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Bounding the Piercing Number*

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Abstract. It is shown that for every k and every $p \geq q \geq d+1$ there is a $c = c(k,p,q,d) < \infty$ such that the following holds. For every family $\mathscr{H}$ whose members are unions of at most k compact convex sets in R^d in which any set of p members of the family contains a subset of cardinality q with a nonempty intersection there is a set of at most c points in R^d that intersects each member of $\mathscr{H}$. It is also shown that for every $p \geq q \geq d+1$ there is a $C = C(p,q,d) < \infty$ such that, for every family $\mathscr{G}$ of compact, convex sets in R^d so that among and p of them some q have a common hyperplane transversal, there is a set of at most C hyperplanes that together meet all the members of $\mathscr{G}$.

1. Introduction

In this paper we study geometric problems of the type introduced in [16] and considered in various subsequent papers. It is convenient, however, to make the required definitions in the more general framework of abstract families of sets. Let $\mathscr{H}$ be a (finite or infinite) family of (finite or infinite) sets, and let $\mathscr{F}$ be another family of sets. For two integers $p \geq q$ we say that $\mathscr{H}$ satisfies *the* (p,q) *property* (with respect to $\mathscr{F}$) if for any p members of $\mathscr{H}$ there is an $F \in \mathscr{F}$ that intersects (at least) q of them. The *piercing number* of $\mathscr{H}$ (with respect to $\mathscr{F}$), denoted by $P(\mathscr{H},\mathscr{F})$, is the minimum number of members of $\mathscr{F}$ that together meet all members of $\mathscr{H}$. Our

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objective is to show that for certain geometrically defined infinite families $\bar{\mathscr{H}}$ and $\mathscr{T}$, and for various values of $p \geq q$, there is a finite constant $c = c(\bar{\mathscr{H}}, \mathscr{T}, p, q)$ so that, for every $\mathscr{H} \subset \bar{\mathscr{H}}$ that satisfies the (p, q) property with respect to $\mathscr{T}$, the piercing number $P(\mathscr{H}, \mathscr{T})$ does not exceed c.

The best-known example of this form is the classical theorem of Helly [18]. In the notation above it can be formulated as follows. Let $\bar{\mathscr{H}}$ be the family of all compact convex sets in the d-dimensional Euclidean space R^d, and let $\mathscr{T}$ be the family of all one-point subsets of R^d. Then, if $\mathscr{H} \subset \bar{\mathscr{H}}$ satisfies the $(d+1, d+1)$ property (with respect to $\mathscr{T}$; that is, if every $d+1$ of the convex sets in $\mathscr{H}$ have a common intersection), then $P(\mathscr{H}, \mathscr{T}) = 1$ (i.e., all the sets have a common intersection). An extension of this statement has been considered by Hadwiger and Debrunner [16]. They conjectured that for every $p \geq q \geq d+1$ there is a $c = c(p, q, d) < \infty$ so that, for $\bar{\mathscr{H}}$ and $\mathscr{T}$ as above, if $\mathscr{H} \subset \bar{\mathscr{H}}$ satisfies the (p, q) property, then $P(\mathscr{H}, \mathscr{T}) \leq c$. This question became known as the (p, q) problem and has been considered in various papers, including the survey articles and books [17], [7], [14], and [10]. Special cases have been proved in [16], [8], [12], [13], [24], and [26], and the general conjecture has recently been proved by Alon and Kleitman in [3].

Another result that can be stated in the above notation is the main result of Eckhoff [11]. Here $\bar{\mathscr{H}}$ is the family of all compact convex sets in the plane and $\mathscr{T}$ is the family of all lines in the plane. It is shown in [11] that if $\mathscr{H} \subset \bar{\mathscr{H}}$ satisfies the $(3,3)$ property with respect to $\mathscr{T}$ (that if, if every three members of $\mathscr{H}$ admit a common line transversal), then $P(\mathscr{H}, \mathscr{T}) \leq 4$, i.e., there are four lines that together meet all the members of $\mathscr{H}$.

In this paper we extend the Alon–Kleitman piercing theorem to families of unions of convex sets and also prove a piercing theorem for hyperplane transversals, which extends Eckhoff's theorem. Let $\mathscr{K}_k^d$ denote the family of all sets in R^d which are the union of at most k convex sets.

Theorem 1.1. *For every k and every $p \geq q \geq d+1$ there is a $c = c(k, p, q, d) < \infty$ such that the following holds. For every family $\mathscr{H} \subset \mathscr{K}_k^d$ that satisfies the (p, q) property with respect to the family $\mathscr{T}$ of all points of R^d, $P(\mathscr{H}, \mathscr{T}) \leq c$.*

The case $k = 1$ of the above is the main result of [3], conjectured in [16]. Note that the assumption $p \geq q \geq d+1$ cannot be improved, as shown by any infinite family whose members are the intersections of hyperplanes in general position with an appropriate box. Such a family satisfies the (p, q) property for all $p \geq q$, $q \leq d$ and yet has an infinite piercing number.

For $k > 1$ Theorem 1.1 is interesting even for $p = q \geq d+1$. It is well known (and quite easy, see [7]) that there is not finite "Helly number" for unions of convex sets, i.e., for every n and $k > 1$ there are examples of families of n sets in $\mathscr{K}_k^d$, such that every $n-1$ of them have a nonempty intersection but all of them have an empty intersection.

An easy consequence of Theorem 1.1 is that $\mathscr{K}_k^d$ has a finite "Helly order." A family of sets $\mathscr{T}$ has *Helly order* t if the following property holds. Let $\mathscr{H}$ be a subfamily of $\mathscr{T}$. If every intersection of members of $\mathscr{H}$ is in $\mathscr{T}$ and if every t

members of $\mathcal{H}$ have a nonempty intersection, then all members of $\mathcal{H}$ have a nonempty intersection.

Corollary 1.2. *Put $h(d,k) = (d+1)\cdot c(k, d+1, d+1, d)$. The Helly order of the family $\mathcal{K}_k^d$ is finite and bounded above by $h(d,k)$.*

For the special case in which every intersection as above is a union of at most k *pairwise disjoint* convex sets, Morris [21] proved in his Ph.D. thesis that the Helly order is $k(d+1)$. (The cases $k = 2, 3$ were already proved by Grübaum and Motzkin [15].) The proofs in [15] and [21] apply in a more general context and use only Helly's theorem and purely combinatorial arguments. (However, Morris's proof is complicated and not fully understood and a simple proof is desirable. See the discussion on p. 399 of [10].)

Another result we prove in this paper is the following.

Theorem 1.3. *For every $p \geq q \geq d+1$ there is a $C = C(p,q,d) < \infty$ such that, for every family $\mathcal{G}$ of compact convex sets in R^d that satisfies the (p,q) property with respect to the family $\mathcal{F}$ of all hyperplanes in R^d, $P(\mathcal{G},\mathcal{F}) \leq C$.*

Here, too, the assumption $p \geq q \geq d+1$ is best possible, as shown by any infinite family $\mathcal{F}$ of points in general position.

It is known (and easy) that there is no finite Helly number for hyperplane transversals. Thus the case $p = q = d+1$ of Theorem 1.3 is nontrivial and of particular interest. For $d = 2$ this special case is a weak form of Eckhoff's theorem on line transversals in the plane.

Our proofs follow the basic approach of [3], but contain several additional ideas. The proofs of both theorems are based on the same general technique, which can be used for proving several additional results of the same type.

The rest of the paper is organized as follows. In Section 2 we describe the general technique applied for proving both theorems above, and present the proofs of a few lemmas that are relevant in both cases. Section 3 contains the proofs of Theorem 1.1 and Corollary 1.2. Theorem 1.3 is proved in Section 4. The final section contains some concluding remarks and open problems.

2. The General Technique

Let $\bar{\mathcal{H}}$ and $\mathcal{F}$ be two families of sets, where $\bar{\mathcal{H}}$ is infinite. Our objective is to show that under suitable assumptions, for every $p \geq q \geq r$ there is a finite constant c (depending only on $\bar{\mathcal{H}}$, $\mathcal{F}$, p, and q) such that, for every $\mathcal{H} \subset \bar{\mathcal{H}}$ that satisfies the (p,q) property (with respect to $\mathcal{F}$), $P(\mathcal{H},\mathcal{F}) \leq c$. Since we are not interested here in finding the best possible estimate for c, we may assume that $q = r$. This is because for $q \geq r$ any $\mathcal{H}$ that satisfies the (p,q) property clearly satisfies the (p,r) property as well. In the cases considered here there is always a simple compactness argument that shows it suffices to (uniformly) bound the piercing numbers $P(\mathcal{H},\mathcal{F})$ for finite subfamilies $\mathcal{H}$ of $\bar{\mathcal{H}}$.

The desired bound for the piercing numbers $P(\mathscr{H},\mathscr{F})$ is derived in three steps, described in the following three subsections.

2.1. A Fractional Piercing Theorem

Let $\bar{\mathscr{H}}$ and $\mathscr{F}$ be families of sets. If $\mathscr{H}' \subset \bar{\mathscr{H}}$ satisfies $P(\mathscr{H}',\mathscr{F}) = 1$, that is, there is an $F \in \mathscr{F}$ that intersects all members of $\mathscr{H}'$, we say that $\mathscr{H}'$ is *pierceable*. We say that $\bar{\mathscr{H}}$ satisfies the *fractional piercing property of order r* (with respect to $\mathscr{F}$) if there is a function $\delta: (0,1] \mapsto (0,1]$ with the following property. For every α, $0 < \alpha \le 1$, and every finite $\mathscr{H} \subset \bar{\mathscr{H}}$, so that at least $\alpha\binom{|\mathscr{H}|}{r}$ subfamilies $\mathscr{H}'$ of cardinality r of $\mathscr{H}$ are pierceable, there is an $F \in \mathscr{F}$ that intersects at least $\delta(\alpha)|\mathscr{H}|$ members of $\mathscr{H}$. In this case we call δ the *fractional piercing function of order r of* $\bar{\mathscr{H}}$.

In Sections 2 and 3 it is shown that various interesting infinite families $\bar{\mathscr{H}}$ satisfy the above property. For our purposes we need the following consequence of this property.

Proposition 2.1. *Suppose $\bar{\mathscr{H}}$ satisfies the fractional piercing property of order r with respect to $\mathscr{F}$ and let δ be the corresponding fractional piercing function. Then, for every $p \ge r$, there is a $\beta = \beta(\delta, p, r) > 0$ with the following property. Suppose $\mathscr{H} = \{H_1, \ldots, H_n\} \subset \bar{\mathscr{H}}$ satisfies the (p,r) property (with respect to $\mathscr{F}$). Assume, further, that each $H_i \in \mathscr{H}$ intersects some member of $\mathscr{F}$. Let $a_1, \ldots, a_n$ be nonnegative integers, define $m = \sum_{i=1}^{n} a_i$, and let $\mathscr{G}$ be the family of cardinality m consisting of a_i copies of H_i for $1 \le i \le n$. Then there is an $F \in \mathscr{F}$ that intersects at least βm members of $\mathscr{G}$.*

Proof. We prove the proposition with

$$\beta = \min\left\{\frac{1}{2p}, \delta\left(\frac{1}{2^p\binom{p}{r}}\right)\right\}.$$

This estimate can be easily improved, but we make no attempt to optimize the constant here and in what follows. If an i with $a_i \ge m/2p$ exists, then, since there is an $F \in \mathscr{F}$ which intersects H_i, the desired result follows, as $\beta \le 1/2p$. We thus assume that $a_i < m/2p$ for all i. Denote the members of $\mathscr{G}$ by $G_{i,j}$, $1 \le i \le n$, $1 \le j \le a_i$, where, for each fixed i, the sets $G_{i,j}$ are the a_i copies of H_i. Let $\mathscr{T}$ be the family of all subsets

$$\{G_{i_1,j_1}, \ldots, G_{i_p,j_p}\}$$

of cardinality p of $\mathscr{G}$ in which $i_u \ne i_v$ for all $1 \le u < v \le p$. Since $a_i \le m/(2p)$ for all i we conclude that

$$|\mathscr{T}| \ge \frac{1}{p!}m\left(m - \frac{m}{(2p)}\right)\left(m - \frac{2m}{(2p)}\right)\cdots\left(m - \frac{(p-1)m}{(2p)}\right) > \frac{1}{p!}\left(\frac{m}{2}\right)^p.$$

Since $\mathscr{H}$ has the (p, r) property, for each member $T = \{G_{i_1, j_1}, \ldots, G_{i_p, j_p}\}$ of $\mathscr{T}$ there is a pierceable subset $S \subset T$ of cardinality r. Moreover, the same subset S is contained in at most $\binom{m-r}{p-r}$ members of $\mathscr{T}$. It thus follows that the number of pierceable subsets of cardinality r of $\mathscr{G}$ is at least

$$\frac{|\mathscr{T}|}{\binom{m-r}{p-r}} \geq \frac{(p-r)!}{2^p p!} m^r \geq \frac{1}{2^p \binom{p}{r}} \binom{m}{r}.$$

By the definition of δ this implies that there is an $F \in \mathscr{F}$ that intersects at least

$$\delta\left(\frac{1}{2p\binom{p}{r}}\right) m \geq \beta m$$

members of $\mathscr{G}$, completing the proof. □

2.2. *Linear Programming Duality and Weighted Piercing*

In this subsection we combine Proposition 2.1 with Linear Programming duality to prove the following.

Proposition 2.2. *Suppose $\bar{\mathscr{H}}$ satisfies the fractional piercing property of order r with respect to $\mathscr{F}$ and suppose $p \geq r$. Assume, further, that each $H \in \bar{\mathscr{H}}$ intersects some $F \in \mathscr{F}$. Let β be any rational positive real satisfying the conclusion of Proposition 2.1. Then the following holds. For every $\mathscr{H} = \{H_1, \ldots, H_n\} \subset \bar{\mathscr{H}}$ that satisfies the (p, r) property, there is a finite (multi)-set $\mathscr{Y}$ of members of $\mathscr{F}$ such that every $H_i \in \mathscr{H}$ intersects at least $\beta|\mathscr{Y}|$ members of $\mathscr{Y}$.*

For the proof we need the following lemma, proved in [3], which follows easily from the min-max Theorem (see, e.g., [22]).

Lemma 2.3. *Let $H = (V, E)$ be a finite hypergraph and let $0 \leq \gamma \leq 1$ be a real. Then the following two conditions are equivalent:*

(i) *A weight function $f: V \mapsto R^+$ satisfying $\sum_{v \in V} f(v) = 1$ and $\sum_{v \in e} f(v) \geq \gamma$ for all $e \in E$ exists.*
(ii) *For every function $g: E \mapsto R^+$ there is a vertex $v \in V$ such that $\sum_{e;\, v \in e} g(e) \geq \gamma \sum_{e \in E} g(e)$.*

Proof of Proposition 2.2. For each of the 2^n subsets $\mathscr{S}$ of $\mathscr{H}$ let $F_{\mathscr{S}}$ be an arbitrarily chosen fixed element F of $\mathscr{F}$ that intersects all members of $\mathscr{S}$, in case there is such a set in $\mathscr{F}$, and let $F_{\mathscr{S}}$ be an arbitrary fixed element of $\mathscr{F}$ otherwise. We define a hypergraph $H = (V, E)$ whose set of vertices V is the set of all those 2^n sets $F_{\mathscr{S}}$. The set of edges of H consists of n edges $e_1, \ldots, e_n$ defined as follows:

$$e_i = \{F_{\mathscr{S}} \in V \colon H_i \cap F_{\mathscr{S}} \neq \emptyset\}.$$

By Proposition 2.1, for every function $g\colon E \mapsto R^+$ for which $g(e_i)$ is rational for all i there is a vertex $F \in V$ such that

$$\sum_{1 \le i \le n;\, F \in e_i} g(e_i) \ge \beta \sum_{i=1}^{n} g(e_i).$$

By continuity, this holds without the rationality assumption as well. Therefore, by Lemma 2.3 there is a weight function $f\colon V \mapsto R^+$ satisfying $\sum_{F \in V} f(F) = 1$ and $\sum_{F;\, F \in e} f(F) \ge \beta$ for all $e \in E$. Since such a function is a solution of a Linear Program with rational constraints there is such a function f for which $f(F)$ is rational for all F. Let l be an integer so that $lf(F)$ is integral for all F, and let $\mathscr{Y}$ consist of $lf(F)$ copies of F for all $F \in V$. The multiset $\mathscr{Y}$ clearly satisfies the conclusion of the proposition. □

2.3. *Weak ε-Nets*

Let $\bar{\mathscr{H}}$ and $\mathscr{F}$ be two families of sets. We say that $\mathscr{F}$ satisfies the *weak ε-nets property* for $\bar{\mathscr{H}}$ if for every $\varepsilon > 0$ there is a finite integer $b = b(\varepsilon)$ with the following property. For every finite multisubset $\mathscr{Y}$ of $\mathscr{F}$ there is a subset $\mathscr{X}$ of cardinality at most b of $\mathscr{F}$ so that every $H \in \bar{\mathscr{H}}$ that intersects at least $\varepsilon|\mathscr{Y}|$ members of $\mathscr{Y}$ intersects at least one member of $\mathscr{X}$.

In the next two sections we describe some geometric examples of $\bar{\mathscr{H}}$ and $\mathscr{F}$ that satisfy this property. The relevance of this property to the problems we consider here is clarified in the following theorem, which is the main result of this section.

Theorem 2.4. *Let $\bar{\mathscr{H}}$ and $\mathscr{F}$ be two families of sets. Suppose that*:

(i) *$\bar{\mathscr{H}}$ satisfies the fractional piercing property of order r with respect to $\mathscr{F}$, and every $H \in \bar{\mathscr{H}}$ intersects some $F \in \mathscr{F}$.*
(ii) *$\mathscr{F}$ satisfies the weak ε-nets property for $\bar{\mathscr{H}}$.*

Then, for very $p \ge q \ge r$, there is a constant $c = c(\bar{\mathscr{H}}, \mathscr{F}, p, q)$ so that for every finite $\mathscr{H} \subset \bar{\mathscr{H}}$ that satisfies the (p, q) property with respect to $\mathscr{F}$ there is a set $\mathscr{X}$ of at most c members of $\mathscr{F}$ that together meet all members of $\mathscr{H}$.

Proof. By (i) and Proposition 2.2 there is a $\beta > 0$ and a multiset $\mathscr{Y}$ of elements of $\mathscr{F}$ so that every member of $\mathscr{H}$ intersects at least $\beta|\mathscr{H}|$ members of $\mathscr{Y}$. By (ii) this implies that there is a subset $\mathscr{X}$ of at most $c = b(\beta)$ elements of $\mathscr{F}$ that together meet all members of $\mathscr{H}$. Since c is a uniform bound that depends only on $\bar{\mathscr{H}}$, $\mathscr{F}$, p, and q (and not on the actual subfamily $\mathscr{H}$) this completes the proof. □

The assertions of Theorems 1.1 and 1.3 can be deduced from the above result by showing that the corresponding $\bar{\mathscr{H}}$ and $\mathscr{F}$ in both cases satisfy properties (i) and (ii). This is done in the next two sections.

3. Unions of Convex Sets

In this section we prove Theorem 1.1. This is done by combining Theorem 2.4 with two known results.

The first known result is the following theorem of Katchalski and Liu [20] which can be viewed as a fractional version of Helly's theorem.

Theorem 3.1 [20]. *For every $0 < \alpha \leq 1$ and for every d there is a $\delta = \delta(\alpha, d) > 0$ such that, for every $n \geq d + 1$, every family of (not necessarily distinct) n convex sets in R^d which contains at least $\alpha\binom{n}{d+1}$ intersecting subfamilies of cardinality $d + 1$ contains an intersecting subfamily of at least δn of the sets.*

Notice that Helly's theorem is equivalent to the statement that in the above theorem $\delta(1, d) = 1$.

A sharp quantitative version of this theorem is proved in [19] and in [9], (see also [2] for a very short proof). All proofs of the sharp result rely on Wegner's theorem [25] that asserts that the nerve of a family of convex sets in R^d is d-collapsible. Note that in our notation the above theorem means that the family of all convex sets in R^d satisfies the fractional piercing property of order $r = d + 1$ with respect to the set of all one-point subsets of R^d.

Another known result we need is the following theorem proved in [1] by applying results from [4] and [23].

Theorem 3.2 [1]. *For every real $0 < \varepsilon < 1$ and for every integer d a constant $b = b(\varepsilon, d)$ exists such that the following holds. For every m and for every multiset Y of m points in R^d, there is a subset X of at most b points in R^d such that the convex hull of any subset of εm members of Y contains at least one point of X.*

In the language of Section 2 this is the assertion that the family of all one-point subsets of R^d satisfies the weak ε-nets property for convex sets in R^d.

Proof of Theorem 1.1. Let d and k be fixed positive integers. Let $\bar{\mathscr{H}} = \mathscr{K}_k^d$ be the family of all unions of k compact convex sets in R^d. Let $\mathscr{F}$ denote the set of all one-point subsets of R^d.

Claim 1. *$\bar{\mathscr{H}}$ satisfies the fractional piercing property of order $d + 1$ with respect to $\mathscr{F}$.*

Proof. Let $\mathscr{H} = \{H_1, \ldots, H_n\}$ be a finite subset of $\bar{\mathscr{H}}$, and suppose that at least $\alpha\binom{n}{d+1}$ subsets of cardinality $d + 1$ of $\mathscr{H}$ are pierceable, i.e., have a nonempty intersection. If $n < d + 1$ there is clearly a point that lies in at least $1/n = 1/(d + 1)$ of the members of $\mathscr{H}$. Otherwise, $n \geq d + 1$ and each member H_i of $\mathscr{H}$ is a union of k compact convex sets $H_{i,1}, \ldots, H_{i,k}$. Let $\mathscr{G}$ be the family of all kn convex sets $H_{i,j}$. By assumption, at least

$$\alpha \binom{n}{d+1} \geq \frac{\alpha}{k^{d+1}} \frac{(d+1)!}{(d+1)^{d+1}} \binom{kn}{d+1}$$

$(d + 1)$ subsets of $\mathscr{G}$ are intersecting (where the last $(d + 1)!/(d + 1)^{d+1}$ term is required only for small values of n ($\geq d + 1$)). By Theorem 3.1 this implies that there is a point that lies in at least δkn members of $\mathscr{G}$ for some $\delta = \delta(\alpha, d, k)$. This point lies in at least δn members of $\mathscr{H}$, completing the proof of the claim. □

Claim 2. *$\mathscr{F}$ satisfies the weak ε-nets property for $\bar{\mathscr{H}}$.*

Proof. Let $\mathscr{Y}$ be a finite multisubset of $\mathscr{F}$, let Y be the corresponding set of points in R^d, and suppose $\varepsilon > 0$. By Theorem 3.2 there is a set of at most $b = b(\varepsilon/k, d)$ points X such that the convex hull of every set of at least $\varepsilon|Y|/k$ points of Y contains a point of X. Let $\mathscr{X}$ be the subfamily of $\mathscr{F}$ consisting of all the sets $\{x\}$ for $x \in X$. Then $|\mathscr{X}| = |X| \leq b(\varepsilon/k, d)$. If $H \in \bar{\mathscr{H}}$ intersects at least $\varepsilon|\mathscr{Y}|$ members of $\mathscr{Y}$, then at least one of the k convex sets whose union is H contains at least $\varepsilon/k|Y|$ members of Y and hence contains a point of X. Since $b(\varepsilon/k, d)$ is only a function of k, d, and ε this completes the proof of Claim 2. □

By Claims 1 and 2 and Theorem 2.4, for every $p \geq q \geq d + 1$ there is a $c = c(k, p, q, d) < \infty$ such that, for every finite $\mathscr{H} \subset \bar{\mathscr{H}}$ that satisfies the (p, q) property with respect to $\mathscr{F}$, $P(\mathscr{H}, \mathscr{F}) \leq c$. This, together with a standard compactness argument, completes the proof of Theorem 1.1. □

Proof of Corollary 1.2. Let $\mathscr{H}$ be a family of sets such that each intersection of members of the family is the union of at most k convex sets, and assume that each $h(d, k) = (d + 1) \cdot c(k, d + 1, d + 1, d)$ of the sets have a nonempty intersection. Consider the family $\mathscr{H}'$ of all intersections of $c = c(k, d + 1, d + 1, d)$ of the sets. Each $d + 1$ members of $\mathscr{H}'$ have a nonempty intersection and therefore, by Theorem 1.1, $\mathscr{H}'$ can be pierced by c points. We claim that at least one of these c points lies in all the sets in $\mathscr{H}$. To see this observe that otherwise for each of the points there is a member of $\mathscr{H}$ that misses it and the intersection of all these c members does not contain any of the points, contradicting their choice. Thus $\mathscr{H}$ is intersecting, as needed. □

4. Piercing by Hyperplanes

In this section we prove Theorem 1.3 using Theorem 2.4. Let $\bar{\mathscr{H}}$ be the family of all compact convex sets in R^d and let $\mathscr{F}$ denote the set of all hyperplanes in R^d.

Proposition 4.1. *$\bar{\mathscr{H}}$ satisfies the fractional piercing property of order $d+1$ with respect to $\mathscr{F}$.*

Proof. We apply induction on the dimension d. The result for $d = 1$ is trivial. Assuming it holds for $d - 1$ we prove it for d.

Let $\mathscr{H} = \{H_1, \ldots, H_n\}$ be a finite subset of $\bar{\mathscr{H}}$, and suppose that at least $\alpha\binom{n}{d+1}$ subsets of cardinality $d + 1$ of $\mathscr{H}$ are pierceable.

Call a d-tuple of members of $\mathscr{H}$ *bad* if it has a $(d-2)$-transversal, that is, a $(d-2)$-flat intersecting all members of the d-tuple. Otherwise call the d-tuple *good*. A subfamily $\mathscr{H}' \subset \mathscr{H}$ is good if all d-tuples of distinct elements of $\mathscr{H}'$ are good.

We need the following known lemma, proved in [5]:

Lemma 4.2. *Let $K_1, K_2, \ldots, K_d$ be convex sets in R^d and assume that there is no $(d-2)$-flat intersecting all of them. Then there are hyperplanes which are common tangents to all the sets and their number is at most 2^d.*

We now proceed with the proof of the proposition and consider two possible cases.

Case 1: There are at least $\beta\binom{n}{d}$ bad d-tuples of elements of $\mathscr{H}$, where $\beta = \alpha/2(d+1)$. In this case choose an arbitrary hyperplane L, and let h_i be the orthogonal projection of H_i on L. Then $h_1, \ldots, h_n$ are convex compact sets in a $(d-1)$-dimensional Euclidean space and at least $\beta\binom{n}{b}$ subsets of cardinality d of them have a $(d-2)$-transversal. By applying the induction hypothesis we conclude that there is a $(d-2)$-flat M in L that intersects at least $\gamma(\beta)n$ of the sets in the projection. The hyperplane containing M which is orthogonal to L intersects all the δn corresponding sets H_i.

Case 2: $\mathscr{H}$ contains less than $\beta\binom{n}{d}$ bad d-tuples of elements of $\mathscr{H}$. In this case there are at least

$$(\alpha - (d+1)\beta)\binom{n}{d+1} \geq \frac{\alpha}{2}\binom{n}{d+1}$$

pierceable *good* subfamilies of cardinality $d + 1$ of $\mathscr{H}$.

Let $\mathscr{G}$ denote the set of all hyperplanes that are common tangents to all members of some good subset of cardinality d of $\mathscr{H}$. By Lemma 4.2,

$$|\mathscr{G}| \leq 2^d\binom{n}{d}.$$

We need the following lemma:

Lemma 4.3. *Let $K_1, K_2, \ldots, K_{d+1}$ be convex sets in R^d such that there is a hyperplane intersecting them all and there is no $(d-2)$-flat intersecting at least d of* *them. Then there is a hyperplane which is tangent to d of the sets and intersects the remaining set.*

Proof. The lemma follows from the following two facts:

(a) Every d members from these $d+1$ sets have a common tangent hyperplane (by Lemma 4.2).
(b) The set of common tangent hyperplanes to any subset of k of the sets, for $k < d$, is connected, as proved in [5]. □

We now return to the proof of the proposition. It follows from the last lemma that every good pierceable subfamily of $d+1$ members of $\mathscr{H}$ is pierced by a member of $\mathscr{G}$. Therefore, an averaging argument shows that some member G of $\mathscr{G}$ intersects at least

$$\frac{\alpha}{2}\binom{n}{d+1}\Big/|\mathscr{G}| \geq \gamma(\alpha, d)n$$

subfamilies of cardinality $d+1$, all containing the d sets used to define G (as one of their common tangents). Here $\gamma = \gamma(\alpha, d) > 0$ depends only on α and d. It follows that G is a hyperplane that intersects at least γn subsets of $\mathscr{H}$, as needed. □

Proposition 4.4. *$\mathscr{F}$ satisfies the weak ε-nets property for $\bar{\mathscr{H}}$ (and in fact even for the family of all connected subsets of R^d).*

This proposition is a simple consequence of the following result proved in [6].

Theorem 4.5 [6]. *For any dimension d, a constant $c(d)$ with the following property exists. For every $r \leq n$ and every family $\mathscr{Y}$ of n (not necessarily distinct) hyperplanes in R^d, a collection of at most $c(d)r^d$ (possibly unbounded) simplices with pairwise disjoint interiors, whose union covers R^d, exists such that the interior of any of them is intersected by at most n/r of the given hyperplanes.*

Proof of Proposition 4.4. Let $\mathscr{Y}$ be a finite multisubset of $\mathscr{F}$, that is, a family of n hyperplanes in R^d, and suppose $\varepsilon > 0$. By Theorem 4.5 with, say, $r = 2/\varepsilon$ there is a collection of at most $c(d)(2/\varepsilon)^d$ simplices satisfying the assertion of the theorem. Let $\mathscr{X}$ be the set of all hyperplanes determined by a facet of at least one of these simplices. Then $|\mathscr{X}| \leq b(\varepsilon, d)$. Moreover, if H is a connected subset of R^d which does not intersect any member of $\mathscr{X}$, then it must be contained in the interior of one of the simplices and hence can meet at most $n/r < \varepsilon n$ members of $\mathscr{Y}$. This completes the proof. □

By Propositions 4.1 and 4.4 and Theorem 2.4, for every $p \geq q \geq d+1$ there is a $C = C(p, q, d) < \infty$ such that, for every finite $\mathscr{H} \subset \bar{\mathscr{H}}$ that satisfies the (p, q) property with respect to $\mathscr{F}$, $P(\mathscr{H}, \mathscr{F}) \leq C$. This, together with a standard compactness argument, completes the proof of Theorem 1.3 □

5. Concluding Remarks and Open Problems

1. The arguments in Section 3 can be easily modified to supply a proof of the following theorem.

Theorem 5.1. *Let $\mathscr{H}$ and $\mathscr{F}$ be two families of sets, and let $k \geq 1$ be an integer. Let $\mathscr{H}(k)$ denote the family of all unions of k members of $\mathscr{H}$. Then*:

(i) *If $\mathscr{H}$ satisfies the fractional piercing property of order r with respect of $\mathscr{F}$, then so does $\mathscr{H}(k)$.*

(ii) *If $\mathscr{F}$ satisfies the weak ε-nets property for $\mathscr{H}$, then it satisfies this property for $\mathscr{H}(k)$ as well.*

This together with Propositions 4.1 and 4.4 imply the following.

Theorem 5.2. *For every $p \geq q \geq d + 1$ and every k there is a $C = C(k, p, q, d) < \infty$ such that, for every family $\mathscr{G}$ whose members are unions of k compact convex sets in R^d that satisfies the (p, q) property with respect to the family $\mathscr{F}$ of all hyperplanes in R^d, $P(\mathscr{G}, \mathscr{F}) \leq C$.*

2. Theorem 1.1 with $k = 1$ and Theorem 1.3 deal with the problem of piercing compact convex sets with flats of dimension 0 and $d - 1$, respectively. It would be interesting to prove an analog of these results for flats of intermediate dimensions i for $1 \leq i \leq d - 2$.

The first open case is that of line transversals in space. At the moment we cannot prove a fractional Helly theorem in this case. In fact, we cannot even answer the following problem.

Problem. Let $r \geq 5$ be an integer. Is it true that if n is sufficiently large, every family of n convex sets in space such that every r of them have a line transversal must contain $r + 1$ sets having a line transversal?

3. The proofs of Theorems 1.1 and 1.3 are constructive. Under suitable assumptions which ensure that the structures described in the conclusions of Theorem 3.1 and Lemma 4.2 for sets in the given families can be found efficiently, these proofs yield, for every fixed d, polynomial-time algorithms for finding the corresponding piercing sets. We omit the details of these algorithmic procedures.

4. It would be interesting to find additional natural families $\mathscr{H}$ and $\mathscr{F}$ for which theorems of the type considered here can be proved. The techniques developed in [3] and in this paper can certainly be applied in additional cases.

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