

Second Derivatives of Circle Packings and Conformal Mappings

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Abstract. William Thurston conjectured that the Riemann mapping function f from a simply connected region Ω onto the unit disk \mathbb{D} can be approximated as follows. Almost fill Ω with circles of radius ε packed in the regular hexagonal pattern. There is a combinatorially isomorphic packing of circles in \mathbb{D} . The correspondence f_ε of ε -circles in Ω with circles of varying radii in \mathbb{D} should converge to f after suitable normalization. This was proved in [RS], and in [H] an estimate was obtained which led to an approximation of $|f'|$ in terms of f_ε ; namely, $|f'|$ is the limit of the ratio of the radii of a target circle of f_ε to its source circle. In the present paper we show how to approximate f' and f'' in terms of f_ε . Explicit rates for the convergence to f , f' , and f'' are obtained. In the special case of convergence to $|f'|$, the estimate in this paper improves the previously known estimate.

Introduction

In [T] Thurston conjectured that the Riemann mapping function from a region Ω onto the unit disk \mathbb{D} can be approximated as follows. Almost fill Ω with circles of radius ε packed in the regular hexagonal pattern. There is a combinatorially isomorphic packing of circles in \mathbb{D} . This correspondence f_ε of ε -circles in Ω with circles of varying radii in \mathbb{D} should converge, with suitable normalizations, to the Riemann map $f: \Omega \rightarrow \mathbb{D}$.

This conjecture was proved in [RS]. There the question was raised of whether the ratio of the radius of a target circle to the radius of its source circle under f_ε converges to $|f'|$. In [H] an estimate was obtained which gave an affirmative answer to this question. The estimate also provided, in conjunction with [R2], an explicit estimate for the rate of convergence $f_\varepsilon \rightarrow f$.

In this paper we approach the problem of approximating f' and f'' in terms of f_ε as follows. A triple of mutually tangent ε -circles in the packing of Ω corresponds under f_ε to a triple of mutually tangent circles in the packing of \mathbb{D} . There is a unique Möbius transformation which maps the first triple onto the second. The first and second derivatives of f_ε in the neighborhood of the three source circles are defined to be the derivatives of this Möbius transformation.

With this definition we show that $f'_\varepsilon \rightarrow f'$ and $f''_\varepsilon \rightarrow f''$ (this method does not work for higher-order derivatives since the Schwarzian derivative of f need not vanish). The method gives explicit estimates for the rates of convergence of $f_\varepsilon \rightarrow f$, $f'_\varepsilon \rightarrow f'$; see Theorem 2.4. The rate which is obtained for $f_\varepsilon \rightarrow f$ is an improvement over the previously known estimate.

In this paper we treat only circle packings having the hexagonal pattern. Similar results but with weaker estimates can be obtained for other classes of circle packings by combining the techniques of this paper with those in [HR].

1. Local Theory

In this section we present results from [H] and derive consequences which are needed later. We consider a base circle and let H_n denote the regular hexagonal circle packing of n generations around c_0 ; by “regular” we mean that all circles have the same radius. Let H'_n denote a circle packing which has the same tangency combinatorics as H_n ; the circles of H'_n need not have equal radii. Let $c \rightarrow c'$ denote the correspondence of circles under the combinatorial isomorphism $H_n \rightarrow H'_n$.

A circle \tilde{c} (resp. \tilde{c}') which passes through the three points of tangency of three mutually tangent circles of H_n (resp. H'_n) is called a *dual circle* of H_n (resp. H'_n), and the disk \tilde{D} (resp. \tilde{D}') which it bounds is called a *dual disk* of H_n (resp. H'_n); see Fig. 1.

Notation. Let c_i, c_j, c_k be three mutually tangent circles of H_n , and let \tilde{D} be the dual disk determined by the three tangency points $c_i \cap c_j, c_j \cap c_k, c_k \cap c_i$. The Möbius transformation which sends $c_i \cap c_j, c_j \cap c_k, c_k \cap c_i$ to the corresponding tangency points $c'_i \cap c'_j, c'_j \cap c'_k, c'_k \cap c'_i$ in H'_n is denoted by $M_{\tilde{D}}$.

Throughout this paper C, C_1, C_2, \dots denote absolute positive constants which may change with the context.

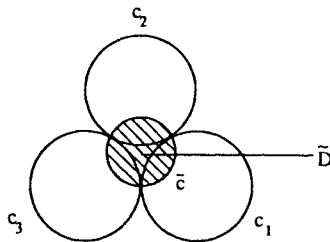


Fig. 1. c_1, c_2, c_3 are mutually tangent circles of H_m and \tilde{c} is the circle dual to c_1, c_2, c_3 . The shaded disk \tilde{D} is a dual disk of H_m .

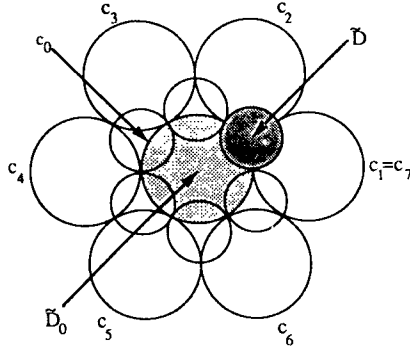


Fig. 2.

Lemma 1.1. *Given H_n and H'_n , where $n \geq 2$, there is a C -quasi-conformal homeomorphism $g: \mathbb{C} \rightarrow \mathbb{C}$ with the following properties:*

- (i) *g maps c_0 and its six neighbors c_i ($1 \leq i \leq 6$) onto the corresponding circles c'_i ($0 \leq i \leq 6$) of H'_n , and maps the center of c_0 to the center of c'_0 .*
- (ii) *Let \tilde{D} be a dual disk of H_n which intersects c_0 . Then $g|_{\tilde{D}}$ is equal to the restriction to \tilde{D} of the Möbius transformation $M_{\tilde{D}}$ (see Fig. 2).*
- (iii) *Let D_0 be the disk bounded by c_0 . Then the $g|_{D_0}$ is conformal except on a "bad" subset of area $\leq (C/n^2) \text{Area}(D_0)$.*

Proof. A detailed proof is given in [H]. Briefly, $g: \mathbb{C} \rightarrow \mathbb{C}$ can be constructed as follows. First define g on each triangular interstice of H_n to be the unique Möbius transformation which maps that interstice onto the corresponding interstice of H'_n . This defines g on each circle of H_n . The Ring Lemma of [RS] shows that the radial extension to the inside of each circle will be quasi-conformal. Redefine g outside the circles and interstices of $H_{[n/2]}$ to obtain a C -quasi-conformal homeomorphism of the entire plane. Finally, modify this map on the inside of the circles so that it becomes equivariant with respect to inversions in the circles of H_n . Since inversions of interstices pave the dual circles, conditions (ii) will be satisfied. The proof of condition (iii) requires delicate estimates of the area of the images of dual circles under inversions. \square

Lemma 1.2. *Given H_n and H'_n , where $n \geq 2$, let P_{n-1} (resp. P'_{n-1}) be the polygon whose boundary consists of line segments joining the centers of pairs of tangent circles of generation $n - 1$ of H_n (resp. H'_n); see Fig. 3. Then there is a C -quasi-conformal homeomorphism $g_n: P_{n-1} \rightarrow P'_{n-1}$ with the following properties:*

- (i) *g_n maps circles or arcs of circles of H_{n-1} onto the corresponding circles or arcs of circles of H'_{n-1} .*
- (ii) *For each dual disk \tilde{D} of H_{n-1} , $g_n|_{\tilde{D}} = M_{\tilde{D}}|_{\tilde{D}}$, where $M_{\tilde{D}}$ is the Möbius transformation defined above.*
- (iii) *For any disk D in $H_k \setminus H_{k-1}$, where $0 \leq k \leq n - 2$ (convention: $H_{-1} = \emptyset$),*

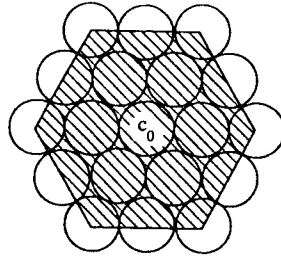


Fig. 3. P_2 = the shaded polygon.

the area of the subset of D where g_n fails to be conformal is bounded from above by

$$\frac{C}{(n - k)^2} \text{Area}(D).$$

Proof. First define g_n in each dual disk \tilde{D} of H_n by $g_n|_{\tilde{D}} = M_{\tilde{D}}|_{\tilde{D}}$. Conditions (i) and (ii) will then be satisfied.

Consider a circle $c \in H_k \setminus H_{k-1}$ ($0 \leq k \leq n - 2$) and let D be the disk it bounds. By Lemma 1.1 there is a C -quasi-conformal mapping g from D to the corresponding disk D' in H'_n such that:

- (a) on each dual disk \tilde{D} of H_n , $g|_{D \cap \tilde{D}} = g_n|_{D \cap \tilde{D}}$, and
- (b) the area of the subset of D on which g fails to be conformal is less than $(C/(n - k)^2) \text{Area}(D)$.

By defining $g_n|_D = g$ we obtain a C -quasi-conformal map g_n defined on the union of disks and dual disks of H_{n-1} which satisfies properties (i)–(iii). The complement in P_{n-1} of this union consists of regions such as those shown in Fig. 4.

It remains to extend g_n uniformly quasi-conformally to these regions so the images are the corresponding regions in P'_{n-1} . All the circles c, c_1, c_2, \dots in Fig. 4 are in H_{n-1} . By the Ring Lemma of [RS], the radii of the corresponding circles in H'_n have uniformly bounded ratios. Thus it is elementary to extend g_n suitably. □

Lemma 1.3. *The map $g_n: P_{n-1} \rightarrow P'_{n-1}$ of Lemma 1.2 is conformal except for a subset whose area is bounded from above by $(C/n) \text{Area}(P_{n-1})$.*

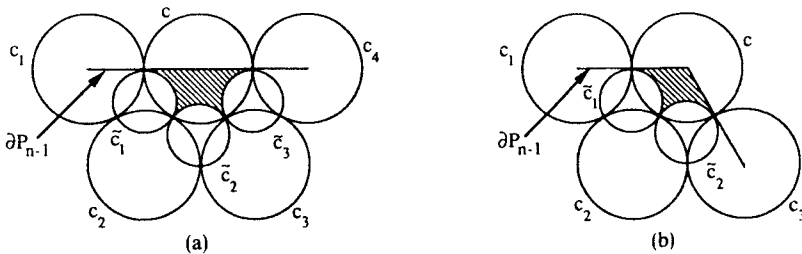


Fig. 4. (a) $c_1, c, c_4 \in H_{n-1} \setminus H_{n-2}$, $c_2, c_3 \in H_{n-2}$, and $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ are dual circles. (b) $c_1, c_3 \in H_{n-1} \setminus H_{n-2}$, $c_2 \in H_{n-2}$, and \tilde{c}_1, \tilde{c}_2 are dual circles.

Proof. The bad subset of g_n is contained in the union of disks bounded by the circles of H_{n-1} . Let r be the radius of the circles of H_n . The number of circles in $H_k \setminus H_{k-1}$ is $6k$ (or 1 if $k = 0$). For $0 \leq k \leq n-2$, Lemma 1.2 implies that the bad area inside the circles of $H_k \setminus H_{k-1}$ is bounded above by $(6kC/(n-k)^2)\pi r^2$ (or $(C/n^2)\pi r^2$ if $k = 0$). This bound also applies for $k = n-1$ if we let $C \geq 1$. Thus the total bad area is bounded by

$$\begin{aligned} \frac{C\pi r^2}{n^2} + \sum_{k=1}^{n-1} \frac{6k}{(n-k)^2} C\pi r^2 &= \frac{C\pi r^2}{n^2} + \sum_{j=1}^{n-1} \frac{6(n-j)}{j^2} C\pi r^2 \\ &\leq \frac{C\pi r^2}{n^2} + 6Cn\pi r^2 \sum_{j=1}^{n-1} \frac{1}{j^2} \\ &\leq C_1 n r^2 = \frac{C_2}{n} \text{Area}(P_{n-1}). \quad \square \end{aligned}$$

If a quasi-conformal self-map φ of the unit disk satisfies $\varphi(0) = 0$, $\varphi(1) = 1$, and is conformal except for a set of measure ε , then $|\varphi(z) - z| < C\sqrt{\varepsilon}$; this fact is referred to as the Grötsch argument for conformal moduli and was proved in [H] as an essential step in the C/n -estimate for s_n . Together with the proof of Lemma 1.2 it gives

Lemma 1.4. *Let $g_n: P_{n-1} \rightarrow P'_{n-1}$ be as in Lemma 1.2. Then there is a euclidean similitude S (that is, a map of the form $z \rightarrow S(z) = z'_0 + re^{i\theta}(z - z_0)$ where z_0 and z'_0 are the centers of c_0 and c'_0) of the inside of c_0 onto the inside of c'_0 such that, for all z inside c_0 ,*

$$|g_n(z) - S(z)| < \left(\frac{C}{n}\right) \text{Radius}(c'_0). \quad (1.1)$$

The following variation of Lemma 1.4 allows a better approximation for g_n in some circumstances.

Lemma 1.5. *Let $g_n: P_{n-1} \rightarrow P'_{n-1}$ be as in Lemma 1.2. Let D be a disk of H_k ($k \leq n-2$) and let \tilde{D} be a dual disk of H_{n-1} which meets D . Then*

$$g_n(D) = M_{\tilde{D}}(D) = D',$$

where $M_{\tilde{D}}$ is the Möbius transformation which was defined above Lemma 1.1 and D' is the disk of H'_{n-1} which corresponds to D , and

$$|g_n(z) - M_{\tilde{D}}(z)| \leq \frac{C}{(n-k)^2} \text{Radius}(D'), \quad \forall z \in \partial D. \quad (1.2)$$

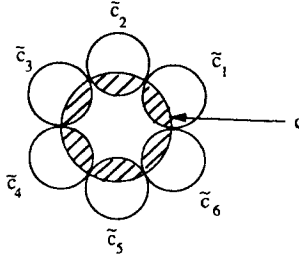


Fig. 5. $\bar{D} \cap \tilde{D}_i$ ($1 \leq i \leq 6$) are the shaded regions.

Proof. The fact that $g_n(D) = M_{\bar{D}}(D) = D'$ follows from Lemma 1.2(i). We may assume that $\text{Radius}(D) = \text{Radius}(D') = 1$. Now $M_{\bar{D}}$ maps \bar{D} to \bar{D}' and carries the three circles of H_{n-1} which determine \bar{D} —say c_i, c_j, c_k —to three circles c'_i, c'_j, c'_k of H'_{n-1} . One of these three circles from H_{n-1} is ∂D . By the Ring Lemma [RS] and the normalization $\text{Radius}(D) = \text{Radius}(D') = 1$, we conclude that the radii of c'_i, c'_j, c'_k are all uniformly bounded from below. Hence $|(d/dz)M_{\bar{D}}|$ is uniformly bounded above in D and therefore $|g_n(z) - M_{\bar{D}}(z)| \leq C_1 |M_{\bar{D}}^{-1} \circ g_n(z) - z|$. Hence it suffices to prove

$$|M_{\bar{D}}^{-1} \circ g_n(z) - z| \leq C(n-k)^{-2}, \quad \forall z \in \partial D. \quad (1.3)$$

Let $\tilde{D}_1 = \bar{D}, \tilde{D}_2, \dots, \tilde{D}_6$, be the six dual disks of H_{n-1} which meet D (see Fig. 5). By Lemma 1.2(ii), the restriction of $h \equiv M_{\bar{D}}^{-1} \circ g_n: \bar{D} \rightarrow \bar{D}$ to each region $\bar{D} \cap \tilde{D}_i$, $1 \leq i \leq 6$, is a Möbius transformation. In particular, the bad subset $B \subset D$ where h (equivalently, g_n) fails to be conformal satisfies

$$B \subset D \setminus \bigcup \{D \cap \tilde{D}_i: 1 \leq i \leq 6\} \quad (1.4a)$$

and

$$\text{Area}(B) \leq C(n-k)^{-2}. \quad (1.4b)$$

Since the restriction of h to $D \cap \tilde{D}_1$ is the identity, (1.3) will follow if we show that

$$|h(z) - z| \leq C(n-k)^{-2}, \quad \forall z \in \partial D \cap \tilde{D}_i \text{ for } i = 2, \dots, 6. \quad (1.5)$$

Since h is a Möbius transformation on $D \cap \tilde{D}_i$, it suffices to prove (1.5) only for three distinct points ζ on the interior of the arc $\partial D \cap \tilde{D}_i$ (for $i = 2, \dots, 6$).

Fix three points w_1, w_2, w_3 on the interior of the arc $\partial D \cap \tilde{D}_1$. They will be left fixed by h . Consider ζ on the interior of the arc $\partial D \cap \tilde{D}_i$ ($i \neq 1$). The quadrilateral $(\bar{D}, w_1, w_2, w_3, \zeta)$ is mapped by h onto the quadrilateral $(\bar{D}, w_1, w_2, w_3, h(\zeta))$. Since the bad set B is bounded away from w_1, w_2, w_3, ζ (see (1.4a)), a Grötsch argument similar to [H] using (1.4b) yields an estimate on the

moduli of these quadrilaterals

$$|\text{mod}(\bar{D}, w_1, w_2, w_3, \zeta) - \text{mod}(\bar{D}, w_1, w_2, w_3, h(\zeta))| \leq C_1(n - k)^{-2}$$

and consequently the desired estimate $|h(\zeta) - \zeta| \leq C_2(n - k)^{-2}$. □

Lemma 1.6. *Let $g_n: P_{n-1} \rightarrow P'_{n-1}$ be as in Lemma 1.2, let $k < n - 1$, let c be a circle in H_k , and let c_0 be the base circle in H_0 . Then*

$$\frac{1}{C^{k/(n-k)}} \leq \frac{\text{Radius}(g_n(c))}{\text{Radius}(g_n(c_0))} \leq C^{k/(n-k)}. \tag{1.6}$$

Proof. The estimate $s_N \leq C_1/N$ in [H] implies that if c' and c'' are tangent circles in H_{n-1} of generation $< k$, then

$$1 - \frac{C_1}{(n - k)} \leq \frac{\text{Radius}(c')}{\text{Radius}(c'')} \leq 1 + \frac{C_1}{(n - k)}. \tag{1.7}$$

Let c_0, c_1, \dots, c be a chain of tangent circles in H_{n-1} of generations $0, 1, \dots$ up to the generation ($\leq k$) of c . Apply (1.7) to adjacent pairs in $g_n(c_0), g_n(c_1), \dots, g_n(c)$ and multiply to get

$$\frac{1}{C^{k/(n-k)}} \leq \left(1 - \frac{C_1}{(n - k)}\right)^k \leq \frac{\text{Radius}(g_n(c))}{\text{Radius}(g_n(c_0))} \leq \left(1 + \frac{C_1}{(n - k)}\right)^k \leq C^{k/(n-k)}. \quad \square$$

It will be convenient to normalize the scale factor so that P_n is a regular hexagon of side length L , where L remains within prescribed bounds $\kappa^{-1} \leq L \leq \kappa$. $O_\kappa(x)$ and $O_{\kappa,t}(x)$ denote functions such that $|O_\kappa(x)/x|$ is bounded by a constant depending only on κ , and $|O_{\kappa,t}(x)/x|$ is bounded by a constant which depends on κ and t . The bounds may change from one occurrence to the next. Constants denoted C_κ (or $C_{\kappa,t}$) depend on κ (or κ and t).

The following corollary follows from Lemmas 1.4–1.6 by taking c_0 and c'_0 to have radii $O_\kappa(1/n)$. Part (iv) is an immediate consequence of part (ii).

Corollary 1.7. *Let κ be a fixed positive constant. Normalize the radii of the circles in H_n and H'_n so that $\kappa/2n \leq \text{Radius}(c_0) \leq (2\kappa n)^{-1}$ and*

$$\kappa/2n \leq \text{Radius}(c'_0) \leq (2\kappa n)^{-1}.$$

For $z \in P_{n-1}$ let M_z denote the Möbius transformation M_B (see the definition preceding Lemma 1.2), where \bar{D} is a dual disk of H_n closest to z . Let $t > 0$ and let

$P(t) = \{z \in P_{n-1} : \text{Dist}(z, \partial P_{n-1}) \geq t\}$. Let $g_n: P_{n-1} \rightarrow P'_{n-1}$ be as in Lemma 1.2:

- (i) For any disk D in H_{n-2} which meets $P(t)$ there is a euclidean similitude S of D onto D' such that

$$g_n(z) - S(z) = O_{\kappa,t}\left(\frac{1}{n^2}\right), \quad \forall z \in \tilde{D}.$$

- (ii) For any disk D in H_{n-2} which meets $P(t)$ and for any $z_0 \in \tilde{D}$ we have

$$g_n(z) - M_{z_0}(z) = O_{\kappa,t}\left(\frac{1}{n^3}\right), \quad \forall z \in \partial D. \quad (1.8)$$

- (ii') For any disk D in H_{n-2} which meets $P(t)$ and for any $z_0 \in \tilde{D}$ we have

$$g_n(z) - M_{z_0}(z) = O_{\kappa,t}\left(\frac{1}{n^2}\right), \quad \forall z \in D. \quad (1.9)$$

- (iii) For any disk D in H_{n-1} which meets $P(t)$ we have

$$\text{Radius}(g_n(D)) = O_{\kappa,t}\left(\frac{1}{n}\right). \quad (1.10)$$

- (iv) For any disk D in H_{n-2} which meets $P(t)$ and any two dual disks \tilde{D}_1, \tilde{D}_2 , which meet D , we have

$$M_{\tilde{D}_1}(z) - M_{\tilde{D}_2}(z) = O_t\left(\frac{1}{n^3}\right), \quad \forall z \in \tilde{D}. \quad (1.11)$$

Lemma 1.8. Let $t > 0$, and normalize the packings H_n and H'_n as in Corollary 1.7. There is a $\delta = \delta(\kappa, t) > 0$ such that, for any $z_0 \in P(t)$,

$$|g_n(z) - M_{z_0}(z)| \leq C_{\kappa,t}|z - z_0|^3 + \frac{C_{\kappa,t}}{n^2}, \quad \forall |z - z_0| \leq \delta. \quad (1.12)$$

If z is in the union of dual disks of H_n and also satisfies $|z - z_0| \leq \delta$, then

$$|g_n(z) - M_{z_0}(z)| \leq C_{\kappa,t}|z - z_0|^3 + \frac{C_{\kappa,t}}{n^3}. \quad (1.13)$$

Proof. Let \tilde{D} be a dual disk of H_{n-2} which intersects $P(t)$ and let D be a disk of H_{n-2} which intersects \tilde{D} . On ∂D , $M_{\tilde{D}}$ is $O_{\kappa,t}(1/n^3)$ -close to g_n , and g_n is $O_{\kappa,t}(1/n^2)$ -close to a similitude S of D onto D' (Corollary 1.7). Therefore the center of D is moved by $M_{\tilde{D}}$ a distance $O_{\kappa,t}(1/n^2)$. By symmetry in ∂D , which has radius $O_{\kappa}(1/n)$, it follows that the pole of $M_{\tilde{D}}$ is at a minimum distance $C_{\kappa,t} > 0$ from the center

of D . Consequently there is a $\delta = \delta(\kappa, t) > 0$ such that, for $|z - \eta| \leq 2\delta$, $M_{\tilde{D}}(z)$ is bounded away from ∞ and $(d/dz)M_{\tilde{D}}(z)$ is bounded away from 0 and ∞ . Choose $\delta < t/3$. (We remark for later reference that the second and third derivatives of $M_{\tilde{D}}$ are also bounded away from ∞ in this neighborhood of η .)

One consequence is that (1.12) holds trivially if n is bounded. We may therefore assume n is large ($> 6t$) and z_0 is the center of some disk D_0 of H_{n-2} .

Given z with $|z - z_0| \leq \delta$, construct an alternating sequence $D_0, \tilde{D}_0, D_1, \dots, D_m, \tilde{D}_m$ of disks and dual disks such that adjacent disks intersect, each disk is at a distance $\leq 2\delta$ from z_0 and hence in $P_{t/3}$, $z \in \tilde{D}_m \cup \bar{D}_m$, and

$$m \leq C_\kappa n |z - z_0|. \quad (1.14)$$

Since $z \in \tilde{D}_m \cup \bar{D}_m$ and $g_n = M_{\tilde{D}_m}$ in \tilde{D}_m , it follows from (1.9) that

$$g_n(z) - M_{\tilde{D}_m}(z) = O_{\kappa, t} \left(\frac{1}{n^2} \right). \quad (1.15)$$

Thus the proof will be complete if we show that

$$|M_{\tilde{D}_m}(z) - M_{\tilde{D}_0}(z)| \leq C_{\kappa, t} |z - z_0|^3. \quad (1.16)$$

By (1.11), for $0 \leq k \leq m - 1$,

$$M_{\tilde{D}_k}(\zeta) - M_{\tilde{D}_{k+1}}(\zeta) = O_{\kappa, t} \left(\frac{1}{n^3} \right), \quad \forall \zeta \in \tilde{D}_k. \quad (1.17)$$

By the remark at the beginning of the proof concerning the boundedness of the derivatives, (1.17) implies

$$M_{\tilde{D}_{k+1}}^{-1} \circ M_{\tilde{D}_k}(\zeta) - \zeta = O_{\kappa, t} \left(\frac{1}{n^3} \right), \quad \forall \zeta \in \tilde{D}_k.$$

Now $T_k = M_{\tilde{D}_{k+1}}^{-1} \circ M_{\tilde{D}_k}$ is a Möbius transformation which leaves D_k invariant and is small on ∂D_k ; we wish to estimate its magnitude at z , a distance

$$\leq C_\kappa (m - k + 1)/n$$

from the center of D_k . For that purpose we make the following observation, the verification of which is left until the end of this proof.

Observation. Let T be a Möbius transformation which leaves invariant a disk of radius r centered at η . Suppose $\|T(z) - z\|_{|z-\eta|=r} < \varepsilon_0$ where $\varepsilon_0 < r/2$. Then $\|T(z) - z\|_{|z-\eta|=R} < 12\varepsilon_0 R^2/r^2$ where $r < R < r^2/2\varepsilon_0$.

In order to apply this observation to $T_k = M_{\tilde{D}_{k+1}}^{-1} \circ M_{\tilde{D}_k}$ set $r = O_\kappa(1/n)$,

$\varepsilon_0 = O_{\kappa,t}(1/n^3)$, $R \leq C_{\kappa,t}(m-k+1)/n$, and obtain

$$|M_{\bar{D}_{k+1}}^{-1} \circ M_{\bar{D}_k}(z) - z| < C_{\kappa,t} \left(\frac{1}{n} \right) \left(\frac{m-k+1}{n} \right)^2,$$

or, by using the boundedness of the derivative again,

$$|M_{\bar{D}_k}(z) - M_{\bar{D}_{k+1}}(z)| < C_{\kappa,t} \left(\frac{1}{n} \right) \left(\frac{m-k+1}{n} \right)^2. \quad (1.18)$$

Summing (1.18) from $k = 0$ to $m-1$ gives

$$|M_{\bar{D}_m}(z) - M_{\bar{D}_0}(z)| \leq C_{\kappa,t} \left(\frac{1}{n^3} \right) (2^2 + 3^2 + \cdots + (m+1)^2) \leq C_{\kappa,t} \left(\frac{m^3}{n^3} \right), \quad (1.19)$$

since $1^2 + 2^2 + \cdots + j^2 = j(j+1)(2j+1)/6$. Since $m/n < C_{\kappa,t}|z-z_0|$ (see (1.14)), (1.19) implies (1.16) which completes the proof of (1.12).

If z is also in the union of dual disks of H_n , then the right-hand side of (1.15) can be replaced by $O_{\kappa,t}(1/n^3)$, and we obtain (1.13).

To verify the observation, assume $\eta = 0$ and write T in the form

$$T(z) = \frac{r^2 e^{i\theta}(z - \alpha)}{r^2 - \bar{\alpha}z}.$$

Note that $|T(0) - 0| = |\alpha| \leq \varepsilon_0$. From $|T(r) - r| < \varepsilon_0$ obtain

$$|r^3(e^{i\theta} - 1) - r^2(e^{i\theta}\alpha - \bar{\alpha})| < \varepsilon_0|r - \bar{\alpha}|r < 2\varepsilon_0r^2,$$

and therefore $|r^3(e^{i\theta} - 1)| < 2\varepsilon_0r^2 + r^2|e^{i\theta}\alpha - \bar{\alpha}| < 4\varepsilon_0r^2$. This shows that

$$|e^{i\theta} - 1| < \frac{4\varepsilon_0}{r}. \quad (1.20)$$

Similarly, $T(z) - z = (r^2z(e^{i\theta} - 1) - r^2e^{i\theta}\alpha - \bar{\alpha}z^2)/(r^2 - \bar{\alpha}z)$. Now

$$|r^2 - \bar{\alpha}z| \geq r^2 - |\alpha|R \geq \frac{r^2}{2},$$

since $|\alpha|R \leq \varepsilon_0R \leq r^2/2$. Using this and (1.20) we obtain

$$\begin{aligned} |T(z) - z| &\leq \left(\frac{2}{r^2} \right) |r^2z(e^{i\theta} - 1) - r^2e^{i\theta}\alpha - \bar{\alpha}z^2| \\ &\leq \left(\frac{2}{r^2} \right) \left[r^2R \left(\frac{4\varepsilon_0}{r} \right) + r^2\varepsilon_0 + \varepsilon_0R^2 \right] \\ &= 8\varepsilon_0 \left(\frac{R}{r} \right) + 2\varepsilon_0 + 2\varepsilon_0 \left(\frac{R}{r} \right)^2 \\ &\leq 12\varepsilon_0 \left(\frac{R}{r} \right)^2. \end{aligned}$$

□

2. Global Theory

We now consider the situation in [RS] of circle packing approximations to the Riemann mapping function. Let Ω be a bounded simply connected region in the plane, let a and b be given points of Ω , and let f be the Riemann map of Ω onto the unit disk \mathbb{D} normalized by the two conditions: $f(a) = 0$, and $f(b)$ lies on the positive real axis.

The circle packing approximations to f are constructed as follows (see [RS] for complete details). Almost fill Ω with circles of radius $\varepsilon > 0$ packed in the regular hexagonal circle packing pattern. Consider an isomorphic circle packing of \mathbb{D} such that two circles of Ω , one nearest a and the other nearest b , are centered at the origin and on the positive real axis.

This correspondence of circles may be considered an approximate mapping f_ε of Ω onto \mathbb{D} . We define the *first derivative and second derivatives* of f_ε at z as follows. Pick a dual circle closest to z , let M_z be the Möbius transformation which maps it and the three distinguished points on it to the corresponding dual circle and three distinguished points in the packing of \mathbb{D} . Set $f'_\varepsilon(z) = M'_z(z)$, $f''_\varepsilon(z) = M''_z(z)$. We shall see that $f'_\varepsilon, f''_\varepsilon$ converge to f', f'' . We also obtain improved estimates of the rate of convergence of f_ε to f .

In order to make convenient use of the local theory in Section 1, let us redefine f_ε so it will be an actual homeomorphism of an approximating subdomain $\Omega_\varepsilon \subset \Omega$ onto an approximating domain $\mathbb{D}_\varepsilon \subset \mathbb{D}$. For a nonboundary disk D in the packing of Ω , there is a maximal H_m configuration of disks surrounding it. There is a corresponding H'_m configuration in \mathbb{D} . If $m \geq 2$, define f_ε in D and the six dual disks intersecting D to be the mapping g of Lemma 1.1.

Lemma 2.1. *Let $K \subset \subset \Omega$. There are constants $C_K > 0$ and $\delta_K > 0$ such that, for any $z_0 \in K$,*

$$|f_\varepsilon(z) - [f_\varepsilon(z_0) + f'_\varepsilon(z_0)(z - z_0) + \frac{1}{2}f''_\varepsilon(z_0)(z - z_0)^2]| \leq C_K|z - z_0|^3 + C_K\varepsilon^2$$

for $|z - z_0| < \delta_K$. If in addition z is in the union of dual disks, then the right-hand side can be improved to $C_K|z - z_0|^3 + C_K\varepsilon^3$.

Proof. If $z_0 \in K$ and D_0 is an ε -disk in the packing of Ω nearest to z_0 , then there is an H_n configuration centered at D_0 , where $n \geq \kappa/\varepsilon$ for $\kappa \equiv \text{Dist}(K, \partial\Omega)/4$. By Lemma 1.8 there is a $\delta_K > 0$ such that $|f_\varepsilon(z) - M_{z_0}(z)| < C_K(|z - z_0|^3 + \varepsilon^2)$ for $|z - z_0| < \delta_K$ (or $C_K(|z - z_0|^3 + \varepsilon^3)$ if in addition z is in the union of dual disks) where we write C_K instead of C_{z_0} . Thus we have

$$\begin{aligned} & |f_\varepsilon(z) - [f_\varepsilon(z_0) + f'_\varepsilon(z_0)(z - z_0) + \frac{1}{2}f''_\varepsilon(z_0)(z - z_0)^2]| \\ & \leq |f_\varepsilon(z) - M_{z_0}(z)| + |M_{z_0}(z_0) - f_\varepsilon(z_0)| \\ & \quad + |M_{z_0}(z) - M_{z_0}(z_0) - M'_{z_0}(z_0)(z - z_0) - \frac{1}{2}M''_{z_0}(z_0)(z - z_0)^2| \\ & \leq C_K|z - z_0|^3 + C_K\varepsilon^2 + C_K\varepsilon^2 + C_K|z - z_0|^3, \end{aligned}$$

where the estimate on the last term rests on the remark on the boundedness of third derivatives in the proof of Lemma 1.8. \square

The next lemma is used to improve the estimate in [R2, Theorem 5] for the rate of convergence of $f_n \rightarrow f$.

Lemma 2.2.¹ *Let $g(z) = z + a_2 z^2 + \dots$ be a schlicht function defined in the unit disk \mathbb{D} and let $\varepsilon > 0$. Then there is a constant C depending only on ε such that, for all measurable subsets $A \subset \mathbb{D}$, $\text{meas}(A) \leq C(\text{meas}(g(A)))^{(1/3)-\varepsilon}$.*

Proof. Let $1/p = (\frac{1}{3}) - \varepsilon$, $1/q = (\frac{2}{3}) + \varepsilon$ so $(1/p) + (1/q) = 1$. Then

$$\begin{aligned} \text{meas}(A) &= \iint_A |g'|^{2/p} |g'|^{-2/p} dx dy \\ &\leq \left\{ \iint_A (|g'|^{2/p})^p dx dy \right\}^{1/p} \left\{ \iint_A (|g'|^{-2/p})^q dx dy \right\}^{1/q}. \end{aligned}$$

Now $2q/p = 1 - 3\varepsilon/((\frac{2}{3}) + \varepsilon) < 1$. By the Koebe Distortion Theorem,

$$|g'(z)| \leq \frac{C}{(1 - |z|)}$$

so

$$\iint_A |g'|^{-2q/p} dx dy \leq C \iint_{\mathbb{D}} (1 - |z|)^{-2q/p} dx dy \leq C.$$

Therefore

$$\text{meas}(A) \leq C \left\{ \iint_A (|g'|^2 dx dy) \right\}^{1/p} = C(\text{meas}(g(A)))^{(1/3)-\varepsilon}. \quad \square$$

Lemma 2.3. *Let $K \subset \subset \Omega$. Then there is a constant C_K such that*

$$\|f_\varepsilon - f\|_K \leq C_K \varepsilon^{0.1428}.$$

Proof. We make use of the proof of Theorem 5 in [R2]. There we wrote W_ε for the domain of f_ε , D_ε for the image of f_ε , and we considered the Riemann mapping functions $G_{W_\varepsilon}: W_\varepsilon \rightarrow \mathbb{D}$ and $G_{D_\varepsilon}: D_\varepsilon \rightarrow \mathbb{D}$ which are normalized by the conditions

¹ Christian Pommerenke (oral communication) has shown that the exponent $(\frac{1}{3}) - \varepsilon$ can be improved to 0.405... by applying the proof given here in conjunction with a result of his (On the integral means of the derivative of a univalent function, *J. London Math. Soc.* (2) **32** (1985), 254–258), and could be replaced by 0.5 if the Brannan conjecture (J. E. Brannan, The integrability of the derivative in conformal mapping, *J. London Math. Soc.* (2) **18** (1978), 261–272) were true.

$G_{W_\varepsilon}(z_0) = 0$, $G_{W_\varepsilon}(z_1) > 0$ and $G_{D_\varepsilon}(0) = 0$, $G_{D_\varepsilon}(f_\varepsilon(z_0)) > 0$. We then estimated each term on the right-hand side of

$$|f(z) - f_\varepsilon(z)| \leq |f(z) - G_{W_\varepsilon}(z)| + |G_{W_\varepsilon}(z) - G_{D_\varepsilon} \circ f_\varepsilon \circ G_{W_\varepsilon}^{-1}(G_{W_\varepsilon}(z))| + |G_{D_\varepsilon}(f_\varepsilon(z)) - f_\varepsilon(z)| \tag{2.1}$$

(see (3.1) and (3.17) in [R2]). The first and last terms on the right were found to be bounded by $C_K \varepsilon^{1/2}$ and $C_K \varepsilon^{1/4}$, respectively. We now obtain an improved estimate of the middle term. Now $h \equiv G_{D_\varepsilon} \circ f_\varepsilon \circ G_{W_\varepsilon}^{-1}$ is a quasi-conformal self-map of \mathbb{D} ; we estimate its bad area and then use the ‘‘Groetsch method’’ to compare h with the identity map.

Let ρ be a positive number which may depend on ε ; we postpone an explicit choice of ρ until the end of the proof. The bad area of f_ε in Ω consists of two parts: the part in $K_\rho \equiv \{z \in \Omega: \text{dist}(z, \partial\Omega) \geq \rho\}$, call it B_1 , and the part B_2 in $\Omega - K_\rho$. The bad area of B_1 lies inside disks of $\Omega(\varepsilon)$, and since each such disk is at a distance ρ or more from $\partial\Omega$, it is the center of at least $\lceil \rho/\varepsilon \rceil$ generations of such disks. Therefore, by Lemma 1.1, the bad area inside it is no greater than $(C/\lceil \rho/\varepsilon \rceil^2)(\pi\varepsilon^2)$. Since the area of Ω is bounded, the total bad area B_1 is no greater than $C\varepsilon^2/\rho^2$.

The bad area of $h \equiv G_{D_\varepsilon} \circ f_\varepsilon \circ G_{W_\varepsilon}^{-1}$ inside \mathbb{D} is contained in the union of $G_{W_\varepsilon}(B_1)$ and the part $G_{W_\varepsilon}(B_2)$. Lemma 2.2 gives $\text{meas}(G_{W_\varepsilon}(B_1)) \leq C(\text{meas}(B_1))^{(1/3)-\beta}$ ($\beta > 0$ but arbitrary). Since we have $\text{meas}(B_1) \leq C\varepsilon^2/\rho^2$, we conclude that

$$\text{meas}(G_{W_\varepsilon}(B_1)) \leq C \left(\frac{\varepsilon^2}{\rho^2} \right)^{(1/3)-\beta} . \tag{2.2}$$

It is well known (see [W] for references) that $G_{W_\varepsilon}(B_2)$ lies in $\{1 - \rho^{1/2} \leq |z| \leq 1\}$. Thus

$$\text{meas}(G_{W_\varepsilon}(B_2)) \leq C\rho^{1/2}. \tag{2.3}$$

We now determine ρ by setting $\rho = \varepsilon^{4/7}$; this transforms (2.2) and (2.3) into

$$\text{bad area of } h \leq C\varepsilon^{2/7-\beta} \approx C\varepsilon^{0.2857}, \tag{2.4}$$

where β is an arbitrary preassigned positive number.

By the Grötsch argument for conformal moduli (see the proof of Lemma 1.3 above) we can conclude that $|h(w) - w| \leq C(\varepsilon^{0.2857})^{1/2} = C\varepsilon^{0.14285}$ for all $w \in \mathbb{D}$. We now return to (2.1) and obtain

$$|f(z) - f_\varepsilon(z)| \leq C_K(\varepsilon^{1/2} + \varepsilon^{1/4} + \varepsilon^{0.14285}) \leq C_K \varepsilon^{0.1428}$$

which completes the proof of Lemma 2.3. □

Theorem 2.4. *Let f_ε be the circle packing approximation to the Riemann mapping function $f: \Omega \rightarrow \mathbb{D}$. Let $f'_\varepsilon, f''_\varepsilon$ be the first and second derivatives of f_ε as in Lemma 2.1. Let $K \subset \subset \Omega$. There exists a constant C_K such that, for all z in K ,*

$$|f_\varepsilon(z) - f(z)| \leq C_K \varepsilon^{0.1428}, \quad (2.5)$$

$$|f_\varepsilon^{(1)}(z) - f'(z)| \leq C_K \varepsilon^{0.1428/2}, \quad (2.6)$$

$$|f_\varepsilon^{(2)}(z) - f''(z)| \leq C_K \varepsilon^{0.1428/4}. \quad (2.7)$$

Proof. The first inequality is Lemma 2.3. Since f is analytic in Ω we have, for all z, ζ_0 in K ,

$$|f(z) - f(\zeta_0) - f'(\zeta_0)(z - \zeta_0)| \leq C_K |z - \zeta_0|^2.$$

By Theorem 2.1,

$$|f_\varepsilon(z) - f_\varepsilon(\zeta_0) - f'_\varepsilon(\zeta_0)(z - \zeta_0)| \leq C_K(|z - \zeta_0|^2 + \varepsilon^2).$$

Combining these with (2.5) we obtain

$$|f'_\varepsilon(\zeta_0)(z - \zeta_0) - f'(\zeta_0)(z - \zeta_0)| \leq C_K(|z - \zeta_0|^2 + \varepsilon^{0.1428}).$$

Hence

$$|f'_\varepsilon(\zeta_0) - f'(\zeta_0)| \leq C_K \inf \left\{ |z - \zeta_0| + \frac{\varepsilon^{0.1428}}{|z - \zeta_0|} : z \in K \right\}.$$

By choosing $z \in K$ to satisfy $|z - \zeta_0| = \varepsilon^{0.1428/2}$ we obtain (2.6). Similarly, (2.7) follows from (2.5), (2.6), and Theorem 2.1. \square

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