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CLT and Other Limit Theorems for Functionals of Gaussian Processes

L. Giraitis and D. Surgailis

Institute of Mathematics and Cybernetics, 232600 Vilnius, Požėlos 54, USSR

Summary. Conditions for the CLT for non-linear functionals of stationary Gaussian sequences are discussed, with special references to the borderline between the CLT and the non-CLT. Examples of the non-CLT for such functionals with the norming factor $\sqrt{N}$ are given.

0. Introduction

In the study of limit theorems for sums of dependent random variables, a particular role has been played by the case when the summands are (non-linear) functionals of a stationary Gaussian process. It was this case which was considered by M. Rosenblatt in his famous example of a non-Gaussian limit law [18]. More recently, the non-central limit theorem (non-CLT) for functionals of Gaussian process was the object of studies by Dobrushin and Major [5], Gordeckii [8], Major [12], Rosenblatt [19, 20], Taqqu [24] and others. On the other hand the CLT for this kind of processes was discussed by Maruyama [15, 16], Breuer and Major [2], Sun [22] and Plikusas [17]. Among more general results on the CLT for dependent random variables which are applicable also in the present situation, we should mention Ibragimov [9], Brillinger [3] and Bentkus [1].

The aim of the present paper is to study the CLT for functionals of Gaussian processes 'in the vicinity of non-CLT'. In order to do that, we also prove some new non-CLT with the norming factor $\sqrt{N}$. To be more explicit, let

$$\xi_t = \sum_{n=1}^{\infty} \int_{\Pi^n} \varphi_n(x)\, e_n(x;t)\, d^n W \equiv \sum_{n=1}^{\infty} \xi_t^{(n)} \tag{0.1}$$

be the Wiener-Ito expansion of a stationary second order process $(\xi_t)_{t\in\mathbb{Z}}$ subordinated to the i.i.d. Gaussian sequence $(X_t)_{t\in\mathbb{Z}}$[13];

$$e_n(x;t) = \exp(i(x_1 + \dots + x_n)t), \qquad x = (x_1, \dots, x_n) \in$$
$$\Pi^n = [-\pi, \pi]^n, \qquad d^n W = W(dx_1) \dots W(dx_n),$$

$W(dx)$ is the random spectral measure of $(X_t)_{t\in\mathbb{Z}}$; $\varphi_n \in L^2(\Pi^n)$. If[1]

$$A_N^2 \equiv \mathrm{Var}\left(\sum_{t=1}^{N} \xi_t^{(n)}\right) \asymp N \tag{0.2}$$

and for any $\varepsilon > 0$

$$\int_{\Pi^n} |\varphi_n|^2 1(x: |x_1 + \ldots + x_n| < 1/N, |\varphi_n| > \varepsilon N^{1/2}) d^n x = o(1/N) \tag{0.3}$$

($1(A)$ is the indicator function of the set A), then $\sum_{t=1}^{N} \xi_t^{(n)}/A_n$ is asymptotically normal (Theorem 1). Of course, conditions (0.2) and (0.3) are not necessary for the CLT, still, condition (0.2) alone (or even a stronger one with '$\sim$' instead of '$\asymp$') is not sufficient. This follows in fact from the existence of subordinated self-similar processes with stationary increments which variance is linear in t; see Major [12], also this paper. As for condition (0.3), if $\varepsilon N^{1/2}$ in it is replaced by $\varepsilon g(N)$, where $g(N)/N^{1/2} \to \infty (N \to \infty)$, then $\sum_{t=1}^{N} \xi_t^{(n)}/\sqrt{N}$ can be asymptotically non-Gaussian (Theorem 7). Theorem 1 (for continuous time processes $\xi_t^{(n)}$ rather than discrete time processes) with $\varepsilon N^{1/6}$ instead of $\varepsilon N^{1/2}$ was obtained earlier by Maruyama [16]. In the case of infinite sum ξ_t (0.1), conditions (0.2) and (0.3) for all $n = 1, 2, \ldots$ do not ensure the CLT in general. The corresponding counterexample as well as a sufficient condition for the CLT in the case of infinite sum (0.1) can be found in Theorems 8 and 2, respectively. Theorems 1 and 2 can be compared with Ibragimov's condition for the CLT ([10], Theorem 18.6.1):

$$\sum_{k=1}^{\infty} E^{1/2}(\xi_0 - E(\xi_0 | X_t, |t| \leqq k))^2 < \infty, \tag{0.4}$$

which is stronger than (0.3) (Theorem 4).

However, condition (0.3) is too restrictive in some cases. In particular, the case

$$\xi_t = H(X_t), \tag{0.5}$$

where $H: \mathbb{R} \to \mathbb{R}$ is a given function and $(X_t)_{t\in\mathbb{Z}}$ is a stationary Gaussian process, deserves a separate treatment. (We call below functionals (0.5) *local*.) Denote $r(t)$, $r_H(t)$ the covariance functions of (X_t), $(H(X_t))$ respectively. According to Theorem 5, if $r(t) \to 0$ $(t \to \infty)$, then conditions

$$\sum_t |r_H(t)| < \infty \tag{0.6}$$

and

$$\sum_t r_H(t) \neq 0 \tag{0.7}$$

imply the CLT for $H(X_t)$. In Theorem 6, the case $r_H(t) = L(|t|)/|t|$, where L is a slowly varying function, is considered. Finally, Theorem 9 discusses a situation

[1] $A_N \sim B_N \Leftrightarrow \lim A_N/B_N = 1$;
$A_N \asymp B_N \Leftrightarrow 0 < \underline{\lim}\, A_N/B_N \leqq \overline{\lim}\, A_N/B_N$

when the non-CLT for local functionals is valid with any norming factor N^{γ}, $0<\gamma<1$ and either (0.6) or (0.7) fails.

Theorems 5 and 6 are related to Theorems 1 and 1′ of Breuer and Major [2], although they were obtained independently of [2]. In Theorem 1 [2], condition (0.6) is replaced by the following one

$$\sum_t |r(t)|^m < \infty, \tag{0.8}$$

where m is the Hermite rank of H. It is easy to show that conditions (0.6) and (0.8) are equivalent (see Lemma 5 below). Still, in our opinion, the proof of Theorem 5 is simpler than that of Theorem 1 [2]. In particular, Lemma 6 (based on Hölder's inequality) permits us to control effectively the semi-invariants of sums of Hermite polynomials of X_t. The proofs of Theorem 1 and 6 are also based on the semi-invariant method, for which estimation the so-called 'diagram formalism' of the multiple integral's calculus [4, 13, 17] is extensively used.

The results of this paper can be extended to continuous time, multivariate time, Fourier coefficients etc. In [6], Theorem 1 was generalized to the case of 2nd order processes, subordinated to non-Gaussian i.i.d. sequence (see Remark 1 below). The CLT for functionals of the form (0.5), where (X_t) is a stationary *linear* process, not necessarily Gaussian, was considered in [7].

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1. CLT for Non-local Functionals

Let $(X_t)_{t\in\mathbb{Z}}$ be a real stationary mean zero Gaussian sequence with covariance $r(t)$, $r(0)=1$ and spectral measure $F(dx)$, $|x|<\pi$, defined on a probability space $(\Omega, \mathscr{F}, P)$, where $\mathscr{F}=\sigma(X_t,\ t\in\mathbb{Z})$. Denote $Z(dx)$ the corresponding Gaussian complex random spectral measure with variance $E|Z(dx)|^2=F(dx)$. Any element $\xi\in L^2(\Omega)=L^2(\Omega, \mathscr{F}, P)$ can be represented uniquely in the form $\xi=\sum_{n=0}^{\infty} I_n(\varphi_n)$, where $I_n(\varphi)=\int_{\Pi^n} \varphi(x)\,d^nZ$, $n\geqq 1$ is the n-fold Ito-Wiener integral, $d^nZ=Z(dx_1)\dots Z(dx_n)$, $\varphi\in L^2(\Pi^n, F^n)=L^2(F^n)$ is symmetric:

$$\varphi=\operatorname{sym}\varphi, \qquad \Pi^n=[-\pi,\pi]^n, \qquad \sum \|\varphi_n\|_n^2\, n!<\infty,$$

and

$$\|\varphi\|_n=\Big(\int_{\Pi^n} |\varphi|^2\, d^nF\Big)^{1/2}; \qquad I_0(\varphi)=\varphi, \qquad \varphi\in\mathbb{C}=L^2(\Pi^0).$$

Moreover, $I_n(\varphi)$ is real if φ is even, i.e. $\overline{\varphi(x)}=\varphi(-x)$, $x\in\Pi^n$, where $\bar{a}$ denotes the complex conjugate of $a\in\mathbb{C}$. The unitary group $(T_t)_{t\in\mathbb{Z}}$ of shift operators $T_tX_s=X_{t+s}$, $s\in\mathbb{Z}$ extends to $L^2(\Omega)$ in a natural way. Random process $(\xi_t)_{t\in\mathbb{Z}}$ defined on $(\Omega, \mathscr{F}, P)$ is called subordinated to (X_t) if $T_t\xi_s=\xi_{t+s}$ $\forall t, s\in\mathbb{Z}$ [13]. Denote by $\mathscr{L}^2(X)$ the vector space of all real subordinated processes (ξ_t) such that $E\xi_t^2 < \infty$. Any $(\xi_t) \in \mathscr{L}^2(X)$ can be represented uniquely as

$$\xi_t = \sum_{n=0}^{\infty} \int_{\Pi^n} \varphi_n(x)\, e_n(x;t)\, d^n Z = \sum_{n=0}^{\infty} \xi_t^{(n)}, \tag{1.1}$$

where $e_n(x;t) = \exp(it(x_1 + \ldots + x_n))$, $n \geqq 1$, $e_0 = 1$, $\varphi_n \in L^2(\Pi^n)$, φ_n are even and symmetric, and $\sum \|\varphi_n\|_n^2\, n! < \infty$. All these preliminary facts as well as other properties of multiple Ito-Wiener integrals can be found e.g. in Ito [11] or Major [13]. In the sequel we'll use the notations

$$S_{N,t} = \sum_{s=1}^{[Nt]} \xi_s, \qquad S_{N,t}^{(n)} = \sum_{s=1}^{[Nt]} \xi_s^{(n)}, \tag{1.2}$$
$$S_N = S_{N,1}, \qquad S_N^{(n)} = S_{N,1}^{(n)}, \qquad A_n^2 = \operatorname{Var} S_N,$$

where $[a]$ is the entire part of $a \in \mathbb{R}$ and $\overset{d}{=}$, $\overset{d}{\Rightarrow}$ denote the equality and the weak convergence of (finite dimensional) distributions, respectively. Also, introduce the Dirichlet kernel

$$D_N(x) = \sin(Nx/2)/\sin(x/2) = \left(\sum_{j=1}^{N} e^{ijx}\right) e^{-i(N+1)x/2}. \tag{1.3}$$

Theorem 1. *Assume that the spectral measure F is absolutely continuous, $F(dx) = f(x)\,dx$ and the series* (1.1) *are finite (i.e. $\varphi_n = 0$ for $n > n_{\max} \geqq 1$), $\varphi_0 = 0$. If, moreover, f is bounded and*

(i) $A_N^2 \asymp N$,

(ii) *for any $\varepsilon > 0$ and $n = 1, \ldots, n_{\max}$, φ_n satisfies* (0.3), *then*

$$A_N^{-1} S_{N,t} \overset{d}{\Rightarrow} W(t), \tag{1.4}$$

where $(W(t))_{t \geqq 0}$ is the standard Wiener process.

Proof. It suffices to show that for any $r \geqq 1$, $0 \leqq t_1 < \ldots < t_r$, $a_1, \ldots, a_r \in \mathbb{R}$ the semi-invariants of order $k \geqq 3$ of $A_N^{-1} \sum_{j=1}^{r} S_{N,t_j} \cdot a_j$ vanish as $N \to \infty$. The proof of this fact below is restricted to the case $r = 1$, $t = 1$ as the general case can be treated analogously[2].

To evaluate the semi-invariants of multiple Ito-Wiener integrals, we shall use the diagram method [4, 13, 14, 17], which we briefly describe below. Denote by $\langle \eta_1, \ldots, \eta_k \rangle$ the semi-invariant of random variables $\eta_1, \ldots, \eta_k$. Let $\varphi_i \in L^2(\Pi^{n_i})$, $i = 1, \ldots, k$ be symmetric and even. Then

$$\langle I_{n_1}(\varphi_1), \ldots, I_{n_k}(\varphi_k) \rangle = \sum_{\gamma} \int_{\Pi^{m/2}} \phi_\gamma\, d^{m/2} F, \tag{1.5}$$

if $n_1 + \ldots + n_k = m$ is even, $= 0$ if m is odd, and the sum (1.5) is taken over all partitions (diagrams) γ of the table

[2] This remark applies also to the proofs of Theorems 5 and 6 below

$$G=\begin{pmatrix}(1,1),\ldots,(1,n_1)\\ \cdots\\ (k,1),\ldots,(k,n_k)\end{pmatrix}, \tag{1.6}$$

by pairs $[(i,j),(i',j')](\in\gamma)$ which we call the edges of γ such that (a) $i\neq i'$ and (b) the rows G_i, $i=1,\ldots,k$ of the table G (1.6) cannot make up two tables each of which is partitioned by the diagram separately. If the set $\{\gamma\}$ of diagrams which satisfy (a) and (b) is empty, the corresponding semi-invariant is zero. (The diagrams γ which satisfy (b) are called *connected* [14].) The function ϕ_γ in (1.5), dependending on $m/2$ variables, is obtained from the tensor product

$$\phi=\bigotimes_{i=1}^{k}\varphi_i,\qquad \phi=\phi(x_{ij},i=1,\ldots,k,j=1,\ldots,n_i)$$

according to the rule

$$x_{ij}=-x_{i'j'}\qquad \forall[(i,j),(i',j')]\in\gamma. \tag{1.7}$$

Lemma 1. (c.f. [17], Lemma 1).

$$\int_{\Pi^{m/2}}|\phi_\gamma|\,d^{m/2}F\leqq\prod_{i=1}^{k}\|\varphi_i\|_{n_i}, \tag{1.8}$$

Proof. Let $f\in L^2(\Pi^n)$, $g\in L^2(\Pi^{n'})$, $f=f(x,y)$, $g=g(x,y')$, $x\in\Pi$, $y\in\Pi^{n-1}$, $y'\in\Pi^{n'-1}$. Then

$$\int_\Pi|f(x,y)\,g(x,y')|\,dF(x)\leqq\Big(\int_\Pi|f(x,y)|^2\,dF\Big)^{1/2}$$
$$\cdot\Big(\int_\Pi|g(x,y')|^2\,dF\Big)^{1/2}\equiv\tilde f(y)\,\tilde g(y'),$$

where $\tilde f\in L^2(\Pi^{n-1})$, $\tilde g\in L^2(\Pi^{n'-1})$. Now, (1.8) follows easily by repeated application of this inequality. □

Let $n_1,\ldots,n_k\in\mathbb{Z}_+$, $1\leqq n_i\leqq n_{\max}$, $n_1+\ldots+n_k=m$ be even. By (1.5),

$$\begin{aligned}J_N&\equiv|\langle S_N^{(n_1)},\ldots,S_N^{(n_k)}\rangle|\\ &\leqq\sum_\gamma\int_{\Pi^{m/2}}\left|\prod_{j=1}^{k}\psi_{N,n_j}(x_{j1},\ldots,x_{jn_j})\right|d^{m/2}F\\ &\equiv\sum_\gamma J_N(\gamma),\end{aligned} \tag{1.9}$$

where $\psi_{N,n}(x_1,\ldots,x_n)=\varphi_n(x_1,\ldots,x_n)\,D_N(x_1+\ldots+x_n)$ and x_{ij} in (1.9) satisfy (1.7). Let

$$V_N=V_{N,K}=\{x\in\Pi^{m/2}:|x_{j1}+\ldots+x_{jn_j}|<K/N,j=1,\ldots,k\},\qquad V_N^c=\Pi^{m/2}\setminus V_N,$$
$$J_N(\gamma)=\int_{V_N}\ldots+\int_{V_N^c}\ldots=J'_N(\gamma)+J''_N(\gamma). \tag{1.10}$$

By (1.8),

$$J''_N(\gamma)\leqq C\sum_{i=1}^{k}\prod_{j\neq i}\|\psi_{N,n_j}\|_{n_j}\Big(\int_{\Pi^{n_i}}d^{n_i}x\,|\psi_{N,n_i}(x)|^2$$
$$\cdot 1(x\in\Pi^{n_i}:|x_1+\ldots+x_{n_i}|>K/N)\Big)^{1/2} \tag{1.11}$$

as $f(x)=dF/dx$ is bounded. Here and below we denote by $C, C(\cdot)$ possibly different constants which may depend on variables in brackets but do not depend on N. Next we need

Lemma 2. *Let* $0 \leqq g \in L^1(\Pi; dx)$ *satisfy the condition*

$$\int_{\Pi} D_N^2(x)\, g(x)\, dx \leqq CN, \qquad N \geqq 1. \tag{1.12}$$

Then $\forall \varepsilon>0\ \exists K>0$ *such that*

$$i(N) \equiv \int_{\pi>|x|>K/N} g(x)\, D_N^2(x)\, dx < \varepsilon N, \qquad N \geqq \max(1, K/N). \tag{1.13}$$

Proof. Set $G(x)=\int_{-x}^{x} g(y)\, dy$. Then G is non-decreasing and bounded in $(0,\pi)$ and $G(1/N) \leqq 2 \int_{\Pi} g(x)\, D_N^2(x)\, dx/N^2 \leqq C/N$, which implies $G(x) \leqq Cx$, $0<x<\pi$. Therefore

$$i(N) \leqq C \int_{K/N<x<\pi} x^{-2}\, dG(x) = C[G(x)\, x^{-2}|_{K/N}^{\pi} + \int_{K/N<x<\pi} x^{-2}\, dx] \leqq CN/K. \quad \square$$

By (1.3), $|D_N(x_1+\ldots+x_n)|$ is periodic in $\mathbb{R}^n$ with the period Π^n. Therefore $\operatorname{Var} S_N^{(n)} = n!\ \|\psi_{N,n}\|_n^2$ can be written as $n! \int_{\Pi} g_n(y)\, D_N^2(y)\, dy$, where

$$g_n(y) = \int_{\Pi^{n-1}} (\tilde{\varphi}_n(x_1, \ldots, x_{n-1}, y-x_1-\ldots-x_{n-1})|^2\, d^{n-1}x$$

and

$$\tilde{\varphi}_n(x_1, \ldots, x_{n-1}, x_n) = \varphi_n(x_1, \ldots, x_{n-1}, x_n'), \qquad x_n \in \mathbb{R}, \qquad x_n' \in \Pi,$$

$x_n = x_n' \pmod{2\pi}$ is the periodic extension of φ_n. As the integral on the right hand side of (1.11) does not exceed

$$\int_{K/N<|y|<\pi} g_{n_i}(y)\, D_N^2(y)\, dy \quad \text{and} \quad \operatorname{Var} S_N^{(n_i)} \leqq CN, \qquad i=1, \ldots, k$$

by (i), Lemma 2 implies that $\forall \varepsilon>0\ \exists K>0$ such that

$$J_N''(\gamma) \leqq \varepsilon N^{k/2}. \tag{1.14}$$

Now, $J_N'(\gamma) = J_N^-(\gamma) + J_N^+(\gamma)$, where

$$J_N^-(\gamma) = \int_{V_N} \prod_{j=1}^{k} |\psi_{N,n_j}^-|\, d^{m/2} F$$

and

$$\psi_{N,n}^- = \psi_{N,n}\, 1(x \in \Pi^n : |\varphi_n| \leqq \varepsilon N^{1/2}).$$

We claim that $\forall \delta>0\ \forall K>0\ \exists \varepsilon>0$ such that

$$J_N^-(\gamma) \leqq \delta N^{k/2}. \tag{1.15}$$

To prove this, we'll need two auxiliary lemmas.

Definition 1. Let γ be a connected diagram of the table G (1.6), and $x_{ij} \in \mathbb{R}$, $(i, j) \in G$ be related by (1.7). We'll say that a row G_p $(1 \leqq p \leqq k)$ is proper if there exist $q \in \{1, \dots, k\}$, $q \neq p$ such that $n_p + k - 2$ variables x_{pj}, $j = 1, \dots, n_p$, $x_{i1} + \dots + x_{in_i}$, $i = 1, \dots, k$, $i \neq p, q$ are linearly independent; in other words, if the relation

$$\sum_{j=1}^{n_p} c_j x_{pj} + \sum_{i=1,\dots,k,\, i \neq p, q} d_i (x_{i1} + \dots + x_{in_i}) \equiv 0 \tag{1.16}$$

(plus (1.7)) implies $c_j = d_i = 0$, $j = 1, \dots, n_p$, $i = 1, \dots, k$, $i \neq p, q$.

Lemma 3. *Let G_p be proper, and V_N be given by* (1.10). *Then*

$$\int_{V_N} |\psi_{N, n_p}|^2 \, d^{m/2} F \leqq C(K/N)^{k-2} \|\psi_{N, n_p}\|^2_{n_p}.$$

Proof. For simplicity of notation, assume that $p = k$ and $q = k - 1$. Identify $\mathbb{R}^{m/2}$ with the $m/2$-dimensional hyperplane in $\mathbb{R}^m = \{x = (x_{ij}, (i, j) \in G\}$, determined by the Eq. (1.7). According to Definition 1, there exist a non-degenerate[3] linear transform T: $\mathbb{R}^{m/2} \to \mathbb{R}^{n_k + k - 2}$ such that

$$(Tx)_j = x_{kj}, \quad j = 1, \dots, n_k, \quad (Tx)_{n_k + i} = x_{i1} + \dots + x_{in_i}, \quad i = 1, \dots, k - 2.$$

This proves Lemma 3. □

Lemma 4. *Let γ and x_{ij} satisfy the conditions of Definition* 1. *There exist at least two distinct proper rows $G_{p'}$, and $G_{p''}$.*

Proof. We say that $G_1, \dots, G_k$ are properly ordered, if for any $i = 1, \dots, k - 1$ there exists an edge $V_i = [(i, j), (i', j')] \in \gamma$ such that $i' > i$. In this case G_1 is proper, with $q = k$. Indeed, let (1.16) hold, and set $i^* = \max(i = 2, \dots, k - 1 \colon d_i \neq 0)$. The available $x_{i^* j}$ connected by V_{i^*} is linearly independent of x_{is}, $i < i^*$, $s = 1, \dots, n_i$, which implies $d_{i^*} = 0$, i.e. we have a contradiction.

It remains to show that there exist two different ways to renumerate the rows of G to get them properly ordered. As γ is connected, there exist $k - 1$ edges $[(i_r, j_r), (i'_r, j'_r)] \in \gamma$, $r = 1, \dots, k - 1$ such that for any $r = 1, \dots, k - 1$,

$$i_r \in \{i_1, i'_1, \dots, i_{r-1}, i'_{r-1}\}, \qquad i'_r \notin \{i_1, i'_1, \dots, i_{r-1}, i'_{r-1}\} \tag{1.17}$$

(the starting row G_{i_1} can be taken arbitrary). Then

$$G'_1 = G_{i'_{k-1}}, G'_2 = G_{i'_{k-2}}, \dots, G'_{k-1} = G_{i'_k}, G'_k = G_{i_1}$$

are properly ordered. If one takes i'_{k-1} as the starting point in (1.17), one gets another properly ordered sequence $G''_1, \dots, G''_k$ such that $G''_1 \neq G''_k = G'_1$. □

Coming back to the proof of (1.15), let G_p, $G_{p'}$ be proper rows for γ. By the definition of $\psi^-_{N,n}$ and the inequalities $|D_N(x)| \leqq CN$, $|x| \leqq K/N$, $|ab| \leqq 1/2(a^2 + b^2)$,

$$J^-_N(\gamma) \leqq C(\varepsilon N^{1/2})^{k-2} N^{k-2} \int_{V_N} (|\psi_{N, n_p}|^2 + |\psi_{N, n_{p'}}|^2) \, d^{m/2} F \leqq C(\varepsilon N^{1/2})^{k-2} N^{k-2} (K/N)^{k-2} N$$

according to Lemma 3. This proves (1.15).

[3] I.e. the rank of the $(m/2, n_k + k - 2)$ matrix corresponding to T is $n_k + k - 2$.

With (1.14) and (1.15) in mind, it remains to verify that $J_N^+(\gamma)=o(N^{k/2})$, $\forall\varepsilon>0$, $\forall K>0$. Again, by using Lemma 1,

$$J_N^+(\gamma)\leqq C\sum_{j=1}^{k}\Big(\prod_{i\neq j}\|\psi_{N,n_i}\|_{n_i}\Big)\,\delta_{N,j}^{1/2},$$

where

$$\delta_{N,j}=N^2\int_{\Pi^{n_j}}|\varphi_{n_j}|^2\,1(|x_1+\ldots+x_{n_j}|<K/N,\ |\varphi_{n_j}|>\varepsilon N^{1/2})\,d^{n_j}x=o(N)$$

according to (ii), which ends the proof. □

Set $S_N^{(\leqq n)}=\sum_{k=1}^{n}S_N^{(k)}$. By Fatou's lemma,

$$\lim_{n\to\infty}\overline{\lim_{N\to\infty}}\ \operatorname{Var}S_N^{(\leqq n)}/\operatorname{Var}S_N\leqq 1. \tag{1.18}$$

In Theorem 8 of Section 3 this limit is zero. It appears that the equality in (1.18) plus the CLT for each $S_N^{(\leqq n)}$ yields the CLT for S_N. Namely, we have

Theorem 2. *Assume that*

$$A_N^{-1}S_N^{(\leqq n)}\overset{d}{\Rightarrow}\mathcal{N}(0,\sigma_n^2) \tag{1.19}$$

for $n\geqq 1$ *sufficiently big, where* $A_N^2=\operatorname{Var}S_N$ *and* $\sigma_n^2\to 1$ $(n\to\infty)$. *Then* $A_N^{-1}S_N\overset{d}{\Rightarrow}\mathcal{N}(0,1)$.

Proof. By (1.18), $\operatorname{Var}(S_N^{(\leqq n)}/A_N)\to\sigma_n^2$ and therefore $\operatorname{Var}((S_N-S_N^{(\leqq n)})/A_N)\to 1-\sigma_n^2$. Together with (1.18) this implies that for any $a\in\mathbb{R}$,

$$\overline{\lim_{N\to\infty}}\,|E\exp(iaA_N^{-1}S_N)-\exp(-a^2/2)|$$
$$\leqq|a|(1-\sigma_n^2)^{1/2}+|\exp(-a^2\sigma_n^2/2)-\exp(-a^2/2)|\to 0\,(n\to\infty).\quad\square$$

In [9] (see also [10], Theorem 18.6.1) Ibragimov obtained a result on the CLT for subordinated processes which we reproduce below in a somewhat less generality.

Theorem 3 (Ibragimov). *Let* $(\xi_t)\in\mathcal{L}^2(X)$ *be stationary process subordinated to i.i.d. sequence*[4] (X_t). *Assume that* (0.4) *holds and* $\sum r_\xi(t)=\sigma^2\neq 0$, *where* $r_\xi(t)$ *is the covariance of* (ξ_t). *Then* $S_N/\sqrt{N}\overset{d}{\Rightarrow}\mathcal{N}(0,\sigma^2)$.

Theorem 4. *Let conditions of Theorem 3 hold and* $(X_t)\in\mathcal{N}(0,1)$ *be Gaussian. Then* (ξ_t) *satisfies the conditions of Theorems 1 and 2.*

Proof. Apart from the 'frequency' representation (1.1), the process (ξ_t) has also the 'moving average' representation

$$\xi_t=\sum_{n=0}^{\infty}\sum_{t_1,\ldots,t_n\in\mathbb{Z}}c_n(t-t_1,\ldots,t-t_n):X_{t_1}\ldots X_{t_n}:$$
$$\equiv\sum_{n=0}^{\infty}\xi_t^{(n)}, \tag{1.20}$$

4 Not necessary Gaussian

where $c_n(t)\in\mathbb{R}$, $t\in\mathbb{Z}^n$, $n\geqq 1$ are the Fourier coefficients of $\varphi_n\in L^2(\Pi^n)$, $\varphi_n(x) = (2\pi)^{-n/2}\sum_t \exp(i(x,t))\,c_n(t)$, and $:X_{t_1}\dots X_{t_n}:$ is the Wick product (invariant with respect to permutations of $t_1,\dots,t_n$) of Gaussian variables $X_{t_1},\dots,X_{t_n}$ [13], i.e.

$$:X_{t_1}\dots X_{t_n}: = H_{k_1}(X_{s_1})\dots H_{k_m}(X_{s_m}) \tag{1.21}$$

if $t_1=\dots=t_{k_1}=s_1,\dots,t_{k_1+\dots+k_{m-1}+1}=\dots=t_n=s_m$, $k_1+\dots+k_m=n$, $s_1<\dots<s_m$ and H_k, $k=0,1,\dots$ are Hermite polynomials. Now, (1.20) follows from the well-known relationship between multiple Ito-Wiener integrals and Hermite polynomials [11, 13]. Note that

$$E(:X_{t_1}\dots X_{t_n}:|X_t, t\in T)=\begin{cases}:X_{t_1}\dots X_{t_n}: & \text{if } t_1,\dots,t_n\in T,\\ 0 & \text{if otherwise}\end{cases} \tag{1.22}$$

and

$$\operatorname{cov}(:X_{t_1}\dots X_{t_n}:, :X_{t'_1}\dots X_{t'_{n'}}:)=\delta(n,n')\cdot\prod_{j=1}^{n}\delta(t_j,t'_j)\prod_{j=1}^{m}k_j!, \tag{1.23}$$

where $t_1\leqq\dots\leqq t_n$, $t'_1\leqq\dots\leqq t'_{n'}$ and $:X_{t_1}\dots X_{t_n}:$ is equal to (1.21). By (1.22) and (1.23),

$$\begin{aligned}\rho(k)&\equiv E(\xi_0-E(\xi_0|X_t,|t|\leqq k))^2\\&=\sum_{n=1}^{\infty}n!\sum_{(t_1,\dots,t_n)\notin[-k,k]^n}c_n^2(t_1,\dots,t_n)\\&\geqq\sum_{t_1,\dots,t_{n-1}\in\mathbb{Z}}(c_n^2(t_1,\dots,t_{n-1},k+1)+c_n^2(t_1,\dots,t_{n-1},-k-1)).\end{aligned} \tag{1.24}$$

To prove condition (ii) of Theorem 1, it suffices to show that for each $n\geqq 1$ there exists $0\leqq\psi_n\in L^2(\Pi^{n-1})$ such that

$$|\varphi_n(x_1,\dots,x_n)|\leqq C\psi_n(x_1,\dots,x_{n-1}) \tag{1.25}$$

a.e. in Π^n. Now, set $\psi_n(x)=\sum_{t_n}\left|\sum_{t_1,\dots,t_{n-1}}c_n(t_1,\dots,t_{n-1},t_n)\cdot\exp\left(i\sum_{j=1}^{n-1}x_jt_j\right)\right|$. Clearly ψ_n satisfies (1.25). By Minkowski's inequality and Parseval's identity,

$$\|\psi_n\|_{n-1}\leqq C\sum_{t_n}\Big(\sum_{t_1,\dots,t_{n-1}}c_n^2(t_1,\dots,t_n)\Big)^{1/2}<\infty$$

according to (1.24) and (0.4).

One can check easily (see also the proof of Theorem 18.6.1 [10]) that

$$|r_\xi(t)|\leqq C\rho^{1/2}(t/2), \tag{1.26}$$

i.e. $\sum_t|r_\xi(t)|<\infty$ by (0.4). Therefore Var $S_N\sim\sigma^2 N$ as $\sigma^2\neq 0$.

Denote $r_\xi^{(\leqq n)}(t)$ the covariance function of $\sum_{k=1}^{n}\xi_t^{(k)}$. Using (1.22), similarly to the proof of (1.26) it can be shown that $r_\xi^{(\leqq n)}(t)$ also satisfies (1.26) with C independent of n (and t). Therefore $\sigma_n^2\equiv\sum_t r_\xi^{(\leqq n)}(t)\to\sigma^2$ $(n\to\infty)$. Together with Theorem 1, this concludes the verification of conditions of Theorem 2. □

Remark 1. Let $(X_t)_{t\in\mathbb{Z}}$ be i.i.d. random variables, not necessary Gaussian, such that there exist orthogonal basis in $L^2(\mathbb{R};\mu)$, $\mu(dx)=P(X_t\in dx)$, consisting of polynomials $P_n(x)=\sum_{j\le n} c_j^{(n)} x^j$, $n=0,1,\ldots$ such that $E\,P_n^2(X_t)=n!$. Let $:X_{t_1}\ldots X_{t_n}:$ be defined by (1.21), with H_k replaced by P_k. It is easy to show that any 2nd order process $(\xi_t)_{t\in\mathbb{Z}}$ subordinated to (X_t) has a unique representation (1.20), where $c_n\in L^2(\mathbb{Z}^n)$ and the series converge in $L^2(\Omega)$ ([6], see also [21]).

Let $\varphi_n\in L^2(\Pi^n)$ denote the Fourier transform of c_n. Assuming that conditions (i) and (ii) of Theorem 1 hold and only a finite number of c_n's in the representation (1.20) do not vanish, one can prove the CLT for (ξ_t) which is a straightforward generalization of Theorem 1 [6].

2. CLT for Local Functionals

Let (X_t) be a real stationary mean zero Gaussian sequence with covariance $r(t)$ such that $r(0)=1$ and

$$r(t)\to 0 \qquad (t\to\infty). \tag{2.1}$$

Any (real) function $H\in L^2(\mathbb{R}, e^{-x^2/2}\,dx)\equiv L^2(X)$ can be represented in the series of Hermite polynomials

$$H(x)=\sum_{k=0}^{\infty} c_k H_k(x), \tag{2.2}$$

where $\sum c_k^2 k!<\infty$. The smallest $k\in\mathbb{Z}_+$ such that $c_k\neq 0$ will be called the *Hermite rank* of H [24]. Given $H\in L^2(X)$ such that $c_0=E(X_0)=0$, denote $r_H(t)$ the covariance of $\xi_t=H(X_t)$ and set again $S_{N,t}=\sum_{s=1}^{[Nt]} H(X_s)$, $S_N=S_{N,1}$.

Theorem 5. *Assume that*

$$\sum_t |r_H(t)|<\infty \tag{2.3}$$

and

$$\sigma^2=\sum_t r_H(t)\neq 0. \tag{2.4}$$

Then

$$N^{-1/2} S_{N,t} \xrightarrow{d} \sigma W(t).$$

Theorem 6. *Let $r_H(t)=L(|t|)/|t|$, where $L\colon[1,\infty)\to\mathbb{R}$ is a slowly varying function, bounded on every finite interval, such that*

$$L_1(N)\to\infty\,(N\to\infty) \tag{2.5}$$

where $L_1(N)=\sum_{t=-N}^{N} r_H(t)$. Then

$$(L_1(N)\,N)^{-1/2} S_{N,t} \xrightarrow{d} W(t).$$

Remark 2. Theorem 9 below shows that conditions on $r_H(t)$ in Theorems 5 and 6 are essential for the CLT. Namely, there exist stationary Gaussian (X_t) with absolutely continuous spectral measure such that $A_N^{-1}\sum_{t=1}^{N} H_n(X_t)$ is asymptotically non-Gaussian and either $\sum|r_{H_n}(t)|<\infty$, $\sum r_{H_n}(t)=0$ (in this case the norming factor $A_N=N^\gamma$, $0<\gamma<1/2$) or the series $\sum|r_{H_n}(t)|$ diverge logarithmically but $r_{H_n}(t)\cdot t$ fails to be slowly varying (and the norming factor is the 'usual' $N^{1/2}$); H_n is any odd ($n=3,5,\ldots$) Hermite polynomial.

Proof of Theorem 5. Let $m\geqq 1$ be the Hermite rank of H. As $EH_k(X_0)H_j(X_t)=\delta(k,j)\,k!\,(r(t))^k$,

$$r_H(t)=r^m(t)\sum_{n=m}^{\infty} c_n^2\, n!\, r^{n-m}(t). \tag{2.6}$$

By (2.1), $|r_H(t)|\geqq|r(t)|^m c_m^2 m!/2$ if t is sufficiently big, hence by (2.3),

$$\sum_t |r^m(t)|<\infty. \tag{2.7}$$

Conversely, (2.7) implies (2.3) by (2.6). This discussion can be summarized in

Lemma 5. *Conditions* (2.3) *and* (2.7) *are equivalent. By Lemma 5,*

By Lemma 5,

$$\operatorname{Var}(N^{-1/2}\sum_{t=1}^{N}\sum_{k\geqq n} c_k H_k(X_t))\leqq\sum_{k\geqq n}\sum_{t=1}^{N} c_k^2\, k!\,|r^k(t)|$$
$$\cdot(N-t)/N\leqq C\sum_{k\geqq n} c_k^2\, k!\to 0 \qquad (n\to\infty). \tag{2.8}$$

According to (2.3) and (2.4), Var $S_N\sim\sigma^2 N$. Together with (2.8) this implies that it suffices to prove Theorem 5 for H whose Hermite series is finite.

Denote $J_N=\langle S_N^{(n_1)},\ldots,S_N^{(n_k)}\rangle$ where $S_N^{(n)}=\sum_{t=1}^{N} H_n(X_t)$ and $n_1\geqq m,\ldots,n_k\geqq m$. We prove that

$$J_N=o(N^{k/2}) \tag{2.9}$$

for $k\geqq 3$. Here $J_N=\sum_\gamma J_N(\gamma)$, where

$$J_N(\gamma)=\sum_{t_1,\ldots,t_k=1}^{N}\prod_{1\leqq i<j\leqq k} r^{l_{ij}}(t_i-t_j), \tag{2.10}$$

the sum $\sum_\gamma$ is taken over all connected diagrams (i.e. partitions of the table G (1.6) which satisfy (a) and (b)), and $l_{ij}=l_{ij}(\gamma)$ is the number of edges between the i-th and j-th row of the table G. The formula above for J_N is a particular case of (1.5), see also [14], Proposition 1.1. By the definition,

$$\sum_{j\neq i} l_{ij}=n_i, \qquad i=1,\ldots,k. \tag{2.11}$$

Write $J_N(\gamma)=J_N'(\gamma)+J_N''(\gamma)$, where $J_N'(\gamma)$ is the sum (2.10) taken over $t_1,\ldots,t_k=1,\ldots,N$ such that $|t_i-t_j|<K$ if $l_{ij}>0$, $i,j=1,\ldots,k$. As γ is connected, without loss of generality we can assume that $G_1,\ldots,G_k$ are properly ordered (see Lemma 4), i.e. for each $i=1,\ldots,k-1$ there exists an edge $[(i,j),(i',j')]\in\gamma$ such that $i'>i$. Set $s_i=t_i-t_{i'}$, $i=1,\ldots,k-1$. Then

$$J_N'(\gamma)\leqq C\sum_{|s_i|\leqq K,\, i=1,\ldots,k-1,\, |t_k|\leqq N} 1\leqq CN.$$

By Lemma 6 below, this concludes the proof. □

Lemma 6. *$J''_N(\gamma) \leqq \varepsilon(K) N^{k/2}$, where $\varepsilon(K) \to 0$ $(K \to \infty)$.*

Proof. By definition, $J''_N(\gamma) = \sum_{1 \leqq i < j \leqq k} I_{ij}$, where

$$I_{12} = \sum_{t_1, \ldots, t_k = \overline{1, N}, |t_1 - t_2| > K} \prod_{1 \leqq i < j \leqq k} r^{l_{ij}}(t_i - t_j)$$

if $l_{12} > 0$, $= 0$ if $l_{12} = 0$ and other I_{ij} are defined analogously. Set $r_{12}(t, s) = r(t - s)$ if $1 \leqq t$, $s \leqq N$, $|t - s| > K$, $= 0$ if otherwise; $r_{ij}(t, s) = r(t - s)$ if $1 \leqq t$, $s \leqq N$, $= 0$ if otherwise, and $(i, j) \neq (1, 2)$, $i, j = 1, \ldots, k$. Then

$$I_{12} = \sum_{t_1, \ldots, t_k} \prod_{1 \leqq i < j \leqq k} r_{ij}^{l_{ij}}(t_i, t_j). \tag{2.12}$$

For any $r_{ij}(t, s) \geqq 0$, $i, j = 1, \ldots, k$, $t, s \in \mathbb{Z}$ and $l_{ij} = l_{ji} \geqq 0$ which satisfy (2.11), the following inequality holds:

$$R \leqq \min\Big(\prod_{1 \leqq i < j \leqq k} R_{ij}, \prod_{1 \leqq i < j \leqq k} R_{ji} \Big), \tag{2.13}$$

where

$$R_{ij} = \Big(\sum_t \Big(\sum_s r_{ij}^{n_i}(s, t)\Big)^{n_j/n_i}\Big)^{l_{ij}/n_j} \tag{2.14}$$

and R denotes the right hand side of (2.12). In fact, by Hölder's inequality:

$$\Big|\int h_1 \ldots h_k\Big| \leqq \prod_j \Big(\int |h_j|^{\beta_j}\Big)^{1/\beta_j}, \qquad 1/\beta_1 + \ldots + 1/\beta_k = 1,$$

we have

$$\begin{aligned} R &\leqq \sum_{t_2, \ldots, t_k} \Big(\sum_{t_1} r_{12}^{n_1}\Big)^{l_{12}/n_1} \ldots \Big(\sum_{t_1} r_{1k}^{n_1}\Big)^{l_{1k}/n_1} \prod_{2 \leqq i < j \leqq k} \ldots \\ &\leqq \sum_{t_3, \ldots, t_k} \Big(\sum_{t_2}\Big(\sum_{t_1} r_{12}^{n_1}\Big)^{n_2/n_1}\Big)^{l_{12}/n_2} \Big(\sum_{t_2} r_{23}^{n_2}\Big)^{l_{23}/n_2} \ldots \\ &\quad \cdot \Big(\sum_{t_2} r_{2k}^{n_2}\Big)^{l_{2k}/n_2} \Big(\sum_{t_1} r_{23}^{n_1}\Big)^{l_{13}/n_1} \ldots \Big(\sum_{t_1} r_{1k}^{n_1}\Big)^{l_{1k}/n_1} \prod_{3 \leqq i < j \leqq k} \ldots \\ &\leqq \ldots \leqq \prod_{1 \leqq i < j \leqq k} R_{ij}; \end{aligned}$$

the other inequality of (2.13) can be proved analogously.

By (2.12), (2.13) and (2.7),

$$I_{12} \leqq C \max\Big\{\Big(\sum_{|t| > K} |r^{n_1}(t)|\Big)^{l_{12}/n_1}, \Big(\sum_{|t| > K} |r^{n_2}(t)|\Big)^{l_{21}/n_2}\Big\} N^{\gamma},$$

where

$$\gamma = \min\Big(\sum_{1 \leqq i < j \leqq k} l_{ij}/n_j, \sum_{1 \leqq i < j \leqq k} l_{ij}/n_i\Big) \leqq k/2;$$

the last inequality follows from (2.11). □

Proof of Theorem 6. Let $m \geqq 1$ denote again the Hermite rank of H. Similarly as in the previous theorem, $|r_H(t)| \geqq C|r^m(t)|$ for t sufficiently big, which implies

$$|r(t)| \leqq C[L(t)/t]^{1/m}. \tag{2.15}$$

It follows from (2.15) that $\operatorname{Var}\left(S_N - \sum_{t=1}^{N} c_m H_m(X_t)\right) \leqq CN$. Together with (2.5) this implies that it suffices to prove Theorem 6 with $H(x)$ replaced by $c_m H_m(x)$.

Let γ be any connected diagram of the table (1.6), where $n_1 = \ldots = n_k = m$ and l_{ij}, $i, j = 1, \ldots, k$ be the same as in the proof of Theorem 5. We'll prove that

$$J_N(\gamma) = \sum_{t_1, \ldots, t_k = 1}^{N} \prod_{1 \leqq i < j \leqq k} (L(|t_i - t_j|)/|t_i - t_j|)^{l_{ij}/m} = o((L_1(N)\,N)^{k/2}). \tag{2.16}$$

In fact, $J_N(\gamma) \leqq C(NL(N))^{k/2} I(\gamma)$, where

$$I(\gamma) = \int_{[0,1]^k} \prod_{1 \leqq i < j \leqq k} |t_i - t_j|^{-(l_{ij}+\varepsilon)/m} d^k t$$

as $L(tN)/L(N) \leqq C(\varepsilon)\, t^{-\varepsilon}$, $1/N \leqq t < 1$ uniformly in N for any $\varepsilon > 0$. By another property of slowly varying functions ([25], Chap. 5.2) $L(N) = o(L_1(N))$. It remains to apply Lemma 7 below. □

Lemma 7. *For $\varepsilon > 0$ sufficiently small, $I(\gamma) < \infty$.*

Proof. Let us prove first that

$$i \equiv \int_0^1 dt \prod_{2 \leqq j \leqq k} |t - t_j|^{-(\varepsilon + l_{1j}/m)} \leqq C(\varepsilon) \sum_{2 \leqq i < j \leqq k} |t_i - t_j|^{-\varepsilon'} \leqq C(\varepsilon) \prod_{2 \leqq i < j \leqq k} |t_i - t_j|^{-\varepsilon'}, \tag{2.17}$$

where $\varepsilon' = \varepsilon'(\varepsilon) \to 0$ $(\varepsilon \to 0)$. In fact, assume that $0 = t_1 < t_2 < \ldots < t_{k+1} = 1$. Then $i = \sum_{j=1}^{k} \int_{tj}^{t_{j+1}} = \sum i_j$, where

$$i_j \leqq \int_{t_j}^{t_{j+1}} dt/|t - t_j|^{\beta_j} |t - t_{j+1}|^{\gamma_j} \leqq C(\varepsilon) |t_{j+1} - t_j|^{-\varepsilon m}$$

as

$$\beta_j = \sum_{i=1}^{j} (\varepsilon + l_{1i}/m) < 1, \gamma_j = \sum_{i=j+1}^{k} (\varepsilon + l_{1i}/m) < 1,$$

$\beta_j + \gamma_j \leqq 1 + \varepsilon m$ and $\varepsilon > 0$ is sufficiently small, $j = 2, \ldots, k-1$, while

$$i_1 \leqq \int_0^{t_2} dt/|t - t_2|^{\varepsilon + l_{12}/m} |t - t_3|^{\gamma_2} \leqq C|t_2 - t_3|^{-2m\varepsilon}$$

due to $|t_3 - t|^{\gamma_2} \geqq |t_3 - t_2|^{2m\varepsilon} |t_2 - t|^{\gamma_2 - 2m\varepsilon}$ and $\gamma_2 - 2m\varepsilon + \varepsilon + l_{12}/m < 1$. Similarly, $i_k \leqq C(\varepsilon)|t_k - t_{k-1}|^{-2m\varepsilon}$. This proves (2.17).

By successive application of (2.17),

$$I(\gamma) \leqq C(\varepsilon) \int_0^1 \int_0^1 dt_{k-1}\, dt_k |t_k - t_{k-1}|^{-(\varepsilon'' + l_{k,k-1}/m)} < \infty$$

as $l_{k,k-1} < m$ and $\varepsilon'' > 0$ is sufficiently small. □

3. Non-central Limit Theorems

Theorems 7–9 below serve as counterexamples to the central limit theorems of Section 1–2, when some of their conditions are violated. This applies to (a) the condition (ii) of Theorem 1, (b) the condition of finiteness of the Ito-Wiener expansion of (ξ_t) in Theorem 1 and (c) the conditions (2.3) and (2.4) of Theorem 5. The variance A_N^2 of $S_N = \sum_{t=1}^{N} \xi_t$ grows linearly in Theorem 7 and 8, while in Theorem 9 it behaves like N^γ, where γ is any number between 0 and 2. The limiting processes in Theorem 7–9 are expressed as multiple stochastic integrals (m.s.i.) with respect to different (or vector) Gaussian measures, which is a simple generalization of m.s.i. of Sect. 1 (see e.g. [20, 30]). Below we recall the basic properties of such integrals.

Let $\mathscr{B}(\mathbb{R})$ denote the Borel subsets of $\mathbb{R}$ with finite Lebesgue measure. By a $\mathbb{C}^m$ valued white noise $W=(W_1, \dots, W_m)$ in $\mathbb{R}$ we mean a (complex) Gaussian family $(W_i(A),\ A\in\mathscr{B}(\mathbb{R}),\ i=1,\dots,m)$, defined on a probability space $(\Omega, \mathscr{F}, P)$ such that $EW_i(A)=0$,

$$EW_i(A)\overline{W_j(B)} = r_{ij} \int_{A\cap B} dx \tag{3.1}$$

and

$$\overline{W_i(A)} = W_i(-A),$$

$i, j=1,\dots,m$, $A, B\in\mathscr{B}(\mathbb{R})$. We assume below that the covariance (matrix) $(r_{ij})_{i,j=\overline{1,m}}$ of W is strictly positive definite. Introduce the Hilbert spaces $L^2(\mathbb{R}^n, (\otimes\mathbb{C}^m)^n) = L^2(\mathbb{R}^n, \cdot)$ $(n=1,2,\dots)$, consisting of all functions f: $\mathbb{R}^n \to (\otimes\mathbb{C}^m)^n$, $f = (f_{i_1,\dots,i_n})_{i_1,\dots,i_n=1,m}$ with finite norm

$$\Big(\int_{\mathbb{R}^n} \sum_{\substack{i_1,\dots,i_n=\overline{1,m}\\ j_1,\dots,j_n=\overline{1,m}}} r_{i_1j_1}\cdots r_{i_nj_n} f_{i_1\dots i_n}(x_1,\dots,x_n)\overline{f_{j_1\dots j_n}(x_1,\dots,x_n)}\, d^nx\Big)^{1/2}.$$

The symmetrization operator sym in $L^2(\mathbb{R}^n, \cdot)$ is given by

$$(\operatorname{sym} f)_{i_1\dots i_n}(x_1,\dots,x_n) = \sum_{(p(1),\dots,p(n))\in\mathscr{P}_n} f_{p(1)\dots p(n)}(x_{p(1)},\dots,x_{p(n)})/n!$$

where $\mathscr{P}_n$ is the set of all permutations $p=(p(1),\dots,p(n))$ of $(1,\dots,n)$.

Proposition (c.f. [23], Theorem 1.1). *Let $W=(W_1,\dots,W_n)$ and (r_{ij}) satisfy the conditions above. For any $n\geqq 1$ and $f\in L^2(\mathbb{R}^n,\cdot)$ there exists random variable*

$$I_n(f) = \int_{\mathbb{R}^n} \sum_{i_1,\dots,i_n=\overline{1,m}} f_{i_1\dots i_n}(x_1,\dots,x_n)\, W_{i_1}(dx_1)\dots W_{i_n}(dx_n)$$

(the m.s.i. of f with respect to W), with the following properties:

(w1) $I_n(f)=I_n(\operatorname{sym} f)\in L^2(\Omega)$; (w2) $EI_n(f)=0$;
(w3) $EI_n(f)\overline{I_k(g)} = \delta_{nk}\, n!\,(\operatorname{sym} f, g)_n$

for any $k\geqq 1$ and $g\in L^2(\mathbb{R}^k,\cdot)$, where δ_{nk} is Kroneker's δ and $(\cdot,\cdot)_n$ is the scalar product in $L^2(\mathbb{R}^n,\cdot)$.

We say that $f \in L^2(\mathbb{R}^n, \cdot)$ is even if $\overline{f_{i_1 \dots i_n}(x_1, \dots, x_n)} = f_{i_1 \dots i_n}(-x_1, \dots, -x_n)$, $i_1, \dots, i_n = 1, \dots, m$, $x_1, \dots, x_n \in \mathbb{R}$. If $f \in L^2(\mathbb{R}^n, \cdot)$ is even, then $I_n(f)$ is real.

Given a function $f \in L^2(\mathbb{R}^n, \mathbb{C}^1)$ and $i_1, \dots, i_n \in \{1, \dots, m\}$, we define

$$\int_{\mathbb{R}^n} f(x_1, \dots, x_n)\, dW_{i_1} \dots dW_{i_n} = I_n(\tilde{f}),$$

where $\tilde{f} \in L^2(\mathbb{R}^n, (\otimes \mathbb{C}^m)^n)$, $\tilde{f}_{j_1 \dots j_n} = f$ if $(j_1, \dots, j_n) = (i_1, \dots, i_n)$, $= 0$ if otherwise. In the case $(r_{ij}) = (\delta_{ij})$, $(w3)$ implies that

$$E \int_{\mathbb{R}^n} f(x_1, \dots, x_n)\, dW_{i_1} \dots dW_{i_n} \cdot \overline{\int_{\mathbb{R}^n} g(x_1, \dots, x_n)\, dW_{j_1} \dots dW_{j_n}}$$
$$= \chi(i_1, \dots, i_n; j_1, \dots, j_n) \int_{\mathbb{R}^n} f \bar{g}\, d^n x \tag{3.2}$$

where $\chi(i_1, \dots, i_n; j_1, \dots, j_n)$ is the number of permutations $p = (p(1), \dots, p(n)) \in \mathscr{P}_n$ such that $(i_{p(1)}, \dots, i_{p(n)}) = (j_1, \dots, j_n)$. In particular,

$$E \left| \int_{\mathbb{R}^n} f(x_1, \dots, x_n)\, dW_1 \dots dW_1\, dW_2 \right|^2 = (n-1)! \int_{\mathbb{R}^n} f \bar{g}\, d^n x.$$

If $A \subset \mathbb{R}^n$ is Borel and $f\colon A \to \mathbb{C}$ is square integrable on A, then $\int_A f\, dW_{i_1} \dots dW_{i_n} = \int_{\mathbb{R}_n} f \cdot 1_A\, dW_{i_1} \dots dW_{i_n}$ by definition. Finally, $2 \operatorname{Re} \int_{\mathbb{R}_n} f\, dW_{i_1} \dots dW_{i_n} = \int_{\mathbb{R}^n_+ \cup \mathbb{R}^n_-} f'\, dW_{i_1} \dots dW_{i_n}$, where $\mathbb{R}^n_\pm = \{x \in \mathbb{R}^n\colon\ x_i \gtrless 0,\ i = 1, \dots, n\}$ and $f'(x) = f(x)$ if $x \in \mathbb{R}^n_+$, $= \bar{f}(-x)$ if $x \in \mathbb{R}^n_-$.

Theorem 7. *Let* $S^{(n)}_{N,t}$ *be defined by* (1.1), (1.2), *where* $F(dx) = dx$ *and*

$$\varphi_n(x_1, \dots, x_n) = \begin{cases} c_n^{-1} |x_1 + \dots + x_n|^{-1/2} & \text{if } \ x_1(x_2 + \dots + x_n) > 0 \\ & \text{and } \ |x_1 + \dots + x_n| < \pi, \\ 0 \ \text{ if otherwise in } \Pi^n, & \end{cases} \tag{3.3}$$

$$c_n = \Big((n-1)! \int_{\Pi^{n-2}} 1(|x_1 + \dots + x_{n-2}| < \pi)\, d^{n-2} x\Big)^{1/2}, \ n > 2, \ \ c_2 = 1.$$

Then

$$N^{-1/2} S^{(n)}_{N,t} \overset{d}{\Rightarrow} \int_{\mathbb{R}^2} \Phi_t(x_1, x_2)\, W_1(dx_1)\, W_2(dx_2) \equiv \zeta_t, \tag{3.4}$$

where

$$\Phi_t(x_1, x_2) = (e^{it(x_1 + x_2)} - 1) / [i(x_1 + x_2) |x_1 + x_2|^{1/2}]$$

if $x_1 \cdot x_2 > 0$, $= 0$ *if otherwise in* $\mathbb{R}^2$, $W = (W_1, W_2)$ *is* $\mathbb{C}^2$*-valued white noise with covariance* $r_{ij} = 1$ $(n = 2)$, $r_{ij} = \delta_{ij} (n > 2)$ *respectively*, $i, j = 1, 2$.

Proof. As before we'll prove the covergence of one-dimensional distributions only. Write $\zeta_1 = \zeta$, $S^{(n)}_{N,1} = S^{(n)}_N$, $\Phi_1(x_1, x_2) = \Phi(x_1, x_2)$. Set also

$$S'_N = \sum_{(j_1, j_2) \in \Lambda;\, j_1 \neq j_2} N^{3/2} \Phi(j_1/m, j_2/m) \int_{B_N(j_1, j_2)} d^n Z, \tag{3.5}$$

$$\zeta' = \sum_{(j_1, j_2) \in \Lambda,\, j_1 \neq j_2} \Phi(j_1/m, j_2/m) \int_{B(j_1, j_2)} dW_1\, dW_2, \tag{3.6}$$

where

$$\Lambda=\{(j_1,j_2)\in\mathbb{Z}^2: |j_i|\leqq Km, i=1,2, j_1\cdot j_2>0\},$$
$$B_N(j_1,j_2)=\{x\in\Pi^n: j_1/m\leqq Nx_1<(j_1+1)/m,$$
$$j_2/m\leqq N(x_2+\ldots+x_n)<(j_2+1)/m\},$$
$$B(j_1,j_2)=[j_1/m,(j_1+1)/m)\times[j_2/m,(j_2+1)/m).$$

Lemma 8. *For any* $\varepsilon>0$, *there exist* $K>0$, $m\geqq 1$ *and* $N_0=N_0(\varepsilon, m, K)$ *such that*

$$E|(S_N^{(n)}-S'_N)/\sqrt{N}|^2+E|\zeta-\zeta'|^2<\varepsilon, \qquad \forall N>N_0.$$

Proof. Set $\alpha_N(x)=e^{ix}(e^{iNx}-1)/(e^{ix}-1)$, $\tilde{\alpha}_N(x_1,\ldots,x_n)=\alpha_N(x_1+\ldots+x_n)$. Then $S_N^{(n)}=\int_{\Pi^n}\varphi_n\,\tilde{\alpha}_n\,d^nZ$ and $E|S_N^{(n)}-S'_N|^2\leqq C(i_1+i_2)$ where

$$i_1=\int|\varphi_n\tilde{\alpha}_N|^2\,1(x\in\Pi^n: K/N<|x_1+\ldots+x_n|<\pi)\,d^nx,$$
$$i_2=\sum_{(j_1,j_2)\in\Lambda}\int_{B_N(j_1,j_2)}d^nx|\varphi_n\tilde{\alpha}_N-N^{3/2}\,\Phi(j_1/m,j_2/m)\,1(j_1\neq j_2)|^2.$$

As $|e^{ix}-1|>C|x|$ for $|x|<\pi$, we have

$$i_1\leqq C\int_0^\pi\int_0^\pi dy_1\,dy_2|y_1+y_2|^{-3}\,1(|y_1+y_2|>K/N)$$
$$\leqq C\int_{K/N}^\pi y^{-2}\,dy\leqq CN/K.$$

By turning to the new coordinates $y_1=Nx_1$, $y_2=N(x_2+\ldots+x_n)$, $y_3=x_3,\ldots,y_n=x_n$, $|d^nx/d^ny|\leqq C/N^2$ one has

$$i_2\leqq CN\int_{(0,K]^2}d^2y|h_{N,m}|^2,$$

where

$$h_{N,m}(y_1,y_2)=(e^{i(y_1+y_2)}-1)/[(y_1+y_2)^{1/2}(e^{i(y_1+y_2)/N}-1)\,N]$$
$$-1(j_1\neq j_2)\cdot\Phi(j_1/m,j_2/m), \qquad (y_1,y_2)\in B(j_1,j_2).$$

Note that $h_{N,m}\to 0$ $(N\to\infty,\ m\to\infty)$ d^2y- a.e. in $(0,K]^2$. Moreover, the double sequence $|h_{N,m}|^2$, $N\geqq 2K$, $m\geqq 1$ is uniformly integrable in $(0,K]^2$. This proves the statement of Lemma 8 about the difference $S_N^{(n)}-S'_N$. The difference $\zeta-\zeta'$ can be discussed analogously. □

By Lemma 8, it remains to prove that

$$\Big(N\int_{B_N(j_1,j_2)}d^nZ,\quad (j_1,j_2)\in\Lambda\Big)\overset{d}{\Rightarrow}\Big(c_n\int_{B(j_1,j_2)}dW_1\,dW_2,\quad (j_1,j_2)\in\Lambda\Big). \tag{3.7}$$

If $n=2$ then (3.7) holds with $\overset{d}{=}$ instead of $\overset{d}{\Rightarrow}$ for N sufficiently large, as $\sqrt{N}\,Z(dx/N)\overset{d}{=}W(dx)$, $|x|<\pi N$. Let $n>2$ and set

$$\beta_{N,1}(j)=\sqrt{N}\,Z(j/mN,(j+1)/mN),$$
$$\beta_{N,2}(j)=\sqrt{N}\int 1(x\in\Pi^{n-1}, j/Nm<x_1+\ldots+x_{n-1}<(j+1)/Nm)\,d^{n-1}Z,$$
$$w_i(j)=W_i([j/m,(j+1)/m)), \qquad i=1,2.$$

As $\int_{B(j_1, j_2)} dW_1\, dW_2 = w_1(j_1)\, w_2(j_2)$ (W_1 and W_2 are independent) and

$$E|N \int_{B_N} d^n Z - \beta_{N,1}(j_1)\, \beta_{N,2}(j_2)|^2 \leqq C/N \tag{3.8}$$

(see below), (3.7) follows from

$$(\beta_{N,i}(j_i), |j_i| \leqq Km,\ i=1,2) \xrightarrow{d} (d_i\, w_i(j_i), |j_i| \leqq Km,\ i=1,2), \tag{3.9}$$

where $d_1=1$, $d_2=c_{n-2}$. To prove (3.8), use the multiplication rule for Ito-Wiener integrals ([13], Proposition 5.1), according to which

$$\beta_{N,1}(j_1)\, \beta_{N,2}(j_2) = N \int_{B_N} d^n Z$$
$$+(n-1) \int_{\Pi^{n-2}} \left\{ \int_{-\pi}^{\pi} f_{j_1}(y)\, g_{j_2}(-y, x_1, \ldots, x_{n-2})\, dy \right\} d^{n-2} Z,$$

where $f_j = \sqrt{N}\, 1(j/Nm \leqq x < (j+1)/Nm)$, $g_j = \sqrt{N}\, 1(j/Nm \leqq x_1 + \ldots + x_{n-1} < (j+1)/Nm)$. It is easy to check that $\int_{\Pi^{n-2}} \left| \int_{-\pi}^{\pi} f_{j_1}(y)\, g_{j_2}(-y, \cdot)\, dy \right|^2 d^{n-2} x \leqq C/N$, which implies (3.8).

Note that the covariances of the left hand side of (3.9) tend to the corresponding covariances of the right hand side, as $N \to \infty$ (In particular, $\beta_{N,1}(j_1)$ and $\beta_{N,2}(j_2)$ are not correlated for any j_1, j_2 as they are given by Ito-Wiener integrals of different multiplicities.) It remains to show that the limit distribution of any linear combinations of $\beta_{N,i}$'s is (complex) Gaussian. This can be done by evaluating the semi-invariants of order $k \geqq 3$ similarly as in Section 1–2. For simplicity, let us consider $\operatorname{Re} \beta_{N,2}(j) = (\beta_{N,2}(j) + \beta_{N,2}(-j-1))/2$. Let $p = k(n-1)$ be even. By (1.5), the k-th semi-invariant of $\operatorname{Re} \beta_{N,2}(j)$ is equal to

$$2^{-k+1} \sum_{\gamma} N^{k/2} \int_{\Pi^{p/2}} \left(\bigotimes_{i=1}^{k} g \right)_{\gamma} d^{p/2} x \equiv \sum_{\gamma} J_{\gamma},$$

where $g = g_j$ (see above) and the sum is taken over all connected diagrams of the table (1.6) with $n_1 = \ldots = n_k = n-1$. Let x_{ij}, $i=1, \ldots, k$, $j=1, \ldots, n-1$ be related by (1.7). Among variables $y_i = x_{i,1} + \ldots + x_{i,n-1}$, $i=1, \ldots, k$, there are $k-1$ linearly independent ones; see Lemma 4 and Definition 1. From here it follows easily that $J_{\gamma} \leqq C N^{k/2} (Nm)^{1-k} \to 0$, if $k \geqq 3$. □

Theorem 8. *Let $\tilde{S}_N^{(n)}$ be defined as in Theorem 7 with the difference that φ_n (3.3) is replaced by*

$$\tilde{\varphi}_n(x) = \varphi_n(x) \cdot 1(B_{n,n}), \tag{3.10}$$

where $B_{n,k} = \{x \in \Pi^k : b(n+1) \leqq |x_1 + \ldots + x_k| < b(n)\}$, $b(n) \downarrow 0$,

$$b(n)\, N(n) \to \infty, \qquad b(n+1)\, N(n) \to 0 \quad (n \to \infty), \tag{3.11}$$

and $1 \leqq N(n) \uparrow \infty$ are integers increasing sufficiently fast with n; $\tilde{S}_N = \sum_{n \geqq 3} \tilde{S}_N^{(n)}$. Then

$$\tilde{S}_{N(n)} / \sqrt{N(n)} \xrightarrow{d} \zeta_1 \quad (n \to \infty),$$

where ζ_t is given by (3.4), with independent W_1 and W_2.

Proof. Let $S_N^{(n)}$ be the same as in the previous theorem. As $S_N^{(n)}/\sqrt{N} \overset{d}{\Rightarrow} \zeta_1$ $(N\to\infty)$ with ζ_1 independent of $n\geqq 3$, this implies that

$$S_{N(n)}^{(n)}/\sqrt{N(n)} \overset{d}{\Rightarrow} \zeta_1 \qquad (n\to\infty) \tag{3.12}$$

if $N(n)$ increase sufficiently fast. With (3.12) in mind, it remains to show that

$$\operatorname{Var}(\tilde{S}_{N(n)} - \tilde{S}_{N(n)}^{(n)}) = o(N(n)), \qquad \operatorname{Var}(\tilde{S}_{N(n)}^{(n)} - S_{N(n)}^{(n)}) = o(N(n)). \tag{3.13}$$

Let us prove the first of the relations (3.13), as the second one can be proved analogously. Consider

$$\begin{aligned}\operatorname{Var}\tilde{S}_N^{(n)} &= n! \int_{\Pi^n} (\operatorname{sym}\tilde{\varphi}_n)^2 D_N^2(x_1+\ldots+x_n)\,d^n x \\ &= (n-1)! \int_{\Pi^n} \tilde{\varphi}_n^2 D_n^2(x_1+\ldots+x_n)\,d^n x + R_{N,n}.\end{aligned}$$

Here,

$$\begin{aligned}\int_{\Pi^n} \tilde{\varphi}_n^2 D_N^2(\ldots)\,d^n x &= c_n^{-2} \int_{B_{n,2}} |y_1+y_2|^{-1} D_N^2(y_1+y_2)\,1(y_1\cdot y_2>0)\,d^2 y \\ &\quad\cdot \int_{\Pi^{n-2}} 1(|y_2-x_1-\ldots-x_{n-2}|<\pi)\,d^{n-2}x.\end{aligned}$$

Denote the last integral by $\theta_n(y_2)$. Then $\theta_n(y)\uparrow c_n^2/(n-1)!$ $(y\downarrow 0)$ and consequently

$$(n-1)! \int_{\Pi^n} \tilde{\varphi}_n^2 D_N^2(\ldots)\,d^n x \leqq \int_{B_{n,1}} D_N^2(u)\,du. \tag{3.14}$$

On the other hand,

$$R_{N,n} = (n-1)!\,(n-1) \int_{\Pi^n} \tilde{\varphi}_n^2 D_N^2(\ldots)\,1(A_1\cap A_2)\,d^n x,$$

where $A_j = \{x\in\mathbb{R}^n : x_j \sum_{i\neq j} x_i > 0\}$. Clearly,

$$\begin{aligned}R_{N,n} &\leqq n!\,c_n^{-2} \int_{B_{n,2}} |y_1+y_2|^{-1} D_N^2(y_1+y_2)\,1(y_1\cdot y_2>0)\,d^2 y \\ &\quad\cdot \int_{\Pi^{n-2}} 1(x\in\Pi^{n-2} : x_1(y_1+y_2-x_1)>0)\,d^{n-2}x \\ &\leqq n!\,c_n^{-2}(2\pi)^{n-2}\,b(n) \int_{B_{n,1}} D_N^2(u)\,du.\end{aligned} \tag{3.15}$$

If $N(n-1)\geqq n!\ c_n^{-2}(2\pi)^{n-2}$, $n\geqq 2$, it follows from (3.11), (3.14) and (3.15) that

$$\operatorname{Var}\tilde{S}_{N(n)}^{(k)} \leqq C \int_{B_{k,1}} D_{N(n)}^2(u)\,du,\ k\geqq 3.$$

Therefore

$$\begin{aligned}\operatorname{Var}(\tilde{S}_{N(n)} - \tilde{S}_{N(n)}^{(n)}) &= \sum_{k\geqq 3,\,k\neq n} \operatorname{Var}\tilde{S}_{N(n)}^{(k)} \\ &\leqq C\left(\int_0^{b(n+1)} D_{N(n)}^2(y)\,dy + \int_{b(n)}^{\pi} D_{N(n)}^2(y)\,dy\right) \equiv C(I'+I'').\end{aligned}$$

Here, $I' \leqq CN^2(n)\, b(n+1) = o(N(n))$, $I'' \leqq C \int_{b(n)}^{\pi} y^{-2}\, dy = C/b(n) = o(N(n))$ according to (3.11), □

Theorem 9. *Let $(X_t)_{t\in\mathbb{Z}}$ be stationary Gaussian process with zero mean, variance* 1 *and the spectral density*

$$f(x)=\begin{cases} c|x-\lambda_1|^{-\beta} & \text{if} \quad |x|\in(\lambda_1,\lambda_1+\varepsilon),\\ c|x-\lambda_2|^{-\beta} & \text{if} \quad |x|\in(\lambda_2-\varepsilon,\lambda_2),\\ 0 \text{ if otherwise in } \Pi, \end{cases} \tag{3.16}$$

where

$$\beta\in(1-2/n,1), \qquad 0<\lambda_1<\lambda_2<\pi,$$
$$\lambda_2=(n-1)\,\lambda_1; \tag{3.17}$$

$\lambda_1=\lambda_1(n)$ and $\varepsilon=\varepsilon(\lambda_1,n)$ are sufficiently small, $n(\geqq 3)$ is odd and $c=(1-\beta)\,\varepsilon^{\beta-1}/4$. Set $S_{N,t}=S_{N,t}^{(n)}=\sum_{s=1}^{[Nt]} H_n(X_s)$. Then

$$A_N^2=\operatorname{Var} S_{N,1}\sim C_1\, N^{2+(\beta-1)n} \tag{3.18}$$

and

$$A_N^{-1} S_{N,t} \overset{d}{\Rightarrow} C_2 \operatorname{Re} \int_{\mathbb{R}_+^n} [(e_n(x;t)-1)/i(x_1+\ldots+x_n)]$$
$$\cdot \prod_1^n x_j^{-\beta/2}\, d^{n-1} W_1\, dW_2, \tag{3.19}$$

where W_1, W_2 are the same as in Theorem 8, *and C_1, C_2 are some constants.*

Proof. Let $Z(dx)$ be the (complex) white noise in Π with variance dx. Then $X_t \overset{d}{=} \int_{\Pi} e^{itx} f^{1/2}\, dZ$,

$$H_n(X_t) \overset{d}{=} \int_{\Pi^n} e_n(x;t) \bigotimes_1^n f^{1/2}\, d^n Z,$$

$$S_{N,1}=S_N \overset{d}{=} \int_{\Pi^n} [(e_n(x,N)-1)/(e_n(x,1)-1)] \bigotimes_1^n f^{1/2}\, d^n Z \tag{3.20}$$

and

$$A_N^2=n! \int_{\Pi^n} D_N^2(x_1+\ldots+x_n) \bigotimes_1^n f\, d^n x. \tag{3.21}$$

Set

$$A(\delta)=\left\{x\in\Pi^n: \left|\sum_1^n x_j (\operatorname{mod} 2\pi)\right|<\delta\right\},$$

$$A^+(\delta)=\left\{x\in\mathbb{R}_+^n: \sum_1^n x_j<\delta\right\},$$

$$V_{\varepsilon,i}^+=\{x\in\mathbb{R}^n:\ x_i\in(-\lambda_2,-\lambda_2+\varepsilon),\ x_j\in(\lambda_1,\lambda_1+\varepsilon),\ j\neq i\},$$

$$V_{\varepsilon,i}=V_{\varepsilon,i}^+\cup V_{\varepsilon,i}^-, \qquad V_{\varepsilon,i}^-=\{x\in\mathbb{R}^n:\ -x\in V_{\varepsilon,i}^+\}.$$

If $\lambda_1=\lambda_1(n)$, $\varepsilon=\varepsilon(n,\lambda_1)$ and $(0<)\,\delta=\delta(n,\lambda_1)$ are sufficiently small, $\lambda_2=(n-1)\lambda_1$, and $n\geqq 3$ is odd, then the relations

$$x=(x_1,\dots,x_n)\in\Lambda(\delta),\qquad |x_i|\in(\lambda_1,\lambda_1+\varepsilon)\cup(\lambda_2-\varepsilon,\lambda_2),\qquad i=1,\dots,n \tag{3.22}$$

imply

$$x\in\bigcup_{i=1}^{n} V_{\varepsilon,i}. \tag{3.23}$$

Write

$$A_N^2=n!\,\Big(\int_{\Lambda(\delta)}\dots+\int_{\Pi^n\setminus\Lambda(\delta)}\dots\Big)=i_N(\delta)+i'_N(\delta).$$

Then $i'_N(\delta)\leqq C$, while

$$\begin{aligned} i_N(\delta)&=C\int_{V^+_{\varepsilon,1}\cap\Lambda(\delta)} D_N^2(x_1+\dots+x_n)\bigotimes_1^n f\,d^n x\\ &=C\int_{\Lambda^+(\delta)} D_N^2(y_1+\dots+y_n)\prod_1^n y_j^{-\beta}\,d^n y \end{aligned} \tag{3.24}$$

according to (3.16, 3.17, 3.22–3.23) and the change of variables

$$y_1=x_1+\lambda_2,\qquad y_j=x_j-\lambda_1,\quad j=2,\dots,n;\qquad \sum_1^n y_j=\sum_1^n x_j.$$

Let $\delta(N)\downarrow 0$, $\delta(1)=\delta$ and $\delta(N)N\to\infty$. By (3.24),

$$\begin{aligned} i_N(\delta(N))&\sim CN^{2+n(\beta-1)}\int_{\Lambda^+(\delta N(\delta))}[\sin(x_1+\dots+x_n)/|x_1+\dots+x_n|]^2\\ &\cdot\prod_1^n x_j^{-\beta}\,d^n x\sim CN^{2+n(\beta-1)} \end{aligned}$$

(the last integral converges as $N\to\infty$). Similarly, $i_N(\delta)-i_N(\delta(N))=o(N^{2+n(\beta-1)})$. This proves (3.18).

Denote by S'_N the stochastic integral in (3.20) with Π^n replaced by $\Lambda(\delta(N))$. By the argument above, $\mathrm{Var}(S_N-S'_N)=o(A_N^2)$. Next, replace the factor $e_n(x,1)-1$ by $i(x_1+\dots+x_n)$ in the integrand of S'_N; the resulting integral denote by S''_N. Again, it is easy to check that $\mathrm{Var}(S'_N-S''_N)=o(A_N^2)$. By (3.22–3.23),

$$S''_N=\sum_{i=1}^{n}\int_{V_{\varepsilon,i}\cap\Lambda(\delta(N))}\dots\equiv\sum_{i=1}^{n}S''_{N,i},$$

where $S''_{N,1}=\dots=S''_{N,n}$ as the integrand of S''_N is symmetric. Now, $S''_{N,n}$ can be rewritten as (c.f. (3.21))

$$\begin{aligned} S''_{N,n}&==2\,\mathrm{Re}\int_{(0,\varepsilon)^n}[(e_n(x,N)-1)/i(x_1+\dots+x_n)]\prod_1^n x_j^{-\beta/2}\\ &\cdot dW_1^{(\varepsilon)}(x_1)\dots dW_1^{(\varepsilon)}(x_{k-1})\,dW_2^{(\varepsilon)}(x_k), \end{aligned}$$

where $dW_1^{(\varepsilon)}(x)=dZ(x+\lambda_1)$, $dW_2^{(\varepsilon)}(x)=dZ(x-\lambda_2)$, $0<x<\varepsilon$ are independent, if $\varepsilon>0$ is sufficiently small. By the change of variables in Ito-Wiener integrals ([13], Theorem 4.4),

$$S''_{N,n} \stackrel{d}{=} N^{1+(\beta-1)n/2}\, 2\,\mathrm{Re} \int_{(0,\varepsilon N)^n} [(e_n(x,1)-1)/i(x_1+\ldots+x_n)] \cdot \prod_1^n x_j^{-\beta/2}\, d^{n-1} W_1\, dW_2,$$

where W_1, W_2 are the same as in (3.19). The last integral converges in $L^2(\Omega)$ as $N\to\infty$. The convergence of general finite dimensional distributions of $S_{N,t}$ can be considered analogously. □

Remark 3. Let $r(t)$, $r_{H_n}(t)=n!\,(r(t))^n$, $f_{H_n}(x)$ be the covariance function of (X_t), the covariance function and the spectral density of $(H_n(X_t))$ in Theorem 9, respectively. It follows from (3.13) that

$$r(t) \sim \mathrm{const}\, t^{\beta-1}(\sin(\pi\beta/2 - t\lambda_1)+\sin(\pi\beta/2+t\lambda_2)) \quad (t\to\infty),$$
$$f_{H_n}(x) \sim \mathrm{const}\, |x|^{n(1-\beta)-1} \quad (|x|\to 0).$$

Consequently, $\sum |r_{H_n}(t)|=\infty$ if $\beta\in(1-1/n,1)$ while $\sum |r_{H_n}(t)|=\infty$ and $\sum r_{H_n}(t) =2\pi f_{H_n}(0)=0$ if $\beta\in(1-2/n,\, 1-1/n)$.

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