

## Lower Bounds on the Length of Monotone Paths in Arrangements

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**Abstract.** We show that the maximal number of turns of an  $x$ -monotone path in an arrangement of  $n$  lines is  $\Omega(n^{5/3})$  and the maximal number of turns of an  $x$ -monotone path in arrangement of  $n$  pseudolines is  $\Omega(n^2/\log n)$ .

Let us consider a set  $L$  of  $n$  lines in the plane. The lines of  $L$  induce a cell complex, called the *arrangement* of  $L$ . The plane is divided into two-dimensional convex *regions*; the sides of these regions are called *edges* and the intersections of the lines are called *vertices*. In the following we assume that none of the lines  $L$  is parallel to the  $y$ -axis.

Combinatorial properties of line arrangements in the plane have been intensively studied in the last few years, mainly in connection with the construction of efficient geometric algorithms (see, e.g., [E]). In this paper we consider one such property, the maximal possible length of an  $x$ -monotone polygonal line (path) composed of edges of the arrangement; the length is measured as the number of turns of the polygonal line plus one.

This problem is posed in [EG]; here an obvious upper bound  $O(n^2)$  and a lower bound  $\Omega(n^{3/2})$  (by M. Sharir) are mentioned. An interesting application of this problem can be found in [YKH].

We give a stronger lower bound:

**Theorem 1.** *The maximal possible length of an  $x$ -monotone path in an arrangement of  $n$  lines is  $\Omega(n^{5/3})$ .*

A *pseudoline* is a connected  $x$ -monotone curve. A *collection of pseudolines* is a finite set of pseudolines, such that any two pseudolines meet at at most one point and cross there. An arrangement of pseudolines and a monotone path in a pseudoline arrangement are defined analogously as in the case of lines.

It is known that there are many more combinatorially distinct arrangements of pseudolines than arrangements of lines [GP]. This contrasts with the fact that most of the results known for arrangements of lines also hold for arrangements of pseudolines. We believe that the maximum monotone path length might be one of the properties which distinguishes arrangements of lines from arrangements of pseudolines. We suspect that the bound in Theorem 1 for line arrangements is tight, while for arrangements of pseudolines we have the following result:

**Theorem 2.** *The maximum length of a monotone path in an arrangement of  $n$  pseudolines is  $\Omega(n^2/\log n)$ .*

Our results immediately give rise to corresponding upper-bound problems. An upper bound for the case of lines of the form  $O(n^{2-\epsilon})$  for some  $\epsilon > 0$  probably requires new and nontrivial techniques, as Theorem 2 indicates. The case of pseudolines might be more tractable; at present no  $o(n^2)$  upper bound is known.

*Proof of Theorem 1.* We choose an integer  $m$  and use the scheme depicted in Fig. 1 (for  $m = 3$ ). The thin lines in this figure are just lines while the thick ones are actually bundles of near parallel lines. These bundles have three directions:

- the horizontal one (direction 1),
- with negative slope (descending; direction 2),
- and with positive slope (ascending; direction 3).

There are  $m^2$  bundles, each consisting of  $m$  lines, in direction 1;  $m$  bundles, each with  $m^2$  lines, in direction 2; and  $(m^2 + m)$  bundles with  $m-1$  lines each, in direction 3. The bundles in direction 2 are further structured, consisting of  $m$

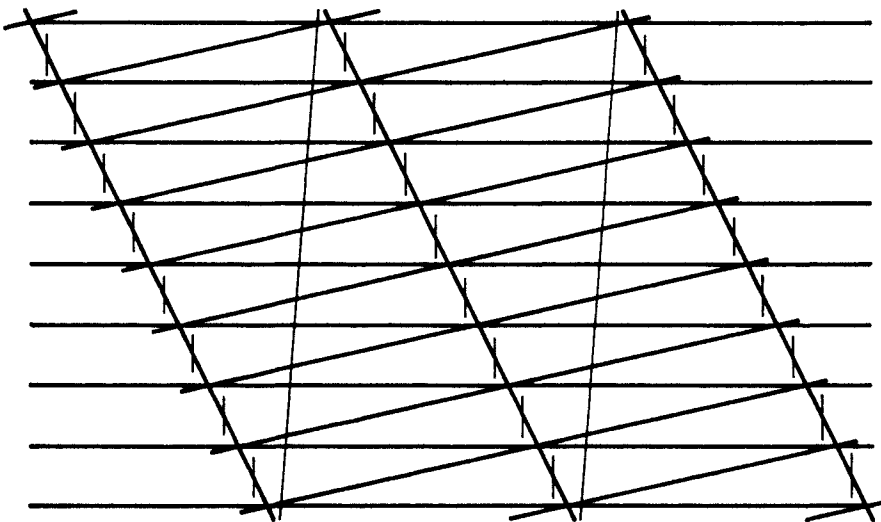


Fig. 1

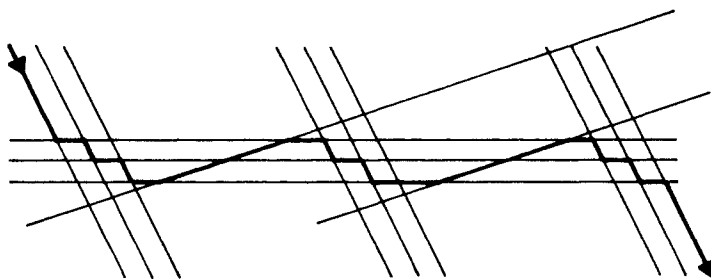


Fig. 2

groups of  $m$  very near lines. Together with  $m \cdot (m^2 - 1)$  short nearly vertical thin lines with negative slope and the  $m-1$  nearly vertical long thin lines with positive slope there are  $O(m^3)$  lines in the arrangement.

One crossing of a triple of bundles, greatly magnified, is depicted in Fig. 2 (all the crossings look the same). The  $x$ -monotone path is depicted by the thick line. It comes to the crossing along direction 2 from above, following the leftmost line in the bundle. It makes at least  $m^2$  turns in the crossing and leaves it again along direction 2 down, following the rightmost line in the bundle. Then it crosses the bundle in direction 2 using the short thin line, and again reaches the leftmost line of this bundle; thus it may continue in the same manner at the next crossing. When it reaches the lowest horizontal bundle, it uses the long thin line to get to the next bundle of direction 2. In this way it passes all  $m^3$  crossings, making  $m^2$  turns in each, which makes  $m^5$  turns in total.

□

*Proof of Theorem 2.* An integer  $r$  will be the parameter of the construction. We put

$$n^2 = 2^{r+1}, \quad q = \lceil 2^{r+1}/(r+1) \rceil.$$

We choose  $\varepsilon > 0$ ,  $\varepsilon < 1/2n$  and define auxiliary points  $A_{ij}$ ,  $i = 1, 2, \dots, n$ ,  $j = 1, \dots, q$ :

$$A_{ij} = (j + \varepsilon(i - 1), i).$$

Further, we choose  $\delta = \varepsilon/2(r + 1)$  and define points

$$X_{i,j,k} = A_{ij} + (k\delta, 0)$$

for  $i = 1, \dots, n, j = 1, \dots, q, k = -r, \dots, r$ .

The arrangement will be constructed so that it allows a monotone path of the following form (illustrated in Fig. 3 for  $r = 2$ ): It starts near the point  $A_{11}$ . It goes upward along the first column of points, always coming to the left of  $A_{i1}$  (to some  $X_{i1,k}, k < 0$ ) and going a small distance horizontally (to some  $X_{i1,k}, k > 0$ ). When

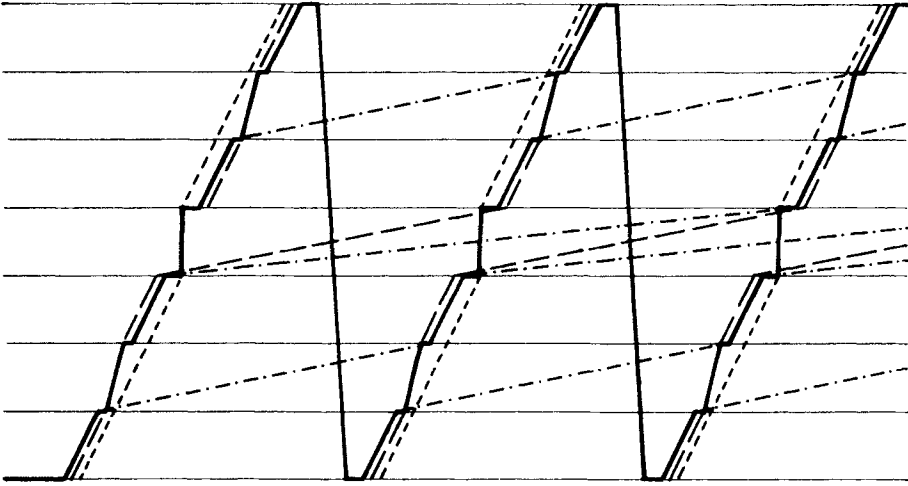


Fig. 3

it reaches the top row, it goes downward along a single straight line with negative slope and the same scheme is repeated along a second column upward, etc. The problem is with the pseudolines which lead the path upward along the columns. In every column, the upward-directed segments of the monotone path will belong to distinct pseudolines, but the same pseudoline will be reused in more columns. We shall have a hierarchy of such pseudolines (grouped in levels); the pseudolines of level  $k + 1$  will help to reuse the pseudolines of level  $k$  for the monotone path.

Let us give a more formal description. We define the following function on natural numbers:

$$\text{tz}(m) = \max\{i; 2^i \text{ divides } m\}$$

(the number of trailing zeros of  $m$  in binary notation).

The arrangement will contain the following pseudolines (actually polygonal lines composed of straight segments):

- Straight horizontal lines with equation  $y = i$  for  $i = 1, \dots, n$  (joining the rows of points  $A_{ij}$ ).
- Straight lines joining  $X_{nj,r}$  to  $X_{1(j+1),-r}$  for  $j = 1, \dots, q - 1$ .

The choice of  $\varepsilon$  guarantees that these lines have negative slope, and they serve for the return of the path to the lowest row.

- Polygonal lines, which are grouped into levels  $0, 1, \dots, r$ .

Level  $k$  consists of polygonal lines of the following form: they start at points  $X_{1j,k}$  ( $j = 1, 2, \dots, q$ ) and at points  $X_{ij,k}$ , where  $\text{tz}(i) > k$  and  $1 \leq j \leq \text{tz}(i) - k$ . They consist of segments with positive slope, each such segment starting at some point  $X_{ij,s}$  and ending one row higher—at some point  $X_{(i+1)j',r}$ . The value of  $s$  is equal to

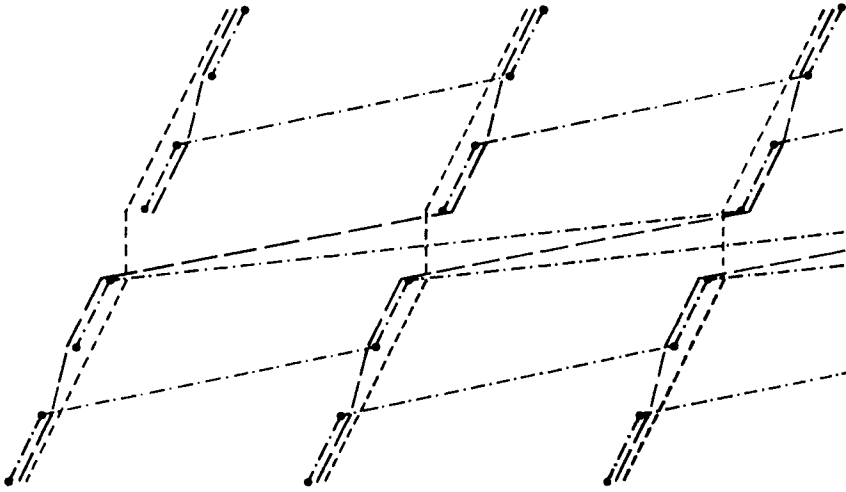


Fig. 4

$-k$  if  $\lfloor (i - 1)/2k \rfloor$  is odd, otherwise it is equal to  $k$ . The value of  $j$  (the column for a given row on the pseudoline) is determined inductively by the following rule: the upper endpoint of a segment with lower endpoint  $X_{ij,s}$  is  $X_{(i+1)(j+d(i)),t}$ , where  $d(i) = \max(0, tz(i) - k)$  (and the value of  $t$  is determined from  $(i + 1)$  as  $s$  was from  $i$ ). Each pseudoline ends as soon as it reaches the topmost row or the rightmost column. Figure 4 shows these pseudolines for  $r = 2$  (the pseudolines of level 0 being indicated by short-dashed lines, of level 1 by long-dashed lines of level 2 by dot-dashed lines and the points  $A_{ij}$  by dots).

It is easily seen that exactly one pseudoline of a given level passes near every  $A_{ij}$ . The slope of all segments used is always positive, by the choice of  $\delta$ ; thus the pseudolines are  $x$ -monotone.

The pseudolines of level  $k$  start at positions  $(i, j)$  with  $i = 1$  or with  $1 \leq j < tz(i) - k$ . It is easy to check that

$$\sum_{i=1}^n \max(tz(i) - k, 0) \leq n/2^k,$$

and therefore there are at most  $q + n/2^k$  pseudolines of level  $k$ . The total number of pseudolines in the arrangement is thus at most  $(r + 1)q + 2n = O(n)$ .

It remains to check that no two pseudolines have more than one point in common. If at least one of the pseudolines is horizontal or has negative slope, or if both the pseudolines have the same level, this is immediate. For a pair of pseudolines of distinct levels, consider the segments joining row  $i$  and row  $i + 1$  on the first and on the second pseudoline. We verify that whenever these segments intersect, the slope of the segment belonging to the pseudoline of the lower level is smaller than the slope of the second segment; thus the two pseudolines can intersect only once.

Finally we define the monotone path. As is described above, it traverses the arrangement “column by column,” i.e., it goes near  $A_{11}$ , near  $A_{21}, \dots, A_{n1}$ ,  $A_{12}, \dots, A_{nq}$ . In the  $j$ th column the rule is the following:

- Start at  $X_{1j, -r}$ .
- Being at some  $X_{ij, s}$ , go horizontally to  $X_{ij, tz(i)}$  and follow the segment of a pseudoline of level  $tz(i)$ , going to  $X_{(i+1)j, -tz(i)}$ .
- Being at some  $X_{nj, s}$ , go horizontally to  $X_{nj, r}$  and then go downward along a straight line to  $X_{1(j+1), -r}$  (to the next column).

For each  $ij$  the monotone path has at least one turn near  $A_{ij}$ , therefore there are at least  $nq = \Omega(n^2/\log n)$  turns.  $\square$

An example very similar to that in our proof of Theorem 1 has been independently obtained by Cole *et al.* [CMS]. They even show that a weight assignment for the lines, such that the  $\Omega(n^{5/3})$ -turn monotone path is the weighted median path, can be found. This has also been observed for the example presented above by Yamamoto [Y].

### Acknowledgment

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