0. Introduction.

In these lectures we discuss a few questions concerning flat manifolds. After establishing terminology, the main result of section 2 is the following STRUCTURE THEOREM 2.2 [45]. Let M be a smooth connected manifold. Then M admits a complete flat connection with parallel torsion if and only if it is the orbitspace of a connected and simply connected Lie group G under a properly discontinuous and free action of a subgroup of the affine group of G.

Then we turn to the study of flat principal bundles. After observing that flatness of a bundle can be expressed in purely topological terms, we discuss the case of $SO(2)$-bundles, where flatness can be completely characterized by the vanishing of the real Euler class. Concerning the real characteristic classes, the main result is the following

THEOREM 4.1[44]. Let X be a CW-complex and ξ a flat principal G-bundle on X. Suppose G has finitely many connected components and is either a compact or a complex and reductive Lie group. Then the characterstic homomorphism $H^*(BG,\mathbb{R}) \longrightarrow H^*(X,R)$ of ξ is zero in positive degrees.

For various other classes of Lie groups G, there are examples of flat G-bundles ξ with non-trivial real characteristic classes, (4.13, 4.14, 4.23). Consequently these characteristic classes are not determined by the curvature form of any connection in ξ ; there is no Chern-Weil theorem in these cases.

In section 5 we study in detail a class of spaces whose ring $K^0(X)$ of complex vector bundles is generated by <u>flat</u> bundles.

4.1 can be applied to prove the following facts.

THEOREM 6.7. Let M be a closed manifold, G as in 4.1 and assume M has a flat G-structure. Then the Euler characteristic $\chi(M)$ of M vanishes.

In 6.9, 6.11 other criteria for the vanishing of $\chi(M)$, M a flat closed manifold, are given. Except in these cases, one does not know very much about the Euler characteristic of closed flat manifolds. However one has

THEOREMS 6.12, 6.13. Let M be a closed flat orientable manifold of dimension $n \equiv 0(4)$. Then the signature of M vanishes. For arbitrary n, $\chi(M) \equiv 0(2)$.

This leads to the problem of computing in general the index of an elliptic complex of differentiable operators on a closed flat manifold. One can prove the following generalization of 6.7 and 6.12.

THEOREM 8.1. Let M be a closed orientable manifold, G as in 4.1, and assume M has a flat G-structure. Let (E,D) be an elliptic complex associated to the flat G-structure on X. Then the index of (E,D) vanishes.