hand, proof theoretic analysis, Gödel's incompleteness results and Skolem's exhibition of nonstandard models of 1st order Peano arithmetic or, even more strikingly, the existence of denumerable models of set theory by Löwenheim's result show *how few* of our intentions we really can express unambiguously, if at all, with our syntactically controlled means, although these controls were developed with clarity and completeness in mind. *On the other hand* our semantical (2nd order) reasoning in classical analysis and, much more strongly, in modern higher set theory, has the character of "informal rigour" (G. Kreisel) and is open to practically no doubt or ambiguity, although intentional interpretations (of 2nd order variables, say, for predicates) are used without syntactical control.

Science has lost a scholar of high rank, and we have lost a warmhearted, unselfish, noble man.

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John E. Littlewood (1885-1977) An Informal Obituary

(Note: Some of the data in this article were taken from *The London Times* Obituary Notice of September 8, 1977, the article on Littlewood in *Who's Who*, and Littlewood's little book *A Mathematician's Miscellany* (1953). We have also benefited from remarks of several people who knew Littlewood or some aspect of his work).

Professor J.E. Littlewood of Cambridge University died on 6 September 1977 at the age of 92. He was one of the outstanding mathematicians of the twentieth century, and his passing marks the end of an era. He was noted for his analytical power, his many research accomplishments in classical analysis and analytic number theory, and his long and fruitful collaboration with G.H. Hardy.

John Edensor Littlewood was born in the Medway town of Rochester, England on June 9, 1885, the son of Edward Thornton Littlewood. (His mother's name does not appear in standard sources.) His father had taken his degree from Peterhouse, Cambridge, and was 9th wrangler in the Mathematical Tripos in 1882. Littlewood spent the years from 1892 to 1900 in South Africa, where his father was a schoolmaster. In 1900 he returned to England, where he attended St. Paul's School and studied with the talented teacher and mathematician F.S. Macaulay. Littlewood entered Cambridge in 1903 as a scholar of Trinity College. He spent his first two years there mainly preparing for Part I of the Tripos Examinations with R.A. Herman, whom he described as the "last of the great coaches." He described this period of his life as a "gloomy" one in which he "wasted (his) time." However, he was preparing himself well for the competitive examinations which were then such an important part of English academic life. As Hardy put it, "he regarded himself as playing a game. It was not exactly the game he would have chosen, but it was the game which the regulations prescribed, and it seemed to him that, if you were going to play the game at all, you might as well accept the situation and play it with all your force." (G.H. Hardy, The Case Against the Tripos, Math. Gazette 13 (1926), 61-71.) In his second year at Cambridge, Littlewood was bracketed as Senior Wrangler with J. Mercer.

In 1906, his third year at Cambridge, he placed in Class I, Division I of Part II of the Tripos. Unlike Part I, this examination, according to Littlewood "dealt in quite genuine mathematics." He began his research later that year on asymptotic formulas for integral functions of order zero, under his tutor and director of studies E.W. Barnes. Barnes had been able to treat functions of positive order by an analytic method. However, Barnes' method would not work in this case, and

Littlewood succeeded in obtaining estimates by an "elementary" method.

This subject was the topic of Littlewood's first paper, submitted to the London Mathematical Society on 1 January 1907. This article, which Littlewood described some 45 years later as being "quite respectable", almost failed to see light of day, for one of the two referees was, in Littlewood's words, "violently unfavorable." Littlewood's future collaborator, G.H. Hardy, was appointed as a third referee and the paper was accepted. 1907 also marked the beginning of Littlewood's 70 year membership in the London Mathematical Society.

After this success, Barnes proposed to Littlewood the task of proving the Riemann hypothesis. This heroic proposal and Littlewood's later account of it are commentaries upon the isolation of British mathematics at that time. Proofs of the prime number theorem, achieved some ten years earlier, were still relatively unknown in England and the monumental *Handbuch der Lehre von der Verteilung der Primzahlen* of E. Landau was not to appear until 1909. For Littlewood the Riemann hypothesis appeared to be an attractive problem in the theory of integral functions. Although he did not succeed in that strenuous assignment, in the next two decades he was to do more than anyone else to render obsolete Landau's exhaustive compendium. We shall discuss his major contributions in this area below.

After receiving his M.A. degree from Cambridge, Littlewood accepted the position of Richardson Lecturer at Manchester University, where he spent the academic years 1907-1910. He won a Smith's Prize in 1908 and was elected a Fellow of Trinity that year. His chief recollection of this period was one of constant lecturing, conferences, paper work, and consequent exhaustion.

In 1910 Littlewood returned to Trinity as a college lecturer. He spent the period 1914-18 doing ballistics in the Royal Artillery. According to C.P. Snow, "Owing to his cheerful indifference he had the distinction of remaining a Second Lieutenant through the four years of war." (Essay on Hardy, *Variety of Men*, 1967). In 1920 he became Cayley lecturer in Cambridge, and in 1928 he was elected to the newly founded Rouse Ball chair of mathematics. He continued in this position until retiring in 1950. A Life Fellow of Trinity, Littlewood remained there and was active until his death.

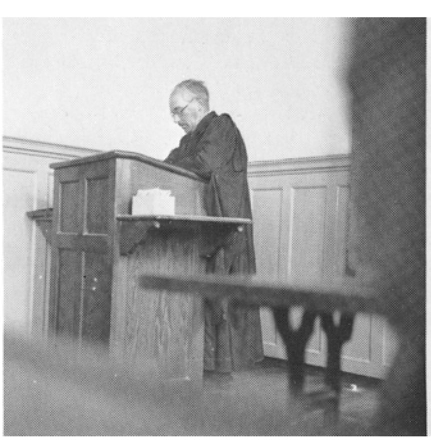
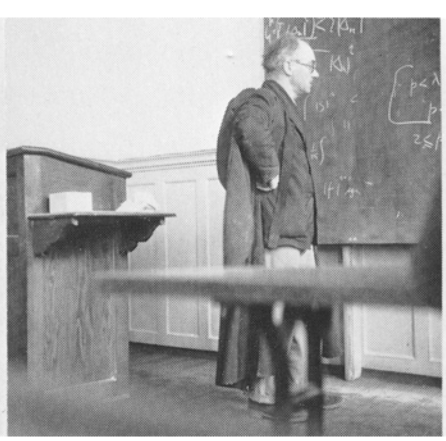
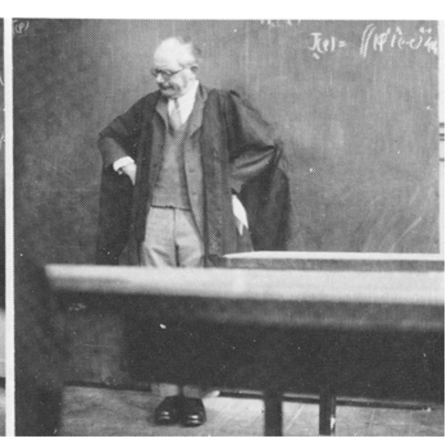
Littlewood's first major achievement upon his return to Cambridge from Manchester was his proof of the deep converse of the well known theorem of Abel that if a series Σa_n converges to a sum S, then

$$\lim_{x \to 1-} \sum a_n x^n$$

exists and has value S. Simple examples show that the direct converse of this theorem is false. Under the additional condition that $n \cdot a_n \to 0$, A. Tauber (1897) succeeded in proving the converse assertion. Littlewood's accomplishment was to prove the far more difficult assertion that the converse of Abel's theorem holds under the weaker side condition that $n \cdot a_n$ is merely bounded. Hardy and Littlewood used the term abelian theorem for any assertion that a certain operation is an averaging process which preserves limits; they immortalized Tauber by using the term tauberian theorem to describe a conditional converse of an abelian theorem, such as the two results just referred to. Wiener remarked with some justice that "it would be far more appropriate to term these theorems Hardy-Littlewood theorems, were it not that usage has sanctioned the other appellation." (Tauberian Theorems, Ann. of Math. (2) 33(1932), 1-100).

"On looking back, "Littlewood wrote many years later, "this time seems to me to mark my arrival at a reasonably assured judgment and taste, the end of my 'education.'" Littlewood at this time entered into his thirty-five year collaboration with G.H. Hardy, the most famous and most powerful such partnership in the history of mathematics.

Their first joint effort appears to have been a paper, "The range of Borel's method of summation of oscillatory series", which they did not publish. They reported on the paper at the June, 1911 meeting of the London Mathematical Society, but subsequently discovered that their proof

of the main theorem was faulty. This theorem asserted that if Σa_n is Borel summable and if $\sqrt{n}\cdot a_n$ is bounded, then Σa_n is convergent. They noted (Proc. London Math. Soc. (2) 11 (1912-13), 3) that "it can hardly be doubted that this result (is true), ... though the difficulties attendant on (Littlewood's) generalization of Tauber's theorem suggest forcibly that the proof may not be at all easy to find." Happily, they overcame these difficulties and finally completed their first project in an article published in Rend. Circ. Mat. Palermo (1) 41 (1916), 36-53.

In their collaboration, which was to last until Hardy's death in 1947, they produced nearly a hundred joint papers, many of outstanding quality. Their joint work covered a wide area of analysis and analytic number theory. In the first area they studied summability and convergence problems, Fourier and Dirichlet series, problems of differentiation, special functions, convex functions, maximal functions, conjugate functions, and inequalities. In number theory they contributed to the theory of the Riemann zeta function and the distribution of prime numbers, diophantine approximation, and additive problems.

In an important series of papers, "Some Problems in Partitio Numerorum," they put forward a new powerful method, now called the Hardy-Littlewood or circle method, for exploiting complex integration to estimate arithmetic quantities. For many arithmetic functions f the associated generating function $F(z) = \Sigma\, f(n)z^n$ behaves in the following way: the series converges inside the unit circle and has the unit circle as a natural boundary. The radial growth of F is strongest along radii terminating at roots of unity of low order. For $r < 1$ we have the Cauchy formula

$$f(n) = \frac{1}{2\pi i} \int_{|z| = r} F(z)\, z^{-n-1}\, dz \quad ,$$

and the crux of the method is to single out the parts of the path of integration near the roots of unity of low order (the so-called major arcs). Separate estimates are given over the remaining ("minor") arcs. This leads to an approximation for f involving a series whose detailed behavior depends on the arithmetical properties of the integer n.

The circle method was an outgrowth of an earlier technique used by Hardy and Ramanujan on the partition problem. Because the generating function of the partition function is a modular form, which has exact transformation formulas, only the major arcs are required for its treatment.

The first major success of the Hardy-Littlewood method was the achievement of quantitative results on Waring's problem, a qualitative solution of which had earlier been obtained by Hilbert by algebraic methods. In due course subsequent work of Vinogradov and his school led to simpler arguments and better estimates than those of Hardy and Littlewood.

According to Harald Bohr, Hardy and Littlewood had a kind of "collaboration contract," at once humorous and revealing: "I should like to tell how Hardy and Littlewood, when they planned and began their far-reaching and intensive team work, still both had some misgivings about it, because they feared that it might encroach on their personal freedom, so vitally important to them. Therefore, as a safety measure, they amused themselves by formulating some so-called axioms for their mutual collaboration. There were in all four such axioms. The first of them said that, when one wrote to the other (they often preferred to exchange thoughts in writing instead of orally), it was completely indifferent whether what they wrote was right or wrong. As Hardy put it, otherwise they could not write completely as they pleased, but would have to feel a certain responsibility thereby. The second axiom was to the effect that, when one received a letter from the other, he was under no obligation whatsoever to read it, let alone to answer it, --because, as they said, it might be that the recipient of the letter would prefer not to work at that particular time, or perhaps that he was just then interested in other problems. And they really observed this axiom to the fullest extent. When Hardy once stayed with me (Bohr) in Copenhagen, thick mathematical letters arrived daily from Littlewood, who was obviously very much in the mood for work, and I have seen Hardy calmly throw the letters unopened into a corner of the room, saying: 'I suppose I shall want to read them some day.' The third axiom was to the effect that, although it did not really matter if they both simultaneously thought about the same detail, still, it was preferable that they should not do so. And, finally, the fourth, and perhaps most important axiom, stated that it was quite indifferent if one of them had not contributed the least bit to the contents of a paper under their common name; otherwise there would constantly arise quarrels and difficulties in that now one, and now the other, would oppose being named co-author. I think one may safely say that seldom - or never - was such an important and harmonious collaboration founded on such apparently negative axioms." (from "Looking Backward," English translation of a talk given by Harald Bohr on his 60th birthday, April 22, 1947, at the University of Copenhagen, pp. XIII to XXXIV of Volume I of Bohr's *Collected Mathematical Papers*.)

In awarding Littlewood the Sylvester medal of the Royal Society Hardy observed: "He is the man most likely to storm and smash a really deep and formidable problem: there is no one else who can command such a combination of insight, technique, and power." In his obituary of Hardy, Norbert Wiener wrote, "I think it is fair to say that throughout their long collaboration the extremes of technical facility belonged to Littlewood, but that much of the nexus of leading ideas and philosophical unity is that of Hardy." (Bull. Am. Math. Soc. 55 (1949), 72-77).

Again according to Bohr (op. cit.) most of the joint Hardy-Littlewood articles received their final form at the hand of Hardy, who displayed a positive talent and enthusiasm for expressing himself well. Littlewood, on the other hand, appeared content if what he had to say was essentially correct. In reference to Hardy's fondness for writing and his own penchant for drinking wine, Littlewood once remarked that Hardy thought no more of ordering a ream of paper than he himself would think of ordering a bottle of sherry.

Similary, Littlewood did not have Hardy's talent or inclination for giving polished lectures. He was often hesitant and his discourse was frequently elliptical, but the lectures teemed with ideas. He believed that lectures should be informal commentaries rather than systematic presentations of a body of mathematics.

Littlewood also engaged in joint work with a number of other mathematicians. In fact well over half of his approximately 200 papers are joint efforts. In collaboration with Offord, for example, he proved the striking theorem that "almost all" entire functions of finite order behave in some sense like the Weierstrass σ-function. In collaboration with Mary L. Cartwright he obtained many deep results concerning the qualitative behavior of solutions of the second-order non-linear differential equation

$$\frac{d^2x}{dt^2} + \mu\, f(x)\, \frac{dx}{dt} + g(x) = \mu\, p(t), \quad \mu > 0 \quad ,$$

which plays an important role in applied mathematics and engineering.

One of Littlewood's most brilliant collaborators was R.E.A.C. Paley (1907-1933). In a series of fundamental papers on Fourier analysis they introduced the function now called after them. During the last few years, at least two books, one by E.M. Stein, the other by R.E. Edwards and G.I. Gaudry, have appeared with "Littlewood-Paley" in their titles, as have numerous papers. The work of Littlewood and Paley combined with that of Hardy and Littlewood on maximal functions has had an enormous and continuing impact on the theory of singular integrals, multiplier theory, and other aspects of modern Fourier analysis, as well as on many other branches of analysis.

Besides engaging in joint work, Littlewood pursued his own research. He made several major contributions to analytic number theory. Littlewood was the first to improve de la Vallée Poussin's error term in the prime number theorem, an effort subsequently carried further by Vinogradov and his school. He showed that the series $\Sigma\, \mu(n)n^{-s}$ for $1/\zeta(s)$ converges if Re s exceeds the least upper bound of the real parts of the zeroes of the zeta function. He also exhibited a modest rate of growth of $1/\zeta(s)$ as a function of Im s, a result closely related to the preceding one.

Littlewood's most spectacular work on the distribution of primes concerned the difference of the function $\pi(x)$, which counts the number of primes not exceeding x, and its prime number theorem approximant

$$\text{li}\ x = \int_2^x (\log t)^{-1}\, dt + 1.04\ldots \quad .$$

Numerical data showed that $\pi(x) < \text{li}\ x$ for $2 \le x \le 10^8$; the same inequality in fact holds for all specific individual values of x for which explicit calculations have been made. Further, a formula of Riemann provided theoretical grounds for believing that li x - 1/2 li $\sqrt{x}$ was a better approximation to $\pi(x)$ than li x itself. Against this background, it came as a complete surprise when Littlewood proved in 1914 that $\pi(x) - \text{li}\ x$ assumes positive as well as negative values for arbitrarily large values of x. This theorem also attracted attention on account of the computationally ineffective nature of its proof. Later on S. Skewes, a former research student of Littlewood, succeeded in exhibiting a number X with the property that $\pi(x) - \text{li}\ x > 0$ for some $x < X$. The fabulous size of Skewes' number, X = exp exp exp exp (7.705), attracted further attention to this problem; a more modest $X = 2 \cdot 10^{1165}$ was later achieved by R. Sherman Lehman. For a more detailed account, see the article of Don Zagier (Math. Intelligencer 0 (1977), 7-19).

Littlewood did not always enjoy writing up his work for publication. In fact, the full details of some of his most famous results were never published as such by Littlewood himself. Thus the full details of his improvement of the error term in the prime number theorem were published only in a paper of Landau (Math. Zeit. 20(1924), 98-125), and in Landau's *Vorlesungen über Zahlentheorie*. (Needless to say, Landau just could not understand Littlewood's reluctance to write up his work.) Details of his sign-change theorem for $\pi(x) - \text{li}\ x$ saw the light of day only as part of a joint paper with Hardy. (Acta. Math. 41(1918), 119-196).

An article of Littlewood which reveals much about his research outlook is his "Quickest Proof of the Prime Number Theorem" (Acta Arith. 18 (1971), 83-86). In it he assembles an arsenal of eight powerful theorems and proceeds to prove the prime number theorem in just over two pages.

Littlewood also made other significant contributions in analysis, applied analysis, and celestial mechanics. He contributed significantly to the problem of escape and capture of a mass particle by an ensemble of gravitating masses. In his later years he studied problems involving trigonometric polynomials and wrote about "pits effects" for various entire functions.

During his long career Littlewood had a number of distinguished research students, including A.O.L. Atkin, S. Chowla, H. Davenport, F.J. Dyson, T.M. Flett, A.E. Ingham, M.J. Lighthill, R.E.A.C. Paley, S.Skewes, D.C.Spencer, and H.P.F.Swinnerton-Dyer. Littlewood's supervision style was to pose a difficult research problem and let the student find a solution. Those who survived this sink-or-swim test were invariably of superior quality.

Littlewood compiled three lists of research problems for his "family", i.e., present and former research pupils. The last list was published as a 57 page book (*Some Problems in Real and Complex Analysis,* 1968). Many of his problems have turned out to be much more difficult than they appear at first glance, and they continue to have a stimulating effect on research.

A particularly well known problem of Littlewood and Spencer is the following: if θ and ϕ are given real numbers, is it necessarily true that

$$\underline{\lim}_{n \to \infty} n|(\sin \pi n \theta)(\sin \pi n \phi)| = 0 \ ?$$

This is equivalent to asking if both θ and ϕ can be simultaneously approximated to a modest degree of accuracy (measured in geometric mean) by rational numbers of the same denominator. (For related almost-everywhere results see Patrick X. Gallagher (J. London Math. Soc. 37(1962), 387-390) and the references quoted therein).

Again, Littlewood conjectured that if $n_1, n_2, .., n_N$ are distinct integers, then

$$(*) \quad \int_0^1 \left| \sum_{j=1}^{N} \exp(2\pi i n_j t) \right| dt > c \log N \quad ,$$

where c is a positive absolute constant. The first result of this type with log N replaced by $(\log N)^{1/8-\varepsilon}$ was obtained by Paul J. Cohen in a paper which won the author the Bôcher prize of the American Mathematical Society. Further progress has been made by H.Davenport and by S.K.Pichorides, but the conjecture in its original form (*) is still open.

Littlewood's *A Mathematician's Miscellany* provides some insight into his education, career, tastes, and ways of thinking about mathematics. Its contents range from humorous anecdotes which he customarily told in his lectures to detailed discussions on celestial mechanics.

He wrote two other books, both based largely on his lectures. *Elements of the Theory of Real Functions* (1926) and *Lectures on the Theory of Functions* (1944). Littlewood's reluctance toward book writing prompted J.C. Burkill in his review of *Real Functions* to hope aloud that "May we suggest to Mr. Littlewood that one book is an unsatisfying contribution from him to mathematical literature and we hope that it is the first of many. Perhaps he will agree that for each 500 pages of work that he contributes to periodicals he will write a book that will appeal to a larger circle of readers." (Math. Gazette 13(1927), 428). However, Littlewood must have understood that he had nowhere near the talent of Hardy in this direction. His books were never to have the impact of Hardy's, and on balance he chose wisely to put his energies into his research.

Aside from a joint problem collection with E. C. Francis (*Examples on Infinite Series with Solutions,* Cambridge, 1928), Littlewood was to have one successful and one unsuccessful venture at coauthorship of a book. He collaborated with Hardy and G. Pólya in writing the book *Inequalities* (1934), which has had a great impact on the development of hard analysis and the way it is now presented. The unsuccessful venture was a joint effort with Harald Bohr, who described it in the following terms:

"During my first long stay in Cambridge, I worked together with Littlewood on a monograph on the theory of the zeta-function and its application to the theory of prime numbers. I have to add, however, that although we succeeded in preparing the complete manuscript, we were so exhausted afterwards that we did not have the strength to send it to the printer, and so it was left for a number of years until, at a later date, when the theory had been developed so much further, we turned it over to two younger English mathematicians, Titchmarsh and Ingham, to use freely in the preparation of their two excellent booklets in the series Cambridge Tracts, about the subjects in question" (from "Looking Backward").

Littlewood's taste in mathematics and his way of expressing himself were admirably distilled into two words in a talk given by G. Pólya (American Math. Monthly 76(1969), 746-753): "Relatively concrete problems, such as the proof of the Riemann hypothesis, are less in vogue nowadays, for reasons partly good and partly bad - 'Mostly bad', Littlewood would interject if he were present."

Littlewood received many honors during his career. He was a Fellow of the Royal Society, Fellow of the Royal Astronomical Society, Fellow of the Institute for Mathematics and its Applications, Corresponding Member of the French and Göttingen Academies, and Foreign Member of the Royal Dutch, Royal Danish, and Royal Swedish Academies. He received the Royal Medal (1929), the Sylvester Medal (1944), and the Copley Medal (1958) from the Royal Society and the DeMorgan Medal (1939) and the Senior Berwick Prize (1960) from the London Mathematical Society. He received honorary doctorates from Cambridge, Liverpool, and St. Andrews Universities.

Littlewood's hobbies included rock climbing and skiing, for which he was well suited by his considerable strength and agility. He frequently spent holidays pursuing these activities in Cornwall, Scotland, and Switzerland. He put into practice his conviction that mathematicians should take a vacation from mathematics of at least twenty one days a year. He followed ball games avidly - shades of Hardy - but unlike Hardy he enjoyed music.

Littlewood did not seek the company of other mathematicians, and he was not highly visible among the mathematicians in Cambridge. His invisibility was not due to essential shyness, but to the fact that he had better things to do than to spend hours in idle chatter. He definitely was not what Hardy would call "a common room mathematician." However, after dinner at Trinity, he was considerably more outgoing. As C.P.Snow put it, "Hardy ... did not really enjoy lingering in the combination-room over port and walnuts. Littlewood ... did."

Littlewood was a rough-hewn earthy person with a charm of his own; he did not impress one as a carbon copy. After Rademacher met Littlewood for the first time in 1962, he remarked to one of us (P.B.) that Littlewood was "a character straight out of Dickens".

Littlewood retired from Cambridge in 1950. Some years later he made his first visit to the United States. This trip, to Chicago, was a great success and was followed by several other visits in the 1950's and 60's. He maintained his ability and his enthusiasm during the years of his retirement and continued to produce significant work.

During the academic year 1973-74 one of us (H.D.) saw Littlewood during a brief visit to Cambridge. The first impression has remained vividly in mind. "Come in!" he shouted in response to a knock on his door. Littlewood was seated in an overstuffed chair, with his back to the door, facing an electric fire. He was reading a newspaper, and the floor around him was littered with the individual sheets of the newspaper which he threw to the floor as he finished them. His room was large and rather dark. Its dominant feature was his filing system for articles. Here and there stalagmites of papers, sometimes the height of a few feet, arose.

He was polite and indeed hospitable to a total stranger, with whom his only connection was a common interest in analytic number theory. Not surprisingly perhaps, he spoke most readily and fluently of things and people of the distant past. We had dinner together at the College, and then he insisted that we visit the combination room for wine and snuff. The result of the snuff was a great sneeze by the visitor, followed by an apology. "Nothing to be sorry for," replied Littlewood, and then a bit sadly he continued, "it doesn't have an effect on me anymore."

A detailed list of Littlewood's publications and further biographical information may be found in a book by Mary Elizabeth Williams (*A Bibliography of John Edensor Littlewood,* Pageant-Poseidon Press, Ltd., Elizabeth, N.J., 1974).

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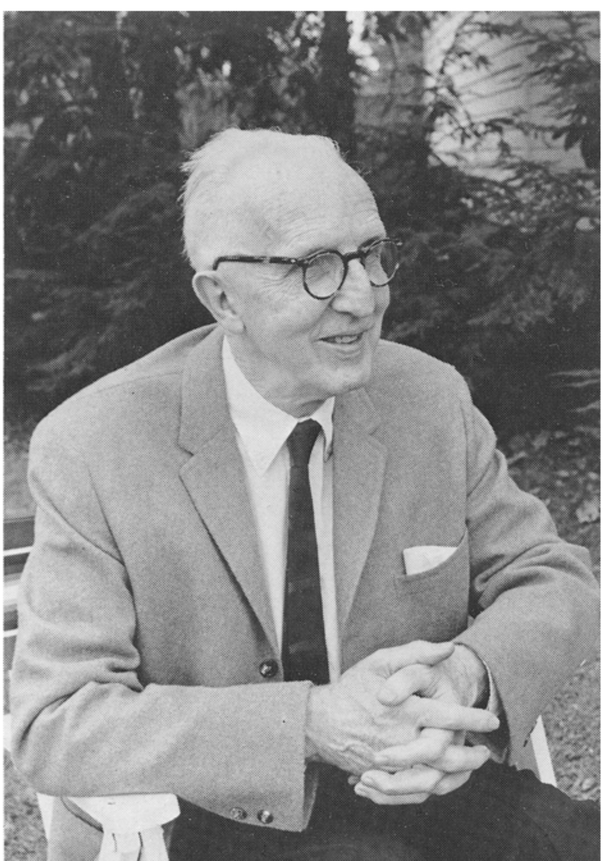

Marston Morse (1892-1977)

In these few words I wish to comment on the place of Marston Morse and his work in American mathematics. My review of Morse's last book, recently printed in the Bulletin of the American Mathematical Society[1], carries a more extensive mathematical and historical discussion of the main work of Morse.

I would place Morse with George D. Birkhoff, Norbert Wiener and Solomon Lefschetz among American (i.e. American trained) nonliving mathematicians as those whose work has had the biggest impact on mathematics. But the nature of the contribution and the style of Morse differs much from the other three in the group.

What distinguished Morse in particular was his single-minded persistence with one theme, now known as Morse theory (or calculus of variations in the large). This character of his work had both a positive and a negative aspect.

Let us deal first with the positive side. I believe that Morse theory is the single greatest contribution of American mathematics (perhaps excluding more recent contributions for which time has been too short to assess sufficiently).

The depth of the contribution of Morse theory is reflected today in the vitality and breadth of what is now called Global Analysis; that is in the study of differential equations, ordinary and partial, from a global or topological point of view. It is natural that this development owes so much to the calculus of variations since the problems of the calculus of variations have an especially global char-

[1] Volume 83, Number 4, July 1977, pp. 683-693