# An invariant for certain smooth manifolds

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*To Enrico Bompiani on his scientific Jubilee*

**Summary.** - *Starting with Hirzebruch's $\widehat{A}$-genus we define a numerical invariant $\mu$ for certain $(4k-1)$-manifolds. We produce examples to show how $\mu$ can distinguish differentiable structures on certain topological manifolds.*

## 1. - Introduction.

In his study [4, 5, 6] of differentiable structures JOHN MILNOR has defined differential invariants for special classes of closed smooth manifolds of dimension $4k-1$, and has used them to distinguish certain differentiable structures on spheres. His construction in each case is based on HIRZEBRUCH'S formula expressing the index of a closed smooth manifold in terms of its PONTRJAGIN classes; see Section 2 below.

The purpose of the present paper is to construct an invariant for a similar class of smooth manifolds, this time based on HIRZEBRUCH'S theorem that the $\widehat{A}$-genus of a closed smooth spin manifold is an integer; see Section 3. We compare our invariant with MILNOR'S invariants in low dimensions and apply it to various differentiable manifolds.

From theories of MILNOR and SMALE we conclude that our invariant $\mu$ determines the complete classification of the differentiable structures on the topological spheres $S^7$ and $S^{11}$ of dimension 7 and 11, but not for $S^{15}$ for example. We also consider inequivalent differentiable $S^3$-fibrations of the usual as well as of the unusual differentiable structures on $S^7$.

Another application is given in [2], where we use the invariant to show that certain closed 3-connected triangulable 8-manifolds (which are like the quaternionic projective plane, in a sense made precise in [2], have homotopy types distinct from that of any closed differentiable manifold. This proves

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again the theorem of KERVAIRE that there exist manifolds which do not admit any differentiable structure.

## 2. - The invariants of Milnor.

Milnor has constructed two differential invariants (equations (2) and (3) below) as follows:

*Conditions* $\lambda$). - Let $M = M^{4k-1}$ be a closed smooth (= differentiable) oriented $(4k-1)$-manifold whose rational cohomology groups satisfy

$$H^{2k}(M;\ Q) = 0, \text{ and } H^{4i}(M;\ \mathbf{Q}) = 0 \text{ for } 0 < i < k;$$

note that by POINCARÉ duality we also have

$$H^{2k-1}(M;\ Q) = 0 \text{ and } H^{4i-1}(M;\ \mathbf{Q}) = 0 \quad (0 < i < k).$$

Suppose there exists a compact, smooth, oriented $4k$-manifold $W = W^{4k}$ having $M$ as boundary; we will speak of $W$ as a *coboundary* of $M$. The cohomology sequence of the pair $(W, M)$ then leads to isomorphisms

$$j:\ H^{4i}(W,\ M;\ Q) \to H^{4i}(W;\ Q) \qquad (0 < i < k).$$

Hence if $p_i(W) \in H^{4i}(W;\ Q)$ is the $i^{\text{th}}$-PONTRJAGIN class of $W$, then $j^{-1}p_i(W)$ is well defined. For any polynomial $K(x_1, \ldots, x_k)$ with rational coefficients, for which $K(x_1, x_2^2, \ldots, x_k^k)$ is homogeneous of degree $k$ in $x_1, \ldots, x_k$, we define the rational number

$$K(p_1, \ldots, p_{k-1}, 0)[W] = K(j^{-1}p_1(W), \ldots, j^{-1}p_{k-1}(W), 0)[W, M] \tag{1}$$

where $[W, M]$ in the right member denotes the fundamental homology class of the oriented pair $(W, M)$.

If $L_k$ denotes the $k^{th}$ polynomial associatod with $z^{1/2}/tghz^{1/2}$ (see HIRZEBRUCH [3, p. 13]) and $s_k = L_k(0, \ldots, 0, 1)$ is the coefficient of $p_k$ in $L_k$, and if $\tau[W]$ is the index of $W$, then MILNOR'S first invariant [5, p. 965] is

$$\lambda(M^{4k-1}) \equiv \{\tau(W) - L_k(p_1, \ldots, p_{k-1}, 0)[W]\}/s_k \text{ mod. } 1. \tag{2}$$

The right member is independent of the choice of the smooth oriented coboundary $W$. That definition of $\lambda(M^{4k-1})$ is a mild modification of MILNOR'S earlier definition in [4, footnote p. 400].

MILNOR'S invariant $\lambda'$ is defined for manifolds satisfying.

*Conditions* $\lambda'$) - $M$ is a smooth oriented homology $(4k-1)$-sphere over the integers for which there exists a compact, smooth, oriented, parallelizable coboundary $W$ (see MILNOR [6]. For any such $W$ define

$$\lambda'(M^{4k-1}) \equiv \tau[W]/8 \qquad \text{mod. } I_k/8 \tag{3}$$

where $I_k$ is the greatest common divisor of the indices $\tau[X]$ of all closed, smooth, almost parallelizable $4k$-manifotds $X$. Then $I_k$ is divisible by 8 by Lemma 3.2 of MILNOR [6]. The value of $I_k$ is known to us for $1 \leq k \leq 5$. Again, the right member of (3) is independent of $W$.

Both of these invariants are based on the Index Theorem

$$L_k(p_1, \ldots, p_k)[X^{4k}] = \tau[X^{4k}]$$

of HIRZEBRUCH [3, p. 85] and the following Lemma ([4] and [6]).

LEMMA. - *Let $W_1$ and $W_2$ be two smooth oriented $4k$-manifolds with coherently oriented boundaries $M_1$ and $M_2$. Let $\iota: M_1 \to M_2$ be a diffeomorphism which carries the orientation of $M_1$ onto that of $M_2$. If $-W_2$ denotes $W_2$ with opposite orientation, then there is a natural orientation and differentiable structure on the identification space $X = W_1 \cup (-W_2)/(\iota)$. For any pairs $(W_1, M_1)$ and $(W_2, M_2)$ satisfying Conditions $\lambda$); and for any polynomial $K$ as above, we have*

$$\tau[X] = \tau[W_1] - \tau[W_2], \tag{4}$$

$$K(p_1, \ldots, p_{k-1}, 0)[X] = K(p_1, \ldots, p_{k-1}, 0)[W_1] - K(p_1, \ldots, p_{k-1}, 0)[W_2].$$

## 3. - The invariant $\mu$.

We will now define an invariant for certain manifolds satisfying Conditions $\mu$) below, based on Lemma 2 and the following theorem (see BOREL-HIRZEBRUCH [1]). Here $\widehat{A}[X^{4k}] = \widehat{A}_k(X^{4k})[X^{4k}] = 2^{-4k}A_k(X^{4k})[X^{4k}]$, where $\widehat{A}_k$ and $A_k$ denote the $k^{th}$-polynomial associated with $\frac{1}{2}z^{1/2}/\sinh \frac{1}{2}z^{1/2}$ and $2z^{1/2}/\sinh 2z^{1/2}$ respectively; see HIRZEBRUCH [3, p. 14]. An orientable manifold $X$ with STIEFEL-WHITNEY class $w_2(X) = 0$ is called a *spin manifold*.

THEOREM (HIRZEBRUCH). - *Let $X$ be a closed, smooth, oriented spin manifold of dimension $4k$. Then the $\widehat{A}$-genus $\widehat{A}[X]$ is an integer; if $k$ is odd, then $\widehat{A}[X]$ is an even integer.*

There are examples [2] to show that $\widehat{A}[X]$ can be odd if $k$ is even.

Elimination of $p_k[X]$ from

$$\tau[X] = L_k(p_1, \dots, p_{k-1}, 0)[X] + L_k(0, \dots, 0,)p_k[X],$$

$$\widehat{A}[X] = \widehat{A}_k(p_1, \dots, p_{k-1}, 0)[X] + \widehat{A}_k(0, \dots, 0, 1)\ p_k[X]$$

gives

$$\widehat{A}[X] = N_k(p_1, \dots, p_{k-1})[X] + t_k\tau[X] \tag{5}$$

where we have set $t_k = \widehat{A}_k(0, \dots, 0, 1))/L_k(0, \dots, 0, 1)$ and

$$N_k(p_1, \dots, p_{k-1})[X] = \widehat{A}_k(p_1, \dots, p_{k-1}, 0)[X] - t_k L_k(p_1, \dots, p_{k-1}, 0)[X].$$

$\mu$ will be defined for $(4k-1)$-manifolds that satisfy the following:

*Conditions* $\mu$). - $M = M^{4k-1}$ is a closed, smooth, oriented $(4k-1)$-manifold, having a compact, smooth, oriented *spin coboundary* $W$, such that

($a$) the homomorphisms in the exact sequence of the pair $(W, M)$ with respect to the field of rational numbers $\boldsymbol{Q}$ (we will omit the field in the notation only if the field is $\boldsymbol{Q}$);

$$j^*: H^{2k}(W, M) \to H^{2k}(W)$$

$$j^*: H^{4i}(W, M) \to H^{4i}(W) \qquad (0 < i < k)$$

are *isomorphisms*.

($b$) The inclusion homomorphism

$$i^*: H^1(W; Z_2) \to H^1(M; Z_2)$$

is an *epimorphiam*.

Condition ($a$) permits the unique pulling back of the PONTRJAGIN classes of $W$ into $H^*(W, M)$ as in the case of MILNOR'S $\lambda$. In view of the exact sequences

$$\dots \to H^{2k-1}(M) \to H^{2k}(W, M) \to H^{2k}(W) \to H^{2k}(M) \to \dots$$

$$\dots \to H^{4i-1}(M) \to H^{4i}(W, M) \to H^{4i}(W) \to H^{4i}(M) \to \dots$$

*the isomorphisms* required under ($a$) *certainly hold in case:*

$$H^{2k-1}(M) = H^{4i-1}(M) = 0,\ 0 < i < k,$$

and (POINCARÉ duality)

$$H^{2k}(M) \quad = H^{4i}(M) \quad = 0,\ 0 < i < k.$$

On the other hand *condition* (b) *holds for every coboundary* $W$ *in casa* $H^1(M;\ Z_2) = 0$.

The Conditions $\mu$) also hold for example for the product space $M = S^4 \times S^3$ of $S^4$ and $S^3$, with their usual differentiable structures. To see this we take $W = D^5 \times S^3$, the product space of the 5-disc and $S^3$. Then in the exact homology sequence $i^*$: $H^3(W) \to H^3(M)$ is an isomorphism, whence $H^4(W) = H^4(W,\ M) = 0$.

Thus the Conditions $\mu$ are satisfied.

The conditions are also satisfied for $M = X^2 \times S^5$ with coboundary $X^2 \times D^6$, where $X^2$ is any oriented closed surface, and for $M = S^1 \times S^2$ with coboundary $W = S^1 \times D^3$. There are other large classes of smooth manifolds satisfying Conditions $\mu$) given by TAMURA [10].

We define the *differential invariant* $\mu(M^{4k-1})$ as the modulo 1 reduction of

$$\mu(W,\ M) = \{N_k(p_1,\ \dots,\ p_{k-1})[W] + t_k\tau[W]\}/a_k, \tag{6}$$

where $a_k = 4/(3 + (-1)^k)$. Thus

$$\mu(M^{4k-1}) \equiv \mu(W^{4k},\ M^{4k-1}) \bmod 1, \tag{7}$$

computed for any spin coboundary $W$ that satisfies Conditions $\mu$).

THEOREM. - *The right member of* (6) *is independent of the choice of* $W$. *Thus* $\mu(M^{4k-1})$ *depends only on the differentiable structure of* $M^{4k-1}$.

PROOF: - Let $(W_1,\ M_1)$ and $(W_2,\ M_2)$ be two pairs for the space $M$, satisfying the Conditions $\mu$), and let $X = \{W_1 \cup (-W_2)\}/(\iota)$ be as in lemma 2. Let $r$ be $2k$ or $4i(0 < i < k)$. In the *commutative* diagram of cohomology groups over $\boldsymbol{Q}$,

$$\begin{array}{ccc} H^r(W_1,\ M) \oplus H^r(W_2,\ M) & \xleftarrow{h} & H^r(X,\ M) \\ \downarrow j_1^* \oplus j_2^* & & \downarrow j^* \\ H^r(W_1) \oplus H^r(W_2) & \xleftarrow{k} & H^r(X) \end{array}$$

$h$ and $j_1^* \oplus j_2^*$ are isomorphisms.

From

$$H^r(M) \xleftarrow{\text{zero}} H^r(W_1) \xleftarrow{\text{isom.}} H^r(W_1,\ M) \xleftarrow{\text{zero}} H^{r-1}(M)$$

it follows by composition with $H^r(X) \to H^r(W_1)$ that $H^r(X) \to H^r(M)$ is the zero homomorphisn.

Consequently in the exact sequence

$$H^r(M) \xleftarrow{\text{zero}} H^r(X) \xleftarrow{j^*} H^r(X,\ M)$$

j* is an epimorphism.

Then all homomorphisms in the above commutative diagram are *isomorphisus.*

As in [4, 7, 10] one concludes that

$$\tau[X] = \tau[W_1] - \tau[W_2]$$

and

$$K(p_1,\ \ldots,\ p_{k-1},\ 0)[X] = K(p_1,\ \ldots,\ p_{k-1},\ 0)[W_1] - K(p_1,\ \ldots,\ p_{k-1},\ 0)[W_2]$$

with for example $K(p_1,\ \ldots,\ p_{k-1},\ 0) = N_k(p_1,\ \ldots,\ p_{k-1})$.

Consequently

$$\begin{aligned}\mu(W_1,\ M_1) - \mu(W_2,\ M_2) &= \{N_k(p_1,\ \ldots,\ p_{k-1})[X] + t_k\tau[X]\}/a_k \\ &= \widehat{A}[X]/a_k,\end{aligned}$$

and this an integer in case $X$ is a spin manifold. So this last fact remains to be proven.

Cosider the exact MAYER-VIETORIS cohomology sequence:

$$\ldots \leftarrow H^2(W_1;\ Z_2) \oplus H^2(W_2;\ Z_2) \xleftarrow{k_1^* \oplus k_2^*} H^2(X;\ Z_2) \xleftarrow{\Delta} H^1(M;\ Z_2)$$

$$\xleftarrow{i_1^* - i_2^*} H^1(W_1;\ Z_2) \oplus H^1(W_2;\ Z_2) \leftarrow \ldots$$

where $i_\alpha \colon M \to W_\alpha$ and $k_\alpha \colon W_\alpha \to X$ are inclusion maps for $\alpha = 1,\ 2$.

The image of the second STIEFEL-WHITNEY class

$$w_2(X) \in H^2(X;\ Z_2) \text{ is } k_1^*(w_2(X)) \oplus k_2^*(w_2(X)).$$

As $k_\alpha$ induces a bundle nap of the tangent bundle of $W_\alpha$ into that of $X$, we find

$$w_2(W_\alpha) = k_\alpha^*(w_2(X))$$

which vanishes because $W_\alpha$ is a spin manifold. Then $(k_1^* \oplus k_2^*)\ w_2(X) = 0$. In view of (*b*), $i_1^* - i_2^*$ is an epinorphism, $\Delta$ is zero, and $k_1^* \oplus k_2^*$ is a monomorphism by exactness. Then $w_2(X) = 0$ , and the theorem is proved.

REMARK. - The invariant $\mu$ has the following three properties, shared by MILNOR's invariants $\lambda$ and $\lambda'$, and proved the same way:

1) $\mu(-M) = -\mu(M)$;

2) $\mu(M_1 \# M_2) = \mu(M_1) + (M_2)$, where $\#$ denotes the smooth connected sum of smooth manifolds introduced by SEIFERT; see MILNOR [6].

3) If $M_1$ and $M_2$ are $J$-equivalent (see [6]), then $\mu(M_1) = \mu(M_2)$.

REMARK. - If $M_1$ and $M_2$ satisfy Conditions $\mu$) then so does $M_1 \# M_2$.

REMARK - The first part of Conditions $\lambda$) can be weakened and replaced by $(a)$ of Conditions $\mu$). This gives a generalization of the demain of definition for $\lambda$.

## 4. - Comparison of $\lambda'$ and $\mu$.

Let us consider these invariants in their common domain of definition. For any such $M$ we have

$$\mu(M^{4k-1}) \equiv t_k \tau[W]/a_k \mod 1,$$

for any parallelizable coboundary $W$. The number $t_k$ can be computed by the method of HIRZEBRUCH [3, p. 13] as follows:

If $K$ is the multiplicative sequence belonging to the function $Q$, then

$$Q(z)\frac{d}{dz}(z/Q\,z)) = \sum_{j=0}^{\infty}(-1)^j s_j z^j$$

with $s_j = K_j(0, \dots, 0, 1)$. Thus for the special case $K = L$ we have

$$s_0 = 1,\ s_k = L_k(0, \dots, 0, 1) = 2^{2k}(2^{2k-1} - 1)B_k/(2k)!,$$

where the $B_k$ are BERNOULLI, numbers, and for the case $K = \widehat{A}$ we have

$$\widehat{s_0} = 1,\ \widehat{s_k} = \widehat{A}_k(0, \dots, 0, 1) = -B_k/2(2k)!$$

As a consequence we find

$$t_k = \widehat{s}_k/s_k = (-1)/(2^{2k+1}(2^{2k-1} - 1)), \tag{9}$$

whence

$t_0 = 1,\ t_1 = -1/8,\ t_2 = -1/2^5.\ 7,\ t_3 = -1/2^7.\ 31,\ t_4 = -1/2^9.\ 127,$

$t_5 = -1/2^{11}.\ 511.$

Thus for the manifolds under consideration we obtain

$$\mu(M^{4k-1}) \equiv -\tau[W]/a_k(2^{2k+1}(2^{2k-1}-1)) \bmod 1.$$

Let us compare this with the expression (3) for $\lambda'(M^{4k-1})$. If we take into account Lemma 3.7 of Milnor [6] we see that $I_k$ is divisible by $a_k 2^{2k+1}(2^{2k-1}-1)$; furthermore, it is known that the quotient is 1 for $1 \leq k \leq 5$ and is greater than one for $k = 6$. Thus we have the

Proposition. - *Let $M$ be a closed, smooth, oriented homology $(4k-1)$-sphere. If $M$ has a parallelizable coboundary $W$, then both $\lambda'(M^{4k-1})$ and $\mu(M^{4k-1})$ are defined. The invariant $\lambda'$ gives a differentiable classification of such spaces $M$ which is at least as fine as that given by $\mu$. For dimensions $4k-1$ with $1 \leq k \leq 5$, both invariants give the same information, and this is expressed by the formula*

$$8\lambda'(M^{4k-1})/I_k \equiv -\mu(M^{4k-1}) \bmod 1$$

*for $k = 1, 2, 3, 4$ and 5.* For $k = 6$, $\mu$ gives less information than $\lambda'$, but it can be shown that the pair $(\lambda, \mu)$ then gives more information than $\lambda'$.

On the other hand $\mu$ can be computed more easily and is defined for a substantially wider class of manifolds.

## 5. - Computations for the caso $k = 1$.

It is known that there is a unique differentiable structure on every 3-manifold. Furthermore, if $M^3$ is a closed oriented 3-manifold with $H^1(M; Z_2) = 0$, then the universal coefficient theorem implies that $H^1(M; Q) = 0$. Poincaré duality shows that we are dealing with a homology 3-sphere relative to $Q$ and $Z_2$.

For any $M^3$ satisfying Conditions $\mu)$ we hawe

$$\mu(M^3) \equiv -\tau[W]/16 \qquad \bmod 1, \tag{10}$$

computed for any spin coboundary $W$ of $M$.

Example 1. - Clearly $\mu(S^3) = 0$; thus if $M^3$ is homeomorphic to $S^3$ then $\mu(M^3)$ is defined and is zero.

Note that to find a counterexample to the Poincaré conjecture it suffices to find a simply connected closed 3-manifold with $\mu \neq 0$.

Example 2. - In [6] Milnor has constructed a homology 3-sphere $M_0^3$ (with $\pi_1(M_0^3 \neq 0!)$ having a parallelizable coboundary $W_0$ with $\tau(W_0) = 8$.

Thus $\mu(M_0^3) = -1/2$.

Let us compare $\mu$ with MILNOR'S invariants $\lambda$ and $\lambda'$ for homotopy 3-spheres. First of all, $\lambda(M^3) = 0$ for any closed 3-manifold by [5, p. 966]. Secondly, any homotopy 3 sphere is the boundary of a parallelizable manifold $W^4$[6, p. 31]; thus both $\lambda'(M^3)$ and $\mu(M^3)$ are defined, and *they provide the same information when computed using parallelizable coboundaries.* On the other hand, $\mu(M^3)$ can be computed using any spin coboundary $W$ of $M^3$.

## 6. Computations for the case $k = 2$.

For any 7-manifold $M$ satisfying Conditions $\mu$) we have

$$(11) \qquad \mu(M^7) \equiv \{p_1^2[W] - 4\tau[W]\}/2^7 . 7 \qquad \text{mod } 1,$$

computed for any oriented spin coboundary $W$ of $M$ satisfying the conditions. We do not know whether every manifold satisfying Conditions $\lambda$) has a spin coboundary; however, for those that do we have a definite refinement over MILNOR'S invariant $\lambda$, which is given by

$$(12) \qquad \lambda(M^7) \equiv \{p_1^2[W] - 4\tau[W]\}/7 \qquad \text{mod } 1$$

for any compact oriented coboundary W.

In [4] MILNOR considered bundles $(\xi_{h,j})$ over the Euclidean 4-sphere $S^4$ with rotation group $R_4$ as structural group and $S^3$ as fibre. (3) Those with Euler class $W_4(\xi_{h,j}) = h + j = 1$ are precisely the bundles with total space $M_k^7$ homeomorphic to the 7-sphere; the subscript $k = h - i = 2h - 1$. If $B_k^8$ is the associated 4-cell bundle with its natural orientation and differentiable structure, then $B_k^8$ is a spin coboundary (because $H^2(B_k^8; Z_2) = 0$); on the other hand, $B_k^8$ is not parallelizable, for $\tau[B_k^8] = 1$ is not divisible by 8. Because $p_1^2\ [B_k^8] = 2^2(2h-1)^2$ (compare (1) for the notation) we have

$$(13) \qquad \mu(M^7_{2h-1}) \equiv h(h-1)/56 \qquad \text{mod } 1$$

In order to proceed we need the

LEMMA. - *The equation*

$$h(h-1) \equiv j \qquad \text{mod } n,$$

*where* $n = p.2^r$ *and* $p$ *is a prime, has solutions for* $(p+1)2^{r-2}$ *different*

(3) Compare the end of section 9.

*values of* $j$ mod $n$. *These values are obtained (for example) by talcing* $h$ mod $n$ *so that*

$$h \equiv 1,\ 2,\ \ldots\ or\ (p+1/2 \qquad \text{mod } p,\ and$$

$$h \equiv 1,\ 2,\ \ldots\ or\ 2^{r-1} \qquad \text{mod } 2^r.$$

PROOF. - The lemma follows from the fact that $h(h-1) \equiv m(m-1)$ mod $n$ or equivalently $(m+h-1)(m-h) \equiv 0$ mod $n$ if and only if

$$(m+h-1)(m-h) \equiv 0 \text{ mod } p \textit{ and } (m+h-1)(m-h) \equiv 0 \quad \text{mod } 2^r.$$

Thus $m \equiv h$ or $1-h$ mod $p$ *and* $m \equiv h$ or $1-h$ mod $2^r$.

If we apply the lemma to our case $p=7$, $r=3$ we conclude that $h(h-1)$ mod 56 assumes $(7+1)2 = 16$ different values. In the following table pairs $(h;\ h(h-1)/2$ mod 28) representing all values of $h(h-1)/2$ mod 28 are given.

| $h \equiv$ | 1 | 2 | 3 | 4 | mod 7 |
|---|---|---|---|---|---|
| 1 | (1;0) | (9;8) | (17;24) | (25;20) | |
| 2 | (50;21) | (2;1) | (10;17) | (18;13) | |
| 3 | (43;7) | (51;15) | (3;3) | (11;27) | |
| 4 | (36;14) | (44;22) | (52;10) | (4;6) | |
| mod 8 | | | | | |

The values of $h(h-1)/2$ mod 28 obtained aro 0, 1, 3, 6, 7, 8, 10. 13, 14, 15, 17, 20, 21, 22, 24, 27.

Taking $h=1$, $\mu=0$ gives the Hopf fibration (STEENROD [9, p. 109]); taking $h=2$ and using the additivity of $\mu$ with respect to connected sums gives

$$\mu(M_3^7 \ \# \ldots \# \ M_3^7) = m/28 \qquad (m \text{ copies } (1 \leq m \leq 28)).$$

Since the connected sum of two smooth manifolds homeomorphic to the sphere is itself a smooth manifold homeomorphic to thesphere, these connected sums provide 28 distinot differentiable structures on the topological 7-sphere. Furthermore, their natural combinatorial structures (*i. e.*, that of the combinatorial structures of the bundle in case of $M_3^7$, and that of the connected sum in the other cases) are all isomorphic.

*This implies that the combinatorial structure on* $S^7$ *is unique*, as pointed out to us by MILNOR and SMALE.

From the theory of $J$-equivalence of MILNOR [6] and SMALE'S theorem [8] that $J$-equivalent $(2m+1)$-spheres $(m \geq 2)$ are diffeomorphic, it follows that there are precisely 28 different (*i. e.*, inequivalent) differentiable structures on the 7-sphere. Thus we obtain the

THEOREM. - *Two smooth 7-manifolds which are homeomorphic to the 7-sphere are diffeomorphic if and only if they have the same* $\mu$ *value. Of the* 28 *diffeomorphism classes, precisely* 16 *occur as total spaces of* $(R_4, S^3)$ *bundles over* $S^4$.

The other 12 differentiable 7-spheres do not possess any such fibration by 3-spheres.

COROLLARY. - $M^7_{2h-1}$ *has the usual differentiable structure of* $S^7$ *if and only if* $(h-1)h \equiv 0 \mod 56$.

EXAMPLE. - An oriented differentiable manifold $M$ homeomorphic to $S^7$, admits an orientation reversing diffeomorphism if and only if $\mu(M) \equiv -\mu(M)$ mod 1. For example $M^7_{41}$, $h = 21$, $k = 2h - 1$, with $\mu(M^7_{41}) = 1/2 \mod 1$.

EXAMPLE. - $M^7_{21}$ obtained for $h = 11$ is diffeomorphic with the homotopy sphere $M_0{}^7$ defined by MILNOR in [6, p. 13] for they both have $\mu = -1/28$. We do not know how to construct such a diffeomorphism.

COROLLARY. - *If* $M^7$ *is any manifold satisfying Conditions* $\mu$), *then the underlying topological manifold admits at least* 28 *different differentiable structures.* Examples; $M = X^2 \times S^5$; $M = S^4 \times S^3$. Compare Section 3.

PROOF. - Each $M^7 \# (M_3{}^7 \# \dots \# M_3{}^7)$with $m$ copies of $M_3{}^7$ in parentheses $(1 \leq m \leq 28)$ has $\mu$ value defined and equal to $\mu(M^7) + m/28$, and there are 28 different values. Furthermore, each such sum is an oriented smooth manifold homeomorphic to $M^7$.

EXAMPLE. - The topological space $S^2 \times S^5$ admits at least 28 different differentiable structures. If $S^2 \times S^5$ denotes the usual differentiable structure then $(S^2 \times S^5) \# M_3{}^7$ is a product space in the homeomorphic sense but it is not a product space in the diffeomorphic sense.

EXAMPLE. - The projective space $P^7(R)$ admits at least 14 different differentiable structures.

PROOF. - Let $P^7$ denote $P_7(R)$ with its usual differentiable structure and form $X = P^7 \# (M_3{}^7 \# \dots \# M_3{}^7)$ with $m$ copies of $M_3{}^7$. If $\tilde{X}$ denotes the universal cover of $X$ with induced differentiable structure, then

$$\tilde{X} = \tilde{P}^7 \# (\tilde{M}_3{}^7 \# \dots \# \tilde{M}_3{}^7) = S^7 \# (M_3{}^7 \# \dots \# M_3{}^7)$$

with $2m$ copies of $M_3{}^7$. Thus $X$ has the invariant $\mu(\tilde{X}) = 2m/28 = m/14$ mod. 1, and it can take 14 different valuss.

EXAMPLE. - It seems that our invariant $\mu$ cannot be computed for the 7-manifold $P^2(C) \times S^3$. Thus no conclusion can be made concerning the diffferentiable structures on this space.

## 7. Smooth fibrations of the 7-sphere.

It is a consequence of Theorem 6 that each of the 16 *differentiable 7-spheres of the form $M_k^7$ admit an infinite number of essentially different differentiable fibrations by differentiable 3-spheres $S_3$*, for if such a manifold has invariant $\mu_0$, then there are infinitely many distinct solutions of $h(h=1)/56 \equiv \mu_0 \mod 1$. Each of these determines a differentiable fibre bundle over the usual 4-sphere with fibre $S^3$ and with total space $M^7_{2h-1}$, differentiably the same space in each case.

G. HIRSCH [12] has proved essentially that if a fibration of the Euclidean 7-sphere $S^7$ is isomorphic to a fibration of $S^7$ by great 3-spheres (which are sections by four dimensional linear subspaces through the center of $S^7$ in Euclidean 8-space), then that fibration is isomorphic to the normal sphere bundle of a quaternionic projective line in the quaternionic projective plane, which is in turn the classical HOPF fibration of $S^7$ (STEENROD [9, p. 108]. It follows that *for $h(h-1) \equiv 0 \mod 56$ and $h \neq 0$ and 1 the corresponding differentiable fibrations of the Euclideam $S^7$ cannot be realized by a fibration with great 3-spheres as fibres.* Furthermore, as we show in [2], *certain of these fibrations are not even of the same fibre homotopy type as the* HOPF *fibration.*

## 8. Computations for the case $k = 3$.

For any 11-manifold $M^{11}$ satisfying Conditions $\mu$) we have

$$\mu(M^{11}) \equiv (4p_2p_1 - 3p_1^3 - 24\tau)[W]/2^{11}\cdot 3\cdot 31 \mod 1, \tag{14}$$

computed for any oriented spin coboundary $W$. This is different from MILNOR'S invariant $\lambda$, which is

$$\lambda(M^{11}) \equiv (13p_2p_1 - 2p_1^3 + 945\tau)[W]/2\cdot 31 \mod 1, \tag{15}$$

computed for any coboundary $W$.

In Theorem 1 of MILNOR [5] the invariant $\lambda$ could not be applied to the case of homotopy 11-spheres. Thus $\mu$ is a better invariant than $\lambda$ for such spaces (and gives the same information as $\lambda'$, as we have already noted in

Section 4. Indeed, any smooth manifold $M$ which is a homotopy 11-sphere bounds parallelizable manifold $W$ (see [6]), whence $p_i[W]=0$ $(0<i<3)$; therefore,

$$\mu(M) \equiv -\tau[W]/2^8 \cdot 31 \qquad \text{mod } 1$$

Again the theory of MILNOR-SMALE shows that there are precisely 992 distinct differentiable stuctures on the topological 11-sphere, and the manifold $M_0^{11}$ of MILNOR [6, p. 13] together with its connected self sums represent all these classes. $\lambda(M_0^{11})=1$ mod 992; $\mu(M_0^{11}) = -1/992$ mod 1.

THEOREM. - *Two smooth* 11-*manifolds which are homeomorphic to the* 11-*sphere are diffeomorphic if and only if they have same* $\mu$ *invariant.*

COROLLARY. - *On any* 11-*manifold satisfying Conditions* $\mu$) *there are at least* 992 *different differentiable structures.*

## 9. Computations for the case $k=4$.

We have

$$\hat{A}_4 = \frac{1}{2^{15}\cdot 3^4 \cdot 5^2 \cdot 7}\left(-192p_4 + 512p_3p_1 + 208p_2^2 - 904p_2p_1^2 + 381p_1^4\right),$$

$$t_4 = -1/2 \cdot 127.$$

For any 15-manifold satisfying Conditions $\mu$) we find:

$$\mu(M^{15}) = 12096p_3p_1 + 5040p_2^2 - 22680p_2p_1^2 + 9639p_1^4 - 181440\tau)[W]\cdot$$
$$\cdot/2^{15}\cdot 3^4 \cdot 5 \cdot 7 \cdot 127 \qquad \text{mod } 1$$

computed for any spin coboundary $W$ satisfying Condition $\mu$). Again $\mu$ provides more information than $\lambda$ in their common domain of definition.

Let us compare $\lambda'$ and $\mu$ when applied to homotopy 15-spheres. First of all, as a consequence of certain calculations of *H.* Toda (kindly supplied to us in a letter of January 17, 1961) we find that $I_4/8 = 8128$. Thus for any homotopy 15 sphere $M^{15}$ which bounds a parallelizable manifotd $W^{16}$ we have $\lambda'(M^{15})$ defined as an element of the cyclic group $Z_{8128}$; also, since $p_i\,[W^{16}]=0$ for $0<i<4$, we have

$$-\lambda'(M^{15})/8128 = \mu(M^{15}) \equiv -\tau[W]/8\cdot 8128 \qquad \text{mod } 1$$

It follows that $\mu$ *and* $\lambda'$ *provide the same information for such homotopy spheres; however, there are homotopy* 15-*spheres which do not bound parallelizable* *manifolds.* [4] In fact, according to recent computations of KERVARIE and MILNOR (unpublished) we have a *split exact sequence* (in the notation of MILNOR [6])

$$0 \to \Theta^{15}(\partial\pi) \to \Theta^{15} \rightleftarrows Z_2 \to 0. \tag{16}$$

Consequently, $\lambda'$ is not defined for all homotopy 15-spheres. In contrast to the 7- and 11-dimensional cases, the 16, 256 *different differentiable structures on the topological* 15-*sphere are not completely distinguished by their* $\mu$ *invariant, although those which bound paral elizable manifolds are.*

In [7] SHIMADA constructed bundles $\xi_{h,j}$ over an 8-sphere, with properties analogous to those of MILNOR discussed in Section 6. Here $\tau[B_k^{16}] = 1$, $h + j = 1$, $p_1(B_k^{16}) = 0$, $p_2[B_k^{16}] = 6^2(2h-1)^2$ and by substitution in the above expression for $\mu$ we find

$$\mu(M_{2h-1}^{15,\sigma}) = h(h-1)/16\ 256 \qquad \text{mod } 1. \tag{17}$$

The symbol $\sigma$ we introduced here has the following meaning. According to KERVAIRE and MILNOR (unpublished) there exist precisely two different differentiable structures on $S^8$; the usual one is denoted by $\sigma = 0$, the other by $\sigma = 1$ In order to explain the index $\sigma$ we give the definition of the relevant fibre bundles with differentiable structure, which definition for a topological base space coincides with the definitions of MILNOR and SHIMADA.

Let $u_1$, $u_2$, $v_1$, $v_2$ be octaves (Cayley numbers) with norm 1 (unit octaves); $r_1$, $r_2$ real numbers; $(u_j, r_j)$ are polar coordinates for a ball in euclideam 8-space with radius 1: $0 \leq r_j \leq 1$, $j = 1$ and 2. Let $\eta$: $S^7 \to S^7$ be a diffeomorphism of the 7-sphere of unit octaves.

The balls with coordinates $(u_1, r_1)$ and $(u_2, r_2)$ can be glued along their boundaries according to the identification of

$$(u_1, 1) \text{ and } (u_2, 1) = (\eta u_1, 1),$$

to give an 8-sphere with a differentiable structure $\sigma = \sigma(\eta)$; $\sigma = 0$ or 1.

A differentiable fibre bundle with total space $M_{2h-1}^{15,\sigma(\eta)}$ over this differentiable manifold $S^{8\sigma(\eta)}$, is now obtained by gluing the product bundle spaces with coordinates

$$(u_1, r_1; v_1) \text{ and } (u_2, r_2; v_2)$$

---

(4) The corresponding subgroup of the group of all diffeomorphism classes $\Theta^{15}$, is denoted by $\Theta^{15}(\partial\pi) \approx Z_{8128}$.

according to the identification of

$$(u_1,\ 1;\ v_2) \text{ and } (u_2,\ 1;\ v_2) = (\eta u_1,\ 1;\ (u_1)^h v_1 (u_1)^j)(h+j=1).$$

Observe that the differentiable structures of $M_{2h-1}^{15,\sigma}$ together with the fibration, determines the differentiable structure of the base space. Observe also that both differentiable structures on $S^8$ determine the same combinatorial structure on $S^8$, and therefore the combinatorial structure of $M_{2h-1}^{15,\sigma}$ and its fibration is independent of $\sigma$.

*We do not know whether $M_{2h-1}^{15,0}$ and $M_{2h-1}^{15,1}$ are diffeomorphic or not*, but the two differentiable fibre bundles are differentiably different as bundles. Both $M_3^{15,0}$ and $M_3^{15,1}$ generate a subgroup of $\Theta^{15}$ of order 8128.

If we apply lemma 6 to the case $n = p.2^7 = 127.2^7 = 16\ 256$ we find that there are $(127+1)2^5 = 4096$ values of $h(h-1)/2$ mod 8128, and these *values* correspond to differentiable 15-spheres which are diffeomorphic to the total space of a 7-sphere bundle over a base space homeomorphic to $S^8$.

## 10. Computations for the manifolds $M(f_1, f_2)$.

MILNOR [5] has defined certain closed, smooth, oriented $(4k-1)$-manifolds $M(f_1, f_2)$ in terms of a bundle over $S^{4r}$ with group the orthogonal group $R_{4(k-r)}$, defined by a map $f_1 \colon S^{4r-1} \to R_{4(k-r)}$, and another $R_{4r}$-bundle over $S^{4(k-r)}$, defined by $f_2 \colon S^{4(k-r)-1} \to R_{4r}$.

There is also a smooth coboundary $W(f_1, f_2)$ of $M(f_1, f_2)$ constructed using these bundles, and $W(f_1, f_2)$ is a spin coboundary since $H^2(W(f_1, f_2)),$ $Z_2) = 0$. We will let $p_r(f_1)$ denote the value on $S^{4r}$ of the PONTRJAGIN class of the bundle determined by $f_1$; MILNOR has computed $\lambda(M(f_1, f_2))$ in most cases in terms of $p_r(f_1)$ and $p_{k-r}(f_2)$, and we can modify those computations to obtain $\mu(M(f_1, f_2))$ as follows:

CASE - $k \neq 2r$. In this case $M(f_1, f_2)$ is a smooth manifold homeomorphic to the $(4k-1)$-sphere, and $\tau[W(f_1, f_2)] = 0$; see Lemma 3 of MILNOR [5]. Then

$$\mu(M(f_1, f_2)) \equiv [(\widehat{s}_r \widehat{s}_{k-r} - s_k) - t_k(s_r s_{k-r} - s_k)] p_r(f_1) p_{k-r}(f_2)/a_k \qquad \text{mod } 1$$

$$\equiv \frac{B_r B_{k-r} \left\{ 1 + \dfrac{2(2^{2r-1}-1)(2^{2k-2r-1}-1)}{2^{2k-1}-1} \right\} p_r(f_1) p_{k-r}(f_2)}{4a_k (2r)!\ (2k-2r)!}$$

CASE - $k = 2r$. In this case (without making the simplifying assumptions made in Lemma 4 of [5]) the expression for $\mu(M(f_1, f_2))$ is more complicated.

In order to derive it we first establish the following straightforward consequence of a result of Thom [11, Th. I.8].

LEMMA. - *Let $\xi$ be an $R_m$-bundle with total space $A$ and fibre the $m$-disc $D^m$ over an oriented closed $n$-manifold $B$. Then $A$ is a bounded oriented $(n+m)$-manifold with boundary $E$, and the zero section $\zeta$ imbeds $B$ in $A$. If $\zeta^{\#}: H^i(B) \to H^{i+m}(A; E)$ denotes the Gysin homomorphism of $\xi$, then the Euler class $w_m(\xi)$ of $\xi$ corresponds to the Poincaré dual of the self intersection class $\zeta(B)$ o $\zeta(B)$ of the zero section:*

$$\mathfrak{D}_A^{-1}(\zeta(B) \circ \zeta(B)) = \zeta^{\#} w_m(\xi). \tag{18}$$

PROOF. - The homomorphism $\zeta^{\#}$ is defined by commutativity of the following diagram (using integer coefficients):

$$\begin{array}{ccc} H^i(B) & \xrightarrow{\zeta} & H^{i+m}(A, E) \\ \downarrow \mathfrak{D}_B & & \downarrow \mathfrak{D}_A \\ H_{n-i}(B) & \xrightarrow{\zeta} & H_{n-i}(A). \end{array}$$

where the vertical arrows are the isomorphisms of POINCARÉ duality. According to Thom [11, Th. I. 8] we have $\zeta^{\#} = \varphi$, where $\varphi$ is the GYSIN-THOM isomorphism of the bundle. On the other hand, as an integral class we have

$$w_m(\xi) = \varphi^{-1}(\varphi(1) \cup \varphi(1))$$

by definition. The lemma follows at once using the relation

$$\zeta(B) \circ \zeta(B) = \mathfrak{D}_A(\mathfrak{D}_A^{-1}(\zeta(B)) \cup \mathfrak{D}_A^{-1}(\zeta(B))),$$

since as a homology class we have $B = \mathfrak{D}_B(1)$.

We are interested in the case $n = m = 4r$ and $B = S^{4r}$. If we set

$$\beta = \mathfrak{D}_A^{-1}(\zeta(B)), \text{ then } \beta \cup \beta = \varphi w_{4r}(\zeta) = \xi^{\#} w_{4r}(\xi),$$

and therefore

$$\beta \cup \beta[A] = \xi^{\#} w_{4r}(\zeta)[A] = w_{4r}(\xi)[B]. \tag{19}$$

For any $f: S^{4r-1} \to R_{4r}$ we define the integer $w_{4r}(f)$ as the value on $S^{4r}$ of the EULER class of the bundle defined by $f$. Then MILNOR [5, p. 969] has shown that (in his notation)

$$p_r(W(f_1, f_2)) = \pm p_r(f_1)j\alpha \pm p_r(f_2)j\beta ,$$

whence

$$j^{-1}p_r(W(f_1, f_2))^2 = p_r(f_1)^2\alpha \cup \alpha \pm 2p_r(f_1)p_r(f_2)\alpha \cup \beta + p_r(f_2)^2\beta \cup \beta ,$$

where $\alpha$, $\beta \in H^{4r}(A, E)$ are the duals of the two imbeddings of $S^{4r}$ in $A$ by the corresponding zero sections. Applying the expression (19) we find (since $\alpha \cup \beta = \pm 1$).

$$(20)\quad j^{-1}p_r(W(f_1, f_2))^2[W(f_1, f_2)] = p_r(f_1)^2 w_{4r}(f_1) \pm 2p_r(f_1)p_r(f_2) + p_r(f_2)^2 w_{4r}(f_2).$$

It follows that $\mu(M(f_1, f_2))$ is the modulo 1 reduction of

$$\frac{B_r^2}{8((2r)!)^2}\left(1 + \frac{2(2^{2r-1}-1)^2}{2^{4r-1}-1}\right)\left(p_r(f_1)^2 w_{4r}(f_1) \pm 2p_r(f_1)p_r(f_2) + p_r(f_2)^2 w_{4r}(f_2)\right)$$
$$+ t_{2r}\tau[W(f_1, f_2)].$$

REMARK. - MILNOR has shown that if $f_1$ maps $S^{4r-1} \to R_{4r-1}$, then $M(f_1, f_2)$ is again homeomorphic to a $(4k-1)$-sphere. In terms of the invariants appearing in (20) this condition implies $w_{4r}(f_1) = 0$ and $\tau[W(f_1, f_2)] = 0$; for then there is a section of the bundle defined by $f_1$, whence its EULER class vanishes. From this we find that the intersection matrix for $W(f_1, f_2)$ in dimension $2r$ is of the form

$$\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & b\cdot b \end{pmatrix},$$

which is easily seen to have index 0.

EXAMPLE - $(r = 2,\ k = 4)$. Then

$$\mu(M(f_1, f_2)) \equiv \{p_2(f_1)^2 w_8(f_1) \pm 2p_2(f_1)p_2(f_2) +$$
$$+ p_2(f_2)^2 w_8(f_2) - 36_t[W(f_1, f_2)]\}/2^{11}\cdot 3^2\cdot 127.$$

If we let $f_1 = f_{h,j}$ and $f_2 = f_{h',j'}$ in the notation of Section 6, then

$$p_2(f_{hj}) = \pm 6(h-j),\ w_8(f_{hj}) = h + j,$$

and similarly for $f_2$. It follows that

$$\mu(M(f_{hj}, f_{h'j'})) \equiv \{(h-j)^2(h+j \pm 2(h-j)(h'-j') +$$
$$(h'-j')^2(h'+j') - t[W]\}/2^9\cdot 127.$$

## BIBLIOGRAPHY

[1] A. BOREL and F. HIRZEBRUCH, *Characteristic classes and homogeneous spaces III* «Am. J. Math.» 82 (1960) 491-504.

[2] J. EELLS and. N. H. KUIPER, *Manifolds which are like projective planes*, (In prepaiation).

[3] F. HIRZEBRUCH, *Neue topologische Methoden in der algebralschen Gemetrie*. Spinger 1956.

[4] J, MILNOR, *On manifolds homeomorphic to the 7-sphere*, «Annals. of Math.» 64 (1956), 399-405.

[5] J. MILNOR, *Differentiable structures on spheres*, «Amer. J. Math.» 81 (1959), 962-972.

[6] J MILNOR, *Differentiable manifolds which are homotopy spheres*, «Mimeo. notes.» Princeten Univ. 1959.

[7] N. SHIMADA, *Differentiable structures on the15-sphere and Pantrjagin classes of certain manifolds*, «Nagoya Math. J.»12 (1957), 59-69.

[8] S. SMALE, *The generalized Poincaré conjecture in higher dimensions*, «Bull. A.M.S.» 66 (1960), 373-375. See also mineo. notes, University of California, Berkeley, 1961.

[9] N. STEERNOD, *The topology of fibre bundles*. Princecton 1951.

[10] I. TAMURA, *Homeomorphy classification of total spaces of sphere bundles over spheres* «J. Math. Soc. Japan» 10 (1958), 29-43.

[11] R THOM, *Espaces fibrés en sphères et carrés de Steenrod*, «Ann. Sci. Ecol. norm. sup.» 69 (1952), 109-182.

[12] G. HIRSCH, *La géomètrie projective et la topologie des espaces fibrés*, «Colloque Inter. de Topologie Algébrique» Paris (1949), 35-42.