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An On-Line Potato-Sack Theorem*

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Abstract. We discuss packings of sequences of convex bodies of Euclidean n-space E^n in a box and particularly in a cube. Following an Auerbach-Banach-Mazur-Ulam problem from the well-known *Scottish Book*, results of this kind are called potato-sack theorems. We consider on-line packing methods which work under the restriction that during the packing process we are given each succeeding "potato" only when the preceding one has been packed. One of our on-line methods enables us to pack into the cube of side $d > 1$ in E^n every sequence of convex bodies of diameters at most 1 whose total volume does not exceed $(d-1)(\sqrt{d}-1)^{2(n-1)}/n!$. Asymptotically, as $d \to \infty$, this volume is as good as that given by the non-on-line methods previously known.

1. Introduction

We denote Euclidean n-space by E^n. Let $(K_m) = K_1, K_2, \ldots$ be a sequence of sets in E^n. We say that (K_m) can be *packed* in a set $K \subset E^n$ if there are rigid motions σ_m such that all the sets $\sigma_m K_m$, where $m = 1, 2, \ldots$, are subsets of K and such that they have pairwise disjoint interiors. The above packing is called *translative* if only translations are allowed here as rigid motions.

In this paper we consider packing methods which work under the restriction that at the beginning we are given only K_1 and that we are given K_m only after K_{m-1} has been packed, $m = 2, 3, \ldots$. Packing methods under this restriction are called *on-line packing methods.*

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The potato-sack theorem of Auerbach *et al.* [1] guarantees that every sequence of convex bodies in E^n of diameters at most 1, the sum of whose volumes is finite, can be packed in a cube. The proofs of Kosiński [5], Moon and Moser [6], and Groemer [2] estimate the total volume of a sequence that can be packed in a cube of side d. There is a survey paper [3] on this subject.

Our aim is to prove the potato-sack theorem under the on-line restriction and to estimate the total volume of convex bodies which can be packed. Thanks to the following result this task can be simplified to the problem of packing sequences of boxes.

Lemma [4], [5]. *Every convex body in E^n of diameter 1 and volume v is a subset of a rectangular parallelotope of volume at most $n!\,v$ and edge lengths at most 1.*

2. On-Line Packing Methods

Let an orthogonal coordinate system in E^n be given. Every set of the form

$$\{(x_1, \ldots, x_n);\ s_i \leq x_i \leq t_i \text{ for } i = 1, \ldots, n\},$$

where $s_i < t_i$ for $i = 1, \ldots, n$, is called a *box*. The number $t_i - s_i$ is called *the* ith *width*, $i = 1, \ldots, n$, and the nth width of this box is called its *height*. The set

$$\{(x_1, \ldots, x_n);\ s_i \leq x_i \leq t_i \text{ for } i = 1, \ldots, n-1 \text{ and } x_n = y_n\}$$

is called the *bottom* of this box for $y_n = s_n$ and the *top* for $y_n = t_n$.

Sequences of boxes of all n widths at most 1 and sequences of convex bodies of diameters at most 1 will be packed in a box

$$C = \{(x_1, \ldots, x_n);\ 0 \leq x_i \leq c_i \text{ for } i = 1, \ldots, n\},$$

where $c_1 > 1, \ldots, c_n > 1$, called the $(c_1, \ldots, c_n)$-*container* or simply a *container*.

If $n > 2$, we define a few additional notions. Let $p_1, \ldots, p_n$ be positive integers such that

$$\prod_{i=2}^{n} p_i < c_1.$$

We dissect C into p_n sets

$$D_j = \left\{(x_1, \ldots, x_n);\ \frac{(j-1)c_n}{p_n} \leq x_1 \leq \frac{jc_n}{p_n} \text{ and } 0 \leq x_i \leq c_i \text{ for } i = 2, \ldots, n\right\},$$

where $j = 1, \ldots, p_n$, called *compartments* of C. Let

$$r = 1 - \frac{1}{\sqrt{p_n c_n - p_n + 1}}. \tag{1}$$

By *layers* of C we understand sets of the form

$$\{(x_1, \ldots, x_n) \in D_j;\ g \leq x_n \leq g + r^k\},$$

where $j \in \{1, \ldots, p_n\}$, k is a nonnegative integer, and where $g \geq 0$ is a number such that $g + r^k \leq c_n$. It is clear what is meant by the height, bottom, and top of a container, a compartment, or a layer because each of them is a box. If the height of a box is greater than r^{k+1} but not greater than r^k, then layers of height r^k are called *proper* for this box.

We are ready to describe on-line methods of packing sequences of boxes of all n widths at most 1 in a container $C \subset E^n$. We proceed by induction with respect to n.

(a_1) *The segment-to-segment method in* E^1. Let α be a sequence of one-dimensional boxes of heights at most 1 (i.e., of closed segments of lengths at most 1). The first box of α is packed in $C = \{x_1; 0 \leq x_1 \leq c_1\}$ such that 0 becomes its left endpoint. Every succeeding box of α is packed such that its left endpoint coincides with the right endpoint of the box packed just before; of course, if there is enough space in C. If there is not enough space in C when a new box Z of α is given, we say that the packing is stopped by Z.

(a_n) *The* $(p_2, \ldots, p_n)$*-method in* E^n, *where* $n \geq 2$. Assume that the method (a_{n-1}) has been described in E^{n-1}. Let α be a sequence of boxes in E^n of all n widths at most 1 and let C be the $(c_1, \ldots, c_n)$-container. Every box A can be packed only in a layer L which is proper for it and with its bottom in the bottom of L. This packing is provided according to the $(p_2, \ldots, p_{n-1})$-method in E^{n-1} applied to the bottom of L as the $(n-1)$-dimensional container and the bottom of A as an $(n-1)$-dimensional box. But if $n = 2$, we use here the segment-to-segment method (a_1). Every box A of α is packed in the last created layer which is proper for A; of course, if such a layer exists and if there is enough space. If in this layer there is not enough space for this, we say that the layer is *full*. In this case, as well as in the case when no proper layer for A has been created yet, we create a new proper layer L_A for A (if this is possible) and we pack A in L_A. The bottom of L_A must coincide with the top of the last layer in a compartment, or with the bottom of this compartment if no layer has been created in this compartment yet. This layer L_A can be created in any compartment where it is possible; the choice of this compartment does not matter. If in every compartment there is not enough space to create a new proper layer when necessary for a succeeding box $Z \in \alpha$, we stop the packing. We say that the packing is *stopped by* Z.

Remark. When discussing the $(p_2, \ldots, p_n)$-method for the $(c_1, \ldots, c_n)$-container we always tacitly assume that $c_1 > 1, \ldots, c_n > 1$, and that $p_2, \ldots, p_n$ are positive integers whose product is smaller than c_1. Moreover, when discussing the $(p_2, \ldots, p_n)$-method in E^n, where $n \geq 1$, we mean the segment-to-segment method when $n = 1$.

The segment-to-segment method in E^1 and the $(1, \dots, 1)$-method in E^n, where $n \geq 2$, are called *one-compartment methods.*

(b_n) *The related combined method* for packing a sequence β of convex bodies of diameters at most 1 in a container $C \subset E^n$. We combine the lemma and method (a_n). The successive body B of β is first packed in a box A according to the lemma and then, if there is enough space in C, the box A is packed in C by method (a_n).

Of course, (a_n) and (b_n) are on-line methods. Observe that (a_n) guarantees a translative packing.

3. How Effective Are the On-Line Methods?

Proposition. *Let α be a sequence of boxes in E^n, each having all n widths at most 1. The $(p_2, \dots, p_n)$-method enables us to pack the next box of α in the $(c_1, \dots, c_n)$-container if the sum of volumes of all boxes packed before does not exceed*

$$w = \left(c_1 - \prod_{i=2}^{n} p_i\right) \prod_{i=2}^{n} \left(\sqrt{c_i - 1 + \frac{1}{p_i}} - \sqrt{\frac{1}{p_i}}\right)^2, \tag{2}$$

where w is understood to be $c_1 - 1$ for $n = 1$.

Proof. We apply induction with respect to n. For the segment-to-segment method in E^1 (see the remark) the proposition is obvious. Assume that the proposition holds true in E^{n-1}, where $n \geq 2$, and consider the packing of α by the $(p_2, \dots, p_n)$-method in the $(c_1, \dots, c_n)$-container $C \subset E^n$. In order to prove the proposition for the above situation it is sufficient to show that if the packing is stopped by a box of α, then the total volume of the preceding boxes of α is greater than w.

Assume that the packing is stopped by a box Z of α. There is a nonnegative integer q such that the height of Z is greater than r^q but not greater than r^{q-1}. Observe that for every nonnegative integer k at most one nonfull layer of height r^k exists. Moreover, no nonfull layer of height r^q exists. Since the packing is stopped by Z, we have

$$h > p_n c_n - p_n r^q - \left[\left(\sum_{k=1}^{\infty} r^k\right) - r^q\right],$$

where h denotes the sum of heights of full layers of C. Thus from

$$\sum_{k=1}^{\infty} r^k = \frac{1}{1-r} = \sqrt{p_n c_n - p_n + 1}$$

and since $r^q \leq 1$ we see that

$$h > p_n c_n - p_n + 1 - \sqrt{p_n c_n - p_n + 1}. \tag{3}$$

By our induction hypothesis, every full layer of C contains boxes of α with total $(n-1)$-dimensional volume of their bottoms greater than

$$b=\left(\frac{c_1}{p_n}-\prod_{i=2}^{n-1} p_i\right)\prod_{i=2}^{n-1}\left(\sqrt{c_i-1+\frac{1}{p_i}}-\sqrt{\frac{1}{p_i}}\right)^2$$

(where b should be understood as c_1/p_2-1 for $n=2$), and thus of total n-dimensional volume greater than br times the volume of the layer. Hence the volume of boxes packed in C before stopping the packing process by the box Z is greater than brh. Using (3) and the equality

$$\frac{r}{p_n}(p_nc_n-p_n+1-\sqrt{p_nc_n-p_n+1})=\left(\sqrt{c_n-1+\frac{1}{p_n}}-\sqrt{\frac{1}{p_n}}\right)^2$$

we see that $brh>w$, which completes the proof. □

Corollary. *Every sequence of boxes of E^n with all n widths at most 1 and the total volume not greater than w can be packed on-line in the $(c_1, \ldots, c_n)$-container by the $(p_2, \ldots, p_n)$-method.*

The estimate given in the proposition and in the corollary is the best possible for the $(p_2, \ldots, p_n)$-method applied to the $(c_1, \ldots, c_n)$-container C. The reason is that for every $\varepsilon>0$ we can construct a sequence of boxes, each with all n widths at most 1, of total volume $w+\varepsilon$, which cannot be packed in C by the $(p_2, \ldots, p_n)$-method. We leave such a construction for the reader as an exercise.

Observe that usually the order of the axes matters for the effectiveness of the $(p_2, \ldots, p_n)$-method; our first axis is special. If before applying the $(p_2, \ldots, p_n)$-method a permutation π re-orders the axes, we call this approach the *$(p_2, \ldots, p_n)$-method under the permutation π*. If for a fixed container C the number w (defined in (2)) is greater for the $(p_2, \ldots, p_n)$-method under a permutation π than for the $(p_2', \ldots, p_n')$-method under a permutation π', then we say that the $(p_2, \ldots, p_n)$-method under π is *more effective* for C than the $(p_2', \ldots, p_n')$-method under π'. Finding a general rule for determining which of the methods is the most effective for the $(c_1, \ldots, c_n)$-container seems to be a difficult task. Some partial results for the case of the cube as the container are presented in the next section.

From the corollary and the lemma we immediately obtain the following on-line potato-sack theorem.

Theorem. *Let $c_1, \ldots, c_n$ be numbers greater than 1 and, if $n \geq 2$, let $p_2, \ldots, p_n$ be positive integers whose product is smaller than c_1. Every sequence of convex bodies in E^n with diameters at most 1 and total volume not greater than*

$$\frac{1}{n!}\left(c_1-\prod_{i=2}^{n} p_i\right)\prod_{i=2}^{n}\left(\sqrt{c_i-1+\frac{1}{p_i}}-\sqrt{\frac{1}{p_i}}\right)^2$$

can be packed on-line in the $(c_1, \ldots, c_n)$-container.

4. Comparison of the Methods for a Cube as a Container

In this section the cube $C_d^n \subset E^n$ of edge length $d > 1$ is considered as the container. We discuss only the case of sequences of boxes, each of all n widths at most 1. The following notation is convenient for the comparison of our methods as $d \to \infty$. If for two functions $f_1(d)$ and $f_2(d)$ we have $\lim_{d\to\infty} f_1(d)/f_2(d) = 1$, then we write $f_1(d) \overset{d}{\sim} f_2(d)$, or simply $f_1 \overset{d}{\sim} f_2$.

From the paper of Groemer [2] it follows that his non-on-line packing method guarantees a packing effectiveness of $g = g_n(d)$ for C_d^n such that $g \overset{d}{\sim} d^n$ for every dimension n. A simple calculation shows that $w \overset{d}{\sim} d^n$ also, where w is defined in (2), for every $(p_2, \ldots, p_n)$-method applied to C_d^n. This means that all our on-line methods are just as effective as the non-on-line method of Groemer, as $d \to \infty$.

An attempt to determine generally which of our on-line methods is the most effective for C_d^n leads to some inequalities which are difficult to solve. For any particular container the most effective method can be easily found by using a computer. For instance, if $n = 2$ and if d is an integer between 2 and 250 the following (p_2)-method is the most effective: the (1)-method for $d \le 5$, the (2)-method for $6 \le d \le 19$, the (3)-method for $20 \le d \le 47$, the (4)-method for $48 \le d \le 97$, the (5)-method for $98 \le d \le 174$, the (6)-method for $175 \le d$. Similarly, if $n = 3$ and if d is an integer between 2 and 500, then the following (p_2, p_3)-method is the most effective: the (1, 1)-method for $d \le 5$, the (1, 2)-method for $6 \le d \le 16$, the (2, 2)-method for $17 \le d \le 68$, the (2, 3)-method for $69 \le d \le 147$, the (3, 3)-method for $148 \le d \le 396$, the (3, 4)-method for $397 \le d$. And, finally, an example of the effectiveness: for C_{100}^3 we have $w = 649539$ for the (1, 1)-method and $w \approx 711786$ for the most effective (2, 3)-method.

We are able to provide an asymptotic comparison of the $(p, \ldots, p)$-methods for C_d^n as $d \to \infty$. For every positive integer $p < d^{1/(n-1)}$ the $(p, \ldots, p)$-method is defined and the number w is a function of two variables d and p:

$$w_n(d, p) = (d - p^{n-1})\left(\sqrt{d - 1 + \frac{1}{p}} - \sqrt{\frac{1}{p}}\right)^{2(n-1)}.$$

In particular, for the one-compartment method this number is

$$(d-1)(\sqrt{d}-1)^{2(n-1)}.$$

For comparison we need a more sensitive tool than the limit $\lim_{d\to\infty} w_n(d, p)/d^n$. For this purpose consider the number

$$u_n(d, p) = d^n - w_n(d, p)$$

which estimates the volume which remains unfilled in C_d^n. It is easy to see that

$$\lim_{d\to\infty} u_n(d, p)/d^{n-1/2} = 2(n-1)/\sqrt{p}. \tag{4}$$

Consequently,

$$u_n(d,p) \overset{d}{\sim} \frac{2(n-1)}{\sqrt{p}} d^{n-1/2}. \tag{5}$$

Moreover, from (4) and from $\sqrt{p} < \sqrt{p+1}$ we see that for every positive integer p there is a number d_p such that for every $d > d_p$ the $(p+1, \ldots, p+1)$-method is more effective than the $(p, \ldots, p)$-method.

A simple calculation shows that, asymptotically as $d \to \infty$, the best choice for p is $\lfloor d^{1/(2n-1)} \rfloor$, where $\lfloor \cdot \rfloor$ means the integer part. From (5) we get

$$u_n(d, \lfloor d^{1/(2n-1)} \rfloor) \overset{d}{\sim} (2n-1) d^{n-(n/(2n-1))}.$$

In particular,

$$u_2(d, \lfloor d^{1/3} \rfloor) \overset{d}{\sim} 3d^{4/3},$$

while for a fixed p we have

$$u_2(d,p) \overset{d}{\sim} \frac{2}{\sqrt{p}} d^{3/2}.$$

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