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A note on the possible existence of an instanton-like self-dual solution to lattice Euclidean gravity

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Abstract: The self-dual solution to lattice Euclidean gravity is constructed. In contrast to the well known Eguchi-Hanson solution to continuous Euclidean Gravity, the lattice solution is asymptotically *globally* Euclidean, i.e., the boundary of the space as $r \longrightarrow \infty$ is $S^3 = \mathrm{SU}(2)$.

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Contents

1 Introduction

Soon after the discovery of the self-dual (instanton) solution to the 4D Euclidean Yang-Mills theory [1], the self-dual solution to 4D Euclidean Gravity has been obtained [2, 3] (see also [4–7]). But the space-time boundary as $r \longrightarrow \infty$ is $S^3/\mathbb{Z}_2$ and not S^3 in the case of Eguchi-Hanson gravitational instanton, in contrast to the Yang-Mills one. Otherwise, a "cone-tipe" singularity (effective delta-function in the curvature at the instanton centre for $r = a$) would be necessary. This means that the space-time topology of the gravitational instanton solution differs crucially from the topology of a real space-time. Therefore, though the action of the Eguchi-Hanson solution is zeroth, the physical meaning of the solution is not clear.

I construct here the analogue of the Eguchi-Hanson self-dual solution to the 4D Euclidean lattice Gravity with zeroth action. The solution transforms locally into the Eguchi-Hanson solution as $r \longrightarrow \infty$. The reason is that the considered lattice theory transforms into Einstein theory for a long-wavelength limit. The remarkable fact is the discrete gravity self-dual solution wipes out a "cone-tipe" singularity at the center of instanton for the case the space-time boundary as $r \longrightarrow \infty$ is S^3. Thus, then the gravitational instantons would exist if the real space-time exhibits the granularity property at super small scales.

A preliminary version of the work has been published in [8].

2 Eguchi-Hanson self-dual solution

First of all it is necessary to describe shortly the Eguchi-Hanson self-dual solution to continuous Euclidean Gravity. Let γ^a, $\gamma^a\gamma^b + \gamma^b\gamma^a = 2\delta^{ab}$, $a = 1, 2, 3, 4$ be the Hermitian

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Dirac matrices 4×4 in spinor representation (σ^α, $\alpha=1,2,3$ are Pauli matrices):

$$\gamma^\alpha = \begin{pmatrix} 0 & i\sigma^\alpha \\ -i\sigma^\alpha & 0 \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\gamma^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^{ab} = \frac{1}{4}[\gamma^a,\gamma^b],$$
$$\sigma^{\alpha 4} = \frac{i}{2}\begin{pmatrix} \sigma^\alpha & 0 \\ 0 & -\sigma^\alpha \end{pmatrix}, \qquad \sigma^{\alpha\beta} = \frac{i\varepsilon_{\alpha\beta\gamma}}{2}\begin{pmatrix} \sigma^\gamma & 0 \\ 0 & \sigma^\gamma \end{pmatrix}. \tag{2.1}$$

Let's consider the pure 4D Euclidean Gravity action in the Palatini form (the independent variables are tetrad and connection):

$$\begin{aligned}
\mathfrak{A} &= -\frac{1}{l_P^2}\int \operatorname{tr}\gamma^5 R\wedge e\wedge e = -\frac{1}{l_P^2}\int \left\{R^\alpha_{(+)}\wedge E^\alpha_{(+)} - R^\alpha_{(-)}\wedge E^\alpha_{(-)}\right\},\\
R &\equiv 2(\mathrm{d}\,\omega + \omega\wedge\omega) = \frac{i\sigma^\alpha}{2}\begin{pmatrix} R^\alpha_{(+)} & 0 \\ 0 & R^\alpha_{(-)} \end{pmatrix},\\
\omega &\equiv \frac{1}{2}\sigma^{ab}\omega^{ab}_\mu\,\mathrm{d}\,x^\mu = \frac{i\sigma^\alpha}{2}\begin{pmatrix} \omega^\alpha_{(+)\mu} & 0 \\ 0 & \omega^\alpha_{(-)\mu} \end{pmatrix}\mathrm{d}\,x^\mu,\\
\omega^\alpha_{(\pm)} &\equiv \left\{\pm\,\omega^{\alpha 4}_\mu + \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\omega^{\beta\gamma}_\mu\right\}\mathrm{d}\,x^\mu,\\
R^\alpha_{(\pm)} &= 2\,\mathrm{d}\,\omega^\alpha_{(\pm)} - \varepsilon_{\alpha\beta\gamma}\omega^\beta_{(\pm)}\wedge\omega^\gamma_{(\pm)},\\
e &\equiv \gamma^a e^a_\mu\,\mathrm{d}\,x^\mu,\\
E^\alpha_{(\pm)} &\equiv \left\{\pm\,(e^\alpha_\lambda e^4_\rho - e^4_\lambda e^\alpha_\rho) + \varepsilon_{\alpha\beta\gamma}e^\beta_\lambda e^\gamma_\rho\right\}\mathrm{d}\,x^\lambda\wedge\mathrm{d}\,x^\rho.
\end{aligned} \tag{2.2}$$

One can take six 1-forms $\omega^\alpha_{(\pm)}$ as independent variables instead of six 1-forms ω^{ab}. Obviously, the representation (2.2) is consistent with the representation of the group $\mathrm{Spin}(4)\approx \mathrm{Spin}(4)_{(+)}\otimes\mathrm{Spin}(4)_{(-)}\approx \mathrm{SU}(2)_{(+)}\otimes\mathrm{SU}(2)_{(-)}$.

The following equations are equivalent

$$\omega = \pm\gamma^5\omega \longleftrightarrow \omega^{ab} = \mp\frac{1}{2}\varepsilon_{abcd}\omega^{cd} \longleftrightarrow \omega^\alpha_{(\pm)} = 0\,. \tag{2.3}$$

Eqs. (2.3) imply the following one:

$$R = \pm\gamma^5 R \longleftrightarrow R^{ab} = \mp\frac{1}{2}\varepsilon_{abcd}R^{cd} \longleftrightarrow R^\alpha_{(\pm)} = 0\,. \tag{2.4}$$

The action stationarity condition relative to the connection gives the equation

$$\delta\,\mathfrak{A}\,/\delta\omega^{ab}_\mu = 0 \longrightarrow \mathrm{d}\,e^a + \omega^{ab}\wedge e^b = 0\,, \tag{2.5}$$

which determines uniquely the connection forms for fixed forms e^a. Additionally, eq. (2.5) imply the algebraic Bianchi identity for Riemannian tensor, the combination of which with eq. (2.4) leads to the Einstein's equation

$$R_{ab} \equiv R^c_{acb} = 0\,. \tag{2.6}$$

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On the other hand, Einstein's equation is equivalent to the action stationarity condition relative to the forms e^a_μ:

$$R_{ab} = 0 \longleftrightarrow \delta\,\mathfrak{A}\,/\delta e^a_\mu = 0\,. \tag{2.7}$$

The question arises: why the additional eq. (2.3) does not come into conflict with eqs. (2.5) and (2.7)? To answer this question let's consider the case with the lower sign in eq. (2.3) when

$$\omega^\alpha_{(-)\mu} = 0\,. \tag{2.8}$$

The combination of the part of eqs. (2.5) $\delta\,\mathfrak{A}\,/\delta\omega^\alpha_{(-)\mu} = 0$ and eq. (2.8) gives the following 12 equations:

$$\varepsilon^{\mu\nu\lambda\rho}\partial_\nu E^\alpha_{(-)\lambda\rho} = 0\,. \tag{2.9}$$

Now we must solve the system of equations (2.7), (2.8), (2.9) and

$$\delta\,\mathfrak{A}\,/\delta\omega^\alpha_{(+)\mu} = 0\,. \tag{2.10}$$

Note that eqs. (2.5) and (2.7) do not fix the variables ω^{ab}_μ, e^a_μ completely but up to the gauge (orthogonal) transformations. Here the gauge group leaves 6 unfixed functions.

Eqs. (2.9) do not fix the quantities $E^\alpha_{(-)\lambda\rho}$ completely but up to summands of the forme $\left(\partial_\lambda\Psi^\alpha_{(-)\rho} - \partial_\rho\Psi^\alpha_{(-)\lambda}\right)$ where $\Psi^\alpha_{(-)\lambda}$ are 3 arbitrary vector fields (12 functions altogether). But each of three vector fields $\Psi^\alpha_{(-)\lambda}$ contains only 3 independent functions due to invariance of the expression $\left(\partial_\lambda\Psi^\alpha_{(-)\rho} - \partial_\rho\Psi^\alpha_{(-)\lambda}\right)$ relative to the changes $\Psi^\alpha_{(-)\lambda} \longrightarrow \Psi^\alpha_{(-)\lambda} + \partial_\lambda\phi^\alpha_{(-)}$. As a result, eqs. (2.9) fix no more than additional $12 - 3\times 3 = 3$ functions. This means that the gauge subgroup $\mathrm{Spin}(4)_{(-)}$ is broken by eq. (2.8). So we see that the system of equations (2.7)–(2.10) is consistent, though it fixes the gauge subgroup $\mathrm{Spin}(4)_{(-)}$. The Eguchi-Hanson solution is the simplest solution of the system. Let's write out this solution [2, 3].

Let $x^i = (r,\,\theta,\,\varphi,\,\psi)$, where $(\theta,\,\varphi,\,\psi)$ be the Euler angles. The cartesian coordinates x^μ in $\mathbb{R}^{\mathbf{4}}$ are connected with the coordinates x^i as follows:

$$\begin{aligned} z_1 &\equiv x^1 + ix^2 = r\cos\frac{\theta}{2}\exp\left[\frac{i}{2}(\psi+\varphi)\right],\\ z_2 &\equiv x^3 + ix^4 = r\sin\frac{\theta}{2}\exp\left[\frac{i}{2}(\psi-\varphi)\right]. \end{aligned} \tag{2.11}$$

There is a one-to-one correspondence between these two coordinate systems if the Euler angles vary in the ranges

$$0 \le \theta \le \pi\,, \qquad 0 \le \varphi \le 2\pi\,, \qquad 0 \le \psi \le 4\pi\,. \tag{2.12}$$

The Eguchi-Hanson solution for the metrics $\mathrm{d}\,s^2 \equiv (e^a_\mu\,\mathrm{d}\,x^\mu)^2$ has the form

$$\begin{aligned} e^a_\mu\,\mathrm{d}\,x^\mu &= \begin{pmatrix} \frac{r}{2}(\sin\psi\,\mathrm{d}\,\theta - \sin\theta\cos\psi\,\mathrm{d}\,\varphi)\\ \frac{r}{2}(\cos\psi\,\mathrm{d}\,\theta + \sin\theta\sin\psi\,\mathrm{d}\,\varphi)\\ -\frac{r}{2}g(r)(\cos\theta\,\mathrm{d}\,\varphi + \mathrm{d}\,\psi)\\ g(r)^{-1}\,\mathrm{d}\,r \end{pmatrix},\\ g(r) &= \sqrt{1 - \frac{a^4}{r^4}}\,, \qquad r \ge a\,. \end{aligned} \tag{2.13}$$

For $a = 0$, the metrics (2.13) transforms to the 4D Euclidean metrics in Euler angles on the S^3 with ranges (2.12). But $0 \leq \psi \leq 2\pi$ for the Eguchi-Hanson solution (when $a \neq 0$) since the points with coordinates ψ and $(\psi + 2\pi)$ and the same (r, θ, φ) are identified. Otherwise, in order for the Chern-Gauss-Bonnet theorem to be satisfied in the case (2.12), a "cone-tipe" singularity (effective delta-function in the curvature at the instanton centre for $r = a$) would be necessary (see [3]).

The connection 1-forms are given by eqs. (2.8) and:

$$\begin{aligned}
\omega^1_{(+)} &= \frac{2g}{r} e^1 = g(\sin\psi\, \mathrm{d}\,\theta - \sin\theta\cos\psi\, \mathrm{d}\,\varphi)\,, \\
\omega^2_{(+)} &= \frac{2g}{r} e^2 = g(\cos\psi\, \mathrm{d}\,\theta + \sin\theta\sin\psi\, \mathrm{d}\,\varphi)\,, \\
\omega^3_{(+)} &= 2\left(\frac{2}{rg} - \frac{g}{r}\right) e^3 = -\left(1 + \frac{a^4}{r^4}\right)(\cos\theta\, \mathrm{d}\,\varphi + \mathrm{d}\,\psi)\,.
\end{aligned} \tag{2.14}$$

We have

$$\int_{r=\mathrm{Const}\to\infty} \left(\frac{1}{2}\omega^1_{(+)}\right) \wedge \left(\frac{1}{2}\omega^2_{(+)}\right) \wedge \left(\frac{1}{2}\omega^3_{(+)}\right) = -\pi^2 \tag{2.15}$$

and

$$\int_{r\longrightarrow a+0} \left(\frac{1}{2}\omega^1_{(+)}\right) \wedge \left(\frac{1}{2}\omega^2_{(+)}\right) \wedge \left(\frac{1}{2}\omega^3_{(+)}\right) = 0 \tag{2.16}$$

for the range $0 \leq \psi \leq 2\pi$ and orientation θ, φ, ψ, r. The integral (2.15) would be equal to $(-\pi^2)$ for any $0 < r = \mathrm{Const} < \infty$ in the case $a = 0$ (the Euclidean metrics in Euler angles). Thus, the boundary conditions (2.15)–(2.16) determine the instanton Eguchi-Hanson solution with the same integration constant a as in the relation (2.16). This means that the system of equations (2.7)–(2.10) together with the boundary conditions (2.15)–(2.16) possess unique solution (2.13) for the centrally symmetrical metrics anzats with the same integration constant a as in the relation (2.16).

Note that

$$\frac{i\sigma^\alpha}{2}\omega^\alpha_{(+)} = U^{-1}\,\mathrm{d}\,U \quad \text{as} \quad r \longrightarrow \infty\,, \tag{2.17}$$

where

$$U = \exp\left(-\frac{i\sigma^3}{2}\varphi\right)\exp\left(\frac{i\sigma^2}{2}\theta\right)\exp\left(-\frac{i\sigma^3}{2}\psi\right). \tag{2.18}$$

According to the eq. (2.17)

$$\frac{1}{12}\,\mathrm{tr}(U^{-1}\,\mathrm{d}\,U)\wedge(U^{-1}\,\mathrm{d}\,U)\wedge(U^{-1}\,\mathrm{d}\,U) = \left(\frac{1}{2}\omega^1_{(+)}\right) \wedge \left(\frac{1}{2}\omega^2_{(+)}\right) \wedge \left(\frac{1}{2}\omega^3_{(+)}\right) \quad \text{as} \quad r \longrightarrow \infty\,. \tag{2.19}$$

Taking into account eqs. (2.15) and (2.19), we obtain for the angle ranges (2.12):

$$\frac{1}{12}\,\mathrm{tr}\int(U^{-1}\,\mathrm{d}\,U)\wedge(U^{-1}\,\mathrm{d}\,U)\wedge(U^{-1}\,\mathrm{d}\,U) = -2\pi^2. \tag{2.20}$$

This equality means that eq. (2.18) gives the smooth mapping of the space-time hypersurface S^3 prescribed by the Euler angles (see eqs. (2.11) with a fixed parameter r) into the SU(2) group space, and the degree of the mapping is equal to (-1).

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Write out also the Riemannian curvature 2-form:

$$
\begin{aligned}
R^{\alpha}_{(-)} &= 0\,,\\
R^{1}_{(+)} &= -\frac{8a^4}{r^6}(e^1\wedge e^4 + e^2\wedge e^3)\,,\\
R^{2}_{(+)} &= \frac{8a^4}{r^6}(e^1\wedge e^3 - e^2\wedge e^4)\,,\\
R^{3}_{(+)} &= \frac{16a^4}{r^6}(e^1\wedge e^2 + e^3\wedge e^4)\,.
\end{aligned}
\tag{2.21}
$$

3 The lattice gravity model

The next step is to adumbrate the model of lattice gravity which is used here. A detailed description of the model is given in [9–11].

The orientable 4-dimensional simplicial complex and its vertices are designated as $\mathfrak{K}$ and $a_{\mathcal{V}}$, the indices $\mathcal{V} = 1, 2, \ldots, \mathfrak{N} \to \infty$ and $\mathcal{W}$ enumerate the vertices and 4-simplices, correspondingly. It is necessary to use the local enumeration of the vertices $a_{\mathcal{V}}$ attached to a given 4-simplex: the all five vertices of a 4-simplex with index $\mathcal{W}$ are enumerated as $a_{\mathcal{W}i}$, $i = 1, 2, 3, 4, 5$. The later notations with extra index $\mathcal{W}$ indicate that the corresponding quantities belong to the 4-simplex with index $\mathcal{W}$. The Levi-Civita symbol with in pairs different indexes $\varepsilon_{\mathcal{W}ijklm} = \pm 1$ depending on whether the order of vertices $s^4_{\mathcal{W}} = a_{\mathcal{W}i}a_{\mathcal{W}j}a_{\mathcal{W}k}a_{\mathcal{W}l}a_{\mathcal{W}m}$ defines the positive or negative orientation of 4-simplex $s^4_{\mathcal{W}}$. An element of the group Spin(4) and an element of the Clifford algebra

$$
\begin{aligned}
&\Omega_{\mathcal{W}ij} = \Omega^{-1}_{\mathcal{W}ji} = \exp(\omega_{\mathcal{W}ij})\,, \qquad \omega_{\mathcal{W}ij} \equiv \frac{1}{2}\sigma^{ab}\omega^{ab}_{\mathcal{W}ij}\,,\\
&\hat{e}_{\mathcal{W}ij} = \hat{e}^{\dagger}_{\mathcal{W}ij} \equiv e^a_{\mathcal{W}ij}\gamma^a \equiv -\Omega_{\mathcal{W}ij}\hat{e}_{\mathcal{W}ji}\Omega^{-1}_{\mathcal{W}ij}
\end{aligned}
\tag{3.1}
$$

are assigned for each oriented 1-simplex $a_{\mathcal{W}i}a_{\mathcal{W}j}$. The lattice analog of the action (2.2) has the form

$$
\mathfrak{A} = \frac{1}{5\times 24}\sum_{\mathcal{W}}\sum_{i,j,k,l,m}\varepsilon_{\mathcal{W}ijklm}\,\mathrm{tr}\,\gamma^5\left\{-\frac{1}{2\,l_P^2}\Omega_{\mathcal{W}mi}\Omega_{\mathcal{W}ij}\Omega_{\mathcal{W}jm}\hat{e}_{\mathcal{W}mk}\hat{e}_{\mathcal{W}ml}\right\}. \tag{3.2}
$$

This action is invariant relative to the gauge transformations

$$
\tilde{\Omega}_{\mathcal{W}ij} = S_{\mathcal{W}i}\Omega_{\mathcal{W}ij}S^{-1}_{\mathcal{W}j}\,, \qquad \tilde{e}_{\mathcal{W}ij} = S_{\mathcal{W}i}\,e_{\mathcal{W}ij}\,S^{-1}_{\mathcal{W}i}\,, \qquad S_{\mathcal{W}i}\in \mathrm{Spin}(4)\,. \tag{3.3}
$$

It is natural to interpret the quantity

$$
l^2_{\mathcal{W}ij} \equiv \frac{1}{4}\,\mathrm{tr}(\hat{e}_{\mathcal{W}ij})^2 = \sum_{a=1}^{4}(e^a_{\mathcal{W}ij})^2 \sim l_P^2 \tag{3.4}
$$

as the square of the length of the edge $a_{\mathcal{W}i}a_{\mathcal{W}j}$, and the parameter l_P is of the order of the lattice spacing. Thus, the geometric properties of a simplicial complex prove to be completely defined.

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Now, let us show in the limit of slowly varying fields, that the action (3.2) reduces to the continuous gravity action (2.2).

Consider a certain $4D$ sub-complex of complex $\mathfrak{K}$ with the trivial topology of four-dimensional disk. Realize geometrically this sub-complex in $\mathbb{R}^4$. Suppose that the geometric realization is an almost smooth four-dimensional surface.[1] Thus each vertex of the sub-complex acquires the coordinates x^μ which are the coordinates of the vertex image in $\mathbb{R}^4$:

$$x^\mu_{\mathcal{W}i} = x^\mu_{\mathcal{V}} \equiv x^\mu(a_{\mathcal{W}i}) \equiv x^\mu(a_{\mathcal{V}})\,, \qquad \mu = 1,\,2,\,3,\,4 \tag{3.5}$$

We stress that these coordinates are defined only by their vertices rather than by the higher dimension simplices to which these vertices belong; moreover, the correspondence between the vertices from the considered subset and the coordinates (3.5) is one-to-one.

We have

$$|\, x^\mu_{\mathcal{W}i} - x^\mu_{\mathcal{W}j} \,| \sim l_P\,. \tag{3.6}$$

It is evident that the four vectors

$$\mathrm{d}\, x^\mu_{\mathcal{W}ji} \equiv x^\mu_{\mathcal{W}i} - x^\mu_{\mathcal{W}j}\,, \qquad i = 1,\,2,\,3,\,4 \tag{3.7}$$

are linearly independent and

$$\begin{vmatrix} \mathrm{d}\, x^1_{\mathcal{W}m1} & \mathrm{d}\, x^2_{\mathcal{W}m1} & \dots & \mathrm{d}\, x^4_{\mathcal{W}m1} \\ \dots & \dots & \dots & \dots \\ \mathrm{d}\, x^1_{\mathcal{W}m4} & \mathrm{d}\, x^2_{\mathcal{W}m4} & \dots & \mathrm{d}\, x^4_{\mathcal{W}m4} \end{vmatrix} \gtrless 0\,, \tag{3.8}$$

depending on whethe the frame $(X^{\mathcal{W}}_{m\,1}, \dots, X^{\mathcal{W}}_{m\,4})$ is positively or negatively oriented. Here, the differentials of coordinates (3.7) correspond to one-dimensional simplices $a_{\mathcal{W}j}a_{\mathcal{W}i}$, so that, if the vertex $a_{\mathcal{W}j}$ has coordinates $x^\mu_{\mathcal{W}j}$, then the vertex $a_{\mathcal{W}i}$ has the coordinates $x^\mu_{\mathcal{W}j} + \mathrm{d}\, x^\mu_{\mathcal{W}ji}$.

In the continuous limit, the holonomy group elements (3.1) are close to the identity element, so that the quantities ω^{ab}_{ij} tend to zero being of the order of $O(\mathrm{d}\, x^\mu)$. Thus one can consider the following system of equation for $\omega_{\mathcal{W}m\mu}$

$$\omega_{\mathcal{W}m\mu}\, \mathrm{d}\, x^\mu_{\mathcal{W}mi} = \omega_{\mathcal{W}mi}\,, \qquad i = 1,\,2,\,3,\,4\,. \tag{3.9}$$

In this system of linear equation, the indices $\mathcal{W}$ and m are fixed, the summation is carried out over the index μ, and index runs over all its values. Since the determinant (3.8) is nonzero, the quantities $\omega_{\mathcal{W}m\mu}$ are defined uniquely. Suppose that a one-dimensional simplex $X^{\mathcal{W}}_{m\,i}$ belongs to four-dimensional simplices with indices $\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_r$. Introduce the quantity

$$\omega_\mu\left(\frac{1}{2}\,(x_{\mathcal{W}m} + x_{\mathcal{W}i})\right) \equiv \frac{1}{r}\,\{\omega_{\mathcal{W}_1 m\mu} + \,\dots\, + \omega_{\mathcal{W}_r m\mu}\}\,, \tag{3.10}$$

[1] Here, by an almost smooth surface, we mean a piecewise smooth surface consisting of flat four-dimensional simplices, such that the angles between adjacent 4-simplices tend to zero and the sizes of these simplices are commensurable.

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which is assumed to be related to the midpoint of the segment $[x^{\mu}_{\mathcal{W}m},\, x^{\mu}_{\mathcal{W}i}\,]$. Recall that the coordinates $x^{\mu}_{\mathcal{W}i}$ as well as the differentials (3.7) depend only on vertices but not on the higher dimensional simplices to which these vertices belong. According to the definition, we have the following chain of equalities

$$\omega_{\mathcal{W}_1\, mi} = \omega_{\mathcal{W}_2\, mi} = \ldots = \omega_{\mathcal{W}_r\, mi}\,. \tag{3.11}$$

It follows from (3.7) and (3.9)–(3.11) that

$$\omega_{\mu}\left(x_{\mathcal{W}m} + \frac{1}{2}\,\mathrm{d}\,x_{\mathcal{W}mi}\right)\mathrm{d}\,x^{\mu}_{\mathcal{W}mi} = \omega_{\mathcal{W}mi}\,. \tag{3.12}$$

The value of the field element ω_{μ} in (3.12) is uniquely defined by the corresponding one-dimensional simplex.

Next, we assume that the fields ω_{μ} smoothly depend on the points belonging to the geometric realization of each four-dimensional simplex. In this case, the following formula is valid up to $O\big((\mathrm{d}\,x)^2\big)$ inclusive

$$\Omega_{\mathcal{W}mi}\,\Omega_{\mathcal{W}ij}\,\Omega_{\mathcal{W}jm} = \exp\left[\frac{1}{2}\,\mathfrak{R}_{\mu\nu}(x_{\mathcal{W}m})\,\mathrm{d}\,x^{\mu}_{\mathcal{W}mi}\,\mathrm{d}\,x^{\nu}_{\mathcal{W}mj}\right], \tag{3.13}$$

where

$$\mathfrak{R}_{\mu\nu} = \partial_{\mu}\omega_{\nu} - \partial_{\nu}\omega_{\mu} + [\omega_{\mu},\, \omega_{\nu}\,]\,. \tag{3.14}$$

When deriving formula (3.13), we used the Hausdorff formula.

In exact analogy with (3.9), let us write out the following relations for a tetrad field without explanations

$$\hat{e}_{\mathcal{W}m\mu}\,\mathrm{d}\,x^{\mu}_{\mathcal{W}mi} = \hat{e}_{\mathcal{W}mi}\,. \tag{3.15}$$

Applying formulas (3.13)–(3.15) to the discrete action (3.2) and changing the summation to integration we find that in the long-wavelength limit the lattice action (3.2) transforms into continuous action (2.2) and any information about lattice is forgotten in the main approximation.

4 The self-dual solutions to lattice gravity

Now let us consider the self-dual solution to lattice gravity. We have the lattice analogue of eqs. (2.5) and (2.7):

$$\delta\,\mathfrak{A}\,/\delta\omega^{\alpha}_{(\pm)\mathcal{W}mi} = 0\,, \tag{4.1}$$

$$\delta\,\mathfrak{A}\,/\delta e^{a}_{\mathcal{W}mi} = 0\,. \tag{4.2}$$

Because of the action (3.2) is a homogeneous quadratic function of the variables $\{e\}$, so

$$2\,\mathfrak{A} = \sum_{\{e\}} e^{a}_{\mathcal{W}mi}(\delta\,\mathfrak{A}\,/\delta e^{a}_{\mathcal{W}mi}) = 0 \tag{4.3}$$

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on the mass shell according to Euler theorem. Let's impose the additional conditions (compare with eqs. (2.8))

$$\omega^{\alpha}_{(-)\mathcal{W}mi} = 0\,. \tag{4.4}$$

Combining eqs. (4.1) with index $(-)$ and (4.4) we obtain:

$$\begin{aligned}
\frac{\delta\,\mathfrak{A}}{\delta\omega^{\alpha}_{(-)\mathcal{W}mi}}\bigg|_{\omega^{\alpha}_{(-)}=0} &= -\frac{2}{l_P^2}\sum_{\mathcal{W}'}\sum_{j,k,l}\varepsilon_{\mathcal{W}'mijkl}\Big\{E^{\alpha}_{(-)\mathcal{W}'[jkl]} \\
&\qquad + \frac{1}{6}\big[\big(E^{\alpha}_{(-)\mathcal{W}'m[kl]} + E^{\alpha}_{(-)\mathcal{W}'m[lj]} + E^{\alpha}_{(-)\mathcal{W}'m[jk]}\big) \\
&\qquad\qquad + (m \leftrightarrow i)\big]\Big\} = 0\,, \\
E^{\alpha}_{*[mkl]} &\equiv \frac{1}{3}\big(E^{\alpha}_{*m[kl]} + E^{\alpha}_{*k[lm]} + E^{\alpha}_{*l[mk]}\big)\,, \\
E^{\alpha}_{(\pm)*m[kl]} &\equiv \mp(e^{\alpha}_{*mk}e^{4}_{*ml} - e^{4}_{*mk}e^{\alpha}_{*ml}) + \varepsilon_{\alpha\beta\gamma}e^{\beta}_{*mk}e^{\gamma}_{*ml}\,.
\end{aligned} \tag{4.5}$$

The index $\mathcal{W}'$ in (4.5) enumerates all 4-simplices which contain a marked 1-simplex $a_{\mathcal{W}m}a_{\mathcal{W}i}$. As in continuous case, the system of equations (4.1), (4.2) and (4.4) is equivalent to the system of equations (4.1) with index $(+)$, (4.2), (4.4) and (4.5). It will be shown that the square brackets give null equation under the sum (4.5). Therefore only the first term in the parentheses in eq. (4.5) is significant.

Equations (4.5) do not fix the quantity $E^{\alpha}_{(-)\mathcal{W}'[jkl]}$ completely but up to summands of the kind

$$E^{\alpha}_{(-)\mathcal{W}'[jkl]} \longrightarrow E^{\alpha}_{(-)\mathcal{W}'[jkl]} + \big(\Psi^{\alpha}_{(-)\mathcal{W}jk} + \Psi^{\alpha}_{(-)\mathcal{W}kl} + \Psi^{\alpha}_{(-)\mathcal{W}lj}\big)\,, \tag{4.6}$$

and the lattice 1-form $\Psi^{\alpha}_{(-)\mathcal{W}mk} = -\Psi^{\alpha}_{(-)\mathcal{W}km}$ can be varied according to

$$\Psi^{\alpha}_{(-)\mathcal{W}mk} \longrightarrow \Psi^{\alpha}_{(-)\mathcal{W}mk} + \big(\phi^{\alpha}_{(-)\mathcal{W}k} - \phi^{\alpha}_{(-)\mathcal{W}m}\big)\,. \tag{4.7}$$

It follows from here that eqs. (4.5) fix no more than 3 additional real-valued parameters at each vertex of complex, leading to the fixation of gauge subgroup $\mathrm{Spin}(4)_{(-)}$.

Let's prove the given statements. For this end one must check that the equation

$$\sum_{\mathcal{W}'}\Big\{\sum_{j,k,l}\varepsilon_{\mathcal{W}'mijkl}\big(\Psi^{\alpha}_{(-)\mathcal{W}'jk} + \Psi^{\alpha}_{(-)\mathcal{W}'kl} + \Psi^{\alpha}_{(-)\mathcal{W}'lj}\big)\Big\} = 0 \tag{4.8}$$

is satisfied identically. It is evident that the braces in eq. (4.8) vanishes identically at each fixed value of index $\mathcal{W}'$ if $\Psi^{\alpha}_{(-)\mathcal{W}'mk} = \big(\phi^{\alpha}_{(-)\mathcal{W}'k} - \phi^{\alpha}_{(-)\mathcal{W}'m}\big)$.

Consider two adjacent positively oriented 4-simplices

$$\begin{aligned}
s^4_{\mathcal{W}} &= a_{\mathcal{W}m}a_{\mathcal{W}i}a_{\mathcal{W}j}a_{\mathcal{W}k}a_{\mathcal{W}l}\,, \\
s^4_{\mathcal{W}'} &= a_{\mathcal{W}'m}a_{\mathcal{W}'i}a_{\mathcal{W}'k}a_{\mathcal{W}'j}a_{\mathcal{W}'l'}\,, \\
a_{\mathcal{W}m} &= a_{\mathcal{W}'m}\,, \qquad a_{\mathcal{W}i} = a_{\mathcal{W}'i}\,, \qquad a_{\mathcal{W}j} = a_{\mathcal{W}'j}\,, \\
a_{\mathcal{W}k} &= a_{\mathcal{W}'k}\,, \qquad a_{\mathcal{W}l} \neq a_{\mathcal{W}'l'}\,,
\end{aligned} \tag{4.9}$$

so that

$$\varepsilon_{\mathcal{W}mijkl} = \varepsilon_{\mathcal{W}'mikjl'} = 1\,. \tag{4.10}$$

Select from the sum (4.8) two summands corresponding to the 4-simplices (4.9):

$$\sum_{j,k,l} \varepsilon_{\mathcal{W}mijkl} \big(\Psi^{\alpha}_{(-)\mathcal{W}jk} + \Psi^{\alpha}_{(-)\mathcal{W}kl} + \Psi^{\alpha}_{(-)\mathcal{W}lj}\big) + \sum_{j,k,l'} \varepsilon_{\mathcal{W}'mikjl'} \big(\Psi^{\alpha}_{(-)\mathcal{W}'kj} + \Psi^{\alpha}_{(-)\mathcal{W}'jl'} + \Psi^{\alpha}_{(-)\mathcal{W}'l'k}\big) \,. \tag{4.11}$$

We see that the quantities $\Psi^{\alpha}_{(-)\mathcal{W}mk}$ belonging to the common 3-simplices of the adjacent 4-simplices cancel on in (4.11) as a consequence of eqs. (4.10) and $\big(\Psi^{\alpha}_{(-)\mathcal{W}jk} + \Psi^{\alpha}_{(-)\mathcal{W}'kj}\big) = 0$. Note that each quantity $\Psi^{\alpha}_{(-)\mathcal{W}'jk}$ in parentheses in (4.8) belongs to the 3-simplex which is common to *two* adjacent 4-simplices. If not, the cavities and boundaries would be in the simplicial complex, but such complexes are not considered here. Thus, the sum (4.8) is equal to zero identically.[2] This means that the system of equations (4.1) with index (+), (4.2), (4.4) and (4.5) is self-consistent and it fixes in part the gauge as well as in continuous case. According to the eqs. (4.2) and (4.3) the action of any solution of this system of equations is equal to zero.

5 Lattice analogue of the Eguchi-Hanson solution

Now proceed to study the lattice analog of the Eguchi-Hanson solution. But it is impossible to give the irregular lattice solution in an explicit form in contrast to the continuous case. Thus, the problem reduces to the solution existence proof and finding of its asymptotics.

Suppose that the complex $\mathfrak{K}$ can be considered as a triangulation of $\mathbb{R}^4$.

Let's introduce the following notations: $\mathfrak{k} \subset \mathfrak{K}$ means a finite sub-complex containing the centre of instanton with the boundary $\partial\,\mathfrak{k} = \mathfrak{S} \approx S^3$; $\mathfrak{S}_\infty$ means the boundary of extra-large sub-complex of complex $\mathfrak{K}$ containing the centre of instanton, so that in a wide vicinity of $\mathfrak{S}_\infty$ the long-wavelength limit is valid and the continuous solution (2.13), (2.14) approximates correctly the exact lattice solution and the hypersurface $\mathfrak{S}_\infty$ is given by the eq. $r = R = \text{Const} \longrightarrow \infty$. Evidently, the Euler characteristics $\chi(\mathfrak{k}) = \chi(\mathfrak{K}) = 1$. We have the exact lattice equivalents of the instanton boundary conditions (2.15) and (2.16):

$$\frac{1}{12\cdot 4!} \sum_{\mathcal{S}(\mathfrak{S}_\infty)} \sum_{ijkm} \varepsilon_{\mathcal{S}ijkm}\, \mathrm{tr}\, \big(\Omega^{-1}_{(+)\mathcal{S}jm}\Omega_{(+)\mathcal{S}ji}\big)\big(\Omega^{-1}_{(+)\mathcal{S}km}\Omega_{(+)\mathcal{S}kj}\big)\big(\Omega^{-1}_{(+)\mathcal{S}im}\Omega_{(+)\mathcal{S}ik}\big) = -2\pi^2, \tag{5.1}$$

$$\sum_{\mathcal{S}(\mathfrak{S})} \sum_{ijkm} \varepsilon_{\mathcal{S}ijkm}\, \mathrm{tr}\, \big(\Omega^{-1}_{(+)\mathcal{S}jm}\Omega_{(+)\mathcal{S}ji}\big)\big(\Omega^{-1}_{(+)\mathcal{S}km}\Omega_{(+)\mathcal{S}kj}\big)\big(\Omega^{-1}_{(+)\mathcal{S}im}\Omega_{(+)\mathcal{S}ik}\big) = 0\,. \tag{5.2}$$

Here the indices $\mathcal{S}(\mathfrak{S}_\infty)$ and $\mathcal{S}(\mathfrak{S})$ enumerate 3-simplices on the boundaries $\mathfrak{S}_\infty$ and $\mathfrak{S}$, correspondingly, and the Levi-Civita symbol $\varepsilon_{\mathcal{S}ijkm} = \pm 1$ depending on whether the order of vertices $s^3_{\mathcal{S}} = a_{\mathcal{S}i}a_{\mathcal{S}j}a_{\mathcal{S}k}a_{\mathcal{S}m}$ defines the positive or negative orientation of this 3-simplex.

[2] Hence the statement that the square brackets give null equation under the sum (4.5) is proved.

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Since the long-wavelength limit is valid as $r \longrightarrow \infty$, one can use the instanton solution for the dynamic variables (2.13)–(2.14) in this region:

$$\Omega_{(+)\mathcal{W}mi} \approx 1 + \frac{i\sigma^\alpha}{2}\omega^\alpha_{(+)\mu}\,\mathrm{d}\,x^\mu_{\mathcal{W}mi}\,.$$

Therefore the sum in (5.1) transforms into integral (2.15) with the only difference that now the angle ψ varies in the interval (2.12), and so the boundary condition (5.1) is just.

To implement the boundary condition (5.2) we suggest the following solutions on the sub-complex $\mathfrak{k}$.

Consider the following solutions of eqs. (4.1), (4.2) and (4.4) on $\mathfrak{k}$:

$$\Omega_{(+)\mathcal{W}ij} = -1\,, \qquad \Omega_{(-)\mathcal{W}ij} = 1\,, \qquad s^4_{\mathcal{W}} \in \mathfrak{k}\,. \tag{5.3}$$

From here it follows that

$$\Omega_{(+)\mathcal{W}mi}\Omega_{(+)\mathcal{W}ij}\Omega_{(+)\mathcal{W}jm} = -1 \quad \text{on } \mathfrak{k}\,. \tag{5.4}$$

The equalities (5.4) hold true if we perform a gauge transformation (see (3.3)) with $S_{(+)\mathcal{W}i} = \pm 1$, $S_{(-)\mathcal{W}i} = 1$. The boundary condition (5.2) is true for each configuration obtained in such a way.

Obviously, 1-form $\hat{e}_{\mathcal{W}ij}$ can be considered as a 1-cochain on complex, and the quantity

$$E^{ab}_{\mathcal{W}m[kl]} = \left(E^\alpha_{(+)\mathcal{W}m[kl]},\ \left(E^\alpha_{(-)\mathcal{W}m[kl]}\right) = \varepsilon_{abcd}e^c_{\mathcal{W}mk}e^d_{\mathcal{W}ml}\,, \tag{5.5}$$

is a 2-cochain which is the superposition of the exterior products of 1-cochains $e^a_{\mathcal{W}ij}$.

It may be verified (compare with (4.5)) that the left hand side of eq. (4.1) is nothing but the exterior lattice derivative of a 2-cochain (5.5), and eq. (4.1) implies that the derivative is equal to zero in the case of (5.4), i.e. the quantity (5.5) is a cocycle:[3]

$$\begin{gathered}\sum_{\mathcal{W}'}\sum_{j,k,l}\varepsilon_{\mathcal{W}'mijkl}E^{ab}_{\mathcal{W}'[jkl]} = 0\,,\\ a_{\mathcal{W}m}a_{\mathcal{W}i} \in s^4_{\mathcal{W}'}\,, \qquad a_{\mathcal{W}m}a_{\mathcal{W}i} \in \mathfrak{k}\,, \qquad a_{\mathcal{W}m}a_{\mathcal{W}i} \notin \partial\,\mathfrak{k}\,.\end{gathered} \tag{5.6}$$

Evidently, eqs. (5.6) are satisfied for

$$e^a_{\mathcal{V}_1\mathcal{V}_2} = \phi^a_{\mathcal{V}_2} - \phi^a_{\mathcal{V}_1}\,, \qquad a_{\mathcal{V}_1}a_{\mathcal{V}_2} \in \mathfrak{k}\,, \qquad a_{\mathcal{V}_1}a_{\mathcal{V}_2} \notin \partial\,\mathfrak{k}\,, \tag{5.7}$$

where $\phi^a_{\mathcal{V}}$ is any scalar field on $\mathfrak{k}$. This statement is checked easily by a direct calculation and it follows from the fact that the cochain (5.5) is the superposition of the exterior products of exact 1-forms in the case (5.7).

Since the second cohomology group $H^2(\mathfrak{k}) = 0$, so the cocycle (5.5) is a lattice coboundary:

$$\begin{gathered}E^{ab}_{\mathcal{W}m[kl]} = \left(\Psi^{ab}_{\mathcal{W}mk} - \Psi^{ab}_{\mathcal{W}ml}\right)\,,\\ \Psi^{ab}_{\mathcal{W}mk} = -\Psi^{ab}_{\mathcal{W}km}\,.\end{gathered} \tag{5.8}$$

[3]It was proved to be the case for the quantity $E^\alpha_{(-)\mathcal{W}[mkl]}$ (see (4.5)). The corresponding provement for the quantity $E^\alpha_{(+)\mathcal{W}[mkl]}$ is the same.

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Eqs. (5.6) fix no more than 6 real numbers at each vertex of the sub-complex $\mathfrak{k}$ for the reason that the lattice 1-form $\Psi^{ab}_{\mathcal{W}mk}$ is determined up to the lattice gradient $(\phi^{ab}_{\mathcal{W}k} - \phi^{ab}_{\mathcal{W}m})$. So it follows from eqs. (5.3) (just like as in the case of (4.4)) that now not only the gauge sub-group $\mathrm{Spin}(4)_{(-)}$ is broken but also the gauge group Spin(4) is broken to on $\mathfrak{k}\backslash\partial\,\mathfrak{k}$ almost wholly except for the center of the sub-group $\mathrm{SU}(2)_{(+)}$ (see eqs. (4.6)–(4.11)).

From here and throughout the following discussions the pairs of indices $(\mathcal{V}_1\mathcal{V}_2)$ enumerate 1-simplexes $a_{\mathcal{V}_1}a_{\mathcal{V}_2} \in \mathfrak{K}$.

The action (3.2) is equal to zero identically for the configuration (5.4) for any values of $e^a_{\mathcal{V}_1\mathcal{V}_2}$, where 1-simplex $a_{\mathcal{V}_1}a_{\mathcal{V}_2} \in \mathfrak{k}$, $a_{\mathcal{V}_1}a_{\mathcal{V}_2} \notin \partial\,\mathfrak{k}$. Therefore, eqs. (4.2) are satisfied automatically in this case, they do not give any constraint onto the corresponding 1-forms $e^a_{\mathcal{V}_1\mathcal{V}_2}$ in addition to eqs. (5.8):

$$\delta\,\mathfrak{A}\,/\delta e^a_{\mathcal{V}_1\mathcal{V}_2} \equiv 0\,, \qquad a_{\mathcal{V}_1}a_{\mathcal{V}_2} \in \mathfrak{k}\,, \qquad a_{\mathcal{V}_1}a_{\mathcal{V}_2} \notin \partial\,\mathfrak{k}\,. \tag{5.9}$$

Now let us prove that the lattice configuration of the variables $\{\omega^{ab}_{\mathcal{V}_1\mathcal{V}_2(\mathrm{inst})},\ e^a_{\mathcal{V}_1\mathcal{V}_2(\mathrm{inst})}\}$ satisfying the system of equations and constraints (4.1), (4.2), (4.4), (5.1) and (5.2) does exist. The configuration is the lattice analogue of the Eguchi-Hanson continuous instanton.

Recall that the sets of equations (4.2) with the sign $(-)$ and (4.4) are equivalent to the set of equations (see eqs. (4.5))

$$\Phi^\alpha_{(\mathcal{V}_1\mathcal{V}_2)} \equiv -\frac{l_P^2}{2}\frac{\delta\,\mathfrak{A}}{\delta\omega^\alpha_{(-)\mathcal{V}_1\mathcal{V}_2}}\bigg|_{\omega_{(-)}=0} = \sum_{\mathcal{W}'}\sum_{j,k,l}\varepsilon_{\mathcal{W}'mijkl}E^\alpha_{(-)\mathcal{W}'[jkl]} = 0 \tag{5.10}$$

and eqs. (4.4). Here 1-simplexes $a_{\mathcal{W}m}a_{\mathcal{W}i}$ are renamed as $a_{\mathcal{V}_1}a_{\mathcal{V}_2}$. Further the set of equations (5.10) is considered as the set of constraints.

At the first stage we shall solve the problem for finite complex $\mathfrak{K}$ with extra-large (though finite) number of vertices $\mathfrak{N}$ and the boundary $\partial\,\mathfrak{K} = \mathfrak{S}_\infty$. There is the estimation

$$l_P^2\mathfrak{N} \sim R^4. \tag{5.11}$$

The constraint (5.2) is realized evidently by taking the variables $\Omega_{(+)\mathcal{V}_1\mathcal{V}_2}$ on the sub-complex $\mathfrak{k}$ as in (5.3). We believe also that the relations (5.7) are valid. The constraint (5.1) also is realized evidently by taking

$$\Omega_{(+)\mathcal{S}mi} = \exp\left\{\frac{i\sigma^\alpha}{2}\omega^\alpha_{(+)\mu}\,\mathrm{d}\,x^\mu_{\mathcal{S}mi}\right\} \approx 1 + \frac{i\sigma^\alpha}{2}\omega^\alpha_{(+)\mu}\,\mathrm{d}\,x^\mu_{\mathcal{S}mi} \tag{5.12}$$

on the boundary $\mathfrak{S}_\infty$, where the field $\omega^\alpha_{(+)\mu}$ is given by eq. (2.17).

So, the problem is equivalent to the searching for the stationary point of the Lagrange function

$$\mathcal{L} = \mathfrak{A} - \sum_{\alpha,\,(\mathcal{V}_1\mathcal{V}_2),\ a_{\mathcal{V}_1}a_{\mathcal{V}_2}\notin\mathfrak{k}\backslash\partial\,\mathfrak{k}} \lambda^\alpha_{(\mathcal{V}_1\mathcal{V}_2)}\Phi^\alpha_{(\mathcal{V}_1\mathcal{V}_2)} \tag{5.13}$$

relative to the variables

$$\{\omega^\alpha_{(+)(\mathcal{V}_1\mathcal{V}_2)},\quad e^a_{(\mathcal{V}_1\mathcal{V}_2)},\quad \lambda^\alpha_{(\mathcal{V}_1\mathcal{V}_2)}\}\,,$$

assuming that the conditions (4.4), (5.3) and (5.12) take place.

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According to (3.1) each of the elements $\Omega_{(+)\mathcal{V}_1\mathcal{V}_2}$ for fixed 1-simplex $a_{\mathcal{V}_1}a_{\mathcal{V}_2}$ is a smooth matrix function on the unit 3D sphere $S^3_{(\mathcal{V}_1\mathcal{V}_2)}$ ($\omega^\alpha_{(+)\mathcal{V}_1\mathcal{V}_2}$ are the coordinates on $S^3_{(\mathcal{V}_1\mathcal{V}_2)}$ which are convenient for the long-wavelength transition). Denote also by S^H the hypersphere which is determined by the equation

$$\sum_{(\mathcal{V}_1\mathcal{V}_2),\ a_{\mathcal{V}_1}a_{\mathcal{V}_2}\notin\mathfrak{k}\setminus\partial\mathfrak{k}} \big\{(e^a_{\mathcal{V}_1\mathcal{V}_2})^2+\big(\lambda^\alpha_{(\mathcal{V}_1\mathcal{V}_2)}\big)^2\big\}=l_P^2\mathfrak{N}\,. \tag{5.14}$$

Let's search for a stationary point for the Lagrange function (5.13) on the compact smooth manifold without boundary:

$$\begin{aligned}\mathcal{C}&=S^H\bigcup\Big(\bigcup_{(\mathcal{V}_1\mathcal{V}_2),\ a_{\mathcal{V}_1}a_{\mathcal{V}_2}\notin\mathfrak{k},\,\mathfrak{S}_\infty} S^3_{(\mathcal{V}_1\mathcal{V}_2)}\Big)\,,\\ \partial\mathcal{C}&=\emptyset\,.\end{aligned} \tag{5.15}$$

Since the Lagrange function (5.13) is a smooth function defined on the compact metric space (5.15), so it is a bounded function and it has the local maximum(s) and minimum(s) at some points $p_\xi\in\mathcal{C}$. Moreover, since the space $\mathcal{C}$ is without boundary, so the total differential of the Lagrange function at the points p_ξ is equal to zero.

It will be appreciated that the Lagrange function variables are constrained due to eq. (5.14). Let's express the only one variable $e^1_{\mathcal{V}_{\mathfrak{N}-1}\mathcal{V}_{\mathfrak{N}}}$ for some $a_{\mathcal{V}_{\mathfrak{N}-1}}a_{\mathcal{V}_{\mathfrak{N}}}\in\mathfrak{S}_\infty$ in terms of the rest variables involved in the constraint (5.14):

$$\begin{aligned}e^1_{\mathcal{V}_{\mathfrak{N}-1}\mathcal{V}_{\mathfrak{N}}}&=\pm f_{\mathfrak{N}}(\ldots)\,,\\ f_{\mathfrak{N}}(\ldots)&=\sqrt{l_P^2\mathfrak{N}-\sum_{(\mathcal{V}_1\mathcal{V}_2),\ a_{\mathcal{V}_1}a_{\mathcal{V}_2}\notin\mathfrak{k}\setminus\partial\mathfrak{k}}\big\{(e^a_{\mathcal{V}_1\mathcal{V}_2})^2+\ldots\big\}}\,.\end{aligned} \tag{5.16}$$

As stated before (see sections 2 and 4), the set of constraints (5.10) fixes at most three real number at each vertex which is equivalent to the gauge fixing. Evidently, in the limit $\omega^\alpha_{(-)\mathcal{V}_1\mathcal{V}_2}\longrightarrow 0$ the Lagrange function (5.13) is obtained from the action (3.2) by replacing $\omega^\alpha_{(-)\mathcal{V}_1\mathcal{V}_2}\longrightarrow(l_P^2/2)\lambda^\alpha_{\mathcal{V}_1\mathcal{V}_2}$. Therefore, the set of Lagrange multipliers $\{\lambda^\alpha_{(\mathcal{V}_1\mathcal{V}_2)}\}$ contains only tree significant real-valued parameters per each vertex. Since the infinitesimal gauge transformations act on the set of Lagrange multipliers according to the rule

$$\delta\lambda^\alpha_{\mathcal{V}_1\mathcal{V}_2}=\psi_{\mathcal{V}_2}-\psi_{\mathcal{V}_1}\,,\qquad \psi_{\mathcal{V}}\longrightarrow 0\,,$$

so one may put

$$\lambda^\alpha_{\mathcal{V}_1\mathcal{V}_2}\big|_p=0 \tag{5.17}$$

at the stationary points. The last equations fix the gauge subgroup $\mathrm{Spin}(4)_{(-)}$. Due to eqs. (5.17) we have

$$\frac{\partial e^1_{\mathcal{V}_{\mathfrak{N}-1}\mathcal{V}_{\mathfrak{N}}}}{\partial\lambda^\alpha_{\mathcal{V}_1\mathcal{V}_2}}\bigg|_p=0\,. \tag{5.18}$$

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Let's scrutinize equations following from the condition $\mathrm{d}\,\mathcal{L}|_p = 0$.

Note firstly that the summand with the sign $(-)$ in the action (3.2) is equal to zero identically for $\Omega_{(-)} = 1$ and any values of the variables e^a:

$$\mathfrak{A}_{(-)}\,|_{\Omega_{(-)}=1} \equiv 0\,, \qquad \left.\frac{\partial\,\mathfrak{A}_{(-)}}{\partial e^a}\right|_{\Omega_{(-)}=1} \equiv 0\,.$$

It is seen from here that eqs. (5.17) do not contradict to the equations cited below.

The conditions

$$\frac{\partial\mathcal{L}}{\partial\omega^{\alpha}_{(+)(\mathcal{V}_1\mathcal{V}_2)}} = 0$$

give the set of eqs. (4.1) with the sign "+".

We have also the set of eqs. (4.4) supplemented by the conditions

$$\frac{\partial\mathcal{L}}{\partial\lambda^{\alpha}_{\mathcal{V}_1\mathcal{V}_2}} = -\Phi^{\alpha}_{(\mathcal{V}_1\mathcal{V}_2)} + \frac{\partial\mathcal{L}}{\partial e^1_{\mathcal{V}_{\mathfrak{N}-1}\mathcal{V}_{\mathfrak{N}}}}\cdot\left.\frac{\partial e^1_{\mathcal{V}_{\mathfrak{N}-1}\mathcal{V}_{\mathfrak{N}}}}{\partial\lambda^{\alpha}_{\mathcal{V}_1\mathcal{V}_2}}\right|_p = -\Phi^{\alpha}_{(\mathcal{V}_1\mathcal{V}_2)} = 0\,. \tag{5.19}$$

Here eqs. (5.18) are taken into account. Eqs. (4.4) and (5.19) are equivalent to eqs. (4.4) and (4.1) with the sign "−".

The last set of equations is (due to eqs. (5.17)) as follows:

$$\frac{\partial\mathcal{L}}{\partial e^a_{\mathcal{V}_1\mathcal{V}_2}} = \frac{\partial\mathfrak{A}_{(+)}}{\partial e^a_{\mathcal{V}_1\mathcal{V}_2}} \pm \frac{\partial\mathfrak{A}_{(+)}}{\partial e^1_{\mathcal{V}_{\mathfrak{N}-1}\mathcal{V}_{\mathfrak{N}}}}\cdot\frac{\partial f_{\mathfrak{N}}}{\partial e^a_{\mathcal{V}_1\mathcal{V}_2}} = 0\,. \tag{5.20}$$

It follows from here that (see (4.3), (5.13) and (5.19))

$$2\,\mathfrak{A} = 2\mathcal{L}|_p = \mp\sum_{(\mathcal{V}_1\mathcal{V}_2)} e^a_{\mathcal{V}_1\mathcal{V}_2}\frac{\partial\mathfrak{A}_{(+)}}{\partial e^1_{\mathcal{V}_{\mathfrak{N}-1}\mathcal{V}_{\mathfrak{N}}}}\cdot\frac{\partial f_{\mathfrak{N}}}{\partial e^a_{\mathcal{V}_1\mathcal{V}_2}} \neq 0\,. \tag{5.21}$$

The quantity $\partial\mathfrak{A}_{(+)}/\partial e^1_{\mathcal{V}_{\mathfrak{N}-1}\mathcal{V}_{\mathfrak{N}}}$ is proportional to the curvature $\mathfrak{R}_{(+)\mu\nu}$ (see (3.13)) in the neighbourhood of the boundary $\mathfrak{S}_\infty$. Due to eqs. (2.21) and (5.11) there is the chain of estimations:

$$\begin{gathered}\frac{\partial\mathfrak{A}}{\partial e^1_{\mathcal{V}_{\mathfrak{N}-1}\mathcal{V}_{\mathfrak{N}}}} \sim R^{-6} \sim \mathfrak{N}^{-3/2},\\ \sum_{(\mathcal{V}_1\mathcal{V}_2)} e^a_{\mathcal{V}_1\mathcal{V}_2}\frac{\partial\mathfrak{A}}{\partial e^1_{\mathcal{V}_{\mathfrak{N}-1}\mathcal{V}_{\mathfrak{N}}}}\cdot\frac{\partial f_{\mathfrak{N}}}{\partial e^a_{\mathcal{V}_1\mathcal{V}_2}} \sim \mathfrak{N}^{-1/2} \sim R^{-2}.\end{gathered} \tag{5.22}$$

According to the eqs. (5.20), (5.21) and estimations (5.22) we obtaine the final result

$$\frac{\partial\mathfrak{A}}{\partial e^a_{\mathcal{V}_1\mathcal{V}_2}} \longrightarrow 0\,, \qquad \mathfrak{A} \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty\,. \tag{5.23}$$

The result can be explained once again as follows.

Let the set of variables $\{\Omega_{\mathcal{V}_1\mathcal{V}_2(\mathrm{inst})}(e)\}$ be such that the boundary conditions (5.4) and (5.12) as well as eqs. (4.1) are fulfilled for any values of variables $\{e\}$. This is possible since the action (3.2) is a smooth real function of the variables $\{\Omega_{\mathcal{V}_1\mathcal{V}_2}\}$ (for fixed set of

values $\{e\}$), i.e. the action (3.2) is a smooth real function on a compact manifold without boundary. Now let's impose the constraints $\Omega_{(-)\mathcal{V}_1\mathcal{V}_2(\mathrm{inst})}(e) = 0$ which are equivalent to the constraints (5.10), and resolve the constraints relative to the a subset of variables $\{e^a_{\mathcal{V}_1\mathcal{V}_2}\}'$. As a result each variable from the subset becomes a homogeneous function of degree 1 of the rest set $\{e^a_{\mathcal{V}_1\mathcal{V}_2}\}''$ mutually independent variables. As a result the action (3.2) becomes the function of the variables $\{e^a_{\mathcal{V}_1\mathcal{V}_2}\}''$ such that eqs. (4.1) are satisfied and, hence

$$2\,\mathfrak{A} = \sum_{\{e\}''} e^a_{\mathcal{V}_1\mathcal{V}_2}(\delta\,\mathfrak{A}\,/\delta e^a_{\mathcal{V}_1\mathcal{V}_2})\,. \tag{5.24}$$

Now let's bound the function $\mathfrak{A}$ on the hypersphere $S^{H''}$ determined by the relation

$$\sum_{\{e\}''}(e^a_{\mathcal{V}_1\mathcal{V}_2})^2 = l_P^2\mathfrak{N}\,.$$

The subsequent consideration and conclusion are identical to those which has been given with the help of eqs. (5.16), (5.20)–(5.23): there is a stationary point of the action (3.2) relative to its variables with the additional equations (4.4) and boundary conditions (5.4), (5.12); the action is equal to zero at the stationary point at the limit $R \longrightarrow \infty$.

6 Conclusion

Aforesaid means that the lattice analogue of the Eguchi-Hanson self-dual solution to continuous Euclidean Gravity does exist for the complex $\mathfrak{K} \approx \mathbb{R}^4$.

It is crucially important that the problem of possible singularities for the curvature tensor does not exist on the lattice gravity. In continuous gravity for the ψ range (2.12), the manifold would have "cone-tip" singularities at $r = a$; this implies the necessity of delta-functions in the curvature at $r = a$ (see [2, 3]). But delta-functions transforms into Kronecker symbol which is of the order of unity in discrete mathematics. The same is true with respect to the lattice analogue of the curvature tensor $\Omega_{\mathcal{W}mi}\Omega_{\mathcal{W}ij}\Omega_{\mathcal{W}jm} \sim \pm 1$. This is the reason why one can take the range of angles (2.12) in a lattice gravity. Moreover, all lattice equations are satisfied and the action for the instanton solution is equal to zero.

Thus, the setting of the problem for finding a self-dual solution to lattice Euclidean gravity is as follows: one must solve (in anti-instanton case) the difference lattice system of equations (4.1) with indices (+), (4.2), (4.4), the constraints (5.10), and the boundary conditions (5.4), (5.12).

Here some questions are not enough clear or remain unclear.

1. Is the offered solution with $\chi(\mathfrak{K}) = 1$ stable or it can be contracted smoothly into the trivial one?

   The answer to this question seems to be as follows.

   Let's consider a smooth path in the configuration space

$$\Omega_{\mathcal{V}_1\mathcal{V}_2}(t)\,, \qquad e^a_{\mathcal{V}_1\mathcal{V}_2}(t)\,, \qquad 0 \le t \le 1\,,$$

such that

$$\Omega_{\mathcal{V}_1\mathcal{V}_2}(0) = \Omega_{\mathcal{V}_1\mathcal{V}_2(\text{inst})}\,, \qquad e^a_{\mathcal{V}_1\mathcal{V}_2}(0) = e^a_{\mathcal{V}_1\mathcal{V}_2(\text{inst})}\,,$$
$$\Omega_{\mathcal{V}_1\mathcal{V}_2}(1) = 1\,, \qquad e^a_{\mathcal{V}_1\mathcal{V}_2}(1) = \phi^a_{\mathcal{V}_2} - \phi^a_{\mathcal{V}_1}\,. \tag{6.1}$$

Here the lower index (inst) designates the discussed self-dual solution. The field configuration $\left\{\Omega_{\mathcal{V}_1\mathcal{V}_2}(1),\ e^a_{\mathcal{V}_1\mathcal{V}_2}(1)\right\}$ is a trivial solution of eqs. (4.1), (4.2). Evidently, the *global* continuous description of this trivial solution is possible. Nevertheless, the considered instanton contraction scenario seems to be unsatisfactory since to do this, one must contract topologically non-trivial connection elements (see (5.1) and (5.12)) into unit in the infinite space-time.

Another instanton contraction scenario which is described in continual limit by limit process $a \longrightarrow 0$ and in the lattice case by reducing the sub-complex $\mathfrak{k}$ up to its disappearance also seems to be unsatisfactory. Indeed, in this case we would have resulted in the failure of the Chern-Gauss-Bonnet theorem in $\mathbb{R}^4$. So, this scenario is also impossible, it can be considered as a decreasing of the instanton scale a up to a lattice scale.

Therefore, the considered instanton solution for the case $\chi(\mathfrak{K}) = 1$ seems to be stable.

2. The case $\chi(\mathfrak{k}) = (2k+1)$, $k = 1, 2, \ldots$ and $\partial\mathfrak{k} = \mathfrak{S} \approx S^3$ is interesting, but it is not considered here. So, the question remains unanswered: for what values of $\chi(\mathfrak{K})$ the lattice self-dual solution does exist and it would be stable?

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