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2D quantum gravity on compact Riemann surfaces with non-conformal matter

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ABSTRACT: We study the gravitational action induced by coupling two-dimensional non-conformal, massive matter to gravity on a compact Riemann surface. We express this gravitational action in terms of finite and well-defined quantities for any value of the mass. A small-mass expansion gives back the Liouville action in the massless limit, the Mabuchi and Aubin-Yau actions to first order, as well as an infinite series of higher-order contributions written in terms of purely geometric quantities.

KEYWORDS: 2D Gravity, Models of Quantum Gravity

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1 Introduction and generalities

1.1 Introduction and motivation

Ever since the seminal paper by Polyakov [1] it has been known that conformal matter coupled to 2D gravity gives rise to an “effective” gravitational action that is the Liouville action

$$S_L[g_0, g] \equiv S_L[g_0, \sigma] = \int d^2x \sqrt{g_0} (\sigma \Delta_0 \sigma + R_0 \sigma), \quad g = e^{2\sigma} g_0. \quad (1.1)$$

More precisely, the Liouville action captures the dependence on the metric of the partition function $Z_{\text{mat}}^{(c)}$ of conformal matter of central charge c . If we consider two metrics g_0 and $g = e^{2\sigma} g_0$ then

$$\ln \frac{Z_{\text{mat}}^{(c)}[g]}{Z_{\text{mat}}^{(c)}[g_0]} = \frac{c}{24\pi} S_L[g_0, g]. \quad (1.2)$$

For more general “matter” (plus ghost) partition functions one defines a general gravitational action as

$$\ln \frac{Z_{\text{mat}}[g]}{Z_{\text{mat}}[g_0]} = -S_{\text{grav}}[g_0, g]. \quad (1.3)$$

Being defined as the logarithm of a ratio of partition functions computed with two different metrics, any gravitational action satisfies a cocycle identity

$$S_{\text{grav}}[g_1, g_2] + S_{\text{grav}}[g_2, g_3] = S_{\text{grav}}[g_1, g_3]. \quad (1.4)$$

The simplest example of a gravitational action satisfying this cocycle identity is the “cosmological constant action”

$$S_c[g_0, g] = \mu_0 \int d^2x (\sqrt{g} - \sqrt{g_0}) = \mu_0(A - A_0) . \quad (1.5)$$

As we will recall below, this action must be present as a counterterm to renormalize the divergences that are present in (1.2) in addition to S_L . Gravitational actions other than the Liouville or cosmological constant actions can be constructed and have been studied mainly in the mathematical literature, like the Mabuchi and Aubin-Yau actions [2–5]. These latter functionals crucially involve not only the conformal factor σ but also directly the Kähler potential ϕ and do admit generalizations to higher-dimensional Kähler manifolds. In the mathematical literature they appear in relation with the characterization of constant scalar curvature metrics [5]. Their rôles as two-dimensional gravitational actions in the sense of (1.3) have been discussed in some detail in [6]. In particular, ref. [6] has studied the metric dependence of the partition function of non-conformal matter like a massive scalar field and shown that a gravitational action defined by (1.3) contains these Mabuchi and Aubin-Yau actions as first-order corrections (first order in $m^2 A$ where m is the mass and A the area of the Riemann surface) to the Liouville action. The partition function of quantum gravity at fixed area, with a gravitational action being a combination of the Liouville and Mabuchi actions, has been studied at one loop in [7] and at two and three loops in [8].

While the results of [6] concerned the first order in an expansion in $m^2 A$, in the present note we will derive a few results that are valid exactly at finite m . Ideally one would like to study some general matter action where non-conformal terms $\sim c_i \mathcal{O}^i$ have been added to some conformal theory and obtain exact results in these couplings c_i . We are going to be much less ambitious and simply study a single massive scalar field with action

$$S_{\text{mat}}[g, X] = \frac{1}{2} \int d^2x \sqrt{g} \left[g^{ab} \partial_a X \partial_b X + m^2 X^2 \right] = \frac{1}{2} \int d^2x \sqrt{g} X (\Delta_g + m^2) X , \quad (1.6)$$

and try to establish some exact results valid for finite m . Here Δ_g is the Laplace operator for the metric g , defined with a minus sign, so that its eigenvalues are non-negative:

$$\Delta_g = -\frac{1}{\sqrt{g}} \partial_a (g^{ab} \sqrt{g} \partial_b) . \quad (1.7)$$

In the remainder of this section, we discuss some basic differences between the massive and massless cases and define the matter partition functions in both cases. Then we briefly recall the Mabuchi and Aubin-Yau actions and their variations. In section 2, we summarize some technical tools involving heat kernels, zeta functions and their perturbation theory. Section 3 then gives the computation of the gravitational action for the massive scalar field, providing explicit formulae for its dependence on the metric in terms of manifestly finite and well-defined quantities. Maybe not too surprisingly, some of our massive formulae will look somewhat similar to those that can be found in [6] for the massless case. However, let us insist that our results are exact in m and valid for any finite mass. Nevertheless, we will write them in a way that immediately allows for a small mass expansion, thus

recovering the Liouville action in the zero-mass limit and the Mabuchi and Aubin-Yau actions as the first-order corrections. The higher-order corrections are similarly expressed in terms of purely geometric objects, but do not seem to have any known counterparts in the mathematical literature, nor is it clear to us whether they can be written as integrals over the manifold of local quantities.

1.2 Massive versus massless matter

One should keep in mind that adding the mass term is *not* just a perturbation by some operator that has a non-zero conformal weight. This is due to the (would-be) zero-mode of the scalar field that is absent from the action for zero mass but obviously plays an important rôle for non-zero mass. In particular, this means that the relevant quantities of the massive theory are not simply given by those of the massless theory plus order m^2 corrections. This is most clearly exemplified by the Green's function $G(x, y)$ of the operator $\Delta_g + m^2$. The latter is hermitian and has a complete set of eigenfunctions φ_n with eigenvalues $\lambda_n \geq m^2$:

$$(\Delta_g + m^2)\varphi_n(x) = \lambda_n\varphi_n(x) \quad , \quad n = 0, 1, 2, \dots \quad (1.8)$$

Clearly, the eigenfunctions φ_n do not depend on m (i.e. $\varphi_n = \varphi_n^{(0)}$), while $\lambda_n = \lambda_n^{(0)} + m^2$. The eigenfunctions which may be chosen to be real, are orthonormalized as

$$\int d^2x \sqrt{g} \varphi_n(x) \varphi_k(x) = \delta_{nk} \quad (1.9)$$

As is clear from (1.7), Δ_g always has a zero mode and, hence,

$$\lambda_0 = m^2 \quad , \quad \varphi_0 = \frac{1}{\sqrt{A}} \quad (1.10)$$

We always refer to φ_0 as the zero-mode, even in the massive case. The Green's function for $m \neq 0$ is given by

$$G(x, y) = \sum_{n>0} \frac{\varphi_n(x)\varphi_n(y)}{\lambda_n} \quad , \quad (\Delta_g + m^2)G(x, y) = \frac{1}{\sqrt{g}}\delta(x - y) \quad (1.11)$$

In general, if B is any quantity defined for $m \neq 0$, we will denote by $B^{(0)}$ the corresponding quantity for $m = 0$. But, for $m = 0$, the Green's function is not simply $G^{(0)}$ since $\lambda_0^{(0)} = 0$ and, obviously, the zero-mode must be excluded from the sum. Then

$$\tilde{G}^{(0)}(x, y) = \sum_{n>0} \frac{\varphi_n(x)\varphi_n(y)}{\lambda_n^{(0)}} \quad , \quad \Delta_g \tilde{G}^{(0)}(x, y) = \frac{1}{\sqrt{g}}\delta(x - y) - \frac{1}{A} \quad (1.12)$$

The subtraction of $\frac{1}{A}$ on the r.h.s. ensures that, when integrated $\int d^2x \sqrt{g} \dots$, one correctly gets zero. We will consistently put a tilde over the various quantities we will encounter if the zero-mode is excluded from the sum.¹ In particular, using (1.10), we can write for the massive Green's function

$$G(x, y) = \frac{1}{m^2 A} + \tilde{G}(x, y) \quad , \quad (\Delta_g + m^2)\tilde{G}(x, y) = \frac{1}{\sqrt{g}}\delta(x - y) - \frac{1}{A} \quad (1.13)$$

¹Except for determinants missing the zero-mode, where we will write \det' following the usual notation.

where \tilde{G} has a smooth limit as $m \rightarrow 0$. Moreover, the smallest eigenvalue contributing in \tilde{G} is $\lambda_1 = \lambda_1^{(0)} + m^2$ with $\lambda_1^{(0)} > 0$ being of order $\frac{1}{A}$. Thus, if $Am^2 \ll 1$, one can expand $\frac{1}{\lambda_n} = \sum_{r=0}^{\infty} (-)^r \frac{m^{2r}}{(\lambda_n^{(0)})^{r+1}}$, resulting in

$$G(x, y) = \frac{1}{m^2 A} + \tilde{G}^{(0)}(x, y) + \sum_{r=1}^{\infty} (-m^2)^r \tilde{G}_{r+1}^{(0)}(x, y), \quad (1.14)$$

where

$$G_r(x, y) = \sum_{n \geq 0} \frac{\varphi_n(x)\varphi_n(y)}{\lambda_n^r}, \quad \tilde{G}_r^{(0)}(x, y) = \sum_{n > 0} \frac{\varphi_n(x)\varphi_n(y)}{(\lambda_n^{(0)})^r}. \quad (1.15)$$

Clearly, the massive Green's function does not equal the massless one plus order- m^2 corrections: there is a crucial $\frac{1}{m^2 A}$ term in (1.13) and (1.14).

The matter partition function is defined with respect to the decomposition $X = \sum_{n \geq 0} c_n \varphi_n$ as

$$Z_{\text{mat}}[g] = \int \mathcal{D}_g X e^{-S_{\text{mat}}[g, X]} = \int \prod_{n=0}^{\infty} \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{1}{2} \sum_{n \geq 0} \lambda_n c_n^2} = (\det(\Delta_g + m^2))^{-1/2}. \quad (1.16)$$

In the massless case, since $\lambda_0^{(0)} = 0$, the integration over c_0 would be divergent and instead one replaces it by a factor \sqrt{A} . Thus

$$Z_{\text{mat}}^{(0)}[g] = \int \mathcal{D}_g^{(0)} X e^{-S_{\text{mat}}^{(0)}[g, X]} = \sqrt{A} \int \prod_{n=1}^{\infty} \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{1}{2} \sum_{n > 0} \lambda_n^{(0)} c_n^2} = \left(\frac{\det' \Delta_g}{A} \right)^{-1/2}. \quad (1.17)$$

Of course, the determinants \det and \det' are ill-defined and need to be regularized. We will use the very convenient regularization-renormalization in terms of the spectral ζ -functions:

$$\zeta(s) = \sum_{n=0}^{\infty} \lambda_n^{-s}, \quad \tilde{\zeta}(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}, \quad (1.18)$$

and similarly for $\tilde{\zeta}^{(0)}(s)$. By Weil's law (see e.g. [9]), the asymptotic behaviour of the eigenvalues for large n is $\lambda_n \sim \frac{n}{A}$ and, hence the spectral ζ -functions are defined by converging sums for $\text{Re } s > 1$, and by analytic continuations for all other values. In particular, they are well-defined meromorphic functions for all s with a single pole at $s = 1$ with residue $\frac{1}{4\pi}$ (see e.g. [9]). A straightforward formal manipulation shows that $\zeta'(0) \equiv \frac{d}{ds} \zeta(s)|_{s=0}$ provides a formal definition of $-\sum_{n \geq 0} \ln \lambda_n$, i.e. of $-\ln \det(\Delta_g + m^2)$:

$$Z_{\text{mat}}[g] = \exp\left(\frac{1}{2} \zeta'(0)\right), \quad Z_{\text{mat}}^{(0)}[g] = A^{1/2} \exp\left(\frac{1}{2} (\tilde{\zeta}^{(0)})'(0)\right). \quad (1.19)$$

There is a slight subtlety one should take into account, see e.g. [9]. While the field X is dimensionless, the φ_n scale as $A^{-1/2} \sim \mu$ where μ is some arbitrary mass scale (even if $m = 0$), and the c_n as μ^{-1} . It follows that one should write $\mathcal{D}_g X = \prod_n \frac{\mu dc_n}{2\pi}$. This results in $Z_{\text{mat}} = \left(\prod_n \frac{\lambda_n}{\mu^2}\right)^{-1/2}$, so that every $\zeta'(0)$ is changed into

$$\zeta'(0) \rightarrow \zeta'(0) + \zeta(0) \ln \mu^2. \quad (1.20)$$

The regularization-renormalization of determinants in terms of the ζ -function may appear as rather ad hoc, but it can be rigorously justified by introducing the spectral regularization [9]. The regularized logarithm of the determinant then equals $\zeta'(0) + \zeta(0) \ln \mu^2$ plus a diverging piece $\sim A\Lambda^2(\ln \frac{\Lambda^2}{\mu^2} + \text{const})$, where Λ is some cutoff. This diverging piece just contributes to the cosmological constant action (1.5), and this is why the latter must be present as a counterterm, to cancel this divergence.

Thus, finally

$$\begin{aligned} S_{\text{grav}}[g_0, g] &= -\frac{1}{2} (\zeta'_g(0) + \zeta_g(0) \ln \mu^2) + \frac{1}{2} (\zeta'_{g_0}(0) + \zeta_{g_0}(0) \ln \mu^2) , \\ S_{\text{grav}}^{(0)}[g_0, g] &= -\frac{1}{2} \ln \frac{A}{A_0} - \frac{1}{2} \left((\tilde{\zeta}_g^{(0)})'(0) + \tilde{\zeta}_g^{(0)}(0) \ln \mu^2 \right) + \frac{1}{2} \left((\tilde{\zeta}_{g_0}^{(0)})'(0) + \tilde{\zeta}_{g_0}^{(0)}(0) \ln \mu^2 \right) , \end{aligned} \tag{1.21}$$

where the first line refers to the massive case and the second line to the massless one.

1.3 Mabuchi and Aubin-Yau actions

Let us briefly recall the basic properties of the known gravitational actions. While the Liouville action (1.1) can be written in terms of g_0 and the conformal factor σ , the Mabuchi and Aubin-Yau actions are formulated using also the Kähler potential ϕ . They are related by

$$g = e^{2\sigma} g_0 \quad , \quad e^{2\sigma} = \frac{A}{A_0} \left(1 - \frac{1}{2} A_0 \Delta_0 \phi \right) , \tag{1.22}$$

where Δ_0 denotes the Laplacian for the metric g_0 with area $A_0 = \int d^2x \sqrt{g_0}$. The Mabuchi action on a Riemann surface of genus h can then be written as [6]

$$S_M[g_0, g] = \int d^2x \sqrt{g_0} \left[2\pi(h-1)\phi \Delta_0 \phi + \left(\frac{8\pi(1-h)}{A_0} - R_0 \right) \phi + \frac{4}{A} \sigma e^{2\sigma} \right] , \tag{1.23}$$

while the Aubin-Yau action takes the form

$$S_{\text{AY}}[g_0, g] = - \int d^2x \sqrt{g_0} \left[\frac{1}{4} \phi \Delta_0 \phi - \frac{\phi}{A_0} \right] . \tag{1.24}$$

As already mentioned, they both satisfy a cocycle identity analogous to (1.4) and were shown [6] to appear in S_{grav} in the term of first order in an expansion in $m^2 A$. Note that $S_M = 8\pi(1-h)S_{\text{AY}} + \int d^2x \sqrt{g_0} \left(\frac{4}{A} \sigma e^{2\sigma} - R_0 \phi \right)$. Eq. (1.22) relates the variations $\delta\sigma$ and $\delta\phi$ as

$$\delta\sigma = \frac{\delta A}{2A} - \frac{A}{4} \Delta \delta\phi \quad \text{and} \quad \delta \left(\frac{e^{2\sigma}}{A} \right) = -\frac{1}{2} \Delta_0 \delta\phi . \tag{1.25}$$

It is then straightforward to show that the variations of the Liouville, Mabuchi and Aubin-Yau actions are given by

$$\begin{aligned} \delta S_L[g_0, g] &= 4\pi(1-h) \frac{\delta A}{A} - \frac{A}{4} \int d^2x \sqrt{g} \Delta R \delta\phi , \\ \delta S_M[g_0, g] &= 2 \frac{\delta A}{A} - \int d^2x \sqrt{g} \left(R - \frac{8\pi(1-h)}{A} \right) \delta\phi , \\ \delta S_{\text{AY}}[g_0, g] &= \frac{1}{A} \int d^2x \sqrt{g} \delta\phi . \end{aligned} \tag{1.26}$$

Thus the Liouville and Mabuchi actions obviously admit the constant scalar curvature metrics as saddle-points at fixed area. Although not obvious from the previous equation, the variation of the Aubin-Yau action when restricted to the space of Bergmann metrics is similarly related to metrics of constant scalar curvature [5].

2 Some technical tools

2.1 The heat kernel

The heat kernel and integrated heat kernel for the operator $\Delta_g + m^2$ are defined in terms of the eigenvalues and eigenfunctions (1.8) as

$$K(t, x, y) = \sum_{n \geq 0} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y) \quad , \quad K(t) = \int d^2x \sqrt{g} K(t, x, x) = \sum_{n \geq 0} e^{-\lambda_n t} . \quad (2.1)$$

The corresponding \tilde{K} , $K^{(0)}$ and $\tilde{K}^{(0)}$ are defined similarly. The heat kernel K is the solution of

$$\left(\frac{d}{dt} + \Delta_g + m^2 \right) K(t, x, y) = 0 \quad , \quad K(t, x, y) \sim \frac{1}{\sqrt{g}} \delta(x - y) \text{ as } t \rightarrow 0 . \quad (2.2)$$

Note that it immediately follows from either (2.1) or (2.2) that the massless and massive heat kernels are simply related by

$$K(t, x, y) = e^{-m^2 t} K^{(0)}(t, x, y) . \quad (2.3)$$

As is also clear from (2.1) (and Weil's law $\lambda_n \sim \frac{n}{A}$), for $t > 0$, $K(t, x, y)$ is given by a converging sum and is finite, even as $x \rightarrow y$. For $t \rightarrow 0$ one recovers various divergences, in particular

$$\int_0^\infty dt K(t, x, y) = G(x, y) \quad (2.4)$$

exhibits the short distance singularity of the Green's function which is well-known to be logarithmic.

The behaviour of K for small t is related to the asymptotics of the eigenvalues λ_n for large n , which in turn is related to the short-distance properties of the Riemann surface. It is thus not surprising that the small- t asymptotics is given in terms of local expressions of the curvature and its derivatives. Indeed, one has the well-known small t -expansion:

$$K(t, x, y) \sim \frac{1}{4\pi t} e^{-(\ell^2/4t) - m^2 t} \left[a_0(x, y) + t a_1(x, y) + t^2 a_2(x, y) + \dots \right] \quad (2.5)$$

where $\ell^2 \equiv \ell_g^2(x, y)$ is the geodesic distance squared between x and y as measured in the metric g . For small t , the exponential forces ℓ^2 to be small (of order \sqrt{t}) and one can use normal coordinates around y . This allows one to obtain quite easily explicit expressions for the $a_r(x, y)$ in terms of the curvature tensor and its derivatives. They can be found e.g. in [9]. Here, we will only need them at coinciding points $y = x$, where

$$K(t, x, x) \sim \frac{1}{4\pi t} \left[1 + \left(\frac{R}{6} - m^2 \right) t + \dots \right] . \quad (2.6)$$

Let us note that in the massless case and if the zero-mode is excluded one has instead $\tilde{K}^{(0)}(t, x, x) \sim \frac{1}{4\pi t} \left[1 + \left(\frac{R}{6} - \frac{4\pi}{A} \right) t + \dots \right]$.

2.2 Local ζ -functions and Green's function at coinciding points

Local versions of the ζ -functions are defined as

$$\zeta(s, x, y) = \sum_{n \geq 0} \frac{\varphi_n(x)\varphi_n(y)}{\lambda_n^s}, \quad (2.7)$$

and similarly for $\tilde{\zeta}(s, x, y)$, etc. Note that $\zeta(1, x, y) = G(x, y)$, while for $s = r = 2, 3, \dots$ these local ζ -functions coincide with the $G_r(x, y)$ defined above in (1.15). They are related to the heat kernel by

$$\zeta(s, x, y) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} K(t, x, y). \quad (2.8)$$

For $s = 0, -1, -2, \dots$, $\Gamma(s)$ has poles and the value of $\zeta(s, x, y)$ is entirely determined by the singularities of the integral over t that arise from the small- t asymptotics of K . As shown above, the latter is given by local quantities on the Riemann surface. In particular,

$$\zeta(0, x, x) = \frac{R(x)}{24\pi} - \frac{m^2}{4\pi} \quad \text{and} \quad \tilde{\zeta}(0, x, x) = \frac{R(x)}{24\pi} - \frac{m^2}{4\pi} - \frac{1}{A}. \quad (2.9)$$

On the other hand, the values for $s = 1, 2, 3, \dots$ or the derivative at $s = 0$ cannot be determined just from the small- t asymptotics and require the knowledge of the full spectrum of $\Delta_g + m^2$, i.e. contain global information about the Riemann surface.

Clearly, $\zeta(1, x, y) = G(x, y)$ is singular as $x \rightarrow y$. For $s \neq 1$, $\zeta(s, x, y)$ provides a regularization of the propagator. More precisely, it follows from (2.8) that $\zeta(s, x, x)$ is a meromorphic function with a pole at $s = 1$ and that the residue of this pole is $\frac{a_0(x, x)}{4\pi} = \frac{1}{4\pi}$. Thus [9]

$$G_\zeta(x) = \lim_{s \rightarrow 1} \left[\mu^{2(s-1)} \zeta(s, x, x) - \frac{1}{4\pi(s-1)} \right] \quad (2.10)$$

is well-defined. (Here μ is an arbitrary scale.) This is an important quantity, called the ‘‘Green’s function at coinciding points’’. One can give an alternative definition of G_ζ by subtracting the short distance singularity from $G(x, y)$ and taking $x \rightarrow y$, see e.g. [10, 11]. More precisely [9] the relation is

$$G_\zeta(y) = \lim_{x \rightarrow y} \left[G(x, y) + \frac{1}{4\pi} \left(\ln \frac{\ell_g^2(x, y)\mu^2}{4} + 2\gamma \right) \right]. \quad (2.11)$$

One can show [9] that both definitions of G_ζ are equivalent and define the same quantity. Again, the same relations hold between $\tilde{G}_\zeta(y)$, $\tilde{G}(x, y)$ and $\tilde{\zeta}(s, x, x)$. Note that $G_\zeta(y)$ contains global information about the Riemann surface and cannot be expressed in terms of local quantities only.

2.3 Perturbation theory

We want to study how the eigenvalues λ_n and eigenfunctions φ_n change under an infinitesimal change of the metric. Since $g = e^{2\sigma}g_0$, the Laplace operator Δ_g and hence also $\Delta_g + m^2$

only depend on the conformal factor σ and on g_0 : $\Delta_g = e^{-2\sigma} \Delta_0$ and thus under a variation $\delta\sigma$ of σ one has

$$\delta\Delta_g = -2\delta\sigma\Delta_g \quad \Rightarrow \quad \langle\varphi_k|\delta\Delta_g|\varphi_n\rangle = -2(\lambda_n - m^2)\langle\varphi_k|\delta\sigma|\varphi_n\rangle, \quad (2.12)$$

where, of course, $\langle\varphi_k|\delta\sigma|\varphi_n\rangle = \int d^2x\sqrt{g}\varphi_k\delta\sigma\varphi_n$. One can then apply standard quantum mechanical perturbation theory. The only subtlety comes from the normalisation condition (1.9) which also gets modified when varying σ [6, 9]. One finds

$$\delta\lambda_n = -2(\lambda_n - m^2)\langle\varphi_n|\delta\sigma|\varphi_n\rangle, \quad (2.13)$$

$$\delta\varphi_n = -\langle\varphi_n|\delta\sigma|\varphi_n\rangle\varphi_n - 2\sum_{k\neq n}\frac{\lambda_n - m^2}{\lambda_n - \lambda_k}\langle\varphi_k|\delta\sigma|\varphi_n\rangle\varphi_k. \quad (2.14)$$

Let us insist that this is first-order perturbation theory in $\delta\sigma$, but it is exact in m^2 . Note the trivial fact that, since $\lambda_0 = m^2$ and $\varphi_0 = \frac{1}{\sqrt{A}}$, one has consistently

$$\delta\lambda_0 = 0 \quad , \quad \delta\varphi_0 = \delta\left(\frac{1}{\sqrt{A}}\right). \quad (2.15)$$

3 The gravitational action

3.1 Variation of the determinant

Recall from (1.21) that the gravitational action is defined as $S_{\text{grav}}[g_0, g] = -\frac{1}{2}(\zeta'_g(0) - \zeta'_{g_0}(0)) - \frac{1}{2}(\zeta_g(0) - \zeta_{g_0}(0))\ln\mu^2$. Our goal is to compute $\delta\zeta'(0) \equiv \delta\zeta'_g(0)$ and $\delta\zeta(0) \equiv \delta\zeta_g(0)$ and express them as “exact differentials” so that one can integrate them and obtain the finite differences $\zeta'_{g_2}(0) - \zeta'_{g_1}(0)$ and $\zeta_{g_2}(0) - \zeta_{g_1}(0)$.

From (2.13) one immediately gets, to first order in $\delta\sigma$,

$$\zeta_{g+\delta g}(s) = \sum_{n\geq 0}\frac{1}{(\lambda_n + \delta\lambda_n)^s} = \zeta_g(s) + 2s\sum_{n\geq 0}\frac{\lambda_n - m^2}{\lambda_n^{s+1}}\langle\varphi_n|\delta\sigma|\varphi_n\rangle, \quad (3.1)$$

As noted before, $\delta\lambda_0 = 0$ and, hence, there is no zero-mode contribution to the second term. Thus²

$$\delta\zeta(s) = \delta\tilde{\zeta}(s) = 2s\int d^2x\sqrt{g}\delta\sigma(x)\left[\tilde{\zeta}(s, x, x) - m^2\tilde{\zeta}(s+1, x, x)\right]. \quad (3.2)$$

For $m \neq 0$, the term in brackets could have been equally well written as $\zeta(s, x, x) - m^2\zeta(s+1, x, x)$, but the writing in terms of the $\tilde{\zeta}$ is valid for all non-zero and zero values of m . It follows that

$$\begin{aligned} \delta\zeta'(0) &= 2\int d^2x\sqrt{g}\delta\sigma(x)\tilde{\zeta}(0, x, x) \\ &\quad - 2m^2\int d^2x\sqrt{g}\delta\sigma(x)\lim_{s\rightarrow 0}\left[\tilde{\zeta}(s+1, x, x) + s\tilde{\zeta}'(s+1, x, x)\right] \\ \delta\zeta(0) &= -2m^2\int d^2x\sqrt{g}\delta\sigma(x)\lim_{s\rightarrow 0}\left[s\tilde{\zeta}(s+1, x, x)\right]. \end{aligned} \quad (3.3)$$

²Note that [6] contains an equation that looks similar but is different since it is obtained by doing a first-order perturbation expansion in m^2 of the determinant.

As recalled above, $\tilde{\zeta}(s, x, x)$ has a pole at $s = 1$ with residue $\frac{1}{4\pi}$, hence

$$\tilde{\zeta}(s, x, x) = \tilde{\zeta}_{\text{reg}}(s, x, x) + \frac{1}{4\pi(s-1)} \Rightarrow \lim_{s \rightarrow 0} [\tilde{\zeta}(s+1, x, x) + s\tilde{\zeta}'(s+1, x, x)] = \tilde{\zeta}_{\text{reg}}(1, x, x). \quad (3.4)$$

From (2.10) one sees that $\tilde{\zeta}_{\text{reg}}(1, x, x) = \tilde{G}_\zeta(x) - \frac{1}{4\pi} \ln \mu^2$. Using also (2.9) we find³

$$\begin{aligned} \delta\zeta'(0) &= \frac{1}{12\pi} \int d^2x \sqrt{g} \delta\sigma(x) R(x) - \frac{\delta A}{A} \\ &\quad - 2m^2 \int d^2x \sqrt{g} \delta\sigma(x) \left(\tilde{G}_\zeta(x) + \frac{1}{4\pi} - \frac{1}{4\pi} \ln \mu^2 \right), \\ \delta\zeta(0) &= -\frac{m^2}{2\pi} \int d^2x \sqrt{g} \delta\sigma(x). \end{aligned} \quad (3.5)$$

Since $G_\zeta = \tilde{G}_\zeta + \frac{1}{m^2 A}$, we arrive at two equivalent expressions for $\delta\zeta'(0) + \delta\zeta(0) \ln \mu^2$:

$$\begin{aligned} \delta\zeta'(0) + \delta\zeta(0) \ln \mu^2 &= \delta\tilde{\zeta}'(0) + \delta\tilde{\zeta}(0) \ln \mu^2 \\ &= \frac{1}{12\pi} \int d^2x \sqrt{g} \delta\sigma(x) R(x) - \frac{\delta A}{A} \\ &\quad - 2m^2 \int d^2x \sqrt{g} \delta\sigma(x) \left(\tilde{G}_\zeta(x) + \frac{1}{4\pi} \right) \\ &= \frac{1}{12\pi} \int d^2x \sqrt{g} \delta\sigma(x) R(x) - 2m^2 \int d^2x \sqrt{g} \delta\sigma(x) \left(G_\zeta(x) + \frac{1}{4\pi} \right). \end{aligned} \quad (3.6)$$

As it stands, this result is exact in m and holds whether $m^2 A$ is small or not. Let us insist that the G_ζ and \tilde{G}_ζ appearing on the right-hand side are the massive ones. The first line is the appropriate way of writing to study the small m^2 asymptotics, as \tilde{G}_ζ has a smooth limit for $m \rightarrow 0$.

3.2 The massless case

Let us quickly show, how in the massless case one recovers the well-known Liouville action. For $m = 0$ one has $\delta\tilde{\zeta}^{(0)}(0) = 0$ and (3.6) immediately gives

$$\delta\tilde{\zeta}^{(0)'}(0) = \frac{1}{12\pi} \int d^2x \sqrt{g} \delta\sigma(x) R(x) - \frac{\delta A}{A}. \quad (3.7)$$

Note that the last term precisely cancels the variation of the $-\frac{1}{2} \ln \frac{A}{A_0}$ in $S_{\text{grav}}^{(0)}[g_0, g]$, cf (1.21), and one gets the well-known result

$$\delta S_{\text{grav}}^{(0)}[g_0, g] = -\frac{1}{24\pi} \int d^2x \sqrt{g} \delta\sigma(x) R(x) = -\frac{1}{24\pi} \delta S_L[g_0, \sigma], \quad (3.8)$$

cf. eq. (1.26). Of course, this is just the contribution of one conformal scalar field with $c = 1$. The determinant that arises from fixing the diffeomorphism invariance gives the same contribution but with a coefficient $+\frac{26}{24\pi}$, so that overall one gets

$$S_{\text{grav}}^{(0)}[g_0, g] \Big|_{\text{ghost} + \text{conf matter}} = \frac{26-c}{24\pi} S_L[g_0, \sigma]. \quad (3.9)$$

³The result for $\delta\zeta(0)$ also follows from (2.9) and the fact that $\int d^2x \sqrt{g} R = 8\pi(1-h)$ is a topological invariant.

3.3 The massive case

Our starting point is obtained by combining eqs (1.21) and (3.6), using also (3.8):

$$\delta S_{\text{grav}}[g_0, g] = -\frac{1}{24\pi} \delta S_L[g_0, g] + m^2 \int d^2x \sqrt{g} \delta\sigma(x) \left(G_\zeta(x) + \frac{1}{4\pi} \right). \quad (3.10)$$

Our task then is to rewrite the second term on the r.h.s. as the variation of some local functional.

Let us compute $\delta G_\zeta(x)$. In order to do so, we first establish a formula for $\delta G(x, y)$ under a variation $\delta g = 2\delta\sigma g$ of the metric and thus under a corresponding variation $\delta\Delta_g = -2\delta\sigma\Delta_g$ of the Laplace operator. One can then either use the definition (1.11) as an infinite sum and the perturbation theory formula (2.13) and (2.14), or directly use the defining differential equation (1.11). In any case one finds

$$\delta G(x, y) = -2m^2 \int d^2z \sqrt{g} G(x, z) \delta\sigma(z) G(z, y). \quad (3.11)$$

To obtain the variation of G_ζ , according to (2.11) one needs to subtract the variation of the short-distance singularity. Now, the geodesic distance $\ell_g(x, y)$ transforms as (see e.g. appendix A1 of [9])

$$\delta\ell_g^2(x, y) = \ell_g^2(x, y) [\delta\sigma(x) + \delta\sigma(y) + \mathcal{O}((x-y)^2)]. \quad (3.12)$$

It follows that

$$\lim_{x \rightarrow y} \delta \ln [\mu^2 \ell_g^2(x, y)] = 2 \delta\sigma(x). \quad (3.13)$$

Plugging (3.11) and (3.13) into (2.11) one gets

$$\delta G_\zeta(x) = -2m^2 \int d^2z \sqrt{g} (G(x, z))^2 \delta\sigma(z) + \frac{\delta\sigma(x)}{2\pi}. \quad (3.14)$$

Upon integrating this over x one encounters

$$\begin{aligned} \int d^2x \sqrt{g} (G(x, z))^2 &= \int d^2x \sqrt{g} \sum_{n, k \geq 0} \frac{\varphi_k(x) \varphi_k(z) \varphi_n(x) \varphi_n(z)}{\lambda_n \lambda_k} \\ &= \sum_{n \geq 0} \frac{\varphi_n(z) \varphi_n(z)}{\lambda_n^2} = \zeta(2, z, z). \end{aligned} \quad (3.15)$$

(Note that $\zeta(2, z, z) = G_2(z, z)$ is finite.) It follows that

$$\begin{aligned} \delta \int d^2x \sqrt{g} G_\zeta(x) &= 2 \int d^2x \sqrt{g} G_\zeta(x) \delta\sigma(x) \\ &\quad - 2m^2 \int d^2z \sqrt{g} \zeta(2, z, z) \delta\sigma(z) + \frac{1}{2\pi} \int d^2x \sqrt{g} \delta\sigma(x). \end{aligned} \quad (3.16)$$

One can then rewrite (3.10) as

$$\delta S_{\text{grav}}[g_0, g] = \delta \left[-\frac{1}{24\pi} S_L[g_0, g] + \frac{m^2}{2} \int d^2x \sqrt{g} G_\zeta(x) \right] + m^4 \int d^2x \sqrt{g} \zeta(2, x, x) \delta\sigma(x). \quad (3.17)$$

Note that in the second term, we can replace G_ζ by \tilde{G}_ζ since the difference is $\frac{m^2}{2} \int d^2x \sqrt{g} \frac{1}{m^2 A} = \frac{1}{2}$, whose variation vanishes.

Next, we use (2.8) to rewrite the last term as $m^4 \int_0^\infty dt t \int d^2x \sqrt{g} K(t, x, x) \delta\sigma(x)$, and establish a formula for the variation of the integrated heat kernel $K(t)$. Since $\delta\lambda_0 = 0$ we have

$$\begin{aligned} \delta K(t) &= \delta \tilde{K}(t) = - \sum_{n>0} t e^{-\lambda_n t} \delta\lambda_n = 2 \sum_{n>0} t e^{-\lambda_n t} (\lambda_n - m^2) \int d^2x \sqrt{g} \varphi_n^2(x) \delta\sigma(x) \\ &= -2t \left(\frac{d}{dt} + m^2 \right) \int d^2x \sqrt{g} \tilde{K}(t, x, x) \delta\sigma(x) \\ &= -2t e^{-m^2 t} \frac{d}{dt} \int d^2x \sqrt{g} \tilde{K}^{(0)}(t, x, x) \delta\sigma(x), \end{aligned} \quad (3.18)$$

where we used (2.13) and (2.3). It then follows that

$$\begin{aligned} \frac{1}{2} \int_0^\infty \frac{dt}{t} (e^{m^2 t} - m^2 t - 1) \delta \tilde{K}(t) &= - \int_0^\infty dt (e^{m^2 t} - m^2 t - 1) \left(\frac{d}{dt} + m^2 \right) \int d^2x \sqrt{g} \tilde{K}(t, x, x) \delta\sigma(x) \\ &= m^4 \int_0^\infty dt t \int d^2x \sqrt{g} \tilde{K}(t, x, x) \delta\sigma(x) = m^4 \int d^2x \sqrt{g} \tilde{\zeta}(2, x, x) \delta\sigma(x) \\ &= m^4 \int d^2x \sqrt{g} \zeta(2, x, x) \delta\sigma(x) - \frac{1}{A} \int d^2x \sqrt{g} \delta\sigma(x), \end{aligned} \quad (3.19)$$

where we integrated by parts and used (2.8). The boundary terms do not contribute⁴ since \tilde{K} vanishes at $t = \infty$ as $e^{-\lambda_1 t}$ and $\lambda_1 - m^2 > 0$. Upon inserting this into (3.17) we finally get

$$\begin{aligned} \delta S_{\text{grav}}[g_0, g] &= \delta \left[- \frac{1}{24\pi} S_L[g_0, g] + \frac{1}{2} \ln \frac{A}{A_0} + \frac{m^2}{2} \int d^2x \sqrt{g} \tilde{G}_\zeta(x) \right. \\ &\quad \left. + \frac{1}{2} \int_0^\infty \frac{dt}{t} (e^{m^2 t} - m^2 t - 1) \tilde{K}(t) \right]. \end{aligned} \quad (3.20)$$

Note that in the last term the t -integral is convergent both at $t = 0$ and at $t = \infty$. This is immediately integrated as

$$\begin{aligned} S_{\text{grav}}[g_0, g] &= - \frac{1}{24\pi} S_L[g_0, g] + \frac{1}{2} \ln \frac{A}{A_0} + \frac{m^2}{2} \int d^2x (\sqrt{g} \tilde{G}_\zeta(x; g) - \sqrt{g_0} \tilde{G}_\zeta(x; g_0)) \\ &\quad + \frac{1}{2} \int_0^\infty \frac{dt}{t} (e^{m^2 t} - m^2 t - 1) (\tilde{K}(t; g) - \tilde{K}(t; g_0)). \end{aligned} \quad (3.21)$$

Thus we have expressed the gravitational action $S_{\text{grav}}[g_0, g]$ in terms of (local) functionals of g and g_0 that are all perfectly well-defined without any need of analytical continuation (contrary to the initial $\zeta'(0)$). Recall also from our remark at the end of section 1.2 that, if we define $S_{\text{grav}}[g_0, g]$ through the spectral cutoff regularization of the logarithm of the

⁴Had we started with δK rather than $\delta \tilde{K}$ and written this equation for K and $\zeta(2, x, x)$, the $-\frac{1}{A} \int \sqrt{g} \delta\sigma$ would have appeared as the boundary term.

determinant, the r.h.s. also involves the variation of the cosmological constant action $\mu_0 A$ with a coefficient $\mu_0 \sim \Lambda^2 (\ln \frac{\lambda^2}{\mu^2} + \text{const})$, to be cancelled by a corresponding counterterm.

With view on the small mass expansion studied below, it will be useful to rewrite (3.14) to obtain the variation of \tilde{G}_ζ in terms of quantities that all have well-defined limits as $m \rightarrow 0$. Recall that $G = \frac{1}{m^2 A} + \tilde{G}$ and $G_\zeta = \frac{1}{m^2 A} + \tilde{G}_\zeta$. Thus (3.14) can be rewritten as

$$\delta \tilde{G}_\zeta(x) = \frac{\delta \sigma(x)}{2\pi} - \frac{4}{A} \int d^2 z \sqrt{g} \tilde{G}(x, z) \delta \sigma(z) - 2m^2 \int d^2 z \sqrt{g} (\tilde{G}(x, z))^2 \delta \sigma(z). \quad (3.22)$$

Using (1.25), we can express $\delta \sigma$ in the second term as $-A \Delta \delta \phi / 4$ (the $\delta A / 2A$ piece doesn't contribute since \tilde{G} has no zero-mode). Using the differential equation (1.13) satisfied by \tilde{G} , one finds for the second term

$$\begin{aligned} -\frac{4}{A} \int d^2 z \sqrt{g} \tilde{G}(x, z) \delta \sigma(z) &= \int d^2 z \sqrt{g} \delta \phi(z) \Delta_z \tilde{G}(x, z) \\ &= \delta \phi(x) - \delta S_{\text{AY}}[g_0, g] - m^2 \int d^2 z \sqrt{g} \tilde{G}(x, z) \delta \phi(z), \end{aligned} \quad (3.23)$$

with δS_{AY} given in (1.26). Thus

$$\delta \tilde{G}_\zeta(x) = \frac{\delta \sigma(x)}{2\pi} + \delta \phi(x) - \delta S_{\text{AY}}[g_0, g] - m^2 \int d^2 z \sqrt{g} \left[\tilde{G}(x, z) \delta \phi(z) + 2(\tilde{G}(x, z))^2 \delta \sigma(z) \right]. \quad (3.24)$$

In exactly the same way we also get

$$\begin{aligned} \delta \tilde{G}(x, y) &= \frac{1}{2} \delta \phi(x) + \frac{1}{2} \delta \phi(y) - \delta S_{\text{AY}}[g_0, g] \\ &\quad - m^2 \int d^2 z \sqrt{g} \left[\frac{1}{2} \tilde{G}(x, z) \delta \phi(z) + \frac{1}{2} \tilde{G}(y, z) \delta \phi(z) + 2 \tilde{G}(x, z) \delta \sigma(z) \tilde{G}(z, y) \right]. \end{aligned} \quad (3.25)$$

While these are exact relations valid for all m , they are written in a way that makes the small mass expansions obvious.

3.4 Small mass expansion

Equations (3.20) and (3.21) are non-perturbative in m . However, they are also written in a way that immediately allows for a perturbative expansion in m , since \tilde{G}_ζ and \tilde{K} have smooth limits as $m \rightarrow 0$. The $m^2 \rightarrow 0$ limits of (3.20) and (3.21) also exhibit an extra term $\frac{1}{2} \ln \frac{A}{A_0}$ which only gets removed for $m^2 = 0$ due to the difference in the definitions (1.21).

Order $m^2 A$ contributions. Let us first give the order $m^2 A$ -correction $S_{\text{grav}}^{(1)}$ to $S_{\text{grav}}^{(0)}$. Since $\tilde{K}(t) = e^{-m^2 t} \tilde{K}^{(0)}(t)$ and $\lambda_n^{(0)} \sim \frac{1}{A}$ it follows that

$$\int_0^\infty \frac{dt}{t} (e^{m^2 t} - m^2 t - 1) \tilde{K}(t) = \mathcal{O}((m^2 A)^2). \quad (3.26)$$

Next, it follows from (1.13) and (1.14), upon letting $x \rightarrow y$ and subtracting the short-distance singularity, that

$$\tilde{G}_\zeta(x) = \tilde{G}_\zeta^{(0)}(x) + \sum_{r=1}^{\infty} (-m^2)^r \tilde{G}_{r+1}^{(0)}(x, x), \quad (3.27)$$

so that $\tilde{G}_\zeta(x) = \tilde{G}_\zeta^{(0)}(x) + \mathcal{O}(m^2 A)$. Thus the term of order $m^2 A$ of the gravitational action can be read from (3.21) as

$$S_{\text{grav}}^{(1)}[g_0, g] = \frac{m^2}{2} \left(A \Psi_G[g] - A_0 \Psi_G[g_0] \right) = \frac{m^2 A}{2} \left(\Psi_G[g] - \Psi_G[g_0] \right) + (A - A_0) \frac{m^2}{2} \Psi_G[g_0], \quad (3.28)$$

where, following [6], we have introduced

$$\Psi_G[g] = \frac{1}{A} \int d^2x \sqrt{g} \tilde{G}_\zeta^{(0)}(x; g). \quad (3.29)$$

The variation of \tilde{G}_ζ was given in (3.24) and that of $\tilde{G}_\zeta^{(0)}$ immediately follow as

$$\delta \tilde{G}_\zeta^{(0)}(x) = \frac{\delta \sigma(x)}{2\pi} + \delta \phi(x) - \delta S_{\text{AY}}[g_0, g], \quad (3.30)$$

so that

$$\tilde{G}_\zeta^{(0)}(x; g) = \tilde{G}_\zeta^{(0)}(x; g_0) + \frac{\sigma(x)}{2\pi} + \phi(x) - S_{\text{AY}}[g_0, g]. \quad (3.31)$$

This relation has been derived before in [6]. Using (1.22), it is then straightforward to obtain ([6])

$$\Psi_G[g] - \Psi_G[g_0] = \frac{1}{8\pi} \int d^2x \sqrt{g_0} \left[\frac{4}{A} \sigma e^{2\sigma} - 2\pi \phi \Delta_0 \phi - 4\pi \phi \Delta_0 \tilde{G}_\zeta^{(0)}(x; g_0) \right]. \quad (3.32)$$

For genus $h = 0$ and choosing g_0 to be the round metric on the sphere, $\tilde{G}_\zeta^{(0)}(x; g_0)$ is a constant, and one directly gets the Mabuchi action for $h = 0$. More generally, for arbitrary genus, [6] show that

$$\Psi_G[g] - \Psi_G[g_0] = \frac{1}{8\pi} S_{\text{M}}[g_0, g] + h \left(S_{\text{AY}}[g_0, g] - \int d^2x \sqrt{g_c} \phi \right), \quad (3.33)$$

where g_c is the canonical metric on the Riemann surface. Finally, (3.28) becomes

$$S_{\text{grav}}^{(1)}[g_0, g] = \frac{m^2 A}{2} \left[\frac{1}{8\pi} S_{\text{M}}[g_0, g] + h \left(S_{\text{AY}}[g_0, g] - \int d^2x \sqrt{g_c} \phi \right) \right] + (A - A_0) \frac{m^2}{2} \Psi_G[g_0]. \quad (3.34)$$

While the last term contributes to the cosmological constant action, the other terms are to be considered as the genuine order $m^2 A$ correction to the gravitational action, and it involves the Mabuchi and Aubin-Yau actions.

Higher-order contributions $S_{\text{grav}}^{(r)}$ for $r \geq 2$. It is straightforward to obtain the expansion in powers of m^2 of the terms in (3.20) or (3.21). Denoting the term $\sim m^{2r}$ by $S_{\text{grav}}^{(r)}$, we obviously have

$$S_{\text{grav}}[g_0, g] = \sum_{r=0}^{\infty} S_{\text{grav}}^{(r)}[g_0, g]. \quad (3.35)$$

In particular, using $\tilde{K}(t) = e^{-m^2 t} \tilde{K}^{(0)}(t)$, one has

$$\begin{aligned} \frac{1}{2} \int_0^\infty \frac{dt}{t} (e^{m^2 t} - m^2 t - 1) \tilde{K}(t) &= \sum_{r=2}^{\infty} (-m^2)^r \frac{(r-1)}{2r!} \int_0^\infty dt t^{r-1} \tilde{K}^{(0)}(t) \\ &= \sum_{r=2}^{\infty} (-m^2)^r \frac{(r-1)}{2r} \tilde{\zeta}^{(0)}(r) \\ &= \sum_{r=2}^{\infty} (-m^2)^r \frac{(r-1)}{2r} \int d^2x \sqrt{g} \tilde{G}_r^{(0)}(x, x), \end{aligned} \quad (3.36)$$

and combining this with (3.27), eq. (3.21) yields

$$S_{\text{grav}}^{(r)}[g_0, g] = \frac{(-)^{r+1}}{2r} m^{2r} (\tilde{\zeta}^{(0)}(r; g) - \tilde{\zeta}^{(0)}(r; g_0)) \quad , \quad r \geq 2 . \quad (3.37)$$

Of course, this coincides with the result of the naive expansion of $\sum_{n \geq 1} \ln \lambda_n = \sum_{n \geq 1} \ln(\lambda_n^{(0)} + m^2)$ in powers of m^2 . One may rewrite $S_{\text{grav}}^{(r)}[g_0, g]$ as a local functional:

$$S_{\text{grav}}^{(r)}[g_0, g] = \frac{(-)^{r+1}}{2r} m^{2r} \left[\int d^2x \sqrt{g} \tilde{G}_r^{(0)}(x, x; g) - \int d^2x \sqrt{g_0} \tilde{G}_r^{(0)}(x, x; g_0) \right] \quad , \quad r \geq 2 . \quad (3.38)$$

It would clearly be desirable to express these $S_{\text{grav}}^{(r)}[g_0, g]$ in terms of more geometric quantities like the conformal factor or the Kähler potential, as was the case for $S_{\text{grav}}^{(0)}[g_0, g]$ and $S_{\text{grav}}^{(1)}[g_0, g]$ with the Liouville, Mabuchi and Aubin-Yau actions. To our knowledge, there does not seem to exist any appropriate functional in the mathematical literature. Nevertheless, since the $\tilde{G}_r^{(0)}$ are entirely determined in terms of the properties of the Riemann surface, they are purely geometric quantities.

It is not clear whether, for $r \geq 2$, these $S_{\text{grav}}^{(r)}[g_0, g]$ can be written as integrals of *local* quantities. Just as the Mabuchi and Aubin-Yau functionals that occur for $r = 1$ can be written in terms of the Kähler potential ϕ that is related to the conformal factor σ of the Liouville action ($r = 0$) by $e^{2\sigma} = \frac{A}{A_0} (1 - \frac{1}{2} A_0 \Delta_0 \phi)$, cf (1.22), that involves a second derivative, a guess would be to introduce local fields ϕ_r such that $\phi_1 = \phi$ and $\phi_{r-1} = f_r(\Delta_0 \phi_r)$ with some function f_r (possibly depending also on A and A_0). We have much tried to make this precise in some satisfactory way, but given our lack of success we must leave this important issue as an open problem for further research.

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