenius algebra structure on the R-charge of ϕ, since in the localization formula the R-charge r only appears in the power of the difference product $\Delta(x, t)$ which can be expanded by the Hall-Littlewood polynomials. It turns out that the structure constants of the algebra are universal in the sense that they are independent of r and the r dependence appears in the metric (topological two point function). Note that the three point functions also depend on r, since it is obtained by contracting the structure constants with the metric. Unfortunately it is difficult to compute the partition function of the $\mathrm{SU}(N)$ theory with $N > 2$ directly. But we can compare our results with those obtained by other methods, after computing the genus g partition function, by using the relation of the $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ partition functions, which we explain shortly below.

Let us propose the relation between $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ partition functions on $\Sigma_g \times S^1$. We decompose the Cartan part of $\mathrm{U}(N)$ Wilson loop $(x_1, \cdots, x_N)$ to central $\mathrm{U}(1)$ Wilson loop y and the Cartan part of $\mathrm{SU}(N)$ Wilson loop $(\tilde{x}_1, \cdots, \tilde{x}_{N-1})$ as

$$x_1 = y\tilde{x}_1, \quad x_2 = y\tilde{x}_1^{-1}\tilde{x}_2, \ \cdots, \ x_{N-1} = y\tilde{x}_{N-2}^{-1}\tilde{x}_{N-1}, \quad x_N = y\tilde{x}_{N-1}^{-1}\,. \tag{4.1}$$

Since the one-loop determinants and H do not depend on y, the center U(1) CS term only depends on y. The integration of y leads to an SU(N) condition for the magnetic charge. There is also a relation between Hessians of U$(N)_\kappa$ and SU$(N)_\kappa$ CS-matter theories:

$$\det_{ab}\left(\frac{(2\pi i)^2\partial^2 W_{\text{eff}}^{\mathrm{U}(N)}}{\partial\log x_a\partial\log x_b}\right)=\frac{\kappa}{N}\det_{ab}\left(\frac{(2\pi i)^2\partial^2 W_{\text{eff}}^{\mathrm{SU}(N)}}{\partial\log\tilde{x}_a\partial\log\tilde{x}_b}\right), \tag{4.2}$$

and we connect SU(N) partition functions with U(N) partition functions. Then we obtain the following relation between SU(N) and U(N) partition functions

$$Z_g^{\mathrm{U}(N)_\kappa}=\left(\frac{\kappa}{N}\right)^g(1-t)^{(g-1)(1-r)}Z_g^{\mathrm{SU}(N)_\kappa}. \tag{4.3}$$

By applying the resumed expression (2.8) to the U$(N)_\kappa$ Chern-Simons theory with an adjoint chiral multiplet with integer R-charge r, the genus g correlation functions are given as a sum over the saddle points of the twisted superpotential W_{eff};

$$\langle\prod_{i=1}^{l}\mathcal{O}_{\mu_i}(x,t)\rangle_g^r=\sum_{x\in\mathrm{Sol}}\prod_{i=1}^{l}\mathcal{O}_{\mu_i}(x,t)\left(\prod_{a\neq b}^{N}(1-x_a x_b^{-1})\prod_{a,b=1}^{N}(1-tx_a x_b^{-1})^{r-1}H^{-1}(x,t)\right)^{1-g}, \tag{4.4}$$

where Sol is given as the set of the roots of the following saddle point equation

$$\exp\left(2\pi i\frac{\partial W_{\text{eff}}}{\partial\log x_a}\right)=x_a^\kappa\prod_{\substack{b=1\\b\neq a}}^{N}\frac{x_a-tx_b}{tx_a-x_b}=1,\quad a=1,\cdots,N. \tag{4.5}$$

As pointed out in [33], (4.5) coincides with the Bethe ansatz equation of the q-boson model with periodic boundary condition. The q-boson model is a one-dimensional quantum integrable lattice model which is regarded as a non-linear deformation of the harmonic oscillator. Especially U(1) flavor Wilson loop t corresponds to the q-deformation parameter q by $t=q^2$. Parameters N and κ correspond to the particle number and the number of lattice sites. Although the Bethe ansatz equation (4.5) cannot be solved explicitly, we will show that three point functions for $r=2$ are explicitly calculable. We summarize important properties of the q-boson model studied in [36] to evaluate the partition functions. Let us introduce $\mathcal{P}_{N,\kappa}$ and $\widetilde{\mathcal{P}}_{N,\kappa}$ as collections of non-negative integers $(\lambda^1,\lambda^2,\cdots,\lambda^N)$

$$\mathcal{P}_{N,\kappa}:=\{\lambda=(\lambda^1,\lambda^2,\cdots,\lambda^N)|\kappa\geq\lambda^1\geq\lambda^2\geq\cdots\lambda^N\geq 1\}, \tag{4.6}$$
$$\widetilde{\mathcal{P}}_{N,\kappa}:=\{\lambda=(\lambda^1,\lambda^2,\cdots,\lambda^N)|\kappa>\lambda^1\geq\lambda^2\geq\cdots\lambda^N\geq 0\}. \tag{4.7}$$

A bijection $\tilde{}:\mathcal{P}_{N,\kappa}\to\widetilde{\mathcal{P}}_{N,\kappa}$ which sends $\lambda\to\tilde{\lambda}$ is defined by deleting all integers equal to κ and supplementing as many zeros as the number of deleted κ's. Another bijection $^*:\mathcal{P}_{N,\kappa}\to\mathcal{P}_{N,\kappa}$ called $*$-involution is defined as some kind of an inverse operation of the bijection $\tilde{}$, namely, λ^* for $\lambda=(\lambda^1,\cdots,\lambda^N)$ is defined by the inverse image of $(\kappa-\lambda^N,\cdots,\kappa-\lambda^1)\in\widetilde{\mathcal{P}}_{N,\kappa}$ by $\tilde{}$. From proposition 7.7 in [36], a basis of the algebra of the Wilson loops can be taken as a set of Hall-Littlewood polynomials $\{P_\lambda(x,t)\}_{\lambda\in\mathcal{P}_{N,\kappa}}$, which means the number of roots of (4.5) equals to the order of the set $\mathcal{P}_{N,\kappa}$. Especially, this means the genus one partition function is given by the number of elements of $\mathcal{P}_{N,\kappa}$;

$$Z_{g=1}^{\mathrm{U}(N)_\kappa} = \frac{(N+\kappa-1)!}{N!(\kappa-1)!}\,, \tag{4.8}$$

and the genus one partition function of the $\mathrm{SU}(N)_\kappa$ theory is written down by using the relation (4.3)

$$Z_{g=1}^{\mathrm{SU}(N)_\kappa} = \frac{(N+\kappa-1)!}{(N-1)!\kappa!}\,. \tag{4.9}$$

This result (4.9) correctly reproduces Witten index of $\mathcal{N}=2$ $\mathrm{SU}(N)_\kappa$ CS-matter theory. Note that we have $Z_{g=1}^{\mathrm{SU}(N)_\kappa} = Z_{g=1}^{\mathrm{U}(\kappa)_N}$, where the correspondence of the states is given by the transpose of the Young diagrams.

The structure constants $C_{\mu\nu}^{\lambda}(t)$ in this basis are defined by the expansion of products of Hall-Littlewood polynomials

$$P_\mu(x,t)P_\nu(x,t) \equiv \sum_{\lambda\in\mathcal{P}_{N,\kappa}} C_{\mu\nu}^{\lambda}(t)P_\lambda(x,t), \quad \mu,\nu\in\mathcal{P}_{N,\kappa} \quad \text{and} \quad x\in\mathrm{Sol}. \tag{4.10}$$

We use "$\equiv$" to emphasize equality up to the Bethe ansatz equation (4.5). An important property of $C_{\mu\nu}^{\lambda}(t)$ is that there exists $S_{\mu\nu}(t)$ which simultaneously diagonalizes the structure constants

$$C_{\mu\nu}^{\lambda}(t) = \sum_{\sigma\in\mathcal{P}_{N,\kappa}} \frac{S_{\mu\sigma}(t)S_{\nu\sigma}(t)S_{\sigma\lambda}^{-1}(t)}{S_{\emptyset\sigma}(t)}\,. \tag{4.11}$$

Here $\emptyset := (\kappa,\cdots,\kappa)$. Note that (4.11) is independent of R-charge. Then the associativity condition (A.6) immediately follows from (4.11). It is also shown in [36] that

$$\sum_{x\in\mathrm{Sol}} \frac{P_\lambda(x,t)P_\mu(x,t)P_\nu(x,t)}{\langle\psi_N(x)|\psi_N(x)\rangle} = \frac{C_{\mu\nu}^{\lambda^*}(t)}{b_\lambda(t)}, \quad \lambda,\mu,\nu\in\mathcal{P}_{N,\kappa}\,. \tag{4.12}$$

Here $|\psi_N(x)\rangle$ and $\langle\psi_N(x)|$ are respectively on-shell Bethe vector of N particles and dual Bethe vector in the q-boson model. $\langle\psi_N(x)|\psi_N(x)\rangle$ is the inner product of these two vectors and $b_\lambda(t)$ is defined by

$$b_\lambda(t) := \prod_{i\geq 1}\prod_{j=1}^{m_i(\lambda)}(1-t^j), \quad m_i(\lambda) := \#\{l|\lambda_l = i\}\,. \tag{4.13}$$

We can relate genus zero three point functions with $r=0$ to the metric or genus zero two point functions with $r=2$ as follows. The correlation functions (4.4) with $r=0$ agree with the correlation functions of the $\mathrm{U}(N)/\mathrm{U}(N)$ gauged WZW-matter model with level κ on genus g Riemann surface introduced in [33]. Then, it was shown in [33] that the genus zero correlation functions with $r=0$ are expressed as

$$\langle\prod_i \mathcal{O}_{\mu_i}\rangle_{g=0}^{r=0} = \sum_{x\in\mathrm{Sol}} \frac{\prod_i \mathcal{O}_{\mu_i}(x,t)}{\langle\psi_N(x)|\psi_N(x)\rangle}\,. \tag{4.14}$$

From (4.12) and (4.14), the genus zero three point functions of $\{P_\lambda(x,t)\}_{\lambda\in\mathcal{P}_{N,\kappa}}$ are given by

$$\langle P_\lambda(x,t)P_\mu(x,t)P_\nu(x,t)\rangle_{g=0}^{r=0} = \frac{C_{\mu\nu}^{\lambda^*}(t)}{b_\lambda(t)}, \quad \lambda,\mu,\nu\in\mathcal{P}_{N,\kappa}. \tag{4.15}$$

Now, we are ready to express the genus zero two point functions of $\{P_\lambda(x,t)\}_{\lambda\in\mathcal{P}_{N,\kappa}}$ for $r=2$ as a linear combination of genus zero three point functions for $r=0$. From (4.4), we have the following relation

$$\langle P_\mu(x,t)P_\nu(x,t)\rangle_{g=0}^{r=2} = (1-t)^{2N}\langle P_\mu(x,t)P_\nu(x,t)\Delta(x,t)^2\rangle_{g=0}^{r=0}, \tag{4.16}$$

where we defined

$$\Delta(x,t) := \prod_{\substack{a,b=1\\a\neq b}}^{N}(1-tx_ax_b^{-1}). \tag{4.17}$$

$\Delta(x,t)^2$ is also written as

$$\Delta(x,t)^2 = \left(\prod_{c=1}^{N} x_c^{2(1-N)}\right)\prod_{a\neq b}(x_b-tx_a)^2. \tag{4.18}$$

Since the factor $\prod_{c=1}^N x_c^{2(1-N)}$ on the right hand side of (4.18) is always rewritten as a symmetric monomial by using the relation $\prod_{a=1}^N x_a^\kappa \equiv 1$ which follows from (B.12) with $\lambda=(\kappa,\cdots,\kappa)$, we find that (4.18) is equal to a symmetric polynomial of x up to the Bethe ansatz equation and can be expanded by $\{P_\lambda(x,t)\}_{\lambda\in\mathcal{P}_{N,\kappa}}$ as

$$\Delta(x,t)^2 \equiv \sum_{\lambda\in\mathcal{P}_{N,\kappa}} g_\lambda(t)P_\lambda(x,t). \tag{4.19}$$

Thus the genus zero two point functions for $r=2$ are written by the structure constant $C_{\mu\nu}^\lambda(t)$ as

$$\begin{aligned}\eta_{\mu\nu} := \langle P_\mu P_\nu\rangle_{g=0}^{r=2} &\equiv (1-t)^{2N}\sum_{\lambda\in\mathcal{P}_{N,\kappa}} g_\lambda(t)\langle P_\lambda P_\mu P_\nu\rangle_{g=0}^{r=0}\\ &= (1-t)^{2N}\sum_{\lambda\in\mathcal{P}_{N,\kappa}}\frac{g_\lambda(t)C_{\mu\nu}^{\lambda^*}(t)}{b_\lambda(t)}.\end{aligned} \tag{4.20}$$

On the right hand side of (4.20), the dependence of R-charge only comes from $(1-t)^{2N}$ and $g_\lambda(t)$. In appendix C we discuss properties of the metric and the coupling in the case of general R-charge. In particular we will show that we have 2d TQFT for any integral charge r. In the following subsections, we will evaluate genus g partition functions of U(2) with level $\kappa=2,3,4$, U(3) with level $\kappa=2,3$ and U(4) with level $\kappa=2$ from (4.10) and (4.20).

4.1 U(2) cases

First we rewrite the insertion factor $\Delta(x,t)^2$ in terms of Hall-Littlewood polynomials which holds for general $\kappa\geq 2$. When $N=2$, the insertion factor is

$$\Delta(x,t)^2 = e_2(x)^{-2}\left((1+t)^2e_2(x)-t(e_1(x))^2\right)^2, \tag{4.21}$$

JHEP02(2019)097

where $e_1(x), e_2(x)$ are the elementary symmetric polynomials defined by

$$e_1(x) := x_1 + x_2, \quad e_2(x) := x_1 x_2. \tag{4.22}$$

Using the relation $e_N(x)^\kappa \equiv 1$ which comes from the Bethe ansatz equation, we may evaluate

$$\Delta(x,t)^2 \equiv e_2(x)^{\kappa-2}\left((1+t)^2 e_2(x) - t(e_1(x))^2\right)^2. \tag{4.23}$$

The relation of the elementary symmetric polynomials and the Schur function is $e_\ell(x) = s_{(1^\ell)}(x)$. Hence we have

$$\Delta(x,t)^2 \equiv s_{(1,1)}^{\kappa-2}\left((1+t)^2 s_{(1,1)} - t(s_{(1)})^2\right)^2. \tag{4.24}$$

From now on we do not write x dependence explicitly. From the composition rule of two SU(2) representations or two spins, we see

$$s_{(1)}^2 = s_{(2)} + s_{(1,1)}, \qquad s_{(2)}^2 = s_{(4)} + s_{(3,1)} + s_{(2,2)}, \tag{4.25}$$

which gives

$$\begin{aligned}\Delta(x,t)^2 &\equiv s_{(1,1)}^{\kappa-2}\left(t^2 s_{(4)} - (2t+t^2+2t^3)s_{(3,1)} + (1+2t+4t^2+2t^3+t^4)s_{(2,2)}\right)\\ &= t^2 s_{(\kappa+2,\kappa-2)} - (2t+t^2+2t^3)s_{(\kappa+1,\kappa-1)} + (1+2t+4t^2+2t^3+t^4)s_{(\kappa,\kappa)}.\end{aligned} \tag{4.26}$$

We want to emphasize that this is a universal formula valid for any level κ. When $N=2$ the relation of the Schur functions and Hall-Littlewood polynomials is[13]

$$\begin{aligned}s_{(\kappa+2,\kappa-2)} &= P_{(\kappa+2,\kappa-2)} + tP_{(\kappa+1,\kappa-1)} + t^2 P_{(\kappa,\kappa)},\\ s_{(\kappa+1,\kappa-1)} &= P_{(\kappa+1,\kappa-1)} + tP_{(\kappa,\kappa)},\\ s_{(\kappa,\kappa)} &= P_{(\kappa,\kappa)}.\end{aligned} \tag{4.27}$$

Hence, we arrive at

$$\Delta(x,t)^2 \equiv t^2 P_{(\kappa+2,\kappa-2)} - (2t+t^2+t^3)P_{(\kappa+1,\kappa-1)} + (1+2t+2t^2+t^3)P_{(\kappa,\kappa)}. \tag{4.28}$$

The next task is to express the right hand side in terms of the Hall-Littlewood polynomials in the fundamental domain $\mathcal{P}_{2,\kappa}$. Here the ideal $\mathcal{I}_{N,\kappa}$ in the ring of the symmetric polynomials $\Lambda_N = \mathbb{R}[x_1, \cdots x_N]^{\mathfrak{S}_N}$ depends on the level κ and we have to consider case by case with level κ. As discussed by Korff in [36] we can obtain any weight $\omega = \sum_{i=1}^N \omega_i \epsilon_i$ in the $\mathfrak{gl}_N$ weight lattice $\mathbb{Z}[\epsilon_1, \cdots \epsilon_N]$ from an appropriate element in $\mathcal{P}_{N,\kappa}$ (hence the name "fundamental domain") by the action of the affine Weyl group $\widetilde{\mathfrak{S}}_{N,\kappa}$ with level κ, which includes the translation of length κ in addition to the usual permutations. We can obtain the necessary relations among the Hall-Littlewood polynomials involved in the process of the action of $\widetilde{\mathfrak{S}}_{N,\kappa}$.

[13]This is *not* true for $N > 2$, since there appears the partition of length greater than two. We have truncated the transition matrix in Macdonald's book [42] by the partitions up to length two.

When $\kappa = 2$ we have

$$\Delta(x,t)^2 \equiv t^2 P_{(4,0)} - (2t + t^2 + t^3) P_{(3,1)} + (1 + 2t + 2t^2 + t^3) P_{(2,2)}. \tag{4.29}$$

The first two terms are outside $\mathcal{P}_{2,2}$, while the last term is already in $\mathcal{P}_{2,2}$. We note $\lambda \cdot \sigma_0 = (4,0)$ for $\lambda = (2,2)$ and $\lambda \cdot \tau = (3,1)$ for $\lambda = (1,1)$. The definition of the actions of σ_0 and τ is given in appendix B. Using (B.8) and (B.11) with the relation of R_λ and P_λ, we see

$$P_{(4,0)} \equiv t(1+t) P_{(2,2)} + (t-1) P_{(3,1)}, \qquad P_{(3,1)} \equiv (1+t) P_{(1,1)}. \tag{4.30}$$

Substituting them, we finally obtain

$$\Delta(x,t)^2 \equiv (1+t)^2 \left((1+t^2) P_{(2,2)} - 2t P_{(1,1)} \right). \tag{4.31}$$

For $\kappa = 3$ we have

$$\Delta(x,t)^2 \equiv t^2 P_{(5,1)} - (2t + t^2 + t^3) P_{(4,2)} + (1 + 2t + 2t^2 + t^3) P_{(3,3)}. \tag{4.32}$$

As before the first two terms are outside $\mathcal{P}_{2,3}$, while the last term is already in $\mathcal{P}_{2,3}$. From $\lambda = (2,1)$, we can obtain $\lambda \cdot \tau = (4,2)$ and $\lambda \cdot \sigma_1 \cdot \tau = (5,1)$, which implies

$$P_{(4,2)} \equiv P_{(2,1)}, \qquad P_{(5,1)} \equiv t P_{(2,1)}. \tag{4.33}$$

Substituting them, we finally obtain

$$\Delta(x,t)^2 \equiv (1+t)(1+t+t^2) P_{(3,3)} - t(2+t) P_{(2,1)}. \tag{4.34}$$

For $\kappa = 4$ we have

$$P_{(6,2)} \equiv (1+t) P_{(2,2)}, \qquad P_{(5,3)} \equiv P_{(3,1)}, \tag{4.35}$$

which lead us to

$$\Delta(x,t)^2 \equiv t^2 (1+t) P_{(2,2)} - (2t + t^2 + t^3) P_{(3,1)} + (1+t)(1+t+t^2) P_{(4,4)}. \tag{4.36}$$

When $\kappa \geq 5$ we see a phenomenon of "stabilization" in the semi-classical limit $\kappa \to \infty$. Namely by acting τ we observe

$$P_{(\kappa+2,\kappa-2)} \equiv P_{(\kappa-2,2)}, \qquad P_{(\kappa+1,\kappa-1)} \equiv P_{(\kappa-1,1)}. \tag{4.37}$$

Thus we obtain a general formula for $\kappa \geq 5$;

$$\Delta(x,t)^2 \equiv t^2 P_{(\kappa-2,2)} - (2t + t^2 + t^3) P_{(\kappa-1,1)} + (1+t)(1+t+t^2) P_{(\kappa,\kappa)}. \tag{4.38}$$

By using the formula of $\Delta(x,t)^2$, we can write down genus zero partition functions in the $\mathrm{U}(2)_\kappa$ models

$$Z_{g=0} = C_{\emptyset\emptyset\emptyset}(t) = (1-t)^4 \frac{g_\emptyset(t)}{b_\emptyset(t)} = \begin{cases} (1-t)(1-t^4) & (\kappa = 2) \\ (1-t)(1-t^3) & (\kappa > 2) \end{cases}. \tag{4.39}$$

Level $\kappa = 2$. We explain how to calculate $C^{\lambda}_{\mu\nu}$ for $\kappa = 2$ in detail . We fix the order of elements of $\mathcal{P}_{2,2}$ to use matrix notation as follows

$$\mathcal{P}_{2,2} = \{(2,2), (2,1), (1,1)\}\,. \tag{4.40}$$

When $\kappa = 2$ for general N the $*$-involution of an element of $\lambda \in \mathcal{P}_{2,N}$ is same as itself; $\lambda^* = \lambda$. Since $P_{(2,2)}(x,t) \equiv 1$ for $x \in \mathrm{Sol}$, we have relations

$$P_{(2,2)}P_{(2,2)} \equiv P_{(2,2)}, \tag{4.41}$$
$$P_{(2,2)}P_{(2,1)} \equiv P_{(2,1)}, \tag{4.42}$$
$$P_{(2,2)}P_{(1,1)} \equiv P_{(1,1)}. \tag{4.43}$$

When $x = (x_1, x_2)$ is a set of generic variables which does not satisfy the Bethe ansatz equation, products of Hall-Littlewood polynomials are expanded as

$$P_{(2,1)}P_{(2,1)} = P_{(4,2)} + (1+t)P_{(3,3)}, \tag{4.44}$$
$$P_{(2,1)}P_{(1,1)} = P_{(3,2)}, \tag{4.45}$$
$$P_{(1,1)}P_{(1,1)} = P_{(2,2)}. \tag{4.46}$$

From an identity (B.8) for $(4,2) = (2,2)\cdot\tau$ and $(3,2) = (2,1)\cdot\tau$, polynomials $P_{(4,2)}, P_{(3,3)}, P_{(3,2)}$ can be expressed as combinations of $\{P_\lambda\}_{\lambda\in\mathcal{P}_{2,2}}$. Then we have

$$P_{(2,1)}P_{(2,1)} \equiv (1+t)P_{(2,2)} + (1+t)P_{(1,1)}, \tag{4.47}$$
$$P_{(2,1)}P_{(1,1)} \equiv P_{(2,1)}. \tag{4.48}$$

The structure constants $C^{\lambda}_{\mu\nu}(t)$'s for $N = 2, \kappa = 2$ in the matrix notation are given by

$$C^{(2,2)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+t & 0 \\ 0 & 0 & 1 \end{pmatrix},\ C^{(2,1)}_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ C^{(1,1)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1+t & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.49}$$

From (4.31) and (4.49), the metric is given by

$$\eta_{\mu\nu} = \begin{pmatrix} (1-t)^2(1+t)(1+t^2) & 0 & -2t(1-t)^2(1+t) \\ 0 & (1-t)^2(1-t^2)^2 & 0 \\ -2t(1-t)^2(1+t) & 0 & (1-t)^2(1+t)(1+t^2) \end{pmatrix}. \tag{4.50}$$

The characteristic polynomial of the handle operator is defined as $\det(yI - H\cdot C) = (y - y_1)^2(y - y_2)$ with y_1 and y_2

$$y_1 = \frac{4}{(1-t)^4(1+t)}, \quad y_2 = \frac{2}{(1-t)^2(1+t)^3}\,. \tag{4.51}$$

So we can write down the genus g partition function in this model

$$Z_g^{\mathrm{U}(2)_{\kappa=2}} = \left(\frac{2}{(1-t)^2(1+t)^3}\right)^{g-1} + 2\left(\frac{4}{(1-t)^4(1+t)}\right)^{g-1}. \tag{4.52}$$

Level $\kappa = 3$. Next we evaluate the model with level $\kappa = 3$, namely, $\mathrm{U}(2)_{\kappa=3}$ Chern-Simons theory with an adjoint chiral multiplet for $r = 2$. In this case, $\mathcal{P}_{2,3}$ consists of six partitions

$$\mathcal{P}_{2,3} = \{(3,3),(3,2),(3,1),(2,2),(2,1),(1,1)\}\,. \tag{4.53}$$

and the $*$-involution acts on these six elements

$$(3,3)^* = (3,3),\ (3,2)^* = (3,1),\ (3,1)^* = (3,2), \tag{4.54}$$

$$(2,2)^* = (1,1),\ (2,1)^* = (2,1),\ (1,1)^* = (2,2). \tag{4.55}$$

By using the relations (B.8)–(B.12) in appendix B, the structure constants are calculated in similar manner as $\kappa = 2$ case. For example,

$$P_{(3,1)}P_{(2,1)} = P_{(5,2)} + (1+t)P_{(4,3)} \equiv (1+t)P_{(2,2)} + P_{(3,1)}\,. \tag{4.56}$$

By using relations $(5,2) = \lambda\cdot\tau$ for $\lambda = (2,2)$ and $(4,3) = \lambda\cdot\tau$ for $\lambda = (3,1)$, we have relations

$$P_{(5,2)} \equiv (1+t)P_{(2,2)},\quad P_{(4,3)} \equiv P_{(3,1)}\,. \tag{4.57}$$

Then structure constants in matrix notation are given by

$$C^{(3,3)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+t & 0 & 0 & 0 \\ 0 & 1+t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1+t & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},\quad C^{(3,2)}_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4.58}$$

$$C^{(3,1)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},\quad C^{(2,2)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1+t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+t & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.59}$$

$$C^{(2,1)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},\quad C^{(1,1)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & t+1 & 0 \\ 0 & 0 & t+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & t+1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.60}$$

Now we introduce the characteristic polynomial

$$\det\left(yI - H\cdot C\right) = (y-y_+)^3(y-y_-)^3\,, \tag{4.61}$$

where

$$y_\pm = \frac{3\left(4t^3+9t^2+9t+5\pm(3t^2+5t+1)\sqrt{4t+5}\right)}{2(1-t)(1-t^2)^3}\,. \tag{4.62}$$

The partition function with genus g can be described by using $y_\pm$

$$Z_g^{\mathrm{U}(2)_{\kappa=3}} = 3y_+^{g-1} + 3y_-^{g-1}\,. \tag{4.63}$$

Level $\kappa = 4$. We fix the order of elements $\mathcal{P}_{2,4}$ as

$$\mathcal{P}_{2,4} = \{(4,4),(4,3),(4,2),(4,1),(3,3),(3,2),(3,1),(2,2),(2,1),(1,1)\}\,. \tag{4.64}$$

By computing the structure constants and the metric, we obtain the characteristic polynomial of the handle operator

$$\det\left(yI - H\cdot C\right) = (y-y_1)^4(y-y_2)^4(y-y_3)^2\,, \tag{4.65}$$

where

$$y_1 = \frac{8}{(1-t)^2(1+t)}, \quad y_2 = \frac{8(t+3)}{(1-t)^4}, \quad y_3 = \frac{2(t+3)}{(1-t)^2(1+t)^3}\,. \tag{4.66}$$

The partiton function with genus g is represented as

$$Z_g^{\mathrm{U}(2)_{\kappa=4}} = 4(y_1^{g-1}+y_2^{g-1}) + 2y_3^{g-1}\,. \tag{4.67}$$

We have computed the genus g partition functions in SU(2) and U(2) models with $\kappa = 2,3,4$ and find these partitions functions actually satisfy the relation (4.3)

$$Z_g^{\mathrm{SU}(2)_\kappa} = \left(\frac{2}{\kappa}\right)^g (1-t)^{g-1} Z_g^{\mathrm{U}(2)_\kappa}\,. \tag{4.68}$$

This result means the U(2) partition functions reproduce the Coulomb branch limit of the lens space index on $S^1\times L(\kappa,1)$ for A_1 class $\mathcal{S}$ theories [25].

4.2 U(3) cases

Let us calculate partition functions for $\kappa = 2,3$. When $N = 3$, the insertion factor $\Delta(x,t)$ reduces to

$$\begin{aligned} P_{(2,2,2)}\Delta(x,t) = (-t^3)P_{(4,2)} &+ t^2(1+t)P_{(4,1,1)} + t^2(1+t)P_{(3,3)} \\ &- t(1+t)P_{(3,2,1)} + (1+t)(1+t+t^2)P_{(2,2,2)}\,. \end{aligned} \tag{4.69}$$

It is amusing that t^4, t^5 and t^6 terms become implicit by the use of t dependent Hall-Littlewood polynomials. At this stage we may use the ideal relations that depend on the level κ. When $\kappa = 2$, the relevant relations from (B.8)–(B.12) are

$$\begin{aligned} P_{(2,2,2)} &\equiv 1\,, \\ P_{(4,1,1)} &\equiv P_{(1,1,2)} \equiv tP_{(1,2,1)} \equiv t^2P_{(2,1,1)}\,, \\ P_{(3,2,1)} &\equiv (1+t)P_{(2,1,1)}\,, \\ P_{(3,3)} &\equiv t^2P_{(2,1,1)}\,, \\ P_{(4,2)} &\equiv t(1+t)(1+t+t^2)P_{(2,2)} - (1-t^2)P_{(2,1,1)}\,. \end{aligned} \tag{4.70}$$

which allow us to write

$$\Delta(t) \equiv (1-t)(1+t)^2(1+t^2)(1+t+t^2)P_{(2,2,2)} - t(1+t)^3(1-t)P_{(2,1,1)}. \tag{4.71}$$

On the other hand when $\kappa = 3$ we can use relations

$$\begin{aligned} (1+t)P_{(4,1,1)} &\equiv (1+t)(1+t+t^2)P_{(1,1,1)}\,, \\ (1+t)P_{(3,3)} &\equiv (1+t)(1+t+t^2)P_{(3,3,3)}\,, \\ P_{(4,2,0)} &\equiv tP_{(3,2,1)}\,, \end{aligned} \tag{4.72}$$

which lead to

$$\begin{aligned} \Delta(t) \equiv\ & (1+t)(1+t+t^2)P_{(2,2,2)} - t(1+t+t^3)P_{(3,2,1)} \\ & + t^2(1+t+t^2)P_{(3,3,3)} + t^2(1+2t)(1+t+t^2)P_{(1,1,1)}. \end{aligned} \tag{4.73}$$

Next we shall consider the structure constants and evaluate partition functions for $\kappa = 2, 3$.

Level $\boldsymbol{\kappa = 2}$. $\mathcal{P}_{3,2}$ consists of following four partitions:

$$\mathcal{P}_{3,2} = \{(2,2,2), (2,2,1), (2,1,1), (1,1,1)\}\,. \tag{4.74}$$

The structure constants for $\kappa = 2$ are shown explicitly

$$C^{(2,2,2)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+t+t^2 & 0 & 0 \\ 0 & 0 & 1+t+t^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C^{(2,2,1)}_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1+t & 0 \\ 0 & 1+t & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4.75}$$

$$C^{(2,1,1)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1+t & 0 & 1 \\ 1 & 0 & 1+t & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad C^{(1,1,1)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1+t+t^2 & 0 \\ 0 & 1+t+t^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.76}$$

Then characteristic polynomial is defined as

$$\det\,(yI - H\cdot C) = (y-y_+)^2(y-y_-)^2\,, \tag{4.77}$$

$$\begin{aligned} y_\pm = \frac{1}{(1-t)^9(1+t)^3(1+t+t^2)^5}\Big(&(5t^2+6t+5)(t^6+t^5+4t^4+4t^3+4t^2+t+1) \\ &\pm (1+t)(t^6+5t^5+6t^4+8t^3+6t^2+5t+1)\sqrt{5t^2+6t+5}\Big)\,. \end{aligned} \tag{4.78}$$

We have the genus g partition function for the $\mathrm{U}(3)_{\kappa=2}$ CS-matter theory

$$Z_g^{\mathrm{U}(3)_{\kappa=2}} = 2(y_+^{g-1} + y_-^{g-1})\,. \tag{4.79}$$

We will show several examples in lower genera

$$Z_{g=0}^{\mathrm{U}(3)_{\kappa=2}} = (1-t)^2(1-t^2)^3(1+t+4t^2+4t^3+4t^4+t^5+t^6)\,, \tag{4.80}$$

$$Z_{g=1}^{\mathrm{U}(3)_{\kappa=2}} = 4\,, \tag{4.81}$$

$$Z_{g=2}^{\mathrm{U}(3)_{\kappa=2}} = \frac{4\left(5t^2+6t+5\right)\left(t^6+t^5+4t^4+4t^3+4t^2+t+1\right)}{(1-t)(1-t^2)^3\left(1-t^3\right)^5}\,, \tag{4.82}$$

$$\begin{aligned} Z_{g=3}^{\mathrm{U}(3)_{\kappa=2}} = \frac{8\left(5t^2+6t+5\right)}{(1-t)^2(1-t^2)^6\left(1-t^3\right)^{10}}&(3t^{14}+14t^{13}+60t^{12}+152t^{11}+309t^{10} \\ &+490t^9+660t^8+720t^7+660t^6+490t^5+309t^4+152t^3+60t^2+14t+3)\,. \end{aligned} \tag{4.83}$$

Level $\kappa = 3$. $\mathcal{P}_{3,3}$ consists of following ten partitions

$$\mathcal{P}_{3,3} = \{(3,3,3),(3,3,2),(3,3,1),(3,2,2),(3,2,1),(3,1,1),(2,2,2),(2,2,1),(2,1,1),(1,1,1)\}. \tag{4.84}$$

By computing the structure constants and the metric, we obtain the characteristic polynomial

$$\det(yI - H\cdot C) = (y-y_1)^6(y-y_2)^3(y-y_3), \tag{4.85}$$

$$y_1 = \frac{9}{(1-t)^4(1-t^2)^3(1-t^3)}, \quad y_2 = \frac{9(t+2)^2}{(1-t)^9(t+1)^3}, \quad y_3 = \frac{(t+2)^2}{(1-t)^3(t^2+t+1)^5}. \tag{4.86}$$

In this model, the partition function is given by

$$Z_g = 6y_1^{g-1} + 3y_2^{g-1} + y_3^{g-1}, \tag{4.87}$$

and we show several examples in lower genera

$$Z_{g=0}^{\mathrm{U}(3)_{\kappa=3}} = (1-t)^3\left(t^8+2t^7+6t^6+6t^5+3t^4+3t^3+3t^2+2t+1\right), \tag{4.88}$$

$$Z_{g=1}^{\mathrm{U}(3)_{\kappa=3}} = 10, \tag{4.89}$$

$$\begin{aligned} Z_{g=2}^{\mathrm{U}(3)_{\kappa=3}} = {} & \frac{1}{(1-t)(1-t^2)^3(1-t^3)^5}(81t^{12}+244t^{11}+1054t^{10}+2746t^9+6071t^8 \\ & +9503t^7+11909t^6+11138t^5+8513t^4+4808t^3+2176t^2+640t+166). \end{aligned} \tag{4.90}$$

By using the relation (4.3) we can obtain $\mathrm{SU}(3)_{\kappa=2,3}$ partition functions from the results (4.79) and (4.87),

$$Z_g^{\mathrm{SU}(3)_\kappa} = \left(\frac{3}{\kappa}\right)^g (1-t)^{g-1} Z_g^{\mathrm{U}(3)_\kappa}. \tag{4.91}$$

We made use of the mathematica notebook file attached to the arXiv version of [25] to compute $\mathrm{SU}(3)_{\kappa=2,3}$ partition functions from the Coulomb branch limit of indices on $S^1 \times L(\kappa, 1)$ associated with T_3 theory. We have found that our results in the $\mathrm{SU}(3)_{\kappa=2,3}$ models agree completely with [25].

4.3 U(4) case

Level $\kappa = 2$. $\mathcal{P}_{4,2}$ consists of following five partitions

$$\mathcal{P}_{4,2} = \{(2,2,2,2),(2,2,2,1),(2,2,1,1),(2,1,1,1),(1,1,1,1)\}. \tag{4.92}$$

By computing the structure constants and the metric, we can write down the characteristic polynomial of the handle operator;

$$\det(yI - H\cdot C) = (y-y_1)^2(y-y_2)^2(y-y_3), \tag{4.93}$$

with

$$\begin{aligned} y_1 &= \frac{4(3t^2+2t+3)}{(1-t)^{16}(1+t)^6(1+t+t^2)^5}, \\ y_2 &= \frac{4}{(1-t)^{10}(1+t)^4(1+t^2)^3(1+t+t^2)^5}, \\ y_3 &= \frac{3t^2+2t+3}{(1-t)^8(1+t)^{10}(1+t^2)^7}. \end{aligned} \tag{4.94}$$

Thus the U(4) partition function is given by using y_1, y_2, and y_3

$$Z_g^{\mathrm{U}(4)_{\kappa=2}} = 2(y_1^{g-1} + y_2^{g-1}) + y_3^{g-1} \,. \tag{4.95}$$

We can also evaluate the SU(4) counterpart from this result $Z_g^{\mathrm{U}(4)_{\kappa=2}}$

$$Z_g^{\mathrm{SU}(4)_{\kappa=2}} = 2^g (1-t)^{g-1} \left(2y_1^{g-1} + 2y_2^{g-1} + y_3^{g-1} \right) \,. \tag{4.96}$$

We expect that $\mathrm{SU}(4)_{\kappa=2}$ partition functions constructed from this U(4) theory reproduce the Coulomb branch limit of indices on $S^1 \times L(2,1)$ associated with T_4 theory [25].

5 Discussions on intriguing aspects of the algebra

5.1 Recurrence formula in genus

Recall the genus g partition functions can be constructed by using the handle operator $Z_g = \mathrm{Tr}\left\{(H\cdot C)^{g-1}\right\} = \sum_{\mu\in\mathcal{P}_{N,\kappa}} \left\{(H\cdot C)^{g-1}\right\}_\mu{}^\mu$ in (A.12). One can evaluate the partition function Z_g by computing eigenvalues y_i of the matrix $(H\cdot C)$ as (A.13). These eigenvalues are calculated as roots of a characteristec equation in (A.14)

$$\det\left[yI - (H\cdot C)\right] = \prod_i (y-y_i)^{m_i} = 0 \quad (m_i \in \mathbb{Z}_{>0}) \,. \tag{5.1}$$

We can also derive the recurrence equation of the partition function Z_g as follows; first we consider the matrix $M = (H\cdot C)$ and look for the minimal polynomial $f(M) := \prod_i (M - y_i I)^{\tilde{m}_i}$ $(\tilde{m}_i \in \mathbb{Z}_{>0})$ which can be evaluated by using roots of the characteristic equation $\det[yI - M] = 0$. In our models ($\mathrm{U}(2)_{\kappa=2,3,4}$, $\mathrm{U}(3)_{\kappa=2,3}$, $\mathrm{U}(4)_{\kappa=2}$), it turns out the minimal polynomials are factorized to first order polynomials, namely $\tilde{m}_i = 1$, which means the matrix $M = (H\cdot C)$ is diagonalizable, or there are no Jordan blocks of size greater than one. In such cases the Frobenius algebra is called semi-simple.

$$\begin{aligned} f(M) &= \prod_i (M - y_i I) = M^K + b_1 M^{K-1} + b_2 M^{K-2} + \cdots + b_{K-1} M + b_K I \,, \\ K &= \begin{cases} 2 \text{ for } \mathrm{U}(2)_{\kappa=2,3},\ \mathrm{U}(3)_{\kappa=2}, \\ 3 \text{ for } \mathrm{U}(2)_{\kappa=4},\ \mathrm{U}(3)_{\kappa=3},\ \mathrm{U}(4)_{\kappa=2} \,. \end{cases} \end{aligned} \tag{5.2}$$

By applying this polynomial equation $f(M = (H\cdot C)) = O$ to the partition function in our models, we can obtain the recurrence formula

$$Z_{g+K} + b_1 Z_{g+K-1} + b_2 Z_{g+K-2} + \cdots + b_{K-1} Z_{g+1} + b_K Z_g = 0 \quad (g = 0,1,2,\cdots) \,.$$

Now we make a comment here: the partiton function can be evaluated by solving the characteristic equation (A.14) associated with the matrix $(H\cdot C)$. In applying this procedure to the partiton functions, the main difficulty to be encountered lies in the evaluation of solutions of the algebraic equation. In such situation that we cannot derive the explicit roots of the algebraic equations, the recursion equation should be very useful. So we will explain another derivation of the recursion equation.

JHEP02(2019)097

We introduce a generating funtion $\mathcal{Z}_{g\geq 2}(s) := \sum_{g\geq 2} s^{g-1} Z_g$ of the genus g partiton functions Z_g $(g \geq 2)$ and rewrite this function

$$\begin{aligned} \mathcal{Z}_{g\geq 2}(s) &= \mathrm{Tr}\left[s\,(H\cdot C)\cdot\{I - s(H\cdot C)\}^{-1}\right] \\ &= -s\frac{d}{ds}\mathrm{Tr}\log\left[I - s(H\cdot C)\right] = -\frac{s\frac{d}{ds}F(s)}{F(s)}. \end{aligned} \tag{5.3}$$

with

$$F(s) := \det\left[I - s(H\cdot C)\right] = 1 + \sum_{m=1}^{l} A_m s^m\,, \tag{5.4}$$

$$l \;=\; \frac{1}{N!}\kappa(\kappa+1)\cdots(\kappa+N-1) \quad \text{for } \mathrm{U}(N)_\kappa\,, \tag{5.5}$$

where the set of coefficients A_m's in $F(s)$ can be expressed by using minors of the matrix $(H\cdot C)$

$$(H\cdot C) := \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,l} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,l} \\ \vdots & \vdots & & \vdots \\ a_{l,1} & a_{l,2} & \cdots & a_{l,l} \end{bmatrix}, \tag{5.6}$$

$$A_m \;=\; (-1)^m \sum_{j_1<j_2<\cdots<j_m} \det \begin{bmatrix} a_{j_1j_1} & a_{j_1j_2} & \cdots & a_{j_1j_m} \\ a_{j_2j_1} & a_{j_2j_2} & \cdots & a_{j_2j_m} \\ \vdots & \vdots & & \vdots \\ a_{j_mj_1} & a_{j_mj_2} & \cdots & a_{j_mj_m} \end{bmatrix}. \tag{5.7}$$

Then we can obtain the partition functions recursively by using (5.3);

$$\begin{aligned} Z_2 &= -A_1\,, \\ Z_{m+1} &= -mA_m - \sum_{k=1}^{m-1} A_k Z_{m+1-k} \quad (m = 2,3,\cdots,l)\,, \\ Z_{m+1} &= -\sum_{k=1}^{l} A_k Z_{m+1-k} \quad (m = l+1, l+2, l+3, \cdots)\,. \end{aligned} \tag{5.8}$$

We show several examples in lower genera,

$$\begin{aligned} Z_{g=2} &= -A_1\,, \qquad Z_{g=3} = -2A_2 + A_1^2\,, \\ Z_{g=4} &= -3A_3 + 3A_1A_2 - A_1^3\,, \\ Z_{g=5} &= -4A_4 + 2A_2^2 + 4A_3A_1 - 4A_1^2A_2 + A_1^4\,. \end{aligned} \tag{5.9}$$

5.2 Fate of level-rank duality

One of the most significant properties of the Verlinde algebra is the level-rank duality [43–47]. Unfortunately it seems that this duality cannot survive after the t-deformation of the algebra. We have obtained the explicit forms of genus g partition function Z_g by solving the characteristic equation for the handle operator for several models. We observe the level-rank duality at $t = 0$ is realized as the agreement of the genus g partition functions between $\mathrm{SU}(N)_\kappa$ and $\mathrm{U}(\kappa)_N$ as follows;

$$Z_g^{\mathrm{SU}(2)_3} = Z_g^{\mathrm{U}(3)_2} = 2 \cdot \left\{(5+\sqrt{5})^{g-1} + (5-\sqrt{5})^{g-1}\right\}, \quad Z_g^{\mathrm{U}(2)_3} = \left(\frac{3}{2}\right)^g \cdot Z_g^{\mathrm{U}(3)_2}, \tag{5.10}$$

$$Z_g^{\mathrm{SU}(2)_4} = Z_g^{\mathrm{U}(4)_2} = 3^{g-1} + 2\cdot 4^{g-1} + 2\cdot 12^{g-1}, \qquad Z_g^{\mathrm{U}(2)_4} = 2^g \cdot Z_g^{\mathrm{U}(4)_2}. \tag{5.11}$$

Note that if we compare $\mathrm{U}(N)_\kappa$ and $\mathrm{U}(\kappa)_N$, we need the additional correction factor $(\kappa/N)^g$, since their Witten indices (the dimensions of the Hilbert space on genus one curve) are different. If we compare (3.25) and (3.29) with (4.78) and (4.94) respectively, we see the t-dependence of the roots of the characteristic polynomial of the handle operator is completely different and there seems to be no simple relations.

5.3 Selection rules in $\mathrm{U}(N)_\kappa$ theory

There is the freedom of the R-charge r of the adjoint matter in our model ($\mathrm{U}(N)_\kappa$ theory). As can be seen from the general formula (4.4), the dependence of the R-charge r only comes from the difference product $\Delta(x,t)$ defined by (4.17) and hence the coefficients $\{g_\lambda(t)\}$ in the expansion (4.19) play an important role in analyzing the r-dependence of the model. In order to investigate properties of the coefficients $\{g_\lambda(t)\}$, we will work out selection rules of these coefficients.

First we decompose the set of partitions $\mathcal{P}_{N,\kappa}$ into κ subsets $\mathcal{P}_{N,\kappa}^{(n)}$ $(n = 0,1,2,\cdots,\kappa-1)$ according to the number of boxes $|\lambda| := \sum_{i=1}^{N} \lambda^i$ of a partition λ;

$$\begin{aligned} \mathcal{P}_{N,\kappa} &= \{\lambda = (\lambda^1,\lambda^2,\cdots,\lambda^N) \,|\, \kappa \geq \lambda^1 \geq \lambda^2 \geq \cdots \geq \lambda^N \geq 1\} = \bigcup_{n=0}^{\kappa-1} \mathcal{P}_{N,\kappa}^{(n)}, \\ \mathcal{P}_{N,\kappa}^{(n)} &:= \{\lambda \in \mathcal{P}_{N,\kappa} \,|\, |\lambda| \equiv n \mod \kappa\} \quad (n = 0,1,2,\cdots,\kappa-1). \end{aligned} \tag{5.12}$$

The Hall-Littlewood polynomials are symmetric homogeneous polynomials of Bethe roots x_a and their degrees are given by $|\lambda|$. We shall take a phase transformation on Bethe roots $x_a \to e^{i\theta} x_a$, $\theta \in \mathbb{R}$ $(a = 1,2,\ldots,N)$. Then the degree $|\lambda|$ of the polynomial can be read from the phase induced from this transformation: $P_\lambda(e^{i\theta}x,t) = e^{i\theta|\lambda|} P_\lambda(x,t)$. For example, $P_{(1,1,\cdots,1)} = x_1 x_2 \cdots x_N$ is a homogeneous polynomial with degree N.

Next we introduce a new polynomial $J(x,t)$ defined by

$$J(x,t) := \{P_{(1,1,\cdots,1)}\}^{\kappa M} \prod_{a,b=1}^{N} (1 - t x_a x_b^{-1}) \qquad (\kappa M - (N-1) > 0\,,\ M \in \mathbb{Z}) \tag{5.13}$$

in order to study properties of $\Delta(x,t)$. The difference product $\Delta(x,t)$ is invariant under the transformation $x_a \to e^{i\theta} x_a$, but $J(x,t)$ is a polynomial with degree $\kappa M N$. Let us expand $\{J(x,t)\}^r$ $(r = 0,1,2,3\cdots)$ in terms of the Hall-Littlewood polynomials $P_\mu(x,t)$;

$$\{J(x,t)\}^r = \sum_{|\mu| = \kappa M N r} a_\mu^{(r)} P_\mu(x,t)\,. \tag{5.14}$$

Because $\{J(x,t)\}^r$ is a homogeneous polynomial with degree (κMNr), the partitions that appear on the right hand side should satisfy $|\mu| = \kappa MNr$, namely, $\mu \notin \mathcal{P}_{N,\kappa}$ in general. But one can rewrite the expansion by using a set of relations summarized in appendix B and restrict the sum of partitions to $\lambda \in \mathcal{P}_{N,\kappa}$, $\{J(x,t)\}^r \equiv (1-t)^{rN} \sum_{\lambda\in\mathcal{P}_{N,\kappa}} g_\lambda^{(r)} P_\lambda(x,t)$. In this reduction, one has to use a set of operations τ, σ_i, σ_0 acting on λ [36] (See also appendix B). In addition, there is an identity $P_{(\kappa,\kappa,\cdots,\kappa)} = 1$ due to the set of Bethe equations (4.5). The number of boxes $|\lambda|$ of partitions is important information in our discussion of selection rules. While the number of boxes $|\lambda|$ may change under these operations, we can see these changes $\delta|\lambda|$ are multiples of κ ($\delta|\lambda| = \pm\kappa$ for τ-operation, $\delta|\lambda| = \pm\kappa N$ when one uses the identity $P_{(\kappa,\kappa,\cdots,\kappa)} = 1$, but $\delta|\lambda| = 0$ for σ_i,σ_0-operations). Hence it is natural to define the number of boxes $|\lambda|$ modulo κ. Since the degree of $\{J(x,t)\}^r$ is a multiple of κ, in the decomposition of $\{J(x,t)\}^r$ there appear only polynomials $P_\lambda(x,t)$ with $|\lambda| \equiv 0 \mod \kappa$, namely $\lambda \in \mathcal{P}^{(0)}_{N,\kappa}$;

$$\{J(x,t)\}^r \equiv (1-t)^{rN} \sum_{\lambda\in\mathcal{P}^{(0)}_{N,\kappa}} g_\lambda^{(r)} P_\lambda(x,t)\,. \tag{5.15}$$

This gives a selection rule of the coefficients $\{g_\lambda^{(r)}\}$ of the theory with the R-charge r. We obtain an important result that nonvanishing expansion coefficients $g_\lambda^{(r)} \neq 0$ should appear only in $\lambda \in \mathcal{P}^{(0)}_{N,\kappa}$;

$$\mathcal{B}^{(r)}_{N,\kappa} := \{\lambda \in \mathcal{P}_{N,\kappa}\,;\, g_\lambda^{(r)} \neq 0\} \to \mathcal{B}^{(r)}_{N,\kappa} \subset \mathcal{P}^{(0)}_{N,\kappa}\,. \tag{5.16}$$

It is possible that the size of the set $\mathcal{B}^{(r)}_{N,\kappa}$ depends on the R-charge r and it is desirable to find a criterion for the equality in (5.16).

Next let us study why the U(1) phase symmetry $e^{i\theta}x_a$ reduces to the discrete one $\mathbb{Z}_\kappa$. We have done the reduction by using the set of Bethe equations (4.5). When one performs the U(1) transformation, the phases $e^{i\kappa\theta}$ appear on the left hand sides in these equations and the set of equations is not invariant in general. However, if the parameter θ satisfies the condition $\theta = \frac{2\pi}{\kappa}m$ ($m \in \mathbb{Z}$), these Bethe equations are invariant. It is the reason why the U(1) symmerty reduces to $\mathbb{Z}_\kappa$.

As an application of this $\mathbb{Z}_\kappa$ charge, we can obtain selection rules for the couplings in $\mathrm{U}(N)_\kappa$ theory. The fusion couplings $C^\lambda_{\mu\nu}$ are defined as the structure constants of Hall-Littlewood polynomials (4.10). In the case of $r=0$, the three point functions $C^{(r=0)}_{\mu\nu\lambda}$ in (C.5) are given by using metric $\eta^{(r=0)}_{\mu\nu}$ in (C.1). By using conservation of $\mathbb{Z}_\kappa$ charge associated with partitions, we can write down the following selection rules

$$\begin{aligned}
(|\mu|+|\nu| \not\equiv |\lambda| \mod \kappa) &\to C^\lambda_{\mu\nu} = 0\,,\\
(|\mu|+|\nu| \not\equiv 0 \mod \kappa) &\to \eta^{(r=0)}_{\mu\nu} = 0\,,\\
(|\mu|+|\nu|+|\lambda| \not\equiv 0 \mod \kappa) &\to C^{(r=0)}_{\mu\nu\lambda} = 0\,,\\
|\lambda^*| \equiv -|\lambda| &\mod \kappa.
\end{aligned}$$

Next let us investigate the case of general R-charge r. In this case, three point functions $C^{(r)}_{\mu\nu\rho}$'s in (C.8) and metric $\eta^{(r)}_{\mu\nu}$ in (C.9) are defined by using $g^{(r)}_\lambda(t)$ in (C.7) and $C^\rho_{\mu\nu}$. By using conservation of $\mathbb{Z}_\kappa$ charge, we can write the following selection rules

$$\begin{aligned} (\lambda \not\equiv 0 \mod \kappa) &\to g_\lambda^{(r)} = 0\,, \\ (|\mu| + |\nu| + |\lambda| \not\equiv 0 \mod \kappa) &\to C_{\mu\nu\lambda}^{(r)} = 0\,, \\ (|\mu| + |\nu| \not\equiv 0 \mod \kappa) &\to \eta_{\mu\nu}^{(r)} = 0\,, \\ (|\mu| + |\nu| \not\equiv 0 \mod \kappa) &\to \eta^{\mu\nu}_{(r)} = 0\,. \end{aligned}$$

We arrange the couplings $C_{\lambda\mu}^{\nu}$ into matrices C_λ whose components are given as $(C_\lambda)_\mu{}^\nu = C_{\lambda\mu}^{\nu}$. In the case $|\lambda| \not\equiv 0 \mod \kappa$, trace part H_λ of the matrix C_λ vanishes, $H_\lambda := \mathrm{Tr}\ C_\lambda = \sum_{\mu \in \mathcal{P}_{N,\kappa}} C_{\lambda\mu}^{\mu} = 0$ and $H^\lambda = \sum_\rho \eta^{\lambda\rho}_{(r)}\ \mathrm{Tr}\ C_\rho = 0$ by using $\eta^{\lambda\rho}_{(r)} = 0$ for $|\lambda| + |\rho| \not\equiv 0 \mod \kappa$.

Now we shall investigate properties of partition functions by using conservation of $\mathbb{Z}_\kappa$ charge. The genus g partition functions Z_g are constructed by using the handle operator $(H \cdot C)_\mu{}^\nu = \sum_{\lambda \in \mathcal{P}_{N,\kappa}} H^\lambda(t) C_{\lambda\mu}^{\nu}$ and they are combined into a generating function $\mathcal{Z}_{g\geq 2}(s)$;

$$\begin{aligned} Z_g &= \sum_{\mu,\nu \in \mathcal{P}^{(0)}_{N,\kappa}} H^\mu(t) \left\{(H \cdot C)^{g-2}\right\}_\mu{}^\nu H_\nu(t) \qquad (g = 2, 3, 4, \cdots)\,, \\ \mathcal{Z}_{g\geq 2}(s) &= \sum_{g\geq 2} s^{g-1} Z_g = \sum_{\mu,\nu \in \mathcal{P}^{(0)}_{N,\kappa}} s\, H^\mu(t) \left[\{I - s(H \cdot C)\}^{-1}\right]_\mu{}^\nu H_\nu(t)\,. \end{aligned} \tag{5.17}$$

When we use the fact $H^\lambda(t) = 0$ for $\lambda \notin \mathcal{P}^{(0)}_{N,\kappa}$, we find that only $(H \cdot C)_\mu{}^\nu$ with $\mu, \nu \in \mathcal{P}^{(0)}_{N,\kappa}$ can contribute to the partition functions Z_g. So we introduce a minor $(\widehat{H \cdot C})_\mu{}^\nu := (H{\cdot}C)_\mu{}^\nu$ $(\mu, \nu \in \mathcal{P}^{(0)}_{N,\kappa})$ and denote their eigenvalues as $\{\hat{y}_i\}$. (We also write the multiplicity of each eigenvalue $\hat{y}_i$ as $\hat{m}_i \in \mathbb{Z}_{>0}$). The generating function $\mathcal{Z}_{g\geq 2}(s)$ in (5.17) is a function of the variable s and the pole structure is determined by $\{\hat{y}_i\}$. On the other hand, we have another expression of the partition function $Z_g = \mathrm{Tr}\left\{(H \cdot C)^{g-1}\right\} = \sum_{\mu \in \mathcal{P}_{N,\kappa}} \left\{(H \cdot C)^{g-1}\right\}_\mu{}^\mu$. These partition functions are expressed by using eigenvalues of the handle operator and are combined into the generating function

$$\mathcal{Z}_{g\geq 2}(s) = \sum_{g\geq 2} s^{g-1} Z_g = \sum_i \frac{m_i s y_i}{1 - s y_i}\,. \tag{5.18}$$

Because our models are topological field theories, two results from (5.17) and (5.18) should agree. By comparing the structure of poles in these equations we find the set of eigenvalues $\{\hat{y}_i\}$ of $(\widehat{H \cdot C})$ and $\{y_i\}$ of $(H \cdot C)$ should match. When one uses this fact, one can obtain the set of eigenvalues $\{y_i\}$ of $(H \cdot C)$ by analysing the minor $(\widehat{H \cdot C})$ and its eigenvalues $\{\hat{y}_i\}$. But the multiplicities of eigenvalues are not equal in general. It means that the essential properties of partition functions are determined by structure of the sector with vanishing $\mathbb{Z}_\kappa$ charge.

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A 2d TQFT and Frobenius algebra

We consider 2d TQFT obtained from supersymmetric field theory by topological twist. Let Q be one of the generators of supersymmetry, which becomes a nilpotent scalar charge after twisting. In the $\mathrm{U}(N)$ or $\mathrm{SU}(N)$ theories discussed in this paper, the equivalence classes of Q-closed operators form a finite dimensional commutative Frobenius algebra $\mathcal{A}$ which is realized as a quotient of the Weyl invariant Laurent polynomial ring

$$\mathcal{A} = \mathcal{R}[x_1, \cdots, x_N, x_1^{-1}, \cdots, x_N^{-1}]^{W(G)}/\mathcal{I}, \tag{A.1}$$

where $\mathcal{I}$ is the ideal generated by the saddle point equation and the coefficient ring $\mathcal{R}$ is generated by flavor Wilson loops. In $\mathcal{N} = 2$ Chern-Simons theories with a single adjoint matter the flavor symmetry is $\mathrm{U}(1)$ and $\mathcal{R} = \mathbb{Z}[[t]]$ can be identified with the ring of formal power series in the $\mathrm{U}(1)$ equivariant parameter t. Precisely speaking, all the rational functions of t in this paper should be expanded as a formal power series around $t = 0$. The ideal $\mathcal{I}$ is generated by the saddle point equation from which the factor $x^\alpha - 1$ is removed.

If we choose a basis $\{\mathcal{O}_\mu\}_{\mu \in L}$ of the algebra $\mathcal{A}$ of dimension $|L|$, the product of Q-closed operators is expanded in this basis

$$\mathcal{O}_\mu \mathcal{O}_\nu = \sum_{\lambda \in L} C^\lambda_{\mu\nu} \mathcal{O}_\lambda\,. \tag{A.2}$$

Here we emphasize that the structure constants $C^\lambda_{\mu\nu} \in \mathcal{R}$ do not depend on R-charges, flavor-flavor, gauge-R-symmetry CS levels, because the saddle point equations, and hence the ideal $\mathcal{I}$, are independent of these parameters. The Frobenius algebra $\mathcal{A}$ is equipped with a non degenerate bilinear form called metric $\eta : \mathcal{A} \otimes \mathcal{A} \to \mathcal{R}$, which is defined by the genus zero two point function.[14] In terms of the basis $\{\mathcal{O}_\mu\}$, the metric is written as

$$\eta_{\mu\nu} = \eta(\mathcal{O}_\mu, \mathcal{O}_\nu) := \langle \mathcal{O}_\mu \mathcal{O}_\nu \rangle_{g=0}\,. \tag{A.3}$$

We also define $\eta^{\mu\nu}$ as the inverse matrix of $\eta_{\mu\nu}$ which corresponds to a sphere with two right oriented holes. Then the bilinear form $\eta^{\mu\nu}$ and the structure constants $C^\lambda_{\mu\nu}$ satisfy the following relation;

$$C^\lambda_{\mu\nu} = \sum_{\rho \in L} C_{\mu\nu\rho} \eta^{\rho\lambda}, \tag{A.4}$$

where $C_{\lambda\mu\nu}$ is the genus zero three point function;

$$C_{\lambda\mu\nu} := \langle \mathcal{O}_\lambda \mathcal{O}_\mu \mathcal{O}_\nu \rangle_{g=0}. \tag{A.5}$$

By definition, $C_{\lambda\mu\nu}$ is totally symmetric under the permutations of λ, μ, ν and is associated to a sphere with three left oriented holes as figure 1. Note that $\eta_{\mu\nu} = C_{\emptyset\mu\nu}$ with

[14] The precise form of $\mathcal{R}$ depends on theories.

JHEP02(2019)097

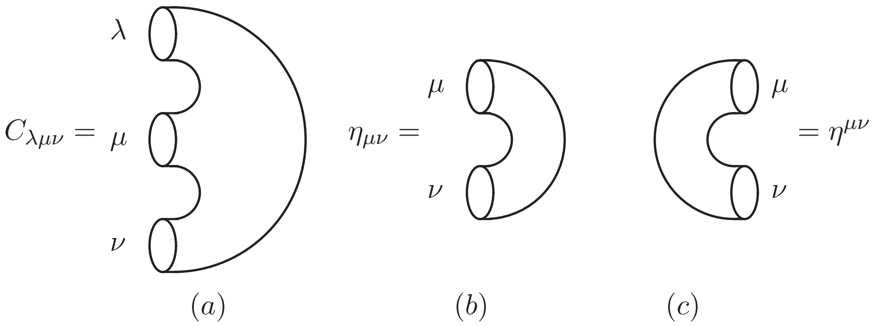


Figure 1. (a): $C_{\lambda\mu\nu}$ corresponds to a sphere with three left oriented holes. (b): $\eta_{\mu\nu}$ corresponds to a sphere with two left oriented holes. (c): $\eta^{\mu\nu}$ corresponds to a sphere with two right oriented holes.

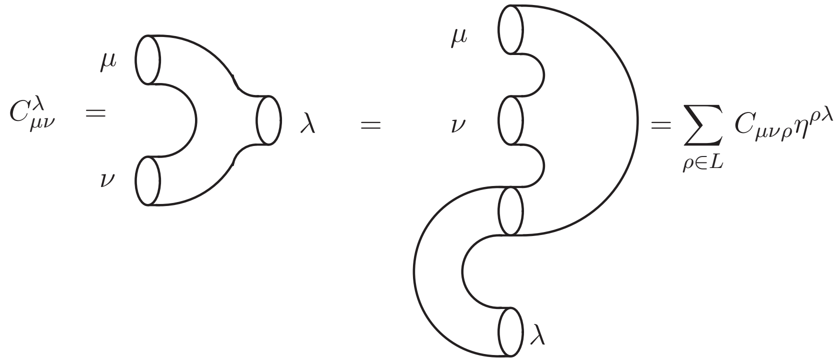


Figure 2. The contraction of indices corresponds to gluing holes on Riemann surfaces.

$\mathcal{O}_{\emptyset} := 1$. The specialization $\mathcal{O}_{\lambda} \to 1$ in the correlation function corresponds to closing a hole with the left orientation and the correlator reduces to the sphere partition function with two left oriented holes as figure 1. The contraction of upper and lower indices corresponds to gluing a left hole and a right hole. For example, see figure 2. The three point functions have to satisfy the associativity condition

$$\sum_{\lambda\in L} C^{\lambda}_{\mu\nu} C^{\sigma}_{\lambda\rho} = \sum_{\lambda\in L} C^{\lambda}_{\nu\rho} C^{\sigma}_{\mu\lambda} \,. \tag{A.6}$$

The associativity corresponds to figure 3. The associativity condition is equivalent to the existence of $S_{\mu\nu}$ such that

$$C^{\lambda}_{\mu\nu} = \sum_{\sigma\in L} \frac{S_{\mu\sigma} S_{\nu\sigma} S^{-1}_{\sigma\lambda}}{S_{\emptyset\sigma}} \,. \tag{A.7}$$

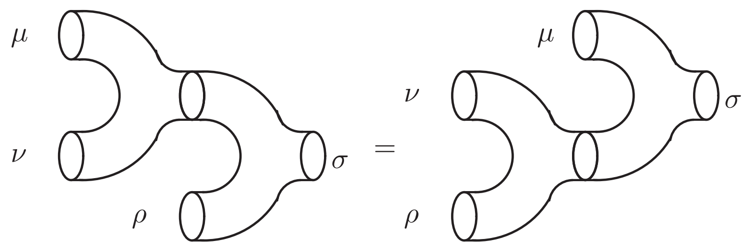
To write down the genus g partition function in a compact form, we introduce H^{λ} and the handle operator $(H \cdot C)$;

$$H^{\lambda} := \sum_{\mu,\nu\in L} \eta^{\mu\nu} C^{\lambda}_{\mu\nu} \,, \tag{A.8}$$

$$(H \cdot C)_{\nu}^{\ \mu} = \sum_{\rho\in L} H^{\rho} C^{\mu}_{\nu\rho} \,. \tag{A.9}$$

As shown in figure 4, H^{λ} corresponds to a genus one surface with a hole with right orientation and $(H \cdot C)_{\nu}^{\ \mu}$ corresponds to a genus one surface with two holes with left and right

JHEP02(2019)097



$$\sum_{\lambda} C^{\lambda}_{\mu\nu} C^{\sigma}_{\lambda\rho} = \sum_{\lambda} C^{\lambda}_{\nu\rho} C^{\sigma}_{\mu\lambda}$$

Figure 3. Associativity of the structure constant.

$$(H \cdot C)_{\nu}^{\ \mu} =$$

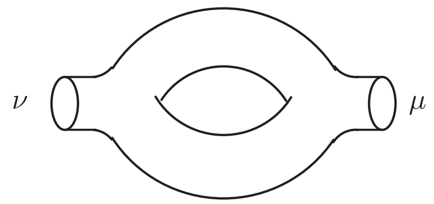


Figure 4. Handle creating operator.

orientation. Then the partition function $Z_g := \langle 1 \rangle_g$ for a closed Riemann surface with genus g is expressed

$$Z_g = \sum_{\nu \in L} \{(H \cdot C)^g\}_{\emptyset}^{\ \nu} \, \eta_{\nu\,\emptyset} \,, \tag{A.10}$$

where we define $(H \cdot C)^g$ as a product of matrices

$$\{(H \cdot C)^g\}_{\mu}^{\ \nu} := \sum_{\mu_1, \cdots \mu_{g-1} \in L} (H \cdot C)_{\mu}^{\ \mu_1} (H \cdot C)_{\mu_1}^{\ \mu_2} \cdots (H \cdot C)_{\mu_{g-2}}^{\ \mu_{g-1}} (H \cdot C)_{\mu_{g-1}}^{\ \nu} \,. \tag{A.11}$$

Since (A.10) is rewritten as

$$Z_g = \mathrm{Tr}(H \cdot C)^{g-1} = \sum_{\mu \in L} \{(H \cdot C)^{g-1}\}_{\mu}^{\ \mu} \,, \tag{A.12}$$

Z_g is expressed in terms of the roots y_i of the characteristic polynomial of the matrix $H \cdot C$;

$$Z_g = \sum_i m_i y_i^{g-1} \,, \tag{A.13}$$

with

$$\det\,(yI - H \cdot C) = \prod_i (y - y_i)^{m_i} \,. \tag{A.14}$$

Here I stands for the unit matrix of size $|L|$ and the integer m_i is the multiplicity of y_i.

Finally since all the higher genus correlation functions are obtained by gluing the genus zero two point function $\eta_{\mu\nu} = C_{\emptyset\mu\nu}$ and structure constants $C^{\lambda}_{\mu\nu}$, any correlation function is expressed in terms of $\eta_{\mu\nu}$ and $S_{\mu\nu}$;

$$\langle \mathcal{O}_{\lambda_1} \mathcal{O}_{\lambda_2} \cdots \mathcal{O}_{\lambda_n} \rangle_g = \sum_{\sigma \in L} \left(\sum_{\mu,\nu \in L} \frac{\eta^{\mu\nu} S_{\mu\sigma} S_{\nu\sigma}}{S_{\emptyset\sigma}^2} \right)^{g-1} \prod_{i=1}^{n} \frac{S_{\lambda_i \sigma}}{S_{\emptyset\sigma}} \,. \tag{A.15}$$

B Hall-Littlewood polynomial

The Hall-Littlewood polynomial $P_\lambda(x,t)$ is an important family of symmetric polynomials, which is regarded as a deformation of the Schur polynomial $s_\lambda(x)$. Let λ be a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N)$ of length (at most) N. We introduce the following polynomial with N variables $x = (x_1, x_2, \cdots, x_N)$;

$$R_\lambda(x,t) = \sum_{\omega \in \mathfrak{S}_N} \left(x_{\omega(1)}^{\lambda_1} \cdots x_{\omega(N)}^{\lambda_N} \prod_{a<b}^{N} \frac{x_{\omega(a)} - t x_{\omega(b)}}{x_{\omega(a)} - x_{\omega(b)}} \right), \tag{B.1}$$

where $\mathfrak{S}_N$ is the symmetric group of N objects and t is an indeterminate (parameter). Then we can define the Hall-Littlewood polynomial by

$$P_\lambda(x,t) = \frac{1}{v_\lambda(t)} R_\lambda(x,t), \qquad v_\lambda(t) := \prod_{i=0}^{\infty} \prod_{j=1}^{m_i(\lambda)} \frac{1-t}{1-t^j}. \tag{B.2}$$

Then $P_\lambda(x,t)$ gives a $\mathbb{Z}[t]$-basis of the ring of the symmetric polynomials $\mathbb{Z}[t][x_1, x_2, \cdots x_N]^{\mathfrak{S}_N}$. Note that $P_\lambda(x,t)$ provides interpolation between the Schur polynomial $s_\lambda(x)$ and the symmetric monomial $m_\lambda(x)$

$$P_\lambda(x,0) = s_\lambda(x), \qquad P_\lambda(x,1) = m_\lambda(x). \tag{B.3}$$

When one changes bases from the Hall-Littlewood polynomials $P_\lambda(x,t)$ to the Schur polynomials $s_\lambda(x)$, its efffect is realised as a matrix $K_{\lambda\mu}$;

$$s_\lambda(x) = \sum_{|\mu|=|\lambda|} K_{\lambda\mu}(t) P_\mu(x,t)\,. \tag{B.4}$$

This matrix has triangular form with respect to the dominance semi-ordering of partitions. $K_{\lambda\mu}(t)$ is called the Kostka polynomial ($K_{\lambda\mu}(1) = K_{\lambda\mu}$ are the Kostka numbers) and is ubiquitous in representation theories and combinatorics. One of the most important properties of $K_{\lambda\mu}(t)$ is that all the coefficients are non-negative integer, which gives us an interpretation of dimensions of appropriate modules.

In our method of computing the structure constants of $\mathrm{U}(N)$ equivariant Verlinde algebra with level κ, after substituting a root of the Bethe ansatz equation to x, we have to reduce the Hall-Littlewood polynomials $P_\lambda(x,t)$ for any partition λ of length N to a linear combination of $P_\mu(x,t)$, where μ runs only in $\mathcal{P}_{N,\kappa}$. We make use of the relations derived in [36] for this purpose. Mathematically these relations generate an ideal $\mathcal{I}_{N,\kappa}$ in the ring of Hall-Littlewood polynomials. This means that we identify the equivariant Verlinde algebra with a quotient of the ring of Hall-Littlewood polynomials by $\mathcal{I}_{N,\kappa}$. This algorithm does work, since any λ regarded as a weight vector of $\mathfrak{gl}(N)$, can be transformed into $\mathcal{P}_{N,\kappa}$ by the affine Weyl group $\widetilde{\mathfrak{S}}_{N,\kappa}$ with level κ. In this sense $\mathcal{P}_{N,\kappa}$ is a fundamental domain for $\widetilde{\mathfrak{S}}_{N,\kappa}$. The group $\widetilde{\mathfrak{S}}_{N,\kappa}$ is generated by σ_i $(1 \leq i \leq N-1)$, σ_0 and τ. The (right) action on a weight λ is defined by

$$\lambda \cdot \sigma_i := (\lambda_1, \lambda_2, \cdots, \lambda_{i+1}, \lambda_i, \cdots, \lambda_N)\,, \tag{B.5}$$

$$\lambda \cdot \sigma_0 := (\lambda_N + \kappa, \lambda_2, \cdots, \lambda_1 - \kappa)\,, \tag{B.6}$$

$$\lambda \cdot \tau := (\lambda_N + \kappa, \lambda_1, \lambda_2, \cdots, \lambda_{N-1})\,. \tag{B.7}$$

JHEP02(2019)097

If we substitute a Bethe root $x=(x_1,\cdots,x_N)$, we have the following identities;[15]

$$R_\lambda(x,t)\equiv R_{\lambda\cdot\tau}(x,t), \tag{B.8}$$

$$R_{\lambda\cdot\sigma_i}(x,t)\equiv tR_\lambda(x,t), \qquad \lambda_i-\lambda_{i+1}=1, \tag{B.9}$$

$$R_{\lambda\cdot\sigma_i}(x,t)\equiv tR_\lambda(x,t)+(t-1)R_{(\lambda_1,\cdots,\lambda_{i-1},\lambda_i+1,\lambda_i+1,\lambda_{i+2},\cdots,\lambda_N)}(x,t), \quad \lambda_i-\lambda_{i+1}=2, \tag{B.10}$$

$$\begin{aligned}R_{\lambda\cdot\sigma_0}(x,t)\equiv{}& tR_\lambda(x,t)-R_{(\lambda_1+1,\lambda_2,\cdots,\lambda_{N-1},\lambda_N-1)}(x,t)\\&+tR_{(\lambda_N-1+\kappa,\lambda_2,\cdots,\lambda_{N-1},\lambda_1+1-\kappa)}(x,t),\end{aligned} \tag{B.11}$$

and

$$P_\lambda(x,t)\equiv P_{\tilde\lambda}(x,t), \quad \lambda\in\mathcal{P}_{N,\kappa}, \tag{B.12}$$

where $\tilde\lambda$ is obtained by deleting all the rows of size κ.

C Couplings $C^{(r)}_{\mu\nu\rho}$ for generic R-charge r

In this appendix, we summarize properties of the three point function $C^{(r)}_{\mu\nu\rho}$ defined by Hall-Littlewood polynomials in the case of generic R-charge r.

First we put $r=0$ for simplicity and study metrics $\eta^{(r=0)}_{\mu\nu}$. The structure constants $C^\lambda_{\mu\nu}$ are defined from the product of Hall-Littlewood polynomials in (4.10), which are independent of R-charge r. The metric in this case $r=0$ is obtained in the paper [36]

$$\eta^{(r=0)}_{\mu\nu}:=\frac{\delta_{\mu\nu^*}}{b_\mu(t)}, \tag{C.1}$$

where $b_\mu(t)$ is defined by (4.13). The structure constants $C^\lambda_{\mu\nu}$ satisfy the following basic properties:

1. Symmetric property
$$C^\lambda_{\mu\nu}=C^\lambda_{\nu\mu}, \qquad \eta^{(r=0)}_{\mu\nu}=\eta^{(r=0)}_{\nu\mu}. \tag{C.2}$$

2. Existence of the unit operator “1” corresponding to $\emptyset=(\kappa,\cdots,\kappa)$
$$C^\nu_{\emptyset\mu}=\delta_\mu{}^\nu. \tag{C.3}$$

3. Associativity relation
$$\sum_{\alpha,\beta}C^\alpha_{\mu_1\mu_2}\eta^{(r=0)}_{\alpha\beta}C^\beta_{\mu_3\lambda}=\sum_{\alpha,\beta}C^\alpha_{\mu_1\mu_3}\eta^{(r=0)}_{\alpha\beta}C^\beta_{\mu_2\lambda}. \tag{C.4}$$

We can also define couplings $C^{(r=0)}_{\mu\nu\rho}$ with three subscripts by

$$C^{(r=0)}_{\mu\nu\rho}:=\sum_\lambda C^\lambda_{\mu\nu}\eta^{(r=0)}_{\lambda\rho}. \tag{C.5}$$

Then they are totally symmetric under the exchange of indices;

$$C^{(r=0)}_{\mu_1\mu_2\mu_3}=C^{(r=0)}_{\mu_2\mu_1\mu_3}, \qquad C^{(r=0)}_{\mu_1\mu_2\mu_3}=C^{(r=0)}_{\mu_1\mu_3\mu_2}, \tag{C.6}$$

[15] We use $\equiv$ to emphasize the equality on the space of Bethe roots.

which can be derived from the existence of the unit operator and the associativity relation above.

Next we shall consider couplings $C^{(r)}_{\mu\nu\rho}$ for integral R-charge r. In order to define them (see the formula (4.4)), we need the expansion of the product $\prod_{a,b}(1-tx_a x_b^{-1})^r$ by the Hall-Littlewood polynomials;

$$\prod_{a,b=1}^{N}(1-tx_a x_b^{-1})^r \equiv (1-t)^{rN}\sum_{\lambda\in\mathcal{P}_{N,\kappa}} g^{(r)}_\lambda(t)P_\lambda(x,t)\,. \tag{C.7}$$

Then the couplings $C^{(r)}_{\mu\nu\rho}$ and metrics $\eta^{(r)}_{\mu\nu}$ are related to the structure constants of the Hall-Littlewood polynomials as follows;

$$C^{(r)}_{\mu\nu\rho}(t) := (1-t)^{rN}\sum_{\lambda,\alpha,\beta} g^{(r)}_\lambda(t)C^\alpha_{\mu\nu}\eta^{(r=0)}_{\alpha\beta}C^\beta_{\rho\lambda} = \sum_\lambda C^\lambda_{\mu\nu}\eta^{(r)}_{\lambda\rho}\,, \tag{C.8}$$

$$\eta^{(r)}_{\mu\nu} := C^{(r)}_{\emptyset\mu\nu}\,, \qquad C^{(r=0)}_{\lambda\mu\nu} := C_{\lambda\mu\nu}\,. \tag{C.9}$$

Note that the R-charge dependence appears only through $(1-t)^{rN}g^{(r)}_\lambda(t)$ which determines the metric $\eta^{(r)}_{\lambda\rho}$. We can prove the fusion couplings $C^{(r)}_{\mu\nu\rho}$ are invariant under the exchange of the subscripts

$$C^{(r)}_{\mu_1\mu_2\mu_3} = C^{(r)}_{\mu_2\mu_1\mu_3}\,, \qquad C^{(r)}_{\mu_1\mu_2\mu_3} = C^{(r)}_{\mu_1\mu_3\mu_2}\,, \tag{C.10}$$

where the first relation is proved by using the symmetry of $C^\alpha_{\mu_1\mu_2}$ and the definition of $C^{(r)}_{\mu_1\mu_2\mu_3}$. The second relation can be shown by using the associativity relation. As a result of this symmetry, the metric $\eta^{(r)}_{\mu\nu} = C^{(r)}_{\emptyset\mu\nu}$ is symmetric $\eta^{(r)}_{\mu\nu} = \eta^{(r)}_{\nu\mu}$ as it should be. We can also show the associativity of the couplings $C^{(r)}_{\mu\nu\rho}$

$$\sum_{\alpha,\beta} C^\alpha_{\mu_1\mu_2}\eta^{(r)}_{\alpha\beta}C^\beta_{\mu_3\lambda} = \sum_{\alpha,\beta} C^\alpha_{\mu_1\mu_3}\eta^{(r)}_{\alpha\beta}C^\beta_{\mu_2\lambda}\,. \tag{C.11}$$

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