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Cube-Tilings of $\mathbb{R}^n$ and Nonlinear Codes

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Abstract. Families of nonlattice tilings of $\mathbb{R}^n$ by unit cubes are constructed. These tilings are specializations of certain families of nonlinear codes over $GF(2)$. These cube-tilings provide building blocks for the construction of cube-tilings such that no two cubes have a high-dimensional face in common. We construct cube-tilings of $\mathbb{R}^n$ such that no two cubes have a common face of dimension exceeding $n - \frac{1}{3}\sqrt{n}$.

1. Introduction

In 1907 Minkowski [9] conjectured that all extremal lattices for the supremum norm were of a certain form, and gave a geometric interpretation of this conjecture: In every lattice tiling of $\mathbb{R}^n$ with unit cubes, there are two cubes that have a complete facet in common. In studying Minkowski's conjecture, in 1930 Keller [5] made the stronger conjecture that in any tiling of $\mathbb{R}^n$ by unit cubes there are two cubes having a complete facet in common. In 1940 Perron [10] proved Keller's conjecture is true in dimensions n up to 6. In 1942 Hajós [2] proved that it holds for all lattice tilings of $\mathbb{R}^n$, which settled Minkowski's conjecture. Subsequently, various reductions of Keller's conjecture were made in [3], [12], and [1]. Recently, we showed that Keller's conjecture is false in all dimensions $n \geq 10$, by construction, see [6]. It remains open in dimensions 7, 8, and 9.

Let K_n denote the largest integer such that every tiling of $\mathbb{R}^n$ by unit cubes contains two cubes that have a common face of dimension K_n. This paper considers the problem of bounding K_n from above. We show by construction that $K_n \leq n - \frac{1}{3}\sqrt{n}$ for all n.

One difficulty with obtaining upper bounds for K_n is that constructions of cube-tilings with no two cubes having a complete facet in common do not carry over in a simple manner from one dimension to another. We have

$$K_{n+1} \le K_n + 1. \tag{1.1}$$

This is easily proved using a "stacking" construction that produces an $(n + 1)$-dimensional tiling from an n-dimensional one, consisting of layers of n-dimensional tilings with successive layers shifted relative to each other to preclude any common faces between cubes in adjacent layers. However, we do not know whether $K_n \le K_{n+1}$. In Appendix A we show $K_{10} \le 7$, but at present we only know that $K_9 \le 8$.

Our construction proceeds in two steps. The first step is to construct a large class of nonlattice cube-tilings, which are combinatorially interesting in their own right. These tilings have a certain "additive" structure and also have the following properties:

(i) They are periodic with period lattice $2\mathbb{Z}^n$, and all cube-centers are in $\frac{1}{2}\mathbb{Z}^n$.
(ii) Each equivalence class $\frac{1}{2}\mathbb{Z}^n$ (mod $\mathbb{Z}^n$) contains exactly zero or two cube-center equivalence classes.

These tilings arise from nonlinear codes in $(\mathbb{Z}/4\mathbb{Z})^n$ having special properties. Property (ii) is a special case of an extremality property, called 2-extremal, which guarantees that such tilings have relatively few cube-pairs having a face in common, as we explain further in Section 2. However, these tilings do contain cube-pairs having a common facet. The second step is a block-substitution construction like those in [6], which eliminates all high-dimensional common faces. The base tilings and the block substitutions used in this construction are derived from two distinct infinite families of "additive" tilings, which have certain extra properties, described in Sections 4 and 5.

A *(nonlinear) code* in $\mathbb{Z}/2\mathbb{Z}$ (resp. $\mathbb{Z}/4\mathbb{Z}$) is simply a finite set of distinct vectors in $(\mathbb{Z}/2\mathbb{Z})^n$ (resp. $(\mathbb{Z}/4\mathbb{Z})^n$). A *linear code* is a linear subspace of $(\mathbb{Z}/2\mathbb{Z})^n$ (resp. $(\mathbb{Z}/4\mathbb{Z})^n$). Coding theory is concerned with the construction of such sets whose vectors are far apart in an appropriate metric, and which correspond to dense packings of space with appropriately scaled unit balls for this metric. Standard references for coding theory include [8] and [13]. Cube-tilings are perfect packings, and are analogous to "perfect codes" in coding theory. We use coding theory terminology to emphasize this analogy, because our constructions may eventually prove useful in constructing codes in other contexts. The proofs in this paper use no results from coding theory, however. Further references on related tiling problems appear in [11].

In Section 2 we define and study 2-extremal cube-tilings.

In Section 3 we describe a class of nonlinear codes produced by an "additive" construction. This construction somewhat resembles that of a linear code in algebraic coding theory, except that it has a nonlinear global constraint on codewords. We show that certain subfamilies of these codes satisfy necessary conditions to give 2-extremal cube-tilings We call these *additive codes*. In fact, these "additive" constructions suggest general methods to produce interesting nonlinear codes, possibly useful for other purposes than cube-tiling.

In Sections 4 and 5 we construct two infinite families of additive codes which give 2-extremal cube-tilings, and prove special properties about their codeword distributions. Then in Section 6 we use these additive codes in a block-substitution construction to construct cube-tilings establishing the bound

$$K_n \leq n - \tfrac{1}{3}\sqrt{n}.$$

This construction generalizes those in [6]. Study of the $n = 10$ construction in that paper led to the discovery of the Construction B tilings detailed in Section 5.

In Section 7 we discuss an approach to strengthen the upper bound for K_n. If a certain kind of 2-extremal cube-tiling exists, then $K_n \leq cn$ for some $c < 1$.

Finally, in Appendix A we construct a ten-dimensional cube-tiling showing that $K_{10} \leq 7$.

2. 2-Extremal Cube-Tilings

Perron [10] showed that if a cube-tiling in $\mathbb{R}^n$ with no two cubes having a common face of dimension d exists, then a periodic cube-tiling with period lattice $2\mathbb{Z}^n$ having the same property exists. (His argument can be easily extended to show that the centers of the cubes in this periodic tiling can be taken in the lattice $(1/2^n)\mathbb{Z}^n$.) Thus $2\mathbb{Z}^n$-periodic cube-tilings need be studied.

A $2\mathbb{Z}^n$-periodic cube-tiling is completely specified by the cubes whose centers lie in the fundamental domain

$$\mathscr{F} = \{(x_1, \ldots, x_n)\colon 0 \leq x_i < 2\}.$$

There are exactly 2^n equivalence classes $\mathbf{v} + 2\mathbb{Z}^n$ of cube-centers, where $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ has $0 \leq v_i < 2$. Two distinct equivalence classes $\mathbf{v} + 2\mathbb{Z}^n$ and $\mathbf{w} + 2\mathbb{Z}^n$ contain cube-pairs sharing a common face (of some dimension ≥ 0) if and only if $\mathbf{v} - \mathbf{w} \in \mathbb{Z}^n$. Call two classes $\mathbf{v}$ and $\mathbf{w}$ *$\mathbb{Z}$-adjacent* if $\mathbf{v} - \mathbf{w} \in \mathbb{Z}^n$. The 2^n equivalence classes in any $2\mathbb{Z}^n$-periodic cube-tiling are divided up into $\mathbb{Z}$-adjacency classes. If $\{m_i\colon 1 \leq i \leq l\}$ are the cardinalities of the $\mathbb{Z}$-adjacency classes in a $2\mathbb{Z}^n$-periodic tiling, then

$$\sum_{i=1}^{l} m_i = 2^n. \tag{2.1}$$

Also $\sum_{i=1}^{l} m_i(m_i - 1)$ counts the number of ordered pairs of equivalence classes containing cube-pairs having a common face.

The fundamental fact about $\mathbb{Z}$-adjacency is that each $\mathbb{Z}$-adjacency class must contain at least two elements. Since two cubes have a common face of some dimension if and only if they have a common corner (0-face), this is equivalent to the following elementary fact.

Lemma 2.1. *In any tiling of $\mathbb{R}^n$ by unit cubes, every corner of every cube touches the corner of at least one adjacent cube.*

Proof. Move the tiling by a Euclidean motion so that the corner is at $\mathbf{0}$, with cubes oriented parallel to the axes. Now assign to each cube touching $\mathbf{0}$ the number counting each orthant in $\mathbb{R}^n$ such that the cube contains an interior point of this orthant. The cube of which $\mathbf{0}$ is a corner counts one orthant, while all cubes touching $\mathbf{0}$, but with $\mathbf{0}$ not being a corner, count an even number of orthants. Since each of the 2^n orthants is counted exactly once, some other cube covers an odd number of orthants. This cube must therefore count one orthant, and has a corner at $\mathbf{0}$. □

Lemma 2.1 supplies the constraint

$$m_i \geq 2 \tag{2.2}$$

on $\mathbb{Z}$-adjacency classes of $2\mathbb{Z}^n$-periodic cube-tilings.

In searching for $2\mathbb{Z}^n$-periodic cube-tilings that do not contain any cube-pairs meeting in a high-dimensional face, it seems reasonable to single out those tilings that have the fewest cube-pairs sharing a common face of any dimension, i.e., those that minimize $\sum_{i=1}^{l} m_i(m_i - 1)$. Now $\sum_{i=1}^{l} m_i(m_i - 1)$ subject to the constraints (2.1) and (2.2) is minimized with all $m_i = 2$ and $l = 2^{n-1}$. We therefore call any $2\mathbb{Z}^n$-cube-tiling having this minimality property (all $m_i = 2$) a *2-extremal tiling*. It is easy to construct examples of 2-extremal tilings in all dimensions.

Another reason to single out 2-extremal tilings for special consideration arises from the problem of obtaining lower bounds for K_n. If a $2\mathbb{Z}^n$-periodic cube-tiling is not 2-extremal, then it must contain two cubes sharing a face of dimension $d \geq n/3$. To see this, consider a $\mathbb{Z}$-adjacency class containing at least three elements $\{\mathbf{v}_i + 2\mathbb{Z}^n\colon 1 \leq i \leq 3\}$. By the pigeonhole principle two of these elements $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ must agree (mod 2) on at least $n/3$ of their coordinates, and these two cubes then share a common face of the required dimension. Thus if K_n were to be smaller than $n/3$, any $2\mathbb{Z}^n$-periodic cube-tiling attaining this bound would be 2-extremal. Hence analysis of 2-extremal cube-tilings seems necessary in obtaining lower bounds for K_n.

2-extremal lattice tilings exist in all dimensions. The next section describes a method to construct nonlinear codes, which we show in Sections 4 and 5 yield nonlattice 2-extremal cube-tilings which have all cube-centers in $\frac{1}{2}\mathbb{Z}^n$.

3. Nonlinear Codes

We construct nonlinear codes which produce codewords in the set $\{0, 1, 2, 3\}^n$. (We can regard these as binary codes by identifying this set with $\{0, 1\}^{2n}$.) The codes are designed to satisfy unusual distance constraints on their codewords, motivated by their application to cube-tilings. These codes always consist of two distinct sets of codewords, which we call *complements*. Under suitable circumstances both complements have cardinality 2^{n-1} each, yielding 2^n codewords in all, see Theorem 3.1.

The construction is based on an $n \times n$ matrix M with entries in $\{0, 1\}$, which we call the *generator matrix*. From it form the matrix

$$\mathsf{A} = \mathsf{A}(\mathsf{M}) := \mathsf{I} + 2\mathsf{M}, \tag{3.1}$$

where I is the identity matrix. Form the set of 2^n vectors $V(\mathsf{A})$ consisting of all 2^n sums of distinct row vectors of A, with entries taken (mod 4). Let $V_{\text{even}}(\mathsf{M})$ consist of all vectors in $V(\mathsf{A})$ containing an even number of entries 3 (mod 4); this is one part of the global constraint. Let

$$\mathsf{A}^T = \mathsf{A}^T(\mathsf{M}) := \mathsf{I} + 2\mathsf{M}^T,$$

and let $V^*_{\text{even}}(\mathsf{M}^T)$ consist of all vectors in the corresponding set $V(\mathsf{A}^T)$ consisting of all vectors containing an odd number of entries that are 0 (mod 4); this is the other part of the global constraint. The nonlinear code is

$$\mathscr{C}(\mathsf{M}) := V_{\text{even}}(\mathsf{M}) \cup (V^*_{\text{even}}(\mathsf{M}^T) + 2\mathbf{e}), \tag{3.2}$$

where $\mathbf{e} = (1, 1, \ldots, 1)$. We call the sets

$$\mathscr{C}^+(\mathsf{M}) := V_{\text{even}}(\mathsf{M}), \qquad \mathscr{C}^-(\mathsf{M}) := V^*_{\text{even}}(\mathsf{M}^T) + 2\mathbf{e}$$

complements, and note that all vectors in $\mathscr{C}^-(\mathsf{M})$ contain an odd number of entries that are 2 (mod 4).

We are interested in generator matrices M that yield $\mathscr{C}(\mathsf{M})$ satisfying the conditions

$$|V_{\text{even}}(\mathsf{M})| = |V^*_{\text{even}}(\mathsf{M}^T)| = 2^{n-1}, \tag{3.3}$$

and

$$V_{\text{even}}(\mathsf{M}) \cap (V^*_{\text{even}}(\mathsf{M}^T) + 2\mathbf{e}) = \varnothing. \tag{3.4}$$

Then $\mathscr{C}(\mathsf{M})$ has exactly 2^n codewords. We call those $\mathscr{C}(\mathsf{M})$ satisfying (3.3) *balanced* and call those $\mathscr{C}(\mathsf{M})$ satisfying (3.4) *additive codes.*

The cube-tiling problem involves a stronger notion of distance between codewords than just being an additive code. We say that the *e-distance* between $\mathbf{v}, \mathbf{w} \in (\mathbb{Z}/4\mathbb{Z})^n$ is

$$d_e(\mathbf{v}, \mathbf{w}) = \#\{i : |v_i - w_i| = 2\}.$$

The discussion in Section 2 and in [6] establishes:

Proposition 3.1. *Let $\mathscr{C}$ be a set of 2^n vectors in $\{0, 1, 2, 3\}^n$. Then $\frac{1}{2}\mathscr{C} + 2\mathbb{Z}^n$ gives the cube-centers of a tiling of $\mathbb{R}^n$ with cubes of sidelength 1 parallel to the axes if and only if $d_e(\mathbf{v}, \mathbf{w}) \geq 1$ for all distinct vectors $\mathbf{v}, \mathbf{w} \in \mathscr{C}$.*

Henceforth, we always use sets $\mathscr{C} \subseteq \{0, 1, 2, 3\}^n$ to specify $2\mathbb{Z}^n$-periodic cube-tilings which have all cube-centers in $\frac{1}{2}\mathbb{Z}^n$. We note the following additional property of such tilings.

Lemma 3.1. *Suppose $\mathscr{C} \subseteq \{0, 1, 2, 3\}^n$ gives a cube-tiling $\frac{1}{2}\mathscr{C} + 2\mathbb{Z}^n$. Let the map $\hat{\varphi}: \mathscr{C} \to \{0, 1\}^n$ be induced coordinatewise from the map $\varphi: \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ given by $\varphi(0) = \varphi(1) = 0$, $\varphi(2) = \varphi(3) = 1$. Then $\hat{\varphi}$ is one-to-one and onto. The same holds for the map $\hat{\psi}$ induced from $\psi: \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ given by $\psi(0) = \psi(3) = 0$, $\psi(1) = \psi(2) = 1$.*

Proof. Since $|\mathscr{C}| = 2^n$, it suffices to prove that $\hat{\varphi}$ is one-to-one. By Proposition 3.1, if $\mathbf{x}_1, \mathbf{x}_2 \in \mathscr{C}$, then $d_e(\mathbf{x}_1, \mathbf{x}_2) \geq 1$, so $\mathbf{x}_1$, $\mathbf{x}_2$ differ by 2 in some coordinate i, hence $\hat{\varphi}(\mathbf{x}_1)$ and $\hat{\varphi}(\mathbf{x}_2)$ differ in this coordinate; similarly for $\hat{\psi}$. □

Lemma 3.1 has a useful consequence concerning the structure of $2\mathbb{Z}^n$-cube-tilings with all cube-centers in $\frac{1}{2}\mathbb{Z}^n$. Partition $\mathscr{C}$ as

$$\mathscr{C} = \mathscr{C}_0 \cup \mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{C}_3,$$

according to whether the value of the first coordinate of each vector in $\mathscr{C}$ is 0, 1, 2, or 3. Lemma 3.1 implies, using $\hat{\varphi}$, that

$$|\mathscr{C}_0 \cup \mathscr{C}_1| = 2^{n-1}, \qquad |\mathscr{C}_2 \cup \mathscr{C}_3| = 2^{n-1}, \tag{3.5}$$

and, using $\hat{\psi}$, that

$$|\mathscr{C}_0 \cup \mathscr{C}_3| = 2^{n-1}, \qquad |\mathscr{C}_1 \cup \mathscr{C}_2| = 2^{n-1}.$$

These equalities imply that any $2\mathbb{Z}^n$-cube-tiling with cube-centers in $\frac{1}{2}\mathbb{Z}^n$ has

$$|\mathscr{C}_0| = |\mathscr{C}_2|, \qquad |\mathscr{C}_1| = |\mathscr{C}_3|. \tag{3.6}$$

A *cube-tiling code* is a balanced additive code $\mathscr{C}(\mathsf{M})$ such that $d_e(\mathbf{v}, \mathbf{w}) \geq 1$ for any two distinct codewords; it yields a $2\mathbb{Z}^n$-periodic bube-tiling via Proposition 3.1.

It seems a hard problem to obtain necessary and sufficient conditions characterizing any of the three properties: balance condition (3.3), additivity condition (3.4), or being a cube-tiling code. In the rest of the section we present sufficient conditions for some of these properties.

We start with the balance condition (3.3). Let the row sums of M be denoted $r_1, r_2, \ldots, r_n$ and the column sums $c_1, c_2, \ldots, c_n$.

Property BC. *The* $n \times n$ *matrix* M *has* $r_i \equiv c_i \pmod 2$ *for* $1 \le i \le n$, *and*

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} c_i \equiv 1 \pmod 2.$$

We prove the following:

Theorem 3.1. *If the* $n \times n$ *matrix* M *has Property BC and* n *is odd, then* $\mathscr{C}(\mathsf{M})$ *is balanced.*

Proof. We first show that $|V_{\text{even}}(\mathsf{M})| = 2^{n-1}$ holds for all n, even or odd. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ denote the rows of A. If $S \subseteq \{1, 2, \dots, n\}$ set

$$\mathbf{w}_S = \sum_{i \in S} \mathbf{v}_i, \qquad \bar{\mathbf{w}}_S = \sum_{i \notin S} \mathbf{v}_i. \tag{3.7}$$

It suffices to show that exactly one of each complementary pair $\mathbf{w}_S$, $\bar{\mathbf{w}}_S$ is in $V_{\text{even}}(\mathsf{M})$.

To do this, let t_S, $\bar{t}_S$ count the number of entries that are 3 (mod 4) in $\mathbf{w}_S$, $\bar{\mathbf{w}}_S$, respectively. Block-partition the entries of $\mathbf{w}_S$ and $\bar{\mathbf{w}}_S$ as pictured:

$$\begin{array}{r|c|c|}
\cline{2-3}
\mathbf{w}_S = & \mathrm{I} & \mathrm{II} \\
\cline{2-3}
\bar{\mathbf{w}}_S = & \mathrm{III} & \mathrm{IV} \\
\cline{2-3}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{i \in S} & \multicolumn{1}{c}{i \in \bar{S}}
\end{array}$$

All entries in I and IV are odd, while all entries in II and III are even. Let a_{I}, a_{II}, a_{III}, a_{IV} denote the sum of the entries in each part, and suppose $|S| = s$. Then

$$a_{\mathrm{I}} = s + 2t_S,$$
$$a_{\mathrm{IV}} = n - s + 2\bar{t}_S.$$

This yields

$$\begin{aligned} a_{\mathrm{I}} - a_{\mathrm{IV}} &= 2s - n + 2(t_S - \bar{t}_S) \\ &\equiv 2s - n + 2(t_S + \bar{t}_S) \pmod 4. \end{aligned} \tag{3.8}$$

However, the definition of $\mathbf{w}_S$ yields

$$a_{\mathrm{I}} + a_{\mathrm{II}} = s + 2\left(\sum_{i \in S} r_i\right).$$

Also, we have

$$a_{\mathrm{II}} + a_{\mathrm{IV}} = (n - s) + 2\left(\sum_{i \notin S} c_i\right).$$

Subtracting these equations yields

$$a_{\mathrm{I}} - a_{\mathrm{IV}} = (2s - n) + 2\left(\sum_{i \in S} r_i\right) - 2\left(\sum_{i \notin S} c_i\right).$$

Comparing this with (3.8) gives

$$2(t_S + \bar{t}_S) \equiv 2\left(\sum_{i \in S} r_i\right) - 2\left(\sum_{i \notin S} r_i\right) \pmod 4,$$

after using the hypothesis $r_i \equiv c_i \pmod 2$ of Property BC. Hence

$$\begin{aligned} t_S + \bar{t}_S &\equiv \sum_{i \in S} r_i - \sum_{i \notin S} r_i \pmod 2 \\ &\equiv \sum_{i \in S} r_i + \sum_{i \notin S} r_i \pmod 2 \\ &\equiv 1 \pmod 2, \end{aligned}$$

using Property BC, so exactly one of $\mathbf{w}_S$, $\bar{\mathbf{w}}_S$ is in $V_{\mathrm{even}}(\mathsf{M})$.

We show that $|V^*_{\mathrm{even}}(\mathsf{M}^T)| = 2^{n-1}$ for n odd by a similar argument. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ now denote the rows of A^T, and define $\mathbf{w}_S$, $\bar{\mathbf{w}}_S$ by (3.7). Let t^*_S, $\bar{t}^*_S$ denote the number of entries that are $0 \pmod 4$ in $\mathbf{w}_S$, $\bar{\mathbf{w}}_S$, respectively. Using the block-partition as above, we have

$$\begin{aligned} a_{\mathrm{II}} &= 2(n - s - t^*_S), \\ a_{\mathrm{III}} &= 2(s - \bar{t}^*_S), \end{aligned}$$

and these yield

$$a_{\mathrm{II}} - a_{\mathrm{III}} = 2n + 2(t^*_S + \bar{t}^*_S) \pmod 4. \tag{3.9}$$

Now

$$a_{\mathrm{I}} + a_{\mathrm{II}} = s + 2\left(\sum_{i \in S} c_i\right),$$

where c_i occurs instead of r_i since we use M^T instead of M. Also

$$a_{\mathrm{I}} + a_{\mathrm{III}} = s + 2\left(\sum_{i \in S} r_i\right),$$

and since $r_i \equiv c_i \pmod 2$, these yield

$$a_{\mathrm{II}} - a_{\mathrm{III}} = 0.$$

Combined with (3.9), this gives

$$t_S^* + \bar{t}_S^* \equiv n \pmod 2,$$

which for $n \equiv 1 \pmod 2$ is the desired parity condition. □

The balancing condition (3.3) apparently holds for a much wider class of M than those satisfying Property BC. In computational experiments on randomly selected M we found, in every case tested, that

$$|V_{\text{even}}(\mathsf{M})| = 2^{n-1} \quad \text{or} \quad 2^{n-1} \pm 2^r, \tag{3.10}$$

where $1 \leq r \leq n-2$, and similarly for $V_{\text{even}}^*(\mathsf{M}^T)$. It seems an interesting combinatorial problem to establish when (3.10) holds and to characterize the values of r that may occur.

Next we study the disjointness condition (3.4) necessary to have an additive code.

Property AC. *Each row of* M *contains an odd number of ones, i.e., all* $r_i \equiv 1 \pmod 2$.

We prove below that Property AC is a sufficient condition for an additive code, which, furthermore, makes the cube-tiling e-distance criterion hold between vectors $\mathbf{v}, \mathbf{w}$ lying in different complements of $\mathscr{C}(\mathsf{M})$.

Theorem 3.2. *If* M *has Property AC, then any* $\mathbf{v} \in V_{\text{even}}(\mathsf{M})$ *and* $\mathbf{w} \in V_{\text{even}}^*(\mathsf{M}^T) + 2\mathbf{e}$ *satisfy*

$$d_e(\mathbf{v}, \mathbf{w}) \equiv 1 \pmod 2. \tag{3.11}$$

In particular, $\mathscr{C}$ *is an additive code containing* $|V_{\text{even}}(\mathsf{M})| + |V_{\text{even}}^*(\mathsf{M}^T)|$ *elements.*

Note that if the code $\mathscr{C}(\mathsf{M})$ has cardinality 2^n, it does not necessarily yield a cube-tiling $\mathscr{C} + 2\mathbb{Z}^n$, because some pair of vectors in $V_{\text{even}}(\mathsf{M})$ (resp. $V^*_{\text{even}}(\mathsf{M}^T) + 2\mathbf{e}$) may have zero e-distance. Extra conditions on M are needed for this not to occur. Two infinite families of such M are constructed in Sections 4 and 5.

Proof of Theorem 3.2. Both complements of the code $\mathscr{C}(\mathsf{M})$ remain unchanged under simultaneous permutations of rows and columns, i.e., replacing M by $\mathsf{M}' = \mathsf{PMP}^T$, where P is a permutation matrix. Also Property AC is preserved under this action, so it suffices to prove the theorem for any such M'.

We choose such a permutation so that $\mathbf{v}$ is a sum of initial rows of $\mathsf{A}' = \mathsf{I} + 2\mathsf{M}'$, and $\mathbf{w} - 2\mathbf{e}$ is the sum of a consecutive set of rows of $(\mathsf{A}')^T$. Then M' has a block-partition

$$\mathsf{M}' = \begin{bmatrix} \mathsf{S}_{11} & \mathsf{S}_{12} & \mathsf{S}_{13} & \mathsf{S}_{14} \\ \mathsf{S}_{21} & \mathsf{S}_{22} & \mathsf{S}_{23} & \mathsf{S}_{24} \\ \mathsf{S}_{31} & \mathsf{S}_{32} & \mathsf{S}_{33} & \mathsf{S}_{34} \\ \mathsf{S}_{41} & \mathsf{S}_{42} & \mathsf{S}_{43} & \mathsf{S}_{44} \end{bmatrix}, \tag{3.12}$$

where the S_{ii} are square matrices of sizes s_1, s_2, s_3, and s_4, respectively, $\mathbf{v}$ is the sum of the first $s_1 + s_2$ rows of A', and $\mathbf{w} - 2\mathbf{e}$ is the sum of the transposes of the middle $s_2 + s_3$ columns of A'. In terms of this block-partition,

$$\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4),$$
$$\mathbf{w} - 2\mathbf{e} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4),$$

with

$$\begin{aligned} \mathbf{v}_i &= 2[\mathbf{e}(s_1)\mathsf{S}_{1i} + \mathbf{e}(s_2)\mathsf{S}_{2i}] + \mathbf{e}(s_i)(\delta(1, i) + \delta(2, i)), \\ \mathbf{w}_i &= 2[\mathbf{e}(s_2)\mathsf{S}_{i2}^T + \mathbf{e}(s_3)\mathsf{S}_{i3}^T] + \mathbf{e}(s_i)(\delta(i, 2) + \delta(i, 3)), \end{aligned} \tag{3.13}$$

where $\mathbf{e}(s_i)$ denotes a row vector of all 1's of size s_i, and $\delta(i, j) = 1$ if $i = j$, and 0 otherwise. The congruence (3.11) is equivalent to showing that $\mathbf{v}$ and $\mathbf{w} - 2\mathbf{e}$ agree (mod 4) in an odd number of coordinates. If d_i denotes the number of entries in which $\mathbf{v}_i$ and $\mathbf{w}_i$ agree (mod 4), then this is equivalent to showing that

$$d_1 + d_2 + d_3 + d_4 \equiv 1 \pmod 2.$$

The matrix A' has all its entries even except for its diagonal entries, which are all odd. Hence all entries of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_2$, and $\mathbf{w}_3$ are odd, while all entries of $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{w}_1$, and $\mathbf{w}_4$ are even. It follows that $d_1 = 0$ and $d_3 = 0$. Thus it remains to prove that

$$d_2 + d_4 \equiv 1 \pmod 2. \tag{3.14}$$

Now let t_{ij} denote the sum of all entries of the 0–1 matrix S_{ij}, i.e., t_{ij} counts the number of ones in S_{ij}. From (3.13),

$$\mathbf{v}_2 - \mathbf{w}_2 = 2[\mathbf{e}(s_1)\mathrm{S}_{12} + \mathbf{e}(s_2)\mathrm{S}_{22} - \mathbf{e}(s_2)\mathrm{S}_{22}^T - \mathbf{e}(s_3)\mathrm{S}_{23}^T]. \tag{3.15}$$

The sum of the entries in $\mathbf{v}_2 - \mathbf{w}_2$ (mod 4) is $2(s_2 - d_2)$ (mod 4), by definition of d_2, since $\mathbf{v}_2 \equiv \mathbf{w}_2$ (mod 2). However, it is also $(\mathbf{v}_2 - \mathbf{w}_2)\mathbf{e}(s_2)^T$ (mod 4), and, using (3.15), this equals $2(t_{12} + t_{22} - t_{22} - t_{23})$ (mod 4). Thus we obtain

$$d_2 + s_2 \equiv t_{12} + t_{23} \pmod 2. \tag{3.16}$$

Likewise, (3.13) gives

$$\mathbf{v}_4 - \mathbf{w}_4 = 2[\mathbf{e}(s_1)\mathrm{S}_{14} + \mathbf{e}(s_2)\mathrm{S}_{24} - \mathbf{e}(s_2)\mathrm{S}_{42}^T - \mathbf{e}(s_3)\mathrm{S}_{43}^T],$$

from which we similarly derive

$$d_4 + s_4 \equiv t_{14} + t_{24} + t_{42} + t_{43} \pmod 2. \tag{3.17}$$

Now recall that since $\mathbf{v} \in V_{\text{even}}(\mathrm{M}')$ it contains an even number of entries $\equiv 3$ (mod 4). It contains exactly $s_1 + s_2$ odd entries, namely, $(\mathbf{v}_1, \mathbf{v}_2)$. The sum of its odd entries must therefore be $s_1 + s_2$ (mod 4), and so

$$\begin{aligned} s_1 + s_2 &\equiv (\mathbf{v}_1, \mathbf{v}_2)\mathbf{e}(s_1 + s_2)^T \pmod 4 \\ &\equiv s_1 + s_2 + 2(t_{11} + t_{21} + t_{12} + t_{22}) \pmod 4. \end{aligned}$$

Hence

$$t_{11} + t_{21} + t_{12} + t_{22} \equiv 0 \pmod 2. \tag{3.18}$$

Next, since $\mathbf{w} - 2\mathbf{e} \in V_{\text{even}}^*((\mathrm{M}')^T)$, it contains an odd number of entries $\equiv 0$ (mod 4). It contains exactly $s_1 + s_4$ even entries, namely, $(\mathbf{w}_1, \mathbf{w}_4)$, so the sum of its even entries is $2 + 2(s_1 + s_4)$ (mod 4). Thus

$$\begin{aligned} 2 + 2(s_1 + s_4) &\equiv (\mathbf{w}_1, \mathbf{w}_4)\mathbf{e}(s_1 + s_4)^T \pmod 4 \\ &\equiv 2(t_{12} + t_{13} + t_{42} + t_{43}) \pmod 4. \end{aligned}$$

This yields

$$1 + s_1 + s_4 \equiv t_{12} + t_{13} + t_{42} + t_{43} \pmod 2. \tag{3.19}$$

Now add (3.16), (3.17), and (3.19) to obtain

$$1 + d_2 + d_4 + s_1 + s_2 \equiv t_{13} + t_{23} + t_{14} + t_{24} \pmod 2. \tag{3.20}$$

Next, adding (3.18) to this gives

$$1 + d_2 + d_4 + s_1 + s_2 \equiv (t_{11} + t_{12} + t_{13} + t_{14}) + (t_{21} + t_{22} + t_{23} + t_{24}) \pmod 2.$$

The right-hand side of this congruence is simply the sum of the first $s_1 + s_2$ rows of M, hence by Property AC it is $s_1 + s_2 \pmod 2$. This yields

$$d_2 + d_4 \equiv 1 \pmod 2,$$

completing the proof. □

In what follows we only consider nonlinear codes $\mathscr{C}(\mathsf{M})$ with $n \equiv 1 \pmod 2$, which have both Property AC and Property BC. The construction of the two complements $\mathscr{C}^+$ and $\mathscr{C}^-$ of a code $\mathscr{C}(\mathsf{M})$ is generally asymmetrical, but in this special case the asymmetry disappears, i.e., we can show that $V_{\text{even}}(\mathsf{M})$ coincides with the set of all vectors in $V(\mathsf{A})$ having an odd number of entries that are $0 \pmod 4$, and $V^*_{\text{even}}(\mathsf{M}^T)$ coincides with the set of all vectors in $V(\mathsf{A}^T)$ containing an even number of entries $3 \pmod 4$. We omit a proof as this fact is not needed in the following.

4. Cube-Tiling Codes: Construction A

In this section only, for a fixed dimension n, let M_n denote the circulant matrix $\mathrm{Circ}(0, 1, 0, \ldots, 0)$. Then $\mathsf{A}(\mathsf{M}_n) = \mathrm{Circ}(1, 2, 0, \ldots, 0)$, so, for example,

$$\mathsf{A}(\mathsf{M}_5) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The set $\mathscr{C}(\mathsf{M}_n)$ is called a Construction A code, and we denote it $\mathscr{C}^n_{\mathsf{A}}$. We prove that $\mathscr{C}^n_{\mathsf{A}}$ is a cube-tiling code when n is odd, and then prove a special property about any two cubes in this tiling having a high-dimensional common face.

Theorem 4.1. *For odd n, $\mathscr{C}^n_{\mathsf{A}}$ is a cube-tiling code which gives a 2-extremal cube-tiling of $\mathbb{R}^n$.*

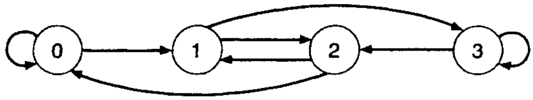

Fig. 4.1. Directed graph $\mathscr{G}$.

Proof. The circulant matrix M_n has Property BC, and since $n \equiv 1 \pmod 2$, $\mathscr{C}_{\mathsf{A}}^n$ is balanced by Theorem 3.1. M_n also has Property AC, hence by Theorem 3.2 it is an additive code with 2^n elements, and

$$d_e(\mathbf{w}, \mathbf{w}') \geq 1$$

holds when $\mathbf{w}$ and $\mathbf{w}'$ lie in different complements of $\mathscr{C}_{\mathsf{A}}^n$.

Now suppose $\mathbf{w}, \mathbf{w}' \in V_{\text{even}}(\mathsf{M})$. Our object is to show that $d_e(\mathbf{w}, \mathbf{w}') \geq 1$. We study the larger set V of 2^n vectors

$$\mathbf{y} = (y_1, \ldots, y_n) = \mathbf{v}_{i_1} + \cdots + \mathbf{v}_{i_m},$$

which are sums of distinct rows $\mathbf{v}_i$ of $\mathsf{A}(\mathsf{M}_n)$. The vectors in V have a simple description, arising because the vectors $\mathbf{v}_i$ have only two nonzero components, which are consecutive, and because the set $\{\mathbf{v}_i\}$ is closed under a cyclic shift of coordinates. They are characterized using the directed graph pictured in Fig. 4.1. The vertices of the graph $\mathscr{G}$ correspond to the possible values of a coordinate y_i of $\mathbf{y}$, and a directed edge gives a transition to an allowed value of y_{i+1}. Then $\mathbf{y} \in V$ if and only if the sequence $(y_1, y_2, \ldots, y_n, y_1)$ describes a closed directed path of length n in $\mathscr{G}$. (That is, they consist of those vectors whose entries, viewed cyclically, consist of blocks of the form $1\ 3^j\ 2$, where $j \geq 0$, separated by blocks of 0's, possibly empty, plus the single vector $3^n = (3, 3, \ldots, 3)$.)

Next, any ordered pair of vectors $\mathbf{w}, \mathbf{w}' \in V$ define a closed directed path in the product graph $\mathscr{G} \times \mathscr{G}$, induced from the paths on $\mathscr{G}$ of $\mathbf{w}$ and $\mathbf{w}'$ separately. Suppose $d_e(\mathbf{w}, \mathbf{w}') = 0$. This means that this closed directed path never visits any of the vertices labeled $\binom{2}{0}$, $\binom{0}{2}$, $\binom{3}{1}$, or $\binom{1}{3}$ in $\mathscr{G} \times \mathscr{G}$. Let $\hat{\mathscr{G}}_2$ denote the subgraph of $\mathscr{G} \times \mathscr{G}$ obtained by deleting these four vertices; it is pictured in Fig. 4.2. Any closed directed path in $\hat{\mathscr{G}}_2$ must lie entirely in one of the two sets of vertices

$$\mathscr{A} = \left\{\binom{0}{0}, \binom{1}{1}, \binom{2}{2}, \binom{3}{3}\right\} \quad \text{or} \quad \mathscr{B} = \left\{\binom{3}{0}, \binom{2}{1}, \binom{1}{2}, \binom{0}{3}\right\}.$$

To see this, note that there are no edges between $\mathscr{A}$ and $\mathscr{B}$, and any directed path entering

$$\mathscr{E} = \left\{\binom{1}{0}, \binom{0}{1}, \binom{3}{2}, \binom{2}{3}\right\}$$

must enter from $\mathscr{A}$ and must exit to $\mathscr{B}$, so cannot be a closed path.

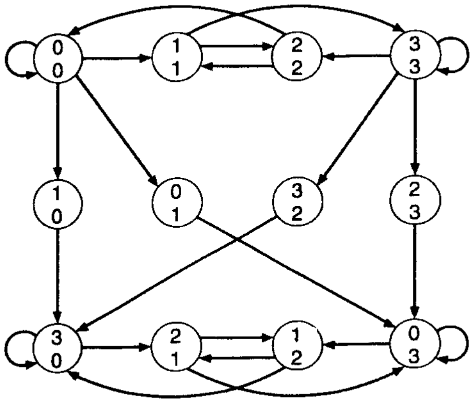

Fig. 4.2. Subgraph $\mathscr{G}_2$ of product graph $\mathscr{G} \times \mathscr{G}$.

If $(\mathbf{w}, \mathbf{w}')$ gives a closed path in $\mathscr{A}$, then all their entries must agree, i.e., $\mathbf{w} = \mathbf{w}'$. Next, suppose $(\mathbf{w}, \mathbf{w}')$ gives a closed directed path in $\mathscr{B}$, see Fig. 4.3. This path necessarily visits vertices $\binom{2}{1}$ and $\binom{1}{2}$ the same number of times, because when it leaves vertex $\binom{1}{2}$ it cannot return to it without visiting vertex $\binom{2}{1}$ first, and vice versa. Since n is odd, this closed directed path must visit vertices $\binom{0}{3}$ and $\binom{3}{0}$ an odd number of times in total. Hence one of $\mathbf{w}$ or $\mathbf{w}'$ contains an odd number of entries equal to 3, so it is impossible that both $\mathbf{w}$, $\mathbf{w}' \in V_{\text{even}}(\mathsf{M}_n)$. Thus if $\mathbf{w}$, $\mathbf{w}' \in V_{\text{even}}(\mathsf{M}_n)$ with $\mathbf{w} \neq \mathbf{w}'$, then

$$d_e(\mathbf{w}, \mathbf{w}') \geq 1.$$

If $\mathbf{w}$, $\mathbf{w}' \in V^*_{\text{even}}(\mathsf{M}_n^T) + 2\mathbf{e}$, then

$$d_e(\mathbf{w}, \mathbf{w}') \geq 1,$$

by an analogous graph-theoretic argument which we omit.

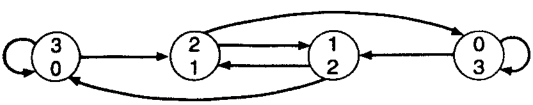

Fig. 4.3. The graph $\mathscr{B}$.

Thus $\mathscr{C}_A^n$ is a cube-tiling code. It is automatically 2-extremal because all vectors in $V_{\text{even}}(\mathsf{M}_n)$ differ (mod 2); in fact the 2^n vectors in V are all incongruent (mod 2). □

We next show that if two cubes in the cube-tiling from $\mathscr{C}_A^n$ have a sufficiently high-dimensional face in common, then the corresponding cube-center vectors in $\mathscr{C}_A^n$ have a positive fraction of matching coordinates having value 0 or 1.

Theorem 4.2. *Suppose that n is odd, and that* $\mathbf{w}$, $\mathbf{w}'$ *are distinct vectors in the cube-tiling code* $\mathscr{C}_A^n$ *with*

$$\mathbf{w} \equiv \mathbf{w}' \pmod 2. \tag{4.1}$$

Suppose that $\mathbf{w}$ *agrees with* $\mathbf{w}'$ *in l coordinate places, and let $l = l_0 + l_1 + l_2 + l_3$, where l_0, l_1, l_2, and l_3 denote the number of matching coordinates equal to* 0, 1, 2, *and* 3, *respectively. Then l is even, and*

$$l_0 = l_2, \qquad l_1 = l_3.$$

In particular, exactly $\frac{1}{2}l$ of the matching coordinates take values 0 *or* 1.

Proof. The proof of Theorem 4.1 indicated that all vectors in $V_{\text{even}}(\mathsf{M}_n)$ correspond to closed directed paths of length n in the graph $\mathscr{G}$ of Fig. 4.1. Similarly the vectors in $V^*_{\text{even}}(\mathsf{M}_n^T) + 2\mathbf{e}$ correspond exactly to those closed directed paths in the graph $\mathscr{G}'$ pictured in Fig. 4.4.

We are interested in pairs $\mathbf{w}$, $\mathbf{w}'$ satisfying (4.1). They must be in opposite complements, hence $(\mathbf{w}, \mathbf{w}')$ corresponds to a closed path in the product graph $\mathscr{G} \times \mathscr{G}'$ that visits only vertices $\binom{i}{j}$ with $i \equiv j \pmod 2$. The restriction $\tilde{\mathscr{G}}$ of $\mathscr{G} \times \mathscr{G}'$ to these vertices is pictured in Fig. 4.5.

In what fashion can a closed path in the restriction $\tilde{\mathscr{G}}$ visit the matching vertices $\left\{\binom{i}{j}: 0 \le i \le 3\right\}$? This is specified by the contracted graph pictured in Fig. 4.6. Any closed path in this graph clearly visits $\binom{0}{0}$ the same number of times as $\binom{2}{2}$, and visits $\binom{1}{1}$ the same number of times as $\binom{3}{3}$. □

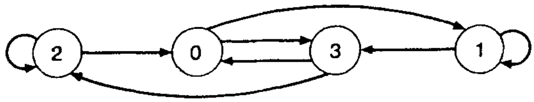

Fig. 4.4. Directed graph $\mathscr{G}'$.

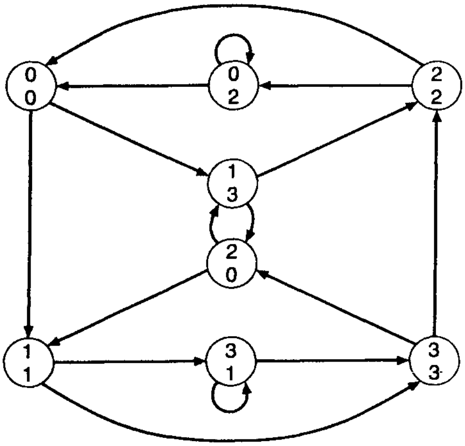

Fig. 4.5. Subgraph $\mathcal{G}$ of $\mathcal{G} \times \mathcal{G}'$.

A strengthening of Theorem 4.2 can be proved when $\mathbf{w}$ and $\mathbf{w}'$ have many matching coordinates, namely, at least $\frac{3}{4}n$. It can be checked that the only ways two vectors $\mathbf{w}$, $\mathbf{w}'$ satisfying (4.1) can have four consecutive matching coordinates is that these coordinates are some cyclic permutation of 0132. Using this fact, we can easily show that

$$l_i \geq l - \tfrac{3}{4}n, \qquad i = 0, 1, 2, 3. \tag{4.2}$$

5. Cube-Tiling Codes: Construction B

In this section only, for all odd n, let M_n denote the matrix with subdiagonal and superdiagonal given by

$$(\mathsf{M}_n)_{i+1,i} = 1, \qquad 1 \leq i \leq n-1, \tag{5.1a}$$

$$(\mathsf{M}_n)_{2i,2i+1} = 1, \qquad 1 \leq i \leq (n-3)/2, \tag{5.1b}$$

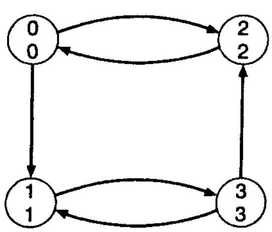

Fig. 4.6. Contraction of $\mathcal{G}$.

with the first row given by

$$(\mathsf{M}_n)_{1,i} = 1, \qquad 3 \le i \le n, \tag{5.1c}$$

with the last column given by

$$(\mathsf{M}_n)_{n-4i,n} = (\mathsf{M}_n)_{n-4i-1,n} = 1, \qquad 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor, \tag{5.1d}$$

also with a 2×2 block checkerboard pattern for $(\mathsf{M}_n)_{i,j}$ in the region $i + 2 \le j \le n - 1$, given by

$$(\mathsf{M}_n)_{i,j} = 1 \qquad \text{if } (i,j) \text{ consists of one element each from } 0, 1 \pmod 4 \text{ and from } 2, 3 \pmod 4, \tag{5.1e}$$

and with all other $(\mathsf{M}_n)_{i,j} = 0$. The resulting matrices

$$\mathsf{A}_n = \mathsf{A}(\mathsf{M}_n) = \mathsf{I} + 2\mathsf{M}_n$$

have slightly different patterns according to whether $n = 1$ or $3 \pmod 4$, see Fig. 5.1. The set $\mathscr{C}(\mathsf{M}_n)$ is called a Construction B code and we denote it $\mathscr{C}_{\mathrm{B}}^n$. We prove that $\mathscr{C}_{\mathrm{B}}^n$ is a cube-tiling code, and later show that codewords congruent (mod 2) must differ in either exactly one coordinate or else in many coordinates.

Before giving the proofs, we point out an important property of the vectors in A_n. Let $\mathbf{v}_i$ denote the ith row of A_n and $\mathbf{v}_i^*$ the ith row of A_n^T, i.e., the ith column of A_n. Let $\oplus$ denote the exclusive-or operation on residues (mod 4) viewed as binary numbers, i.e., $0 \oplus 0 = 1 \oplus 1 = 2 \oplus 2 = 3 \oplus 3 = 0$, $1 \oplus 3 = 3 \oplus 1 = 2$, and

$$\mathbf{A}_{13} = \begin{bmatrix} 1&0&2&2&2&2&2&2&2&2&2&2&2 \\ 2&1&2&2&2&0&0&2&2&0&0&2&0 \\ 0&2&1&0&2&0&0&2&2&0&0&2&0 \\ 0&0&2&1&2&2&2&0&0&2&2&0&2 \\ 0&0&0&2&1&0&2&0&0&2&2&0&2 \\ 0&0&0&0&2&1&2&2&2&0&0&2&0 \\ 0&0&0&0&0&2&1&0&2&0&0&2&0 \\ 0&0&0&0&0&0&2&1&2&2&2&0&2 \\ 0&0&0&0&0&0&0&2&1&0&2&0&2 \\ 0&0&0&0&0&0&0&0&2&1&2&2&0 \\ 0&0&0&0&0&0&0&0&0&2&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&2&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&2&1 \end{bmatrix}, \quad \mathbf{A}_{15} = \begin{bmatrix} 1&0&2&2&2&2&2&2&2&2&2&2&2&2&2 \\ 2&1&2&2&2&0&0&2&2&0&0&2&2&0&2 \\ 0&2&1&0&2&0&0&2&2&0&0&2&2&0&2 \\ 0&0&2&1&2&2&2&0&0&2&2&0&0&2&0 \\ 0&0&0&2&1&0&2&0&0&2&2&0&0&2&0 \\ 0&0&0&0&2&1&2&2&2&0&0&2&2&0&2 \\ 0&0&0&0&0&2&1&0&2&0&0&2&2&0&2 \\ 0&0&0&0&0&0&2&1&2&2&2&0&0&2&0 \\ 0&0&0&0&0&0&0&2&1&0&2&0&0&2&0 \\ 0&0&0&0&0&0&0&0&2&1&2&2&2&0&2 \\ 0&0&0&0&0&0&0&0&0&2&1&0&2&0&2 \\ 0&0&0&0&0&0&0&0&0&0&2&1&2&2&0 \\ 0&0&0&0&0&0&0&0&0&0&0&2&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&2&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&2&1 \end{bmatrix}$$

Fig. 5.1. Matrices from Construction B.

$i \oplus j = i + j \pmod 4$ otherwise. The important property of A_n is that $\mathbf{v}_i \oplus \mathbf{v}_i^*$ takes only three possible values. Namely, if $n \equiv 1 \pmod 4$, then

$$\mathbf{v}_i \oplus \mathbf{v}_i^* = \begin{cases} 0\,2^{n-1} & \text{if } i = 1, \\ 2(0022)^{(n-5)/4}\,0020 & \text{if } i \equiv 2 \text{ or } 3 \pmod 4 \quad \text{or if} \quad i = n, \\ 2(2200)^{(n-5)/4}\,2202 & \text{if } i \equiv 0 \text{ or } 1 \pmod 4 \quad \text{and} \quad i \neq 1 \text{ or } n, \end{cases} \tag{5.2a}$$

while if $n \equiv 3 \pmod 4$, then

$$\mathbf{v}_i \oplus \mathbf{v}_i^* = \begin{cases} 0(2)^{n-1} & \text{if } i = 1, \\ 2(0022)^{(n-3)/4}\,02 & \text{if } i \equiv 2, 3 \pmod 4 \quad \text{and} \quad i \neq n, \\ 2(2200)^{(n-3)/4}\,20 & \text{if } i \equiv 0, 1 \pmod 4 \quad \text{and} \quad i \neq 1, \quad \text{or} \quad i = n. \end{cases} \tag{5.2b}$$

Formulae (5.2) are easily verified by direct calculation from (5.1). Also note that the three vectors on the right-hand side of (5.2a) (resp. (5.2b)) together with the vector 0^n, form a set closed under the $\oplus$ operation. Formulae (5.2) prove extremely important in studying the structure of codewords in $\mathscr{C}_{\mathrm{B}}$. They are particularly useful in studying codewords that are congruent (mod 2), due to the identity that, if $\mathbf{v}_i \equiv \mathbf{v}_i' \pmod 2$,

$$\mathbf{v}_i - \mathbf{v}_i' \equiv \mathbf{v}_i \oplus \mathbf{v}_i' \pmod 4, \tag{5.3}$$

see Theorem 5.2.

We also note that the first vector $\mathbf{v}_1$ in the matrix A_n has a special structure different from the other vectors in A_n, which is reflected in (5.2) and also in the different e-distance behavior of the codewords in $\mathscr{C}_{\mathrm{B}}^n$ depending on the parity of their first coordinate (Theorem 5.2(ii)).

Theorem 5.1. *For odd n, $\mathscr{C}_{\mathrm{B}}^n$ is a cube-tiling code which gives a 2-extremal cube-tiling of $\mathbb{R}^n$.*

Proof. For odd n, the matrices M_n have odd row and column sums, hence have Property BC and Property AC. Consequently, $\mathscr{C}_{\mathrm{B}}^n$ is balanced by Theorem 3.1, and is an additive code by Theorem 3.2, and satisfies

$$d_e(\mathbf{w}, \mathbf{w}') \geq 1,$$

when $\mathbf{w}$ and $\mathbf{w}'$ are in different complements of $\mathscr{C}_{\mathrm{B}}^n$.

Now suppose $\mathbf{w}, \mathbf{w}' \in \mathscr{C}^+ = V_{\text{even}}(\mathsf{M}_n)$. We wish to show that they differ by 2 in some coordinate. It proves convenient to study the larger set V consisting of all 2^n possible sums of rows of A_n. So suppose

$$\mathbf{y}_1 = \sum_{i \in I_1} \mathbf{v}_i, \qquad \mathbf{y}_2 = \sum_{i \in I_2} \mathbf{v}_i$$

are arbitrary members of V. Just as in the proof of Theorem 3.1, our object is to show that if $d_e(\mathbf{y}_1, \mathbf{y}_2) = 0$, then $\mathbf{y}_1$ and $\mathbf{y}_2$ have between them an odd number of coordinates equal to 3. This will show that at least one of $\mathbf{y}_1$ and $\mathbf{y}_2$ is not in $\mathscr{C}^+$.

So suppose $d_e(\mathbf{y}_1, \mathbf{y}_2) = 0$, and partition $I_1 = R_1 \cup R_3$, $I_2 = R_2 \cup R_3$, where $R_3 = I_1 \cap I_2$. It is easy to see that $d_e(\mathbf{y}_1, \mathbf{y}_2) = 0$ implies that

$$\mathbf{w}_{R_1 \cup R_2} := \sum_{i \in R_1 \cup R_2} \mathbf{v}_i$$

must contain no 2 in any coordinate position. □

This motivates the study of sets R such that $\sum_{i \in R} \mathbf{v}_i$ contains no 2's, which we now characterize. Given R, let χ_R be its characteristic function, i.e., the 0–1 vector having 1's corresponding exactly to $i \in R$. In the following lemma we regard χ_R as specified by a string of 0's and 1's. Also in what follows $\{A_1, \ldots, A_n\}^*$ denotes the set of all words formed by concatenation from the finite strings $A_1, \ldots, A_n$ of 0's and 1's, as in the theory of regular expressions, see [4] and [7].

Lemma 5.1. *For odd n, the set*

$$\mathscr{S}_n := \left\{ \chi_R \colon \sum_{i \in R} \mathbf{v}_i \text{ contains no 2's} \right\}$$

consists exactly of the words of length n in the language

$$\mathscr{L} = \{0\}^* \cup 1\{11, 0011, 100001, 00100001\}^*.$$

Proof. Certainly $\varnothing \in \mathscr{S}_n$, corresponding to the vector 0^n. So suppose $R \in \mathscr{S}_n$ is nonempty. If $1 \notin R$, then the first nonzero coordinate in

$$\mathbf{w}_R := \sum_{i \in R} \mathbf{v}_i$$

is 2, a contradiction showing $1 \in R$. Next, if $3 \in R$, then $2 \in R$, otherwise, the second coordinate of $\mathbf{w}_R$ is 2. If $2 \in R$ and $3 \notin R$, then we must have $4, 5, 6 \notin R$ and $7 \in R$ in order not to have any 2's in positions 1–6. If $2 \notin R$ and $3 \notin R$, then we are similarly forced to have $4 \in R$, and then either $5 \in R$ or else $5, 6, 7, 8 \notin R$ and $9 \in R$. Thus always $1 \in R$ and χ_R begins with one of the patterns

$$1\{11, 0011, 100001, 00100001\}.$$

Consider the first case 111. We have

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 331022\ldots2.$$

Deleting the first two coordinates of this gives the first row of A_{n-2}, while if we delete the first two columns and three rows of A_n we get the remaining rows of A_{n-2}. Since all vectors $\mathbf{v}_4, \ldots, \mathbf{v}_n$ in A_n are 0 in their first two coordinates, the problem of extending $111\ldots$ to an element of $\mathcal{S}_n$ is equivalent to the problem of extending $1\ldots$ to an element of $\mathcal{S}_{n-2}$.

Consider the second case 10011. We have

$$\mathbf{v}_1 + \mathbf{v}_4 + \mathbf{v}_5 = 10011022\ldots2.$$

Deleting the first four coordinates of this gives row 1 of A_{n-4}, while deleting the first four columns and five rows of A_n gives the rest of A_{n-4}. The problem of extending $10011\ldots$ to an element of $\mathcal{S}_n$ is thus equivalent to the problem of extending $1\ldots$ to an element of $\mathcal{S}_{n-4}$.

Similarly, the third and fourth cases 1100001 and 100100001 reduce to $\mathcal{S}_{n-6}$ and $\mathcal{S}_{n-8}$, respectively.

Now the lemma follows by induction on odd n, after an easy check of the base cases $n = 1$, 3, 5, and 7. □

Lemma 5.1 implies, in particular, that, except for the empty set, all elements of $\mathcal{S}_n$ contain the first and last rows $\mathbf{v}_1$ and $\mathbf{v}_n$.

We now continue the proof of Theorem 5.1 for $\mathscr{C}^+$. It proceeds in three steps.

(a) For nonempty $R \in \mathcal{S}_n$, $\mathbf{w}_R := \sum_{i\in R} \mathbf{v}_i$ contains an odd number of 3's.

(b) If $R_1 \cup R_2 \in \mathcal{S}_n$ and $R_1 \cap R_2 = \varnothing$, then

$$\mathbf{w}_1 = \sum_{i\in R_1} \mathbf{v}_i, \qquad \mathbf{w}_2 = \sum_{i\in R_2} \mathbf{v}_i$$

contain an odd number of 3's between them.

(c) If $R_1 \cup R_2 \in \mathcal{S}_n$ and R_1, R_2, R_3 are pairwise disjoint, then

$$\mathbf{y}_1 = \sum_{i\in R_1\cup R_3} \mathbf{v}_i, \qquad \mathbf{y}_2 = \sum_{i\in R_2\cup R_3} \mathbf{v}_i$$

contain an odd number of 3's between them.

We introduce some notation. Given $R \subseteq \{1, \ldots, n\}$, let B_R be the square submatrix of A_n given by

$$\mathsf{B}_R = [b_{ij}: i, j \in R] := [(\mathsf{A}_n)_{i,j}: i, j \in R]. \tag{5.4}$$

To prove (a) observe that, for any R, $\mathbf{w}_R$ has an odd number of 3's if and only if B_R contains an odd number of 2's. (The sum of all rows of B_R (mod 4) gives the set of coordinates of $\mathbf{w}_R$ that are odd.) Now suppose $R \in \mathcal{S}_n$ is nonempty, so $1 \in R$ and $n \in R$ by Lemma 3.1. Divide B_R into the sets $E_i = \{b_{ij}: j > i\} \cup \{b_{ji}: j > i\}$ for $i \in R$, plus its diagonal. The diagonal is all 1's, and contributes no 2's. We will show that E_1 and E_n each contain an even number of 2's, while all other E_i each contain an odd number of 2's. Since R has odd cardinality by Lemma 3.1, B_R will contain an odd number of 2's and (a) will follow.

The number of 2's in E_i is even or odd according to whether the sum σ_i of all elements in E_i is 0 or 2 (mod 4). We have

$$b_{ij} + b_{ji} \equiv b_{ij} \oplus b_{ji} = (\mathbf{v}_i \oplus \mathbf{v}_i^*)_j \pmod 4$$

because each element in E_i is 0 or 2, hence

$$\sigma_i = \sum_{\substack{j \in R \\ j > i}} (\mathbf{v}_i \oplus \mathbf{v}_i^*)_j.$$

The set E_1 has all $(\mathbf{v}_i \oplus \mathbf{v}_i^*)_j = 2$ by (5.2), and the set $R - \{1\}$ has even cardinality, so contains an even number of 2's. The set E_n is empty. To analyze the other sets E_i, we use Lemma 5.1. Now $R \in 1\{11, 0011, 100001, 00100001\}^*$. Each $i > 1$ in R lies in a block of two coordinates $\{j_1, j_2\}$ with $j_1 < j_2$, according to the decomposition of R into blocks in the language $\mathscr{L}$. We always have, for $i = j_1$ within a block,

$$(\mathbf{v}_{j_1} \oplus \mathbf{v}_{j_1}^*)_{j_2} = 0, \tag{5.5a}$$

except for the last block ($j_2 = n$), where

$$(\mathbf{v}_{j_1} \oplus \mathbf{v}_{j_1}^*)_{j_2} = 2. \tag{5.5b}$$

For $i > 1$ in R and a block $\{j_1, j_2\}$, with $i < j_1$, we have

$$(\mathbf{v}_i \oplus \mathbf{v}_i^*)_{j_1} \equiv (\mathbf{v}_i \oplus \mathbf{v}_i^*)_{j_2} \pmod 4, \tag{5.5c}$$

except for the last block ($j_2 = n$), where

$$(\mathbf{v}_i \oplus \mathbf{v}_i^*)_{j_1} \equiv (\mathbf{v}_i \oplus \mathbf{v}_i^*)_{j_2} + 2 \pmod 4. \tag{5.5d}$$

These facts are proved by induction on n, by the method of Lemma 3.1. They imply

$$\sigma_i \equiv 2 \pmod 4$$

for all $i \in R$ such that $1 < i < n$, completing (a).

To prove (b) we proceed by induction on the size $|R_2|$ of R_2. The base case $R_1 = R$, $R_2 = \varnothing$ is already established by (a). We analyze the effect of shifting a single element $\mathbf{v}_i$ from R_1 to R_2. Set $R_1' = R_1 - \{\mathbf{v}_i\}$, $R_2' = R_2 \cup \{\mathbf{v}_i\}$, and $R = R_1 \cup R_2 = R_1' \cup R_2'$. Now

$$\mathbf{w}_{R_1'} = \mathbf{w}_{R_1} - \mathbf{v}_i,$$

$$\mathbf{w}_{R_2'} = \mathbf{w}_{R_2} + \mathbf{v}_i.$$

If π_{R_1,R_2} denotes the parity of the total number of 3's in $\mathbf{w}_{R_1}$ and $\mathbf{w}_{R_2}$, then we claim that

$$\pi_{R_1',R_2'} \equiv \pi_{R_1,R_2} + \#(\text{2's in } i\text{th row of } \mathsf{B}_R) + \#(\text{2's in } i\text{th column of } \mathsf{B}_R) \pmod 2. \tag{5.6}$$

To see this, note that if $j \in R$ with $j \neq i$, then exactly one of $\mathbf{w}_{R_1}$ and $\mathbf{w}_{R_2}$ has an odd jth coordinate value, and if $\mathbf{v}_i$ has value 2 in its jth coordinate, this value switches from 1 to 3 or vice versa. Hence the parity of the change in the number of 3's in those coordinates $j \neq i$ is $\#$(2's in ith row of B_R). For the ith coordinate

$$(\mathbf{w}_{R_1'})_i \equiv \mathbf{w}_{R_1} - 1 \pmod 4,$$
$$(\mathbf{w}_{R_2'})_i \equiv \mathbf{w}_{R_2} + 1 \pmod 4,$$

where $(\mathbf{w}_{R_1})_i$ and $(\mathbf{w}_{R_2'})_i$ are odd, whence

$$\begin{aligned}(\mathbf{w}_{R_1})_i - (\mathbf{w}_{R_2'})_i &\equiv (\mathbf{w}_{R_1})_i - (\mathbf{w}_{R_2})_i - 1 \pmod 4\\ &\equiv 2 \cdot \#(\text{2's in } i\text{th column of } \mathsf{B}_R) \pmod 4.\end{aligned}$$

Thus the parity of the change in the number of 3's in coordinate i is equal to $\#$(2's in ith column of B_R) (mod 2). This proves (5.6).

It now suffices to show that

$$\#(\text{2's in } i\text{th row of } \mathsf{B}_R) \equiv \#(\text{2's in } i\text{th column of } \mathsf{B}_R) \pmod 2. \tag{5.7}$$

If so, then (5.6) yields

$$\pi_{R_1',R_2'} \equiv \pi_{R_1,R_2} \pmod 2, \tag{5.8}$$

which will complete the induction step for (b). Note that (5.7) says that corresponding row sums and column sums in B_R are congruent (mod 4). Now (5.7) reduces to showing that

$$(\mathbf{v}_i \oplus \mathbf{v}_i^*) \cdot \chi_R \equiv 0 \pmod 4, \tag{5.9}$$

since the parity of the difference is equal to the left-hand side of (5.9). The congruence (5.9) holds because, using Lemma 3.1, the decomposition of R into blocks using $\mathscr{L}$ shows that $\mathbf{v}_i \oplus \mathbf{v}_i^*$ has an even number of 2's in each block, except the first and last, where it always has an odd number of 2's, see (5.5). In fact, this argument shows that (5.9) holds for *all* rows $1 \le i \le n$. Thus (5.7) holds and (b) follows.

To prove (c) we proceed by induction on the size of $|R_3|$. The base case $R_3 = \varnothing$

holds by (b). We consider the effect of adding a new vector $\mathbf{v}_i$ to R_3. Set $R = R_1 \cup R_2$, $S_1 = R_1 \cup R_3$, $S_2 = R_2 \cup R_3$, $S_1' = S_1 \cup \{\mathbf{v}_i\}$, and $S_2' = S_2 \cup \{\mathbf{v}_i\}$. Then

$$\pi_{S_1', S_2'} \equiv \pi_{S_1, S_2} + \#\{2\text{'s in } \mathbf{v}_i \text{ lying in } R\} + \#\{2\text{'s in } \mathbf{v}_i^* \text{ lying in } R\} \pmod 2, \tag{5.10}$$

by a proof similar to case (b). Note that the last term in (5.10) occurs because both $\mathbf{w}_{S_1}'$ and $\mathbf{w}_{S_2}'$ have an odd ith coordinate, and these agree (mod 4) if and only if $\#\{2\text{'s in } \mathbf{v}_i^* \text{ lying in } R\}$ is even. Finally, we claim that

$$\#\{2\text{'s in } \mathbf{v}_i \text{ lying in } R\} \equiv \#\{2\text{'s in } \mathbf{v}_i^* \text{ lying in } R\} \pmod 2, \tag{5.11}$$

which with (5.10) yields

$$\pi_{S_1', S_2'} \equiv \pi_{S_1, S_2} \pmod 2,$$

completing the induction step. Claim (5.11) is proved by reducing it to (5.9), which is valid for all i, so (c) follows.

This proves that $\mathbf{w}, \mathbf{w}' \in \mathscr{C}^+$ differ by 2 in some coordinate. It remains to do the same for $\mathbf{w}, \mathbf{w}' \in \mathscr{C}^-$. This has a similar proof. The set $\mathscr{S}_n^*$ of sets of columns of A_n that sum to a vector containing no 2 is exactly the same as $\mathscr{S}_n$, and is proved by a similar induction. All the subsequent arguments for $\mathscr{C}^+$ depended on conditions which are symmetric with respect to rows and columns of A_n, so carry over identically to the $\mathscr{C}^-$ case. □

A crucial feature of the Construction B codes is that vectors with $\mathbf{w} \equiv \mathbf{w}' \pmod 2$ in $\mathscr{C}_{\mathrm{B}}^n$ are either close or widely separated in e-distance.

Theorem 5.2. *Suppose that n is odd, and that $\mathbf{w}$, $\mathbf{w}'$ are distinct vectors in the cube-tiling code $\mathscr{C}_{\mathrm{B}}^n$ with*

$$\mathbf{w} \equiv \mathbf{w}' \pmod 2. \tag{5.12}$$

Then:

(i) $d_e(\mathbf{w}, \mathbf{w}')$ *is* 1, $(n-3)/2$, $(n+1)/2$, *or* n *if* $n \equiv 1 \pmod 4$, *and is* 1, $(n-1)/2$, *or* n *if* $n \equiv 3 \pmod 4$.

(ii) *If* $d_e(\mathbf{w}, \mathbf{w}') = 1$, *then* $\mathbf{w}$ *and* $\mathbf{w}'$ *disagree in their first coordinate, and this coordinate is odd.*

Proof. The condition $\mathbf{w} \equiv \mathbf{w}' \pmod 2$ puts $\mathbf{w}$ and $\mathbf{w}'$ in opposite complements, say $\mathbf{w} \in \mathscr{C}^+$ and $\mathbf{w}' \in \mathscr{C}^-$. Write

$$\mathbf{w} = \sum_{i \in I} \mathbf{v}_i, \tag{5.13a}$$

where I is a subset of the rows of A_n. Then (5.12) forces

$$\mathbf{w}' = 2\mathbf{e} + \sum_{i \in I} \mathbf{v}_i^*. \tag{5.13b}$$

To prove (i), we use the fact that (5.12) also implies that

$$\mathbf{w} - \mathbf{w}' \equiv \mathbf{w} \oplus \mathbf{w}' \pmod 4. \tag{5.14}$$

Since $\oplus$ is commutative and associative

$$\mathbf{w} - \mathbf{w}' \equiv (2\mathbf{e}) \oplus \left(\bigoplus_{i \in I} (\mathbf{v}_i \oplus \mathbf{v}_i^*) \right) \pmod 4. \tag{5.15}$$

Now, assuming $n \equiv 3 \pmod 4$, (5.2b) gives

$$\bigoplus_{i \in I} (\mathbf{v}_i \oplus \mathbf{v}_i^*) \equiv 0^n \quad \text{or} \quad 0(2)^{n-1} \quad \text{or} \quad 2(0022)^{(n-3)/4}02 \quad \text{or} \quad 2(2200)^{(n-3)/4}20.$$

Hence

$$\mathbf{w} - \mathbf{w}' = 2^n \quad \text{or} \quad 2(0)^{n-1} \quad \text{or} \quad 0(2200)^{(n-3)/4}20 \quad \text{or} \quad 0(0022)^{(n-3)/4}02,$$

so

$$d_e(\mathbf{w}, \mathbf{w}') = 1, \frac{n-1}{2}, \text{ or } n.$$

The result for $n \equiv 1 \pmod 4$ follows similarly from (5.2a), and (i) is proved.

To prove (ii), we observe that (5.15) gives

$$(\mathbf{w} - \mathbf{w}') \oplus 2\mathbf{e} = \bigoplus_{i \in I} (\mathbf{v}_i \oplus \mathbf{v}_i^*),$$

and the right-hand side takes only four possible values, by (5.2). The only one of these allowing $d_e(\mathbf{w}, \mathbf{w}') = 1$ is

$$\bigoplus_{i \in I} (\mathbf{v}_i \oplus \mathbf{v}_i^*) = 0(2)^{n-1}. \tag{5.16}$$

Thus $\mathbf{w}$ and $\mathbf{w}'$ must differ on their first coordinate.

Now divide the set $\{2, 3, \ldots, n\}$ into two subsets P and Q according to whether

$$\mathbf{v}_i \oplus \mathbf{v}_i^* = 22200\ldots \qquad \text{for} \quad i \in P,$$
$$\mathbf{v}_i \oplus \mathbf{v}_i^* = 20022\ldots \qquad \text{for} \quad i \in Q.$$

(Then P is all $i \equiv 0$ or $1 \pmod 4$ and Q is all $i \equiv 2$ or $3 \pmod 4$, except that n goes in the opposite subset from what its congruence class (mod 4) indicates.)

Suppose that $\mathbf{w} = \sum_{i \in I} \mathbf{v}_i$ satisfies (5.16). There are two possibilities.

Case (*a*). $1 \in I$ and $|I \cap P| \equiv |I \cap Q| \equiv 0 \pmod 2$.

Case (*b*). $1 \notin I$ and $|I \cap P| \equiv |I \cap Q| \equiv 1 \pmod 2$.

These cases correspond to the first coordinate of $\mathbf{w}$ being odd or even, respectively. We will establish (ii) by showing that if $\mathbf{w}$ falls in Case (b), then $\mathbf{w}$ contains an odd number of 3's, contradicting $\mathbf{w} \in \mathscr{C}^+$.

Consider the square submatrix $\mathsf{B}_I = [b_{ij}: i, j \in I]$ of A_n consisting of the rows and columns of I. Then $\mathbf{w}$ has an odd number of 3's if and only if B_I contains an odd number of 2's, i.e., if and only if

$$\pi_I := \sum_{\substack{i, j \in I \\ i \neq j}} b_{ij} \equiv 2 \pmod 4. \tag{5.17}$$

Now (5.2) yields, for $2 \le i < j \le n$, that

$$b_{ij} + b_{ji} \equiv \begin{cases} 0 \pmod 4 & \text{if} \quad i, j \in P \quad \text{or} \quad i, j \in Q, \\ 2 \pmod 4 & \text{if} \quad i \in P, \quad j \in Q, \quad \text{or} \quad i \in Q, \quad j \in P. \end{cases} \tag{5.18}$$

In Case (b), $1 \notin I$, and

$$\pi_I = \sum_{\substack{i, j \in I \\ 2 \le i < j}} (b_{ij} + b_{ji}) \equiv 2 \pmod 4$$

using (5.17), because

$$|\{(i, j): i < j \text{ and } i \in P, j \in Q \text{ or } i \in Q, j \in P\}| = |\{(i, j): i \in P \text{ and } j \in Q\}|$$

has odd cardinality, since $|I \cap P| = |I \cap Q| \equiv 1 \pmod 2$. □

6. Cube-Tilings Without High-Dimensional Common Faces

We now construct cube-tilings of $\mathbb{R}^n$ having no two cubes with a common face of dimension exceeding $n - \frac{1}{3}\sqrt{n}$. The construction uses the block-substitution method of Lagarias and Shor [6].

Theorem 6.1. *For each integer $k \geq 1$, and $n = 8k^2 + 24k + 10$, a tiling of $\mathbb{R}^n$ by unit cubes exists such that:*

(1) *The centers of all cubes are in $\frac{1}{4}\mathbb{Z}^n$.*
(2) *The tiling is periodic with period lattice $2\mathbb{Z}^n$.*
(3) *No two cubes have a complete d-dimensional face in common, for $d > n - (2k + 1)$.*

Proof of Theorem 6.1. Let $k \geq 0$ be a fixed integer. We start with a Construction A cube-tiling $\mathscr{C}_{\mathrm{A}}^m$ with $m = 2k + 5$. Then $\mathscr{C}_{\mathrm{A}}^m$ is 2-extremal, and has complements $\mathscr{C}_m^+$, $\mathscr{C}_m^-$, with $|\mathscr{C}_m^+| = |\mathscr{C}_m^-| = 2^{m-1}$. Now form a new set $\hat{\mathscr{C}}_{\mathrm{A}}^m := \hat{\mathscr{C}}_m^+ \cup \hat{\mathscr{C}}_m^-$, where $\hat{\mathscr{C}}_m^+ = \mathscr{C}_m^+$ and $\hat{\mathscr{C}}_m^-$ consists of the vectors in $\mathscr{C}_m^-$ with each 0 coordinate replaced by a new symbol $0'$.

The desired cube-tiling is a collection of vectors $\mathscr{I}_k \subseteq \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}\}^n$, where $n = (2k + 5)(4k + 2) = 8k^2 + 24k + 10$. $\mathscr{I}_k$ consists of all possible substitutions for each of the symbols $0, 0', 1, 2, 3$ in all vectors in $\hat{\mathscr{C}}_{\mathrm{A}}^m$ with vectors from certain corresponding sets S_0, S_0', S_1, S_2, S_3 contained in $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}\}^{4k+2}$. These sets are produced from the Construction B cube-tiling code $\mathscr{C}_{\mathrm{B}}^{4k+3}$, as follows. Partition the vectors in $\mathscr{C}_{\mathrm{B}}^{4k+3}$ according to the value of their first coordinate

$$\mathscr{C}_{\mathrm{B}}^{4k+3} = \mathscr{C}_0 \cup \mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{C}_3,$$

and write, symbolically,

$$\mathscr{C}_0 = 0X, \qquad \mathscr{C}_1 = 1Y, \qquad \mathscr{C}_2 = 2Z, \qquad \mathscr{C}_3 = 3W, \tag{6.1}$$

where X, Y, Z, W are sets of $(4k + 2)$-vectors. Lemma 3.1 gives the information that

$$|X| = |Z|, \qquad |Y| = |W|, \tag{6.2}$$

see (3.6). Then take

$$\begin{aligned} S_0 &= X, & S_1 &= X + \tfrac{1}{2}\mathbf{e} \pmod 4, \\ S_0' &= Z, & S_1' &= Z + \tfrac{1}{2}\mathbf{e} \pmod 4, \\ S_2 &= Y, & S_3 &= Y + \tfrac{1}{2}\mathbf{e} \pmod 4, \end{aligned} \tag{6.3}$$

where $\mathbf{e} = (1, 1, \ldots, 1)$. Note that $S_0 \cup S_0' \cup S_2$ is disjoint from $S_1 \cup S_1' \cup S_3$ because all vectors in the first set have all integer coordinates, while those in the second set have all half-integer coordinates.

To prove the theorem it suffices to establish the following three facts:

(a) $\mathscr{I}_k$ consists of 2^n distinct vectors.
(b) $\frac{1}{2}\mathscr{I}_k + 2\mathbb{Z}^n$ is a tiling of $\mathbb{R}^n$ by unit cubes, with centers in $\frac{1}{4}\mathbb{Z}^n$.
(c) For distinct $\mathbf{z}$, $\mathbf{z}'$ in $\mathscr{I}_k$ with $\mathbf{z} \equiv \mathbf{z}' \pmod 2$, we have

$$d_e(\mathbf{z}, \mathbf{z}') \geq 2k + 1.$$

Fact (c) asserts that if two cubes from $\frac{1}{2}\mathscr{I}_k + 2\mathbb{Z}^n$ have a common face, then it has dimension at most $n - (2k + 1)$.

To prove (a) we begin by showing that X, Y, and Z are pairwise disjoint. Indeed, $X \cap Y = Z \cap Y = \varnothing$ because $\mathscr{C}_{\mathrm{B}}^{4k+3}$ is a cube-tiling (hence these elements differ by 2 in some coordinate), while $X \cap Z = \varnothing$ follows from Theorem 5.2(ii), because any common vector $\mathbf{x}$ would produce $0\mathbf{x}$ and $2\mathbf{x}$ in $\mathscr{C}_{\mathrm{B}}^{4k+3}$.

In consequence, S_0, S_0', S_1, S_1', S_2, S_3 are all pairwise disjoint, hence all vectors produced by the block-substitution construction are distinct. To count these, observe that

$$|S_0| = |S_0'| = |S_1| = |S_1'| = |X|,$$
$$|S_2| = |S_3| = |Y|.$$

Now an element $\mathbf{w} = (w_1, \ldots, w_m) \in \mathscr{C}_{\mathrm{A}}^m$ yields $|X|^{n_0}|Y|^{n_1}$ vectors in $\mathscr{I}_k$, where n_0 and n_1 are determined using the mapping $\varphi\colon \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ with $\varphi(0) = \varphi(1) = 0$, $\varphi(2) = \varphi(3) = 1$, namely, $n_0 = \#\{w_i\colon \varphi(w_i) = 0\}$ and $n_1 = \#\{w_i\colon \varphi(w_i) = 1\}$. Applying Lemma 3.1 to $\mathscr{C}_{\mathrm{A}}^m$, the total number of such elements in

$$|\mathscr{I}_k| = \sum_{j=0}^{m} \binom{m}{j} |X|^j |Y|^{m-j} = (|X| + |Y|)^m.$$

However Lemma 3.1 also implies for $\mathscr{C}_{\mathrm{B}}^{4k+3}$ that

$$|X| + |Y| = 2^{4k+2},$$

by (3.5). Thus $|\mathscr{I}_k| = 2^{(2k+5)(4k+2)}$, proving (a).

To prove (b), observe that all centers of $\frac{1}{2}\mathscr{I}_k + 2\mathbb{Z}^n$ lie in $\frac{1}{4}\mathbb{Z}^n$. To prove that it is a cube-tiling, it suffices to show no two cubes overlap.[1] which is equivalent to showing that any distinct vectors $\mathbf{z}, \mathbf{z}' \in \mathscr{I}_k$ differ by 2 (mod 4) in some coordinate position. Suppose first that $\mathbf{z}$, $\mathbf{z}'$ were produced from the same vector $\mathbf{w}$ in $\mathscr{C}_{\mathrm{A}}^m$. Then they differ in some block coming from the ith coordinate position w_i of $\mathbf{w}$, say. Thus they have different blocks $\mathbf{z}_i, \mathbf{z}_i' \in S_{w_i}$. However, any two vectors in S_{w_i} differ by 2 (mod 4) in some coordinate, using the fact that $\mathscr{C}_{\mathrm{B}}^{4k+3}$ is a cube-tiling code, so all vectors in it have e-distance at least 1. Now suppose $\mathbf{z}$, $\mathbf{z}'$ were produced from different vectors $\mathbf{w}$, $\mathbf{w}'$ in $\mathscr{C}_{\mathrm{A}}^m$. Since $\mathscr{C}_{\mathrm{A}}^m$ is a cube-tiling code, $\mathbf{w}$, $\mathbf{w}'$ differ by

[1] If no cubes overlap, a volume-argument shows that the density of space covered by cubes is 1. If there were any uncovered space, it would be $2\mathbb{Z}^n$-periodic, hence have positive density.

2 in some position, say $w_i \equiv w_i' + 2 \pmod 4$. Then all blocks $\mathbf{z}_i$ in S_{w_i} differ from all blocks $\mathbf{z}_i'$ in $S_{w_i'}$ by 2 (mod 4) in some coordinate, again from the cube-tiling code property of $\mathscr{C}_{\mathrm{B}}^{4k+3}$. This proves (b).

To prove (c) we treat three cases. The first case is that $\mathbf{z}, \mathbf{z}' \in \mathscr{I}_k$ with $\mathbf{z} \equiv \mathbf{z}'$ (mod 2) were produced from the same vector $\mathbf{w} \in \mathscr{C}_{\mathrm{A}}^m$. Then there is some coordinate w_i, where $\mathbf{z}$ and $\mathbf{z}'$ have distinct blocks $\mathbf{z}_i, \mathbf{z}_i' \in S_{w_i}$. Since $\mathbf{z}_i \equiv \mathbf{z}_i' \pmod 2$, Theorem 5.2 implies that

$$d_e(\mathbf{z}, \mathbf{z}') \geq d_e(\mathbf{z}_i, \mathbf{z}_i') \geq 2k + 1.$$

Next suppose that $\mathbf{z}$, $\mathbf{z}'$ arise from distinct vectors $\mathbf{w}, \mathbf{w}' \in \mathscr{C}_{\mathrm{A}}^m$. The condition $\mathbf{z} \equiv \mathbf{z}' \pmod 2$ requires $\mathbf{w} \equiv \mathbf{w}' \pmod 2$, by (6.3). The second case is when $d_e(\mathbf{w}, \mathbf{w}') \geq 2k + 1$. Then $d_e(\mathbf{z}, \mathbf{z}') \geq 2k + 1$, because the sets S_i are pairwise disjoint. The third case is when $d_e(\mathbf{w}, \mathbf{w}') \leq 2k$. Then $\mathbf{w}$ and $\mathbf{w}'$ must agree in at least five coordinates, hence Theorem 4.2 guarantees that $\mathbf{w}$ and $\mathbf{w}'$ have at least two matching coordinates that take the values 0 or 1. Now $\mathbf{w} \equiv \mathbf{w}' \pmod 2$ implies they are in opposite complements of $\mathscr{C}_{\mathrm{A}}^m$, hence, for matching 0 coordinates, one of these has the value 0 and the other has the value 0′, and for matching 1 coordinates one has the value 1 and the other 1′. However, using Theorem 5.2(ii), any $\mathbf{z}_i \in S_0$ and $\mathbf{z}_i' \in S_0'$ have

$$d_e(\mathbf{z}_i, \mathbf{z}_i') \geq 2k,$$

because they come from vectors $0\mathbf{z}_i \in \mathscr{C}_0$ and $2\mathbf{z}_i' \in \mathscr{C}_2$ in $\mathscr{C}_{\mathrm{B}}^{4k+3}$ with $\mathbf{z}_i \equiv \mathbf{z}_i' \pmod 2$, and similarly for any $\mathbf{z}_i \in S_1$ and $\mathbf{z}_i' \in S_1'$. Hence

$$d_e(\mathbf{z}, \mathbf{z}') \geq 4k \geq 2k + 1, \tag{6.4}$$

proving (c). □

Theorem 6.1 used $2\mathbb{Z}^n$-periodic cube-tilings with centers in $\frac{1}{4}\mathbb{Z}^n$, rather than in $\frac{1}{2}\mathbb{Z}^n$, in order to construct conveniently sets of blocks S_1, S_1', S_3 disjoint from $S_0 \cup S_0' \cup S_2$. It should be possible to construct similar tilings with all cube-centers in $\frac{1}{2}\mathbb{Z}^n$, by instead finding S_1, S_1', S_3 in $\{0, 1, 2, 3\}^n$ disjoint from $S_0 \cup S_0' \cup S_2$, using the large group of automorphisms of $(\mathbb{Z}/4\mathbb{Z})^n$, as was done in the constructions in [6].

Corollary 6.1. *For all n a tiling of $\mathbb{R}^n$ by unit cubes exists such that no two cubes have a common face of dimension exceeding $n - \frac{1}{3}\sqrt{n}$.*

Proof. Suppose that $m_k \leq n < m_{k+1}$, where $m_k = 8k^2 + 24k + 10$. Theorem 6.1 together with $K_{n+1} \leq K_n + 1$ gives

$$K_n \leq (n - m_k) + K_{m_k} \leq n - (2k + 1). \tag{6.5}$$

The example in Appendix A gives $K_n \le n - 3$ for $n \ge 10$. Next we have $K_n \le n - 4$ for $n \ge 50$, using the case $k = 2$ of a construction $n = (2k + 1)(4k + 2)$ which has no $(n - 2k)$-dimensional face, proved exactly as in Theorem 6.1, except that the bound (6.4) is weakened to $d_e(\mathbf{z}, \mathbf{z}') \ge 2k$. This covers all $n \le 90$.

Finally, the corollary holds for all $n \ge 90$ using (6.5) and the fact that

$$\tfrac{1}{3}\sqrt{m_{k+1}} < 2k + 1$$

holds for all $k \ge 2$. □

Theorem 6.1 gives, for any $\varepsilon > 0$, that

$$K_n \le n - \frac{1}{(\sqrt{2} + \varepsilon)}\sqrt{n}$$

for all sufficiently large n.

7. Upper Bounds for the Cube-Tiling Constant K_n

To what extent can the block-substitution construction of section 6 be improved? One possibility is to find a better block-substitution construction.

A *special tiling* is a 2-extremal cube-tiling $\mathcal{I} = 0X \cup 1Y \cup 2Z \cup 3W$ of $\mathbb{R}^n$ such that:

(i) X, Y, Z, and W each consist of distinct elements (mod 2).
(ii) $X \cap Z = \varnothing$.

It is unknown whether or not any special tilings exist. The relevance of special tilings to K_n is that their existence would give a nontrivial linear upper bound for K_n.

Theorem 7.1. *If a special tiling in $\mathbb{R}^d$ exists, then*

$$K_n \le \left(1 - \frac{1}{3d}\right)n$$

for all $n \ge 12d^3$.

Proof. Any special tiling $\mathcal{I}$ in $\mathbb{R}^d$ must have

$$|X| = |Y| = |Z| = |W| = 2^{d-2}. \tag{7.1}$$

This holds since $|X| + |Y| + |Z| + |W| = 2^d$, and if any sets, say X, had $|X| > 2^{d-2}$, then by (i) it would contain two complementary $(d - 1)$-vectors $\mathbf{x}$

and $\bar{\mathbf{x}} \equiv \mathbf{x} + \mathbf{e}$ (mod 2), whence $0\mathbf{x}$ and $0\bar{\mathbf{x}}$ would not differ by 2 in any coordinate, contradicting $\mathscr{S}$ being a cube-tiling.

Now we imitate the block-substitution construction of Theorem 6.1, but use the special tiling $\mathscr{S}$ to construct the blocks.

Choose a Construction A tiling $\mathscr{C}_{\mathrm{A}}^m$ with $m = 4k + 1$, and replace all values 0 and 1 in the complement $\mathscr{C}^-$ by $0'$ and $1'$, respectively. Then make the block-substitution

$$\begin{aligned} S_0 &= X, & S_1 &= X + \tfrac{1}{2}\mathbf{e} \pmod 4, \\ S'_0 &= Z, & S'_1 &= Z + \tfrac{1}{2}\mathbf{e} \pmod 4, \\ S_2 &= Y, & S_3 &= Y + \tfrac{1}{2}\mathbf{e} \pmod 4. \end{aligned}$$

The set $\mathscr{S}_k^*$ resulting from this block-substitution lies in $\mathbb{R}^n$, where $n = (4k + 1)(d - 1)$. It has cardinality 2^n, as can be proved using (7.1). Also $\frac{1}{2}\mathscr{S}_k^* + 2\mathbb{Z}^n$ is a tiling of $\mathbb{R}^n$ by unit cubes, having all cube-centers in $\frac{1}{4}\mathbb{Z}^n$, by a similar proof to that of Theorem 6.1.

We show that if $\mathbf{z} \equiv \mathbf{z}'$ (mod 2) in $\mathscr{S}_k^*$, then

$$d_e(\mathbf{z}, \mathbf{z}') \geq \tfrac{4}{3}k. \tag{7.2}$$

The key role of a special tiling is that property (i) guarantees that any two distinct $\mathbf{z}, \mathbf{z}' \in \mathscr{S}_k^*$ arising from the same vector $\mathbf{w} \in \mathscr{C}_{\mathrm{A}}^m$ have $\mathbf{z} \not\equiv \mathbf{z}'$ (mod 2). Thus any $\mathbf{z}, \mathbf{z}' \in \mathscr{S}_k^*$ with $\mathbf{z} \equiv \mathbf{z}'$ (mod 2) arise from $\mathbf{w}, \mathbf{w}' \in \mathscr{C}_{\mathrm{A}}^m$ with $\mathbf{w} \equiv \mathbf{w}'$ (mod 2) and $\mathbf{w} \neq \mathbf{w}'$. There are two cases. First, if

$$d_e(\mathbf{w}, \mathbf{w}') \geq \tfrac{4}{3}k,$$

then (7.2) is inherited, because the e-distance between elements in different S_i, S_j with $i \not\equiv j$ (mod 2) is at least 1. Second, if

$$d_e(\mathbf{w}, \mathbf{w}') < \tfrac{4}{3}k,$$

then $\mathbf{w}, \mathbf{w}'$ agree on at least $\frac{8}{3}k + 1$ coordinates, so by Theorem 4.2 they have at least $\frac{4}{3}k$ matching coordinates that are 0 or 1. Since $\mathbf{w}, \mathbf{w}'$ are in opposite complements of $\mathscr{C}_{\mathrm{A}}^m$, all their 0 and 1 coordinates are labeled 0 and $0'$, and 1 and $1'$, respectively. Again (7.2) holds since every element in S_0 is at e-distance at least 1 from every element of S'_0, and similarly for S_1 and S'_1.

Thus, for $n_k = (4k + 1)(d - 1)$, we have

$$K_{n_k} \leq n_k - \tfrac{4}{3}k,$$

and this implies that

$$K_{n_k} \leq n_k - \frac{n_k}{3d} - 4d$$

holds for all $k \geq 3d^2 + d$. Since these n_k are spaced at intervals of $4(d-1)$, we get

$$K_n \leq \left(1 - \frac{1}{3d}\right)n$$

valid for all $n \geq 12d^3 \geq n_{3d^2+d}$. □

We doubt the existence of special tilings. In this regard, we formulate:

Rigidity Conjecture for 2-Extremal Cube-Tilings. *In any 2-extremal cube-tiling $\mathscr{C}$ of $\mathbb{R}^n$, knowledge of one vector of $\mathscr{C}$ in each* (mod 2) *equivalence class determines $\mathscr{C}$ uniquely.*

The rigidity conjecture implies that no special tilings exist. Given a special tiling $\mathscr{I}$, the set $\mathscr{I}' = 0X \cup 1Y \cup 2X \cup 3Y$ is easily checked to be a 2-extremal cube-tiling also. By the Rigidity Conjecture, $0X \cup 1Y$ uniquely specifies $\mathscr{I}$, hence $\mathscr{I} = \mathscr{I}'$, so $X = Z$, a contradiction.

Appendix A. A 10-Dimensional Cube-Tiling With No Common Faces of Dimension Exceeding 7

We use a construction similar to one in [6]. Consider the sets of vectors S_0, S_0', S_1, S_1', S_2, S_3 in $\{0, 1, 2, 3\}^4$, given in Table A.1. The sets S_0, S_0', S_2 are the same as in [6], but the sets S_1, S_1', S_3 are derived from them by adding the vector $(1, 1, 1, 1)$ to S_0, S_0', and S_2, respectively, instead of by adding $(1, 0, 0, 0)$, as in the earlier construction.

Table A.1. Blocks used in construction.

S_0	S_0'	S_2	S_1	S_1'	S_3
0000	0303	0211	1111	1010	1322
0012	1011	1132	1123	2122	2203
0213	1113	2303	1320	2220	3010
0230	1130	3020	1301	2201	0131
0332	1323		1003	2030	
1020	1331		2131	2002	
2100	2211		3211	3322	
2112	3001		3223	0112	
2220	3022		3331	0133	
2301	3103		3012	0210	
2322	3223		3033	0330	
3132	3231		0203	0302	

The construction uses partial block-substitution into the set $\mathscr{S} = S_0 \cup S_2^*$, where S_2^* is obtained from S_2 by replacing 0 and 1 in the middle two coordinates by $0'$ and $1'$, i.e.,

$$S_2^* = \begin{cases} 0 & 2 & 1' & 1, \\ 1 & 1' & 3 & 2, \\ 2 & 3 & 0' & 3, \\ 3 & 0' & 2 & 0. \end{cases}$$

Make the block substitutions $0 \to S_0, 0' \to S_0', 1 \to S_1, 1' \to S_1', 2 \to S_2$, and $3 \to S_3$, for all elements in just the second and third coordinates of $\mathscr{S}$. This gives a cube-tiling code $\mathscr{I} \subset \{0, 1, 2, 3\}^{10}$. The proof that $\mathscr{I}$ is a cube-tiling code is the same as in [6].

It remains to show that if $\mathbf{z}, \mathbf{z}' \in \mathscr{I}$ satisfy

$$\mathbf{z} \equiv \mathbf{z}' \pmod{2},$$

then

$$d_e(\mathbf{z}, \mathbf{z}') \geq 3.$$

The sets $(S_0 \cup S_2) \pmod 2$ and $(S_1 \cup S_3) \pmod 2$ are easily seen to be disjoint, so if $\mathbf{z}, \mathbf{z}' \in \mathscr{I}$ satisfy $\mathbf{z} \equiv \mathbf{z}' \pmod 2$, then the vectors $\mathbf{w}, \mathbf{w}' \in \mathscr{S}$ that they derive from must also satisfy

$$\mathbf{w} \equiv \mathbf{w}' \pmod{2}, \tag{A.1}$$

where we consider $0 \equiv 0' \pmod 2$ and $1 \equiv 1' \pmod 2$. Since $S_0, S_0', S_1, S_1', S_2, S_3$ are pairwise disjoint, if

$$d_e(\mathbf{z}, \mathbf{z}') \leq 2,$$

then

$$d_e(\mathbf{w}, \mathbf{w}') \leq 2, \tag{A.2}$$

where 0 and 1 are regarded as at e-distance 1 from $0'$ and $1'$, respectively. This gives two conditions on the pair $(\mathbf{w}, \mathbf{w}')$, namely, (A.1) and (A.2), that must be satisfied in order to give rise to a bad pair $(\mathbf{z}, \mathbf{z}')$. It is easy to check that these conditions leave only four possible pairs for $(\mathbf{w}, \mathbf{w}')$, namely, $(0213, 021'1)$, $(1020, 30'20)$, $(2301, 230'3)$, and $(3132, 11'32)$. It now suffices to verify that there is no pair of vectors $\mathbf{x} \in S_0$, $\mathbf{x}' \in S_0'$ (or $\mathbf{x} \in S_1$, $\mathbf{x}' \in S_1'$) such that $\mathbf{x} \equiv \mathbf{x}' \pmod 2$ and $d_e(\mathbf{x}, \mathbf{x}') \leq 1$. This is easily checked for $\mathbf{x} \in S_0$, $\mathbf{x}' \in S_0'$, and then follows for $\mathbf{x} \in S_1$, $\mathbf{x}' \in S_1'$, because $S_1 = S_0 + \mathbf{e}$ and $S_1' = S_0' + \mathbf{e}$.

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