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SU(1,2) invariance in two-dimensional oscillator

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Abstract: Performing the Hamiltonian analysis we explicitly established the canonical equivalence of the deformed oscillator, constructed in arXiv:1607.03756, with the ordinary one. As an immediate consequence, we proved that the $\mathrm{SU}(1,2)$ symmetry is the dynamical symmetry of the ordinary two-dimensional oscillator. The characteristic feature of this $\mathrm{SU}(1,2)$ symmetry is a non-polynomial structure of its generators written in terms of the oscillator variables.

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Contents

1 Introduction

It is a well-known fact that the invariance with respect to the $\ell > 1/2$-conformal Galilei algebra [1–3] demands the appearance of high-derivative terms in the Lagrangians of the corresponding mechanical systems [4–9]. The important fact is that standard methods of nonlinear realizations [10–13] work quite nicely for these algebras. Indeed, within the nonlinear realizations approach one can easily construct the Pais-Uhlenbeck oscillators [14]. The exceptional case with $\ell = 1/2$ corresponds to the Shrödinger algebra, and the mechanical system possessing this symmetry is just a standard d-dimensional oscillator [15, 16]. It was demonstrated in the recent paper [14] that the $su(1,2)$ algebra admits a contraction to the two-dimensional Shrödinger algebra and, therefore, the system possessing the $\mathrm{SU}(1,2)$ symmetry reduces to the ordinary two-dimensional oscillator. Such a deformed oscillator has been constructed in [14] within the Lagrangian formalism.

As for a possible relation of the deformed oscillator with the ordinary one, one should note that it seems to be impossible to relate these systems within the Lagrangian approach. On the contrary, within the Hamiltonian approach the freedom to relate these systems is much wider, because the admitted change of variables includes arbitrary (but invertible) functions defined on the phase-space. That is why we provide a Hamiltonian description of the $su(1,2)$ oscillator in the present paper. It turns out that the standard procedure to pass to the Hamiltonian formalism is not much convenient for the present case, resulting in a rather complicated Hamiltonian. The basic explanation of this fact is that the canonically defined momenta have rather complicated transformation properties with respect to the $\mathrm{SU}(1,2)$ group. On the other hand, within the nonlinear realization approach applied to this system [14], there are coset space variables $v, \bar{v}$ with transparent transformation properties, which can be used as proper momenta . Interestingly enough, one of the Cartan forms, used as the Lagrangian in [14], is capable of providing the symplectic structure as well as the Hamiltonian in terms of the initial variables $u, \bar{u}, v, \bar{v}$. The complicated structure of the Poisson brackets in this basis is compensated by the simple form of the Hamiltonian and the generators of the $su(1,2)$ algebra.

Having at hand all ingredients in the initial variables, we succeeded in finding the new variables in which the Hamiltonian of the deformed $su(1,2)$-invariant oscillator coincides with the Hamiltonian of the ordinary two-dimensional harmonic oscillator. Thus, we proved that these two systems are canonically equivalent. However, the generators of the $su(1,2)$ algebra have a non-polynomial structure in these new variables; so it is problematic to state about quantum equivalence of the deformed and ordinary oscillators.

The deformed oscillator is a solvable system possessing a quite high $\mathrm{SU}(1,2)$ symmetry. The harmonic oscillator also possesses this symmetry. It should be stressed that the $\mathrm{SU}(1,2)$ group can be viewed as the simplest example of quasi-superconformal algebras - algebras which have a 5-grading structure [17]. Just this 5-grading structure is the key feature for application of the nonlinear realization approach [14]. Funnily enough, for any simple Lie algebra there is a noncompact real form that possesses 5-grading decomposition. Therefore, one may expect that for an arbitrary simple Lie algebra there exist deformed versions of the oscillators, which are invariant with respect to the corresponding symmetry.

2 Deformed oscillator in the Lagrangian approach

In [14], the Lagrangian of the deformed oscillator

$$\mathcal{L} = \frac{\dot{u}\dot{\bar{u}} - \omega^2 u\bar{u}}{1 + \frac{\mathrm{i}\gamma}{2}\left(u\dot{\bar{u}} - \dot{u}\bar{u}\right) + \frac{\gamma^2\omega^2}{4} u^2\,\bar{u}^2} \tag{2.1}$$

was constructed within the nonlinear realization of the $\mathrm{SU}(1,2)$ group. The structure relations of the corresponding algebra $su(1,2)$ were chosen as

$$\begin{aligned}
&\mathrm{i}\left[L_n, L_m\right] = (n-m)L_{n+m}, && \mathrm{i}\left[L_n, G_r\right] = \left(\frac{n}{2} - r\right)G_{n+r},\\
&\mathrm{i}\left[L_n, \overline{G}_r\right] = \left(\frac{n}{2} - r\right)\overline{G}_{n+r}, &&\\
&\left[U, G_r\right] = G_r, && \left[U, \overline{G}_r\right] = -\overline{G}_r,\\
&\mathrm{i}\left[G_r, \overline{G}_s\right] = \gamma\left(\frac{3}{2}(r-s)U - \mathrm{i}L_{r+s}\right), && n, m = -1, 0, 1,\ r, s = -1/2, 1/2.
\end{aligned} \tag{2.2}$$

In this form, in the limit $\gamma = 0$ these relations coincide with the relations of the $\ell = \frac{1}{2}$ conformal Galilei algebra in three dimensions [1, 2], and so they can be viewed as the deformation of the conformal Galilei algebra with the parameter of deformation γ. The exact value of γ is inessential: if nonzero, it can be put to unity by a re-scaling of the generators G_r and $\overline{G}_r$.

The group $\mathrm{SU}(1,2)$ itself was realized by the left multiplication of the coset $g = \mathrm{SU}(1,2)/H$ with the stability subgroup $H \propto (U, L_0, L_1)$ parameterized as

$$g = e^{\mathrm{i}t\left(L_{-1} + \omega^2 L_1\right)}\, e^{\mathrm{i}\left(uG_{-1/2} + \bar{u}\overline{G}_{-1/2}\right)}\, e^{\mathrm{i}\left(vG_{1/2} + \bar{v}\overline{G}_{1/2}\right)}. \tag{2.3}$$

Using the Cartan forms, defined in a standard way as

$$g^{-1}\,d\,g = \mathrm{i}\sum_{n=-1}^{1}\Omega_n L_n + \mathrm{i}\sum_{\alpha=-1/2}^{1/2}\left(\omega_\alpha G_\alpha + \bar\omega_\alpha \overline{G}_\alpha\right) + \mathrm{i}\,\Omega_U U, \quad u^* = \bar u,\ v^* = \bar v,\ (g)^\dagger = g^{-1}, \tag{2.4}$$

one may eliminate the inessential Goldstone fields $v,\bar v$ via the fields $u,\bar u$ by imposing the constraints[1]

$$\omega_{-1/2} = \bar\omega_{-1/2} = 0 \qquad \Rightarrow \qquad \begin{aligned} v &= \frac{\dot u + \mathrm{i}\frac{\gamma\,\omega^2}{2} u^2\,\bar u}{1 + \mathrm{i}\frac{\gamma}{2}\left(u\dot{\bar u} - \bar u \dot u\right) + \frac{\gamma^2\,\omega^2}{4} u^2\,\bar u^2}, \\ \bar v &= \frac{\dot{\bar u} - \mathrm{i}\frac{\gamma\,\omega^2}{2} u\,\bar u^2}{1 + \mathrm{i}\frac{\gamma}{2}\left(u\dot{\bar u} - \bar u \dot u\right) + \frac{\gamma^2\,\omega^2}{4} u^2\,\bar u^2}. \end{aligned} \tag{2.5}$$

The action for the deformed oscillator is provided by the Cartan form Ω_U (2.4), which explicitly reads [14]

$$\begin{aligned} \Omega_U = \frac{3}{2}\gamma \Bigg[& v\,\bar v \left(\left(1 + \frac{1}{4}\gamma^2\,\omega^2 u^2\,\bar u^2\right) dt + \frac{\mathrm{i}}{2}\gamma\left(u\,d\bar u - \bar u\,du\right)\right) \\ & -v\left(d\bar u - \frac{\mathrm{i}}{2}\gamma\,\omega^2 u\,\bar u^2\,dt\right) - \bar v\left(du + \frac{\mathrm{i}}{2}\gamma\,\omega^2 u^2\,\bar u\,dt\right) + \omega^2 u\,\bar u\,dt\Bigg]. \end{aligned} \tag{2.6}$$

Finally, upon the substitution of (2.5) into (2.6) one may get

$$S = -\frac{2}{3\gamma}\int \Omega_U = \int \mathcal{L}\,dt \tag{2.7}$$

where the Lagrangian is given by (2.1). One has to stress again that the action (2.7) is invariant with respect to the SU$(1,2)$ symmetry.

Finally, note that the action

$$S = -\frac{2}{3\gamma}\int \Omega_U \tag{2.8}$$

with the form Ω_U given by expression (2.6) is sufficient to describe the deformed oscillator without any references to the inverse Higgs constraints (2.5). Indeed, varying the action (2.8) over the variables $v,\bar v$ we immediately reproduce the constraints (2.5). Thus, the action (2.8) contains all needed information to describe the deformed oscillator.

3 Hamiltonian formulation

To provide the Hamiltonian description of the deformed oscillator with the Lagrangian (2.1), one may perform Legendre transformation and get the system with the canonical Poisson brackets with the momenta $\pi,\bar\pi$ canonically conjugated with $u,\bar u$ variables. However, the Hamiltonian written in the canonical variables is not very convenient for further analyses.

[1]This is the particular case of the Inverse Higgs phenomenon conditions [18].

Interestingly enough, the non-linear realization approach allows the Hamiltonian formulation of the system in suitable phase space coordinates, without referring to Legendre transformation. The key observation is that the form Ω_U (2.6) provides us with the first-order Lagrangian which is variationally equivalent to (2.1):

$$\widetilde{\mathcal{L}}\,dt = -\frac{2}{3\gamma}\Omega_U = \boldsymbol{\alpha} - \mathcal{H}\,dt, \tag{3.1}$$

where

$$\boldsymbol{\alpha} = v d\bar{u} + \bar{v} du + \mathrm{i}\,\frac{\gamma}{2} v\bar{v}\,(\bar{u}du - u\,d\bar{u})\,, \tag{3.2}$$

is the symplectic one-form and

$$\mathcal{H} = v\,\bar{v} + \omega^2\,u\,\bar{u}\left(1+\frac{\mathrm{i}}{2}\gamma\,\bar{u}\,v\right)\left(1-\frac{\mathrm{i}}{2}\gamma\,u\,\bar{v}\right). \tag{3.3}$$

is the Hamiltonian. The external differential of the symplectic one-form yields the symplectic structure

$$\Omega = d\boldsymbol{\alpha} = \left(1-\frac{\mathrm{i}\gamma}{2}u\,\bar{v}\right)dv\wedge d\bar{u} + \left(1+\frac{\mathrm{i}\gamma}{2}\bar{u}\,v\right)d\bar{v}\wedge du$$
$$+\frac{\mathrm{i}\gamma}{2}\left(\bar{u}\,\bar{v}\,dv\wedge du - u\,v\,d\bar{v}\wedge d\bar{u}\right) + \mathrm{i}\gamma\,v\,\bar{v}\,d\bar{u}\wedge du. \tag{3.4}$$

The respective Poisson brackets are defined by the following non zero relations:

$$\{v,\bar{u}\} = \frac{1+\mathrm{i}\,\frac{\gamma}{2}v\bar{u}}{1-\mathrm{i}\,\frac{\gamma}{2}(u\bar{v}-\bar{u}v)},\quad \{v,\bar{v}\} = -\mathrm{i}\,\gamma\frac{v\bar{v}}{1-\mathrm{i}\,\frac{\gamma}{2}(u\bar{v}-\bar{u}v)}\quad \{v,u\} = \frac{\mathrm{i}\,\frac{\gamma}{2}\,vu}{1-\mathrm{i}\,\frac{\gamma}{2}(u\bar{v}-\bar{u}v)}, \tag{3.5}$$

and their complex conjugated ones. Let us notice that from the (3.2) one can immediately get the expressions for the canonical momenta $\pi, \bar{\pi}$

$$\pi = v - \mathrm{i}\,\frac{\gamma}{2}v\bar{v}u,\quad \bar{\pi} = \bar{v} + \mathrm{i}\,\frac{\gamma}{2}v\bar{v}\bar{u}\;:\qquad \{\pi,\bar{u}\} = \{\bar{\pi},u\} = 1. \tag{3.6}$$

To complete this section, let us write down the Hamiltonian realization of the $su(1,2)$ generators in terms of $u, \bar{u}, v, \bar{v}$:

$$\begin{aligned}
&L_{-1} = v\,\bar{v}, \qquad L_0 = -\frac{1}{2}\left(u\,\bar{v} + \bar{u}\,v\right), \qquad L_1 = u\,\bar{u}\left(1+\frac{\mathrm{i}}{2}\gamma\,\bar{u}\,v\right)\left(1-\frac{\mathrm{i}}{2}\gamma\,u\,\bar{v}\right),\\
&U = \mathrm{i}\left(\bar{u}\,v - u\,\bar{v}\right) + \gamma\,u\,\bar{u}\,v\,\bar{v},\\
&G_{-1/2} = -\,\bar{v}\left(1+\mathrm{i}\gamma\,\bar{u}\,v\right), \qquad \overline{G}_{-1/2} = -\,v\left(1-\mathrm{i}\gamma\,u\,\bar{v}\right),\\
&G_{1/2} = \bar{u}\left(1+\mathrm{i}\gamma\,\bar{u}\,v\right)\left(1-\frac{\mathrm{i}}{2}\gamma\,u\bar{v}\right), \qquad \overline{G}_{1/2} = u\left(1-\mathrm{i}\gamma\,u\,\bar{v}\right)\left(1+\frac{\mathrm{i}}{2}\gamma\,\bar{u}v\right).
\end{aligned} \tag{3.7}$$

These generators form the $su(1,2)$ algebra with respect to the Poisson brackets (3.5)

$$\begin{aligned}
&\{L_n, L_m\} = (n-m)L_{n+m}, \quad \{L_n, G_r\} = \left(\frac{n}{2}-r\right)G_{n+r}, \quad \{L_n, \overline{G}_r\} = \left(\frac{n}{2}-r\right)\overline{G}_{n+r},\\
&\{U, G_r\} = \mathrm{i}\,G_r, \qquad \{U, \overline{G}_r\} = -\,\mathrm{i}\,\overline{G}_r,\\
&\{G_r, \overline{G}_s\} = -\mathrm{i}\gamma L_{r+s} + \frac{3}{2}\gamma\,(r-s)\left(U + \frac{2}{3\gamma}\right),
\end{aligned} \tag{3.8}$$

where $n, m = -1, 0, 1,\ r, s = -1/2, 1/2.$

Noteworthy is the appearance of the constant central charge in the Poisson brackets $\{G_r, \overline{G}_s\}$. If $\gamma \neq 0$, it can be absorbed in the generator U by its redefinition $U \to \widetilde{U} = U + \frac{2}{3\gamma}$. But if $\gamma = 0$, this central charge survives and we have at hand the central charge extension of the $\ell = 1/2$ conformal Galilei algebra.

4 Canonical variables

Within the Hamiltonian description of the given system, we have much more possibilities to redefine the phase space variables than in the Lagrangian approach. In this section, we will demonstrate that the deformed oscillator with the Hamiltonian (3.3) and the symplectic structure (3.4) is canonically equivalent to the ordinary oscillator. To simplify the presentation, we start with the deformed free particle (i.e. with $\omega = 0$) and then will consider the deformed oscillator in a full generality.

4.1 Free particle

In the free particle case, i.e. when $\omega = 0$, the Hamiltonian is given by the generator L_{-1}

$$\mathcal{H}_0 = v\bar{v}. \tag{4.1}$$

The system has three constants of motion given by the generators $G_{-1/2}, \overline{G}_{-1/2}$ and U (3.7)

$$\{G_{-1/2}, \mathcal{H}_0\} = \{\overline{G}_{-1/2}, \mathcal{H}_0\} = \{U, \mathcal{H}_0\} = 0 \tag{4.2}$$

The commutation relations between these generators follow from (2.2)

$$\{G_{-1/2}, \overline{G}_{-1/2}\} = -\mathrm{i}\gamma L_{-1} = -\mathrm{i}\gamma \mathcal{H}_0, \quad \{U, G_{-1/2}\} = \mathrm{i}\, G_{-1/2}, \; \{U, \overline{G}_{-1/2}\} = -\mathrm{i}\, \overline{G}_{-1/2}. \tag{4.3}$$

The Hamiltonian (4.1) can be written in terms of these constants of motion

$$\mathcal{H}_0 = \frac{G_{-1/2}\, \overline{G}_{-1/2}}{1 + \gamma U}. \tag{4.4}$$

Note that the expression in the denominator is strictly positive, because in virtue of (3.7) we have

$$1 + \gamma U = (1 + \mathrm{i}\, \gamma\, \bar{u} v)\,(1 - \mathrm{i}\, \gamma\, u \bar{v})\,. \tag{4.5}$$

It is slightly unexpected that the evident definitions of the new variables $p, \bar{p}$

$$p = -\frac{G_{-1/2}}{\sqrt{1 + \gamma U}}, \quad \bar{p} = -\frac{\overline{G}_{-1/2}}{\sqrt{1 + \gamma U}}, \qquad \mathcal{H}_0 = p\,\bar{p}, \qquad p^* = \bar{p}, \tag{4.6}$$

provide us with proper momenta because $\{p, \bar{p}\} = 0$. To get complete correspondence with the free particle, we have to find the coordinates x, $\bar{x}$ canonically conjugated with the momenta $p, \bar{p}$. Explicitly, they read

$$x = \bar{u}\, \frac{2 + \gamma\, U + \mathrm{i}\, \gamma\, \bar{u}\, v}{2\sqrt{1 + \gamma\, U}}, \quad \bar{x} = u\, \frac{2 + \gamma\, U - \mathrm{i}\, \gamma\, u\, \bar{v}}{2\sqrt{1 + \gamma\, U}}, \qquad x^* = \bar{x}, \tag{4.7}$$

$$\{p, \bar{x}\} = \{\bar{p}, x\} = 1, \qquad \{p, \bar{p}\} = \{p, x\} = \{\bar{p}, \bar{x}\} = \{x, \bar{x}\} = 0. \tag{4.8}$$

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The variables $x, \bar{x}, p, \bar{p}$ (4.7), (4.6) are related with the canonical variables $u, \bar{u}, \pi, \bar{\pi}$ (3.6) by the canonical transformation whose explicit form can be easily obtained, if needed.

Hence, we have shown that the deformed free particle introduced in [14] is canonically equivalent to the ordinary free particle. Respectively, the actions of both systems admit SU(1, 2) invariance, which is reduced to the $\ell = 1/2$ conformal Galilei group in the $\gamma = 0$ limit [15, 16].

It is instructive to write the explicit realization of the $su(1,2)$ generators in terms of the canonical variables $x, \bar{x}, p, \bar{p}$:

$$\begin{aligned} L_{-1} &= \mathcal{H}_0 = p\,\bar{p}, \qquad & L_0 &= -\frac{1}{2}\left(p\,\bar{x} + \bar{p}\,x\right), \qquad L_1 = x\,\bar{x}, \\ U &= \mathrm{i}\left(x\,\bar{p} - \bar{x}\,p\right), \\ G_{-1/2} &= -\,p\sqrt{1+\gamma U}\,, & \overline{G}_{-1/2} &= -\,\bar{p}\sqrt{1+\gamma U}, \\ G_{1/2} &= x\sqrt{1+\gamma U}, & \overline{G}_{1/2} &= \bar{x}\sqrt{1+\gamma U}. \end{aligned} \tag{4.9}$$

The time-dependent extensions of these generators, defining the isometries of the Lagrangian, are given by the expressions

$$\begin{aligned} L^t_{-1} &= L_{-1}, & L^t_0 &= L_0 + t\,L_{-1}, \qquad L^t_1 = L_1 + 2t\,L_0 + t^2\,L_{-1}, \quad U^t = U, \\ G^t_{-1/2} &= G_{-1/2}, & \overline{G}^t_{-1/2} &= \overline{G}_{-1/2}, \\ G^t_{1/2} &= G_{1/2} + t\,G_{-1/2}, & \overline{G}^t_{1/2} &= \overline{G}_{1/2} + t\,\overline{G}_{-1/2}. \end{aligned} \tag{4.10}$$

The respective Hamiltonian vector fields restricted to the Lagrangian surface parameterized by $x, \bar{x}$, define the following symmetry transformations:

$$\delta x = \mathcal{G}^t x, \qquad \delta\bar{x} = \mathcal{G}^t \bar{x}, \qquad \mathcal{G}^t \in \left\{\mathbf{L}^t_{\pm 1}, \mathbf{L}^t_0, \mathbf{U}^t, \mathbf{G}^t_{\pm 1/2}, \overline{\mathbf{G}}^t_{\pm 1/2}\right\}, \tag{4.11}$$

where

$$\begin{aligned} \mathbf{L}^t_{-1} &= \dot{x}\frac{\partial}{\partial x} + \dot{\bar{x}}\frac{\partial}{\partial \bar{x}}, \\ \mathbf{L}^t_0 &= \left(-\frac{1}{2}\,x + t\,\dot{x}\right)\frac{\partial}{\partial x} + \left(-\frac{1}{2}\,\bar{x} + t\,\dot{\bar{x}}\right)\frac{\partial}{\partial \bar{x}}, \\ \mathbf{L}^t_1 &= \left(-t\,x + t^2\,\dot{x}\right)\frac{\partial}{\partial x} + \left(-t\,\bar{x} + t^2\,\dot{\bar{x}}\right)\frac{\partial}{\partial \bar{x}}, \\ \mathbf{U}^t &= \mathrm{i}\,x\frac{\partial}{\partial x} - \mathrm{i}\,\bar{x}\frac{\partial}{\partial \bar{x}}, \\ \mathbf{G}^t_{-1/2} &= -\frac{1 + \mathrm{i}\,\gamma\,\bar{x}\,\dot{x} - \frac{3\mathrm{i}}{2}\,\gamma\,x\,\dot{\bar{x}}}{\sqrt{1 + \mathrm{i}\,\gamma\,\bar{x}\dot{x} - \mathrm{i}\,\gamma\,x\dot{\bar{x}}}}\frac{\partial}{\partial x} - \frac{\mathrm{i}\,\gamma\,\bar{x}\,\dot{\bar{x}}}{2\sqrt{1 + \mathrm{i}\,\gamma\,\bar{x}\,\dot{x} - \mathrm{i}\,\gamma\,x\dot{\bar{x}}}}\frac{\partial}{\partial \bar{x}}, \\ \overline{\mathbf{G}}^t_{-1/2} &= \left(\mathbf{G}^t_{-1/2}\right)^*, \\ \mathbf{G}^t_{1/2} &= -\frac{2\,t + \mathrm{i}\,\gamma\,\bar{x}x + 2\,\mathrm{i}\,\gamma\,t\,\bar{x}\dot{x} - 3\,\mathrm{i}\,\gamma\,t\,x\dot{\bar{x}}}{2\sqrt{1 + \mathrm{i}\,\gamma\,\bar{x}\dot{x} - \mathrm{i}\,\gamma\,x\dot{\bar{x}}}}\frac{\partial}{\partial x} + \frac{\mathrm{i}\,\gamma\,\bar{x}(\bar{x} - t\,\dot{\bar{x}})}{2\sqrt{1 + \mathrm{i}\,\gamma\,\bar{x}\dot{x} - \mathrm{i}\,\gamma\,x\dot{\bar{x}}}}\frac{\partial}{\partial \bar{x}}, \\ \overline{\mathbf{G}}^t_{1/2} &= \left(\mathbf{G}^t_{1/2}\right)^*. \end{aligned} \tag{4.12}$$

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These transformations indeed preserve the standard free particle action

$$S_0 = \int dt\, \dot{x}\, \dot{\bar{x}} \tag{4.13}$$

The crucial observation is that they form the $su(1,2)$ algebra only on mass shell, i.e. the algebra closed modulo the equations of motion, only. Thus, being canonically equivalent at the Hamiltonian level, the deformed and the free particle are not equivalent in the Lagrangian formalism. The off shell $su(1,2)$ symmetry of the deformed free particle becomes the on shell symmetry of the ordinary free particle.

It is worth noting that the explicit realization of the $su(1,2)$ algebra (4.9) makes evident the statement that the $su(1,2)$ algebra as well as its $\gamma = 0$ reduction (i.e. the Schrödinger algebra) can be constructed in terms of the two one-dimensional oscillators. Thus, both these algebras lie in the enveloping algebra of two oscillators [19].

4.2 Oscillator

Now let us consider the deformed oscillator with the Hamiltonian $\mathcal{H} = L_{-1} + \omega^2 L_1$ (3.3). In contrast with the free particle case, the Hamiltonian of the deformed oscillator does not commute with the generators $G_{\pm 1/2}$. Nevertheless, in addition to the constant of motion U (3.7), the deformed oscillator possesses the hidden symmetries given by the generalization of the Fradkin tensor [20]

$$A = G^2_{-1/2} + \omega^2\, G^2_{1/2}, \quad \bar{A} = \overline{G}^2_{-1/2} + \omega^2\, \overline{G}^2_{1/2}, \qquad \{\mathcal{H}, A\} = \{\mathcal{H}, \bar{A}\} = 0. \tag{4.14}$$

These constants of motion $A, \bar{A}$, together with the generator U, form the deformation of the $su(2)$ algebra

$$\{A, \bar{A}\} = 4\mathrm{i}\,\omega^2\,(U + 3\,\gamma\, U^2 + 2\,\gamma^2\, U^3) - 4\,\mathrm{i}\,\gamma\,(1+\gamma\, U)\mathcal{H}^2, \quad \{U, A\} = 2\mathrm{i}A, \quad \{U, \bar{A}\} = -2\mathrm{i}\bar{A}. \tag{4.15}$$

The Hamiltonian is the Casimir operator of this algebra. It can be expressed through the generators $A, \bar{A}$ and U as follows:

$$\mathcal{H}^2 = \frac{A\,\bar{A}}{(1+\gamma U)^2} + \omega^2 U^2. \tag{4.16}$$

To get the canonical formulation of a symmetry algebra, we redefine the Fradkin tensors as

$$\begin{gathered}\mathcal{A} = \frac{A}{1+\gamma U} = p^2 + \omega^2 x^2, \qquad \overline{\mathcal{A}} = \frac{\bar{A}}{1+\gamma U} = \bar{p}^2 + \omega^2\, \bar{x}^2, \\ \{\mathcal{A}, \overline{\mathcal{A}}\} = 4\mathrm{i}\,\omega^2\, U, \qquad \{U, \mathcal{A}\} = 2\,\mathrm{i}\,\mathcal{A}, \qquad \{U, \overline{\mathcal{A}}\} = -2\,\mathrm{i}\,\overline{\mathcal{A}}.\end{gathered} \tag{4.17}$$

where $p, \bar{p}, x, \bar{x}$ are given in (4.6), (4.7). In terms of these tensors the Hamiltonian of the deformed oscillator reads

$$\mathcal{H}^2 = \mathcal{A}\,\overline{\mathcal{A}} + \omega^2 U^2. \tag{4.18}$$

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One may directly check that the Hamiltonian (3.3), being rewritten in terms of the canonical variables $x, \bar{x}, p, \bar{x}$ (4.6), (4.7), acquires the form

$$\mathcal{H} = p\,\bar{p} + \omega^2 x\,\bar{x}, \tag{4.19}$$

as it should be.

The time-dependent extensions of the generators defining the isometries of the oscillator Lagrangian are given by the expressions

$$\begin{aligned}
L^t_{-1} &= \cos^2(\omega t)\,\big(L_{-1} + \omega^2 L_1\big) - \omega\sin(2\omega t)\,L_0 - \omega^2\cos(2\omega t)\,L_1,\\
L^t_0 &= \cos(2\omega t)\,L_0 + \frac{\sin(2\omega t)}{2\omega}\,\big(L_{-1} - \omega^2 L_1\big)\,,\\
L^t_1 &= \frac{\sin^2(\omega t)}{\omega^2}\,\big(L_{-1} + \omega^2 L_1\big) + \cos(2\omega t)\,L_1 + \frac{\sin(2\omega t)}{\omega}\,L_0, \qquad U^t = U,\\
G^t_{-1/2} &= \cos(\omega t)\,G_{-1/2} - \omega\,\sin(\omega t)\,G_{1/2},\\
\overline{G}^t_{-1/2} &= \cos(\omega t)\,\overline{G}_{-1/2} - \omega\,\sin(\omega t)\,\overline{G}_{1/2},\\
G^t_{1/2} &= \cos(\omega t)\,G_{1/2} + \frac{(\sin\omega t)}{\omega}\,G_{-1/2},\\
\overline{G}^t_{1/2} &= \cos(\omega t)\,\overline{G}_{1/2} + \frac{(\sin\omega t)}{\omega}\,\overline{G}_{-1/2}.
\end{aligned} \tag{4.20}$$

Again, the corresponding transformations form a closed algebra only on shell.

Hence, the deformed oscillator is (classically) canonically equivalent to the non deformed one. Since the deformed oscillator admits the $su(1,2)$ symmetry, we conclude that the ordinary harmonic oscillator possesses the same invariance, as well.

5 Conclusion

In this paper, we provided the Hamiltonian description of the deformed two-dimensional oscillator possessing the dynamical $\mathrm{SU}(1,2)$ symmetry [14]. The generators of the dynamical symmetry do not commute with the Hamiltonian, as it happens in the case of the hidden symmetries. Instead, the dynamical symmetry is the symmetry of the action.

One of the interesting features of this system is the fact that its first-order Lagrangian is nothing but one of the Cartan forms defined on the coset $\mathrm{SU}(1,2)/H$ with a quite unusual choice of the stability subgroup H, which includes the dilatation and conformal boosts together with $\mathrm{U}(1)$ rotation. On the other hand, this one-form is a source of the symplectic form and the Hamiltonian, both being written in terms of the initial coset space variables. In this basis, the Hamiltonian of the deformed oscillator is simple, while the Poisson brackets are more involved. Analysing the structure of the Hamiltonian, we succeeded in finding the new variables in which the Hamiltonian of the deformed oscillator coincides with the Hamiltonian of the ordinary two-dimensional oscillator. Thus, we proved that these two systems, deformed and ordinary two dimensional oscillators, are canonically equivalent at the Hamiltonian level.

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Proving the canonical equivalence of these systems, we have explicitly constructed the generators spanning the $su(1,2)$ algebra in terms of the ordinary oscillator variables. The main feature of this realization is a non polynomial structure of the $su(1,2)$ generators. Probably just this property was the obstacle preventing from immediate visualization of the $su(1,2)$ algebra within the enveloping algebra of the two-dimensional oscillator. Note that the $su(3)$ algebra can also be constructed through the oscillator variables [19]. However, in contrast with the SU(1, 2) group, the SU(3) group cannot be the symmetry of the harmonic oscillator, because it does not contain the SU(1, 1) subgroup, which is the standard conformal symmetry of the harmonic oscillator.

The established equivalence of the deformed and ordinary oscillators within the Hamiltonian approach does not mean their equivalence as the Lagrangian systems. Indeed, the transformations between these two systems depend on the velocities and, therefore, they are forbidden at the Lagrangian level. Moreover, at the Lagrangian level the generators of the $su(1,2)$ symmetry are closed only on shell. Thus, the Hamiltonian formulation is more suitable for analysis of this type of the systems, as compared to the Lagrangian one.

Let us briefly discuss the quantization issues. Since the $su(1,2)$ generators are expressed via the coordinates $p, \bar{p}, x, \bar{x}$ in a non polynomial way, the canonical quantization scheme seems to be useless for the quantum realization of this algebra. Instead, the geometric quantization in the coordinates $u, v, \bar{u}, \bar{v}$ seems to be a relevant tool for this purpose. Following the general prescription, to get the quantum-mechanical representation of the $su(1,2)$ operators, one should introduce the "pre-quantum operators" which will obey the same quantum $su(1,2)$ algebra (see e.g. [21]),

$$\widehat{\mathcal{G}} = \{\mathcal{G},\ \} + \imath \boldsymbol{\alpha}(\{\mathcal{G},\ \}) + \imath \mathcal{G},$$

and then restrict their action to the Hilbert space parameterized by $u, \bar{u}$. Here $\boldsymbol{\alpha}$ is symplectic one-form (3.1) and $\mathcal{G} \in \left\{L_{\pm 1}, L_0, U, G_{\pm 1/2}, \bar{G}_{\pm 1/2}\right\}$. Though we did not consider the geometric quantization of the system, it seems there are no any visible obstacles in its realization.

Concerning further developments, one has to note that the $su(1,2)$ algebra is not a unique one which admits reduction to the conformal Galilei algebra and, thus, can be viewed as its deformation. The immediate example of possible algebras having the proper structure is provided by the wedge subalgebras in the U(n) quasi-superconformal algebras [22]. A preliminary analysis shows that the extension of the approaches of [14] and the Hamiltonian formalism of the present paper to these algebras will result in the SU($n+1, 1$) invariant $d = n + 1$ dimensional oscillators. Another interesting algebra is $so(2,3)$ which may be viewed as a deformation of the three-dimensional $\ell = 1$ conformal Galilei algebra.

More generally, the established equivalence opens a wide area of applications of the numerous tools and results obtained for the standard oscillator to the issues related with the deformations of the l-extended conformal Galilei algebra and the corresponding deformed oscillators.

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