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# Computing the Link Center of a Simple Polygon*

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**Abstract.** The link center of a simple polygon **P** is the set of points $x$ inside **P** at which the maximal link-distance from $x$ to any other point in **P** is minimized. Here the link distance between two points $x$, $y$ inside **P** is defined to be the smallest number of straight edges in a polygonal path inside **P** connecting $x$ to $y$. We prove several geometric properties of the link center and present an algorithm that calculates this set in time $O(n^2)$, where $n$ is the number of sides of **P**. We also give an

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$O(n \log n)$ algorithm for finding an approximate link center, that is, a point $x$ such that the maximal link distance from $x$ to any point in **P** is at most one more than the value attained from the true link center.

## 1. Introduction

This paper concerns itself with analysis and calculation of the *link center* of a simple planar polygon **P** having $n$ sides. The notion of a *link distance* between two points $x$, $y$ inside **P** has been recently introduced in [13]; it is defined as the smallest number of "links" (i.e., straight segments) in a polygonal path connecting $x$ and $y$ within **P**, and is a useful metric for path planning within **P** when straight motion is cheap but turns are expensive.

Suri [13] presents a linear-time algorithm for calculating the link distance between any two given points within **P**, provided a triangulation of **P** is given (such a triangulation can be calculated in time $O(n \log n)$ by the technique of [6], or in almost linear time by the recent algorithm of [15]). In fact, the technique in [13] provides a linear-time algorithm for decomposing (a triangulated) **P** into *k-visibility regions* for a fixed source point $v$, where the $i$th visibility region consists of all points within **P** whose link distance from $v$ is exactly $i$. (In particular, the first visibility region is just the set of all points in **P** *visible* from $v$; see [2], [4], [5], and [8] for analysis of visibility within a simple polygon.)

Suri [14] has also considered the problem of calculating the *link diameter* of **P**, where this quantity is defined as the maximal link distance between any two points in **P**. He presents an algorithm for calculating this diameter in time $O(n \log n)$, using an interesting divide-and-conquer approach on the set of vertices of **P**. In this paper we consider the somewhat more difficult problem of calculating the link center of **P**, defined to be the set of all points within **P** whose maximal link distance to any point of **P** is the smallest possible. As will be shown below, the link center is in general not a singleton; nevertheless it is connected and has certain convexity-like properties. From the point of view of geometric location theory, any point in the link center can serve as a location for a mobile unit that has to reach any point within **P** so that the maximal number of turns it needs to perform is minimized; alternatively, such a point can serve as a location for a transmitter that can broadcast to any point within **P** (along a path fully contained in **P**) such that the maximal number of relays necessary to reach any point in **P** is minimized.

Section 2 of this paper introduces the notion of geodesic and link geodesic paths along with their associated metrics and establishes several useful results about these concepts. Section 3 contains an analysis of various properties of the link center and some additional related results needed for the calculation of the link center. Section 4 presents an $O(n^2)$-time algorithm for calculating the link center. Section 5 gives an $O(n \log n)$-time algorithm for finding a point in an "approximate" link center, in the sense that its maximal link distance to any point inside **P** is at most one larger than the smallest possible such value, attained at points in the actual link center.

## 2. Geodesics and Link Geodesics

Let **P** be (the set of points lying in the interior or on the boundary of) a simple polygon in the plane. A polygonal path between points $x$ and $x'$ in **P** will be denoted by listing the endpoints of the segments that constitute the path, for example, $\bar{x} = (x = x_0, \ldots, x_k = x')$. If $v$ and $w$ are two vertices of **P** then $b[v, w]$ will refer to the polygonal path obtained by traversing the boundary of **P** clockwise from $v$ to $w$. Two different metrics will be used in what follows: the *Euclidean* metric and the *link* metric. Given points $x$ and $x'$ in a polygon **P** and a continuous path $\bar{x}$ from $x$ to $x'$ (in fact, it suffices to consider only polygonal paths), we denote the usual *Euclidean* length of the path as $l_{\mathrm{E}}(\bar{x})$. Given a polygonal path $\bar{x} = (x = x_0, \ldots, x_k = x')$ from $x$ to $x'$, the *link* length of the path is defined to be the number $k$ of segments in the path and is denoted by $l_{\mathrm{L}}(\bar{x})$. The Euclidean distance $d_{\mathrm{E}}(x, x')$ between points $x$ and $x'$ in a polygon **P** is defined by

$$d_{\mathrm{E}}(x, x') = \min\{l_{\mathrm{E}}(\bar{x})\}$$

over all continuous (polygonal) paths $\bar{x}$ from $x$ to $x'$. The link distance $d_{\mathrm{L}}(x, x')$ between $x$ and $x'$ is given by

$$d_{\mathrm{L}}(x, x') = \min\{l_{\mathrm{L}}(\bar{x})\}$$

over all polygonal paths $\bar{x}$ from $x$ to $x'$. (It is easily checked that $d_{\mathrm{L}}$ is indeed a metric on **P**.) We will call a path $\bar{x}$ connecting points $x$ and $x'$ within **P** *geodesic* if $l_{\mathrm{E}}(\bar{x}) = d_{\mathrm{E}}(x, x')$. Similarly, we call $\bar{x}$ *link-geodesic* if $l_{\mathrm{L}}(\bar{x}) = d_{\mathrm{L}}(x, x')$. Any geodesic or link-geodesic path must be simple. Between any two points $x$ and $x'$ in **P** there is a unique geodesic path. This path is polygonal and will be denoted by $g(x, x')$. On the other hand, there may be many link-geodesic paths between two points. A path $\bar{x}$ from $x$ to $x'$ is called an *optimal* link-geodesic path if $\bar{x}$ is a link-geodesic path and $l_{\mathrm{E}}(\bar{x})$ is minimum over all link-geodesic paths from $x$ to $x'$.

Concerning these concepts, note that the number of links on a geodesic path between two points $x$, $x'$ in **P** can be considerably larger than the link distance between $x$ and $x'$ (see Fig. 1). It also turns out that there can be more than one

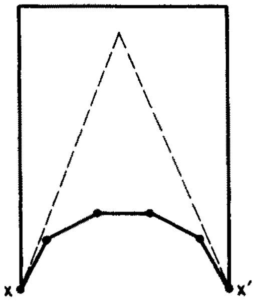



Fig. 1

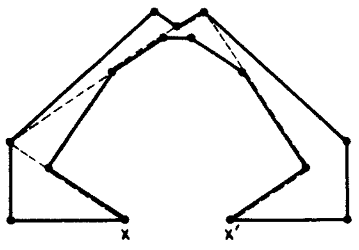


Fig. 2

optimal link-geodesic path between two points. Moreover, two such paths may fail to have a deformation that takes one of them to the other through a family of optimal link-geodesic paths (see Fig. 2).

In the following we give a few lemmas about basic properties of geodesic and link-geodesic paths.

**Lemma A.** *Given line segments $xy$ and $x'y'$ in $\mathbf{P}$ and optimal link-geodesic paths $\bar{x}=(x=x_0,\ldots,x_k=x')$ and $\bar{y}=(y=y_0,\ldots,y_m=y')$, any point $q$ on $\bar{x}$ can see some point $r$ on $\bar{y}$ (i.e., the segment $qr$ is contained in $\mathbf{P}$).*

*Proof.* Let $x_ix_{i+1}$ be a link of $\bar{x}$; if $\bar{y}$ intersects $x_ix_{i+1}$ then every point on $x_ix_{i+1}$ sees a point on $\bar{y}$. Assuming that $\bar{y}$ does not intersect $x_ix_{i+1}$, consider the chain $y_0, x_0, x_1, \ldots, x_k, y_m$. Form a simple polygon $\mathbf{P}_1$ as follows: starting at $x_i$, traverse the chain in each direction until a point on $\bar{y}$ is reached, then connect these two points using the portion of $\bar{y}$ between them. The polygon $\mathbf{P}_1$ contains $x_ix_{i+1}$ and every interior point of this segment must see some vertex of $\mathbf{P}_1$ other than $x_i$ and $x_{i+1}$. By the optimality of $\bar{x}$, no interior point of $x_ix_{i+1}$ can see another vertex of $\bar{x}$, so the vertex of $\mathbf{P}_1$ seen by an interior point of $x_ix_{i+1}$ must be a point on (although not necessarily a vertex of) $\bar{y}$.

What about the endpoint $x_i$ (or $x_{i+1}$)? If the neighbor of $x_i$ along the boundary of $\mathbf{P}_1$ is a point of $\bar{y}$ we are done. Otherwise we consider two cases: either $x_i$ is a reflex vertex or a convex vertex of $\mathbf{P}_1$ (the interior angle at $x_i$ is greater or less than $\pi$): in the reflex case $x_i$ must see a vertex of $\mathbf{P}_1$ other than $x_{i-1}$ and $x_{i+1}$ and because of the optimality of $\bar{x}$ this vertex must lie on $\bar{y}$; in the convex case the optimality of $\bar{x}$ implies that the segment $x_{i-1}x_{i+1}$ is not contained in $\mathbf{P}_1$, i.e., the triangle $x_{i-1}x_ix_{i+1}$ contains some other vertex of $\mathbf{P}_1$ that is visible from $x_i$; again by the optimality of $\bar{x}$ that vertex must lie on $\bar{y}$. □

**Lemma B.** *The conclusion of Lemma A holds if $\bar{x}$ and $\bar{y}$ are geodesic paths (in the Euclidean metric) instead of optimal link-geodesic paths.*

*Proof.* Same as above. □

**Lemma C.** *Given any points $x$, $x'$, $y \in \mathbf{P}$ and a geodesic (resp. an optimal link-geodesic) path $\bar{x}=(x=x_0,\ldots,x_k=x')$ in $\mathbf{P}$ and given any point $q$ on $\bar{x}$, we have*

$$d_L(q, y) \leq \max\{d_L(x, y), d_L(x', y)\}.$$

*Proof.* Assume that $\bar{x} = (x = x_0, \ldots, x_k = x')$ is a geodesic (resp. an optimal link-geodesic) path from $x$ to $x'$ and let

$$d = \max\{d_L(x, y), d_L(x', y)\}.$$

We proceed by induction on $d$. The case $d = 1$ simply reduces to Lemma B (resp. Lemma A) where $xy$ and $x'y$ play the roles of the two segments, and $\bar{x}$ and the trivial path $\bar{y} = (y)$ are the two paths connecting their respective endpoints.

Thus assume that $d > 1$. Assume also without loss of generality that $d_L(x, y) = d$ and that $d_L(x', y) = k$ where $1 \leq k \leq d$ (the case $k = 0$ is trivial) and let $\bar{u} = (y = u_0, \ldots, u_d = x)$ and $\bar{v} = (y = v_0, \ldots, v_k = x')$ be paths realizing these distances. In addition, let $\bar{w}$ be a geodesic (resp. an optimal link-geodesic) path from $u_{d-1}$ to $v_{k-1}$. By induction, every point $r$ on $\bar{w}$ satisfies

$$d_L(r, y) \leq \max\{d_L(u_{d-1}, y), d_L(v_{k-1}, y)\} = d - 1.$$

By Lemma B (resp. Lemma A), each point $q$ on $\bar{x}$ sees some $r$ on $\bar{w}$ and so

$$d_L(q, y) \leq d_L(q, r) + d_L(r, y) \leq d. \qquad \square$$

**Lemma D.** *For any points $y, y' \in \mathbf{P}$ there exists a vertex $v$ of $\mathbf{P}$ such that $d_L(y, y') \leq d_L(y, v)$.*

*Proof.* Let $k = d_L(y, y')$ and let $\bar{y} = (y = y_0, \ldots, y_k = y')$ be an optimal link-geodesic path from $y$ to $y'$. Let $w$ be the point obtained by intersecting the boundary of $\mathbf{P}$ with the ray emanating from $y_{k-1}$ which passes through $y_k = y'$, and let $v$ and $v'$ be the endpoints of the boundary edge that contains $w$.

The segment connecting $y_{k-1}$ and $w$ is certainly the optimal link-geodesic path between these two points. Applying Lemma C to this path (with $y'$ playing the role of $q$) implies that $d_L(y, w) \geq d_L(y, y')$. Again applying Lemma C, but now to the optimal link-geodesic path formed by the edge connecting $v$ and $v'$ (with $w$ playing the role of $q$), then yields

$$\max\{d_L(y, v), d_L(y, v')\} \geq d_L(y, w) \geq d_L(y, y'),$$

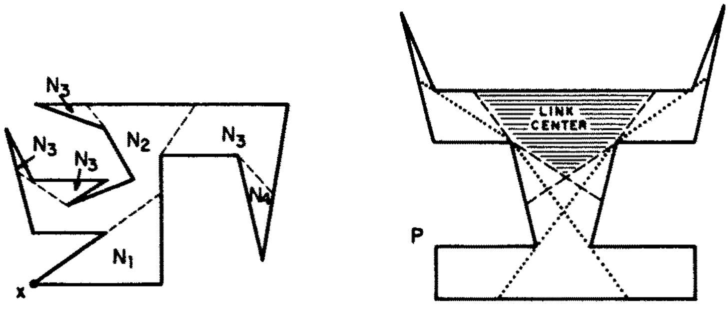
as desired. $\square$

## 3. The Link Center

In this section we analyze several properties of the link center that will be useful for the calculation of this set, to be presented in the following section.

The *k-neighborhood* or *k-disk* about a point $x \in \mathbf{P}$ (closely related to the notion of $k$th visibility regions mentioned in the introduction) is defined by

$$N_k(x) = \{x' \in \mathbf{P} \mid d_L(x, x') \leq k\}$$




(a) (b)

Fig. 3

and the *covering radius* $c(x)$ of $x$ is the smallest $k$ such that $\mathbf{P} \subset N_k(x)$. (Note that $c(x) = 1$ if and only if $\mathbf{P}$ is star-shaped with respect to $x$.) The *link diameter* of $\mathbf{P}$ is defined by $D_L(\mathbf{P}) = \max_{x \in \mathbf{P}} c(x)$ and the *link radius* is defined by $r_L(\mathbf{P}) = \min_{x \in \mathbf{P}} c(x)$. The link center of $\mathbf{P}$ is defined by

$$C_L(\mathbf{P}) = \{x \in \mathbf{P} \mid c(x) = r_L(\mathbf{P})\}.$$

See Fig. 3 for an illustration of these concepts.

Using the Euclidean metric we can similarly define, for any nonnegative number $\alpha$, the $\alpha$-neighborhood about a point, the covering radius of a point, and the geodesic diameter, radius, and center of a polygon. As it turns out, these concepts for the continuous Euclidean metric are simpler to analyze and calculate, as demonstrated, e.g., in [1], [12], and [14].

We will now establish some properties of the $k$-disks $N_k(x)$ for points $x \in \mathbf{P}$ and of the link center of $\mathbf{P}$.

**Theorem 1.** *Let $k \geq 1$ and let $v$ be a point of $\mathbf{P}$. The neighborhood $N_k(v)$ is a subpolygon of $\mathbf{P}$ all of whose corners lie on the boundary of $\mathbf{P}$ and each of whose edges is one of two types*:

(a) *a (portion of a) side of P*;
(b) *a segment $wz$ connecting a reflex vertex $w$ of $\mathbf{P}$ to a point $z$ on the boundary of $\mathbf{P}$.*

*Proof* (by induction on $k$). For the base case $k = 1$ note that $N_1(v)$ is nothing but the visibility polygon of the vertex $v$, which obviously has the properties prescribed in the theorem (see, e.g., [4]). Now let $k > 1$ and assume $N_{k-1}(v)$ has the prescribed properties. Let $e$ be a type (b) edge of $N_{k-1}(v)$. It cuts $\mathbf{P}$ into two parts, one of which does not contain $N_{k-1}(v)$. Call that part $\mathbf{P}_e$. Let $V_e$ be the weak visibility polygon of edge $e$ in polygon $\mathbf{P}_e$, i.e., $V_e$ comprises all points $x \in \mathbf{P}_e$ for which the straight segment $xy$ is contained in $\mathbf{P}_e$ for some $y \in e$. Again, it is well known [4] that $V_e$ is a subpolygon of $\mathbf{P}_e$ that has the structure prescribed by the theorem (with respect to $\mathbf{P}_e$). We claim that $N_k(v) = N_{k-1}(v) \cup \bigcup \{V_e | e$ is a type (b) edge of $N_{k-1}(v)\}$. This immediately implies that $N_k(v)$ has the desired properties, since the boundaries of $N_{k-1}(v)$ and the visibility polygons $V_e$ consist only of type (a) and (b) edges and since $V_e$ and $N_{k-1}(v)$ intersect in $e$. For a proof of the claim note that obviously $N_{k-1}(v) \subset N_k(v)$ and $V_e \subset N_k(v)$ for each type (b) edge $e$ of $N_{k-1}(v)$. On the other hand, let $x \in N_k(v)$: if $d_L(v, x) < k$, then $x \in N_{k-1}(v)$. If $d_L(v, x) = k$, then let $\bar{x} = (v = x_0, \ldots, x_k = x)$ be a link-geodesic path from $v$ to $x$. Clearly, $x_{k-1} \in N_{k-1}(v)$ and $x \notin N_{k-1}(v)$. Thus the link $x_{k-1}x$ has to intersect the boundary of $N_{k-1}(v)$, and, since $\bar{x}$ lies entirely within $\mathbf{P}$, this intersection has to be at a type (b) edge $e$ of $N_{k-1}(v)$. Obviously, $x \in V_e$, which concludes the proof of the claim and the theorem. □

Call a subset $S \subset \mathbf{P}$ *geodesically convex* if for any points $x, x' \in S$, the geodesic path from $x$ to $x'$ is contained in $S$ (see also [12]); similarly, call $S$ *link convex* if any optimal link-geodesic path from $x$ to $x'$ is contained in $S$. Note that these notions of convexity are preserved under intersection. It can be shown that link convexity implies geodesic convexity, whereas the converse is not true (see Fig. 4).

**Lemma 2.** *For any $y \in \mathbf{P}$ and any $k \geq 0$, $N_k(y)$ is both geodesically convex and link convex.*

*Proof.* This is an immediate consequence of Lemma C. □

Since the link center of $\mathbf{P}$ is just the intersection of the sets $N_r(x)$ over all $x \in \mathbf{P}$ where $r = r_L(\mathbf{P})$ and since, by definition, the intersection of link convex (geodesically convex) sets is link convex (geodesically convex), we obtain the following result:

**Theorem 3.** *The link center of a polygon* $\mathbf{P}$ *is link convex, geodesically convex and thus connected.*

As was mentioned above, the link center of a polygon $\mathbf{P}$ consists of the intersection of the sets $N_r(x)$ over all $x \in \mathbf{P}$. In fact, a slightly stronger result can be proven:

**Theorem 4.** *The link center of a polygon* $\mathbf{P}$ *is the intersection of the sets $N_r(v)$ over all convex vertices $v$ of* $\mathbf{P}$, *where $r = r_L(\mathbf{P})$.*

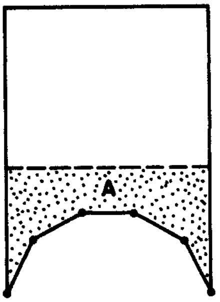



Fig. 4

*Proof.* Lemma $D$ implies that $C_L(\mathbf{P})$ is the intersection of $N_r(v)$ for *all* vertices $v$ of $\mathbf{P}$. However, Lemma C is easily seen to imply that for a reflex vertex $v$ the neighborhood $N_r(v)$ contains the intersection of $N_r(v_1)$ and $N_r(v_2)$, where $v_1$ and $v_2$ are the two convex vertices at the ends of the chain of reflex vertices of $\mathbf{P}$ that contains $v$. □

In the remainder of this section we establish a relationship between the link radius and the link diameter of a polygon. To accomplish this we use the following lemmas:

**Lemma 5.** *Let $T$ be a finite free tree and let $\mathbf{T}$ be a family of subtrees of $T$. If $S \cap S' \neq \varnothing$ for every $S$, $S' \in \mathbf{T}$, then $\bigcap \{s \in \mathbf{T}\} \neq \varnothing$.*

*Proof.* See Proposition 4.7, p. 92, of [7]. □

**Lemma 6.** *Let $k \geq 1$ and let $S$ be a finite set of points of polygon $\mathbf{P}$. If $N_k(x) \cap N_k(y) \neq \varnothing$ for every $x$, $y \in S$, then $\bigcap \{N_{k+1}(x) | x \in S\} \neq \varnothing$.*

*Proof.* Let $\Delta$ be an arbitrary but fixed triangulation of $\mathbf{P}$. The triangles of $\Delta$ with the relation "sharing an edge" form a graph that is a tree $T$. For every $x \in S$ let $T_x$ be the triangles of $\Delta$ that intersect $N_k(x)$. $T_x$ viewed as a graph must be a subtree of $T$. The assumptions of the lemma imply that $T_x$ and $T_y$ have a nonempty intersection for every $x$, $y \in S$. Lemma 5 now implies that $\bigcap \{T_x | x \in S\} \neq \varnothing$, i.e., there must be a triangle in $\Delta$ that intersects $N_k(x)$ for all $x \in S$. Clearly, this triangle is contained in $N_{k+1}(x)$ for each $x \in S$, which proves our lemma. □

We are now ready to prove that the link radius of a simple polygon is approximately half of the link diameter.

**Theorem 7.** *For any simple polygon* $\mathbf{P}$

$$\left\lceil \frac{D_L(\mathbf{P})}{2} \right\rceil \leq r_L(\mathbf{P}) \leq \left\lceil \frac{D_L(\mathbf{P})}{2} \right\rceil + 1.$$

*Proof.* Let $k = \lceil D_L(\mathbf{P})/2 \rceil$. By the triangle inequality $D_L(\mathbf{P}) \leq 2r_L(\mathbf{P})$, i.e., $k \leq r_L(\mathbf{P})$. We claim that $N_k(x) \cap N_k(y) \neq \varnothing$ for any pair of points $x$, $y \in \mathbf{P}$, which by virtue of Lemma 6 and Theorem 4 implies the desired $r_L(\mathbf{P}) \leq k+1$. For a proof of the claim let $x$ and $y$ be two points in $\mathbf{P}$. Let $\bar{w} = (x = w_0, \ldots, w_s = y)$ be a link-geodesic path connecting $x$ and $y$ and let $i = \lceil s/2 \rceil$. Since $s$ cannot exceed the diameter $D_L(\mathbf{P})$ we have $i \leq k$. Consider $w_i$: obviously $d_L(x, w_i) \leq i$ and $d_L(w_i, y) \leq s - i \leq i \leq k$. Thus the neighborhoods $N_i(x)$ and $N_i(y)$ intersect and so do $N_k(x)$ and $N_k(y)$, which proves the claim and the theorem. □

It is possible to strengthen the result of Theorem 7 somewhat. For polygons with *odd* link-diameter $D_L(\mathbf{P})$ we will now show that the link radius $r_L(\mathbf{P})$ is always equal to $\lceil D_L(\mathbf{P})/2 \rceil$. The proof of this fact is very similar to the proof given above but it involves more geometry.

At first we give an analogue of Lemma 5. It is a topological version of Helly's theorem for convex sets, first shown by Helly himself [9] and then in a slightly stronger form by Molnár [11]. For our purposes a *cell* in the plane is a simply connected compact subset of the plane.

**Lemma 8.** *Let* $\mathbf{C}$ *be a set of cells in the plane. If* $C \cap C'$ *is a cell for every* $C$, $C' \in \mathbf{C}$ *and* $C \cap C' \cap C'' \neq \emptyset$ *for* $C$, $C'$, $C'' \in \mathbf{C}$, *then* $\bigcap \{C \in \mathbf{C}\} \neq \emptyset$.

With this lemma we can prove a slightly stronger version of Lemma 6.

**Lemma 9.** *Let* $k \geq 1$ *and let* $S$ *be a finite set of points of polygon* $\mathbf{P}$. *If* $N_k(x) \cap N_{k+1}(y) \neq \emptyset$ *for every* $x$, $y \in S$, *then* $\bigcap \{N_{k+1}(x) | x \in S\} \neq \emptyset$.

*Proof.* First note that clearly all neighborhoods $N_k(x)$ are cells. Any two such neighborhoods, if they have nonempty intersection, intersect in a cell (because they are link convex). Thus by Lemma 8 it suffices to show that the assumptions of this lemma imply that $N_{k+1}(x) \cap N_{k+1}(y) \cap N_{k+1}(z) \neq \emptyset$ for $x$, $y$, $z \in S$.

Let $x$, $y$, $z$ be three points in $S$ and consider the neighborhoods $N_{k+1}(x)$, $N_{k+1}(y)$, and $N_k(z)$. If $N_{k+1}(x) \cap N_{k+1}(y) \cap N_k(z) \neq \emptyset$, then obviously also $N_{k+1}(x) \cap N_{k+1}(y) \cap N_{k+1}(z) \neq \emptyset$, as desired. So assume that $N_{k+1}(x) \cap N_{k+1}(y) \cap N_k(z) = \emptyset$. The assumptions of this lemma imply that $N_{k+1}(x) \cap N_{k+1}(y) \neq \emptyset$. Because of link convexity (Lemma 2), this intersection must be simply connected. Thus $N_{k+1}(x) \cup N_{k+1}(y)$ forms a subpolygon of $\mathbf{P}$ that is simple, i.e., it has no holes. By the assumptions of this lemma $N_k(z)$ intersects both $N_{k+1}(x)$ and $N_{k+1}(y)$. Since we assume that $N_{k+1}(x) \cap N_{k+1}(y) \cap N_k(z) = \emptyset$, it follows that $N_k(z)$ intersects $N_{k+1}(x) \cup N_{k+1}(y)$ in at least two different connected components. Thus the complement of $N_{k+1}(x) \cup N_{k+1}(y) \cup N_k(z)$ has a bounded connected component (i.e., the polygon formed by this union has a hole). Call it $X$.

Consider the boundary of $X$. Since $X$ lies completely inside the polygon $\mathbf{P}$, the structure Theorem 1 implies that each of $N_{k+1}(x)$, $N_{k+1}(y)$, and $N_k(z)$ can contribute at most one edge to the boundary of $X$. It follows that $X$ must be a triangle and the edge contributed by $N_k(z)$ can "see" all of $X$, in particular also its opposite corner (which is in $N_{k+1}(x) \cap N_{k+1}(y)$). Thus this corner is in $N_{k+1}(z)$ and it follows that $N_{k+1}(x) \cap N_{k+1}(y) \cap N_{k+1}(z) \neq \emptyset$, as desired. □

With this lemma the following theorem can be proven in the same manner as Theorem 7.

**Theorem 10.** *For a simple polygon* $\mathbf{P}$ *with odd link diameter* $D_L(\mathbf{P})$ *the link radius is given by*

$$r_L(\mathbf{P}) = \left\lceil \frac{D_L(\mathbf{P})}{2} \right\rceil.$$

## 4. Calculating the Link Center

The results of the preceding section lead to a rather simple approach to the calculation of the link center of a given simple polygon **P** with $n$ sides, which can be implemented to run in $O(n^2)$ time. Specifically, we first calculate the link diameter of **P** in time $O(n \log n)$ by the technique in [13]. Theorems 7 and 10 then give at most two possible values for the link-radius $r = r_L(\mathbf{P})$: it can be $\lceil D_L(\mathbf{P})/2 \rceil$ or, in the case of even link diameter, it can also be $\lceil D_L(\mathbf{P})/2 \rceil + 1$.

By Theorem 4 we need to calculate the intersection $\bigcap N_r(v)$ over all convex vertices $v$ of **P**. We shall try this for the smaller of the two candidate values for $r$ first. If the intersection is nonempty, then it is the link-center of **P**. If it is found to be empty, then Theorem 7 guarantees that the intersection $\bigcap N_r(v)$ will be nonempty for the larger candidate value of $r$ and this intersection will constitute the link-center. Thus it suffices to present an $O(n^2)$ algorithm for the construction of $\bigcap N_k(v)$ over all convex vertices $v$ of **P**, where $k$ is any fixed integer between 1 and $n$.

Theorem 1 implies that each $N_k(v)$ is a subpolygon of **P** each of whose sides is either (a portion of) an edge of **P** or a segment $e$ connecting a reflex vertex $w$ of **P** to the boundary of **P** (called a type (b) edge in Theorem 1). Denote such a segment by $e_v(w)$. A type (b) segment $e_v(w)$ cuts the polygon **P** into two parts, one of which contains the vertex $v$. Denote this subpolygon by $\overline{e_v(w)}$. Let us assume the notational convention $\overline{e_v(w)} = \mathbf{P}$ if for some convex vertex $v$ the reflex vertex $w$ is not incident to a type (b) edge of $N_k(v)$. With this convention, clearly

$$N_k(v) = \bigcap \{\overline{e_v(w)} \mid w \text{ reflex vertex of } \mathbf{P}\}.$$

We want to compute

$$\begin{aligned}
&\bigcap \{N_k(v) \mid v \text{ convex vertex of } \mathbf{P}\}\\
&\quad = \bigcap \{\overline{e_v(w)} \mid v \text{ convex vertex and } w \text{ reflex vertex of } \mathbf{P}\}\\
&\quad = \bigcap \{P(w) \mid w \text{ reflex vertex of } \mathbf{P}\},
\end{aligned}$$

where $P(w) = \bigcap \{\overline{e_v(w)} \mid v \text{ convex vertex of } \mathbf{P}\}$. The structure of a subpolygon $P(w)$ is particularly simple: its boundary consists of two segments $e_1, e_2$ emanating from $w$, each of which is either one of the segments $e_v(w)$ or a side of **P** incident to $w$, and of a portion $b[q_1, q_2]$ of the boundary of **P** between the two other endpoints $q_1, q_2$ of $e_1, e_2$ respectively.

We can therefore proceed as follows: we first triangulate **P** in time $O(n \log n)$ as in [6]. For each convex vertex $v$ of **P** we compute $N_k(v)$ in $O(n)$ time using Suri's algorithm [13], and for each reflex vertex $w$ we collect the (at most $n$) segments $e_v(w)$. It is then straightforward to compute each $P(w)$ in $O(n)$ time, respectively. (Note that if **P** has no reflex vertices then the link center of **P** is clearly **P** itself.)

We are thus left with the problem of calculating $\bigcap_w \mathbf{P}(w)$. Note that the total number of edges in the boundaries of the subpolygons $\mathbf{P}(w)$ is only $O(n)$, so that the intersection of these polygons can be calculated trivially in time $O(n^2)$ as follows. We calculate the $O(n^2)$ intersection points between edges of (distinct) $\mathbf{P}(w)$'s. For each edge $e$ of such a polygon we have $O(n)$ intersection points of this kind along $e$, so that $e \cap [\bigcap_w \mathbf{P}(w)]$ is equal to the intersection of half-segments of $e$, each delimited at one of these intersection points. The intersection of these half-segments is trivial to calculate in $O(n)$ time. Thus all the edges along the boundary of the link center can be calculated in overall $O(n^2)$ time. Since there are only $O(n)$ such edges, the final step of finding the circular order of these edges along the boundary of the link center is now easy to accomplish in subquadratic time. We have thus shown:

**Theorem 11.** *The link center of a simple polygon* $\mathbf{P}$ *having n edges can be calculated in* $O(n^2)$ *time.*

## 5. Finding an Approximate Link Center

The result of the preceding section immediately raises the question whether the link center of a simple polygon $\mathbf{P}$ with $n$ sides can be calculated in subquadratic time. This problem is also motivated by the fact (used above) that the link diameter (and an approximation of the link radius) can be calculated in $O(n \log n)$ [14], and that the (unique) geodesic center of $\mathbf{P}$ can be calculated in time $O(n \log^2 n)$ [12]. Even if all we seek is just one point in the link center, we still do not know how to find it in subquadratic time. However, as we show in this section, if we are willing to compromise, and are satisfied with finding an *approximate link center*, by which we mean a point whose covering radius exceeds the link radius at most by one, then we can achieve this goal in time $O(n \log n)$.

Let $\Delta$ be an arbitrary triangulation of $\mathbf{P}$ and let $r = r_L(\mathbf{P})$ be the link radius of $\mathbf{P}$. The following observation is almost too trivial to state (see also the proof of Lemma 6):

**Lemma 12.** *There is a triangle t in the triangulation* $\Delta$ *such that every point in t is an approximate link center of* $\mathbf{P}$.

In other words every $x \in t$ has covering radius $c_L(x) \leq r+1$. Let us call such a triangle a *center triangle*. That a center triangle must exist is clear: any triangle that contains a point of the true link center of $\mathbf{P}$ is a center triangle.

How can we find such a center triangle quickly? Some sort of "binary search" on the tree formed by the triangles of $\Delta$ seems to be called for. Recall that, as already pointed out in the proof of Lemma 6, the triangles of $\Delta$ form a graph $T$ that is a tree. In this graph the $n-2$ nodes are the triangles of $\Delta$ and two nodes are adjacent iff the corresponding triangles share an edge. Obviously no node in $T$ has degree exceeding 3. It is well known that, using only linear time, we can always find in such a tree an edge whose removal would leave two subtrees, each containing no more than two-thirds of the nodes of the original tree. If we could also determine in $O(n)$ time which of the two subtrees contains a node corresponding to a center triangle in $\Delta$, then recursive application of this tree-splitting idea would yield an $O(n \log n)$ algorithm for finding such a center triangle. (See [3] and [8] for other algorithms based on this idea.)

Thus we want to solve the following problem in linear time: given an edge $e$ of the triangulation $\Delta$ that cuts the polygon $\mathbf{P}$ into two polygons $\mathbf{P}_1$ and $\mathbf{P}_2$, which of $\mathbf{P}_1$ or $\mathbf{P}_2$ contains a triangle of $\Delta$ that is a center triangle of $\mathbf{P}$?

First a few more definitions: in the usual way we can generalize the distance function $d_L$ so that one of the arguments is a subset of $\mathbf{P}$:

$$d_L(A, x) = \min\{d_L(y, x) | y \in A\}.$$

The *k-neighborhood* of a set $A \subset \mathbf{P}$ in $\mathbf{P}$ is then defined as

$$N_k(A, \mathbf{P}) = \{x \in \mathbf{P} | d_L(A, x) \leq k\},$$

and the *covering radius* of $A$ is defined as

$$c(A, \mathbf{P}) = \min\{k | \mathbf{P} \subset N_k(A, \mathbf{P})\}.$$

With these definitions we can state the following characterization:

**Lemma 13.** *Let $e$ be a diagonal of a triangulation $\Delta$ of a simple polygon $\mathbf{P}$ that cuts $\mathbf{P}$ into two simple polygons $\mathbf{P}_1$ and $\mathbf{P}_2$. Let $c_1 = c(e, \mathbf{P}_1)$ and $c_2 = c(e, \mathbf{P}_2)$.*

(i) *If $c_1 = c_2$, then both triangles of $\Delta$ that contain $e$ are center triangles for $P$.*
(ii) *If $c_1 < c_2$, then $\mathbf{P}_2$ contains a triangle of $\Delta$ that is a center triangle for $P$.*

*Proof.* Let $r = r_L(\mathbf{P})$ be the link radius of the polygon $\mathbf{P}$.

(i) Let $z$ be a point of the link center of $\mathbf{P}$. Without loss of generality assume $z$ lies in $\mathbf{P}_1$. Let $p$ be a point in $\mathbf{P}_2$ for which $d_L(p, e) = c_2$. Since a link-geodesic path connecting $p$ and $z$ must cross the segment $e$, we have $c_1 = c_2 \leq d_L(p, z) \leq r$. From this it follows immediately that both triangles of $\Delta$ that contain $e$ must be center triangles of $\mathbf{P}$ since any point $q \in \mathbf{P}$ can be connected to any point $x$ in these two triangles by a link path with at most $r+1$ segments (at most $r$ segments from $q$ to some point on $e$ and one more segment to $x$).

(ii) If $\mathbf{P}_2$ contains a point of the true link center of $\mathbf{P}$, then there is nothing to prove since the triangle of $\Delta$ that contains that point will be a center triangle of $\mathbf{P}$ contained in $\mathbf{P}_2$. So assume $\mathbf{P}_1$ contains a point $z$ of the link center of $\mathbf{P}$. Using exactly the same argument as in (i) we get the inequalities $c_1 < c_2 \leq r$, and as in (i) we can conclude that the triangle of $\Delta$ that contains the edge $e$ and is contained in $\mathbf{P}_2$ is a center triangle. □

The only question that remains is whether the covering radii $c(e, \mathbf{P}_1)$ and $c(e, \mathbf{P}_2)$ can be computed in linear time. Using the techniques presented in [13] this is indeed possible if a triangulation $\Delta$ of $\mathbf{P}$ is available. We construct in linear time what Suri calls the "window tree" for $e$ in the polygon $\mathbf{P}_1$. The depth of that tree is then the covering radius $c(e, \mathbf{P}_1)$. The covering radius $c(e, \mathbf{P}_2)$ can be computed analogously. Thus we summarize the result of this section:

**Theorem 14.** *For a simple polygon* $\mathbf{P}$ *with* $n$ *vertices it is possible to compute in* $O(n \log n)$ *time a triangle contained in* $\mathbf{P}$ *all of whose points are approximate link centers of* $\mathbf{P}$, *i.e., they have a covering radius that exceeds the link radius of* $\mathbf{P}$ *at most by one.*

(*Note*: The results of this section provide an alternative method for obtaining an approximation for the link radius with error at most one, instead of calculating the link diameter and using Theorem 7 or Theorem 10.)

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