Discrete Comput Geom 3:97–102 (1988)

Discrete & Computational Geometry

Arrangements of Lines with a Minimum Number of Triangles Are Simple

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Abstract. Levi has shown that for every arrangement of n lines in the real projective plane, there exist at least n triangular faces, and Grünbaum has conjectured that equality can occur only for simple arrangements. In this note we prove this conjecture. The result does not hold for arrangements of pseudolines.

An *arrangement of lines* is a finite collection $\mathscr{A}$ of lines in the real projective plane $\mathbb{P}$, such that there is no point of $\mathbb{P}$ contained in all the lines of $\mathscr{A}$. In the case where no point of $\mathbb{P}$ belongs to more than two lines of $\mathscr{A}$, we say that $\mathscr{A}$ is *simple*. An arrangement of lines decomposes $\mathbb{P}$ into a cell complex. The three-sided faces are called *triangles* and the number of triangles is denoted by p_3.

In [3] we have obtained a nontrivial upper bound for p_3 (as a function of the number n of lines). Here, we are concerned with the lower bound of p_3, which was first determined by Levi:

Theorem 1 [2], [1, p. 25].

(1) *Any arrangement $\mathscr{A}$ of n lines satisfies $p_3 \geq n$.*
(2) *More precisely, for every line L of $\mathscr{A}$, there exist at least three triangles having an edge on L.*

In particular, if $p_3 = n$, then for every line L of $\mathscr{A}$ there exist exactly three triangles having an edge on L. For each $n \geq 6$, there exist several nonisomorphic arrangements of n lines satisfying $p_3 = n$ [1], [2], hence Theorem 1 is best possible. One example consists in taking the lines defined by the edges of a convex n-gon.

In his monograph Grünbaum has proposed the following conjecture:

Conjecture 1 [1, p. 25]. *If $\mathscr{A}$ is an arrangement of n lines with $p_3 = n$, then $\mathscr{A}$ is a simple arrangement.*

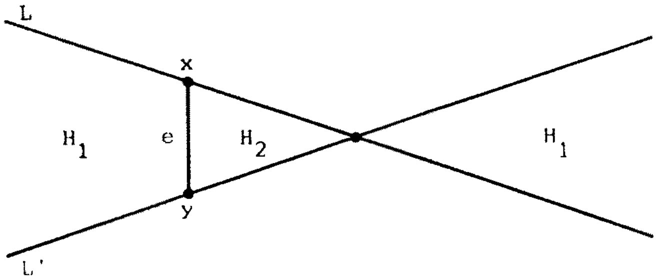

Fig. 1

It is the purpose of this note to prove this conjecture. The idea of the proof is to show that if a nonsimple arrangement of n lines with $p_3 = n$ exists, then it would necessarily contain a configuration which cannot be realized with straight lines.

We first need a definition and two lemmas. Let $\mathscr{A}$ be an arrangement of lines, $e = [x, y]$ an edge of $\mathscr{A}$, and L, L' lines of $\mathscr{A}$ such that $x \in L$, $y \notin L$, $x \notin L'$, and $y \in L'$. The edge e and the two lines L and L', define in a natural way two three-sided closed regions H_1 and H_2 of $\mathbb{P}$. We call these two regions the *half-lenses* determined by (e, L, L') (see Fig. 1).

Lemma 1. *Let H be a half-lens of $\mathscr{A}$ determined by (e, L, L'). Then, there exists a triangle of $\mathscr{A}$ having an edge on L and contained in H.*

Proof. By induction on the number f of faces of $\mathscr{A}$ that are contained in H. If $f = 1$, the result is trivial. If $f \geq 2$, there is an edge $e' = [x', y']$ of $\mathscr{A}$, $e' \neq e$, which is included in H and such that $e' \cap L = \{x'\}$. Let L'' be a line of $\mathscr{A}$ such that $e' \cap L'' = \{y'\}$. Clearly one (and in fact exactly one) half-lens H' determined by (e', L, L'') is included in H. We conclude by applying the induction hypothesis to H'. □

Lemma 2. *Let $\mathscr{A}$ be an arrangement of n lines with $p_3 = n$, and let T denote a triangle of $\mathscr{A}$. Then, every vertex of T belongs to exactly two lines of $\mathscr{A}$.*

Proof. We assume that there is a triangle T of $\mathscr{A}$ with a vertex x belonging to at least three lines of $\mathscr{A}$. Let e_1, e_2 be the edges of T that contain x, L the line of $\mathscr{A}$ defined by the third edge of T, and L' a line of $\mathscr{A}$ with $x \in L'$, $e_1 \not\subset L'$, $e_2 \not\subset L'$. Let also $e_3 = [y, z]$ be an edge on L' with $y = L \cap L'$, and L'' be a line of $\mathscr{A}$ such that $z \in L''$ and $y \notin L''$. It is easy to show that there exist three half-lenses H_1, H_2, H_3, determined, respectively, by (e_1, L, L'), (e_2, L, L'), and (e_3, L, L'') which have pairwise no face in common and do not contain T (see Fig. 2). By Lemma 1, there is a triangle having an edge on L in each H_i. Adding T, we get four triangles having an edge on L: a contradiction. □

Proof of Grünbaum's Conjecture. We assume that $\mathscr{A}$ is an arrangement of n lines with $p_3 = n$ and with a vertex x incident to three lines D_1, D_2, D_3 of $\mathscr{A}$. Let $[x, a_1]$, $[x, a_2]$, $\ldots$, $[x, a_6]$ be the edges on D_1, D_2, D_3 that contain x, with the notation of Fig. 3. Let $L_1, L_2, \ldots, L_6$ denote lines of $\mathscr{A} \backslash \{D_1, D_2, D_3\}$ that,

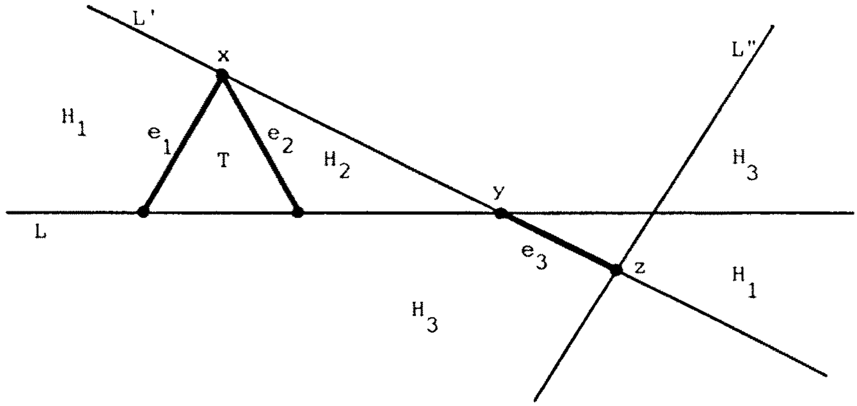

Fig. 2

respectively, contain $a_1, a_2, \ldots, a_6$. The L_i's are pairwise distinct, otherwise there would exist a triangle of $\mathscr{A}$ having x as one of its vertices, thus contradicting Lemma 2.

We now restrict our attention to the subarrangement $\mathscr{A}' = \{D_1, D_2, D_3, L_1, L_2, \ldots, L_6\}$ of $\mathscr{A}$. Let $b_{i,j} = D_i \cap L_j$ for all $i \in \{1, 2, 3\}$ and $j \in \{1, 2, \ldots, 6\}$. In Fig. 3 we have chosen the line at infinity in such a way that in the corresponding affine plane, x is between $b_{1,2}$ and $b_{1,3}$. Suppose that $b_{1,5}$ does not belong to the interval $]b_{1,2}, b_{1,3}[$. We then have the situation shown in Fig. 4. By Lemma 1, there is a triangle having an edge on D_1 in each of the four half-lenses H_1, H_2, H_3, H_4 represented on that figure. Since these half-lenses have pairwise no face in common, we get a contradiction. The same argument applies to $b_{1,6}$. More precisely:

(1) If $b_{1,5}$ belongs to $]b_{1,2}, x[$, then $b_{1,6}$ also belongs to $]b_{1,2}, x[$, otherwise L_6 would cut the edge $[x, a_5]$.

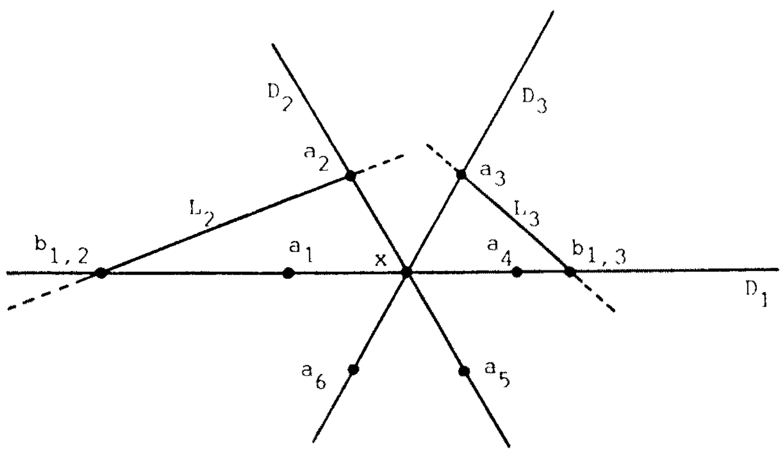

Fig. 3

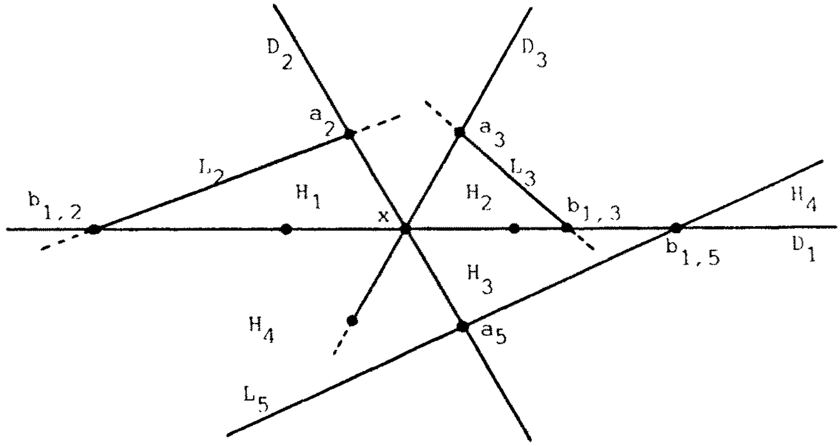

Fig. 4

(2) If $b_{1,5}$ belongs to $]x, b_{1,3}[$, then $b_{1,6}$ also belongs to $]x, b_{1,3}[$, for if $b_{1,6}$ belongs to $]b_{1,2}, x[$, we would be in the case of Fig. 5 and there would again exist four triangles having an edge on D_1 (one in each of the half-lenses H_1, H_2, H_3', H_4' with the notation of Fig. 5).

By symmetry, we may assume that we are in the first case, i.e., both $b_{1,5}$ and $b_{1,6}$ belong to $]b_{1,2}, x[$. We have then reconstructed a part of the arrangement $\mathscr{A}'$ (see Fig. 6). From Fig. 6 we immediately get that $b_{3,2} \notin]x, b_{3,5}[$. The preceding argument applied to D_3 then shows that both $b_{3,2}$ and $b_{3,1}$ belong to $]b_{3,4}, x[$ and we can continue the reconstruction of $\mathscr{A}'$ (see Fig. 7). Similarly, $b_{2,4} \notin]x, b_{2,1}[$. We deduce, always with the same arguments, that both $b_{2,4}$ and $b_{2,3}$ belong to $]b_{2,6}, x[$. We have then completely reconstructed $\mathscr{A}'$ and we obtain the arrangement shown in Fig. 8. But this arrangement cannot be realized with straight lines, as proved in [1, pp. 42 and 55] ($\mathscr{A}'$ fails to satisfy the Pappus–Pascal theorem), which completes the proof. □

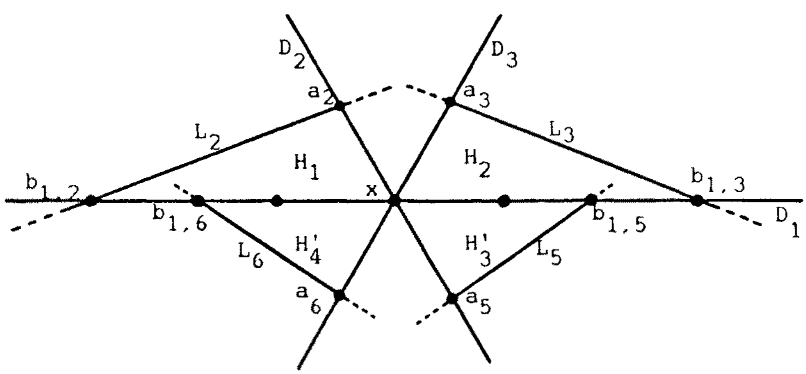

Fig. 5

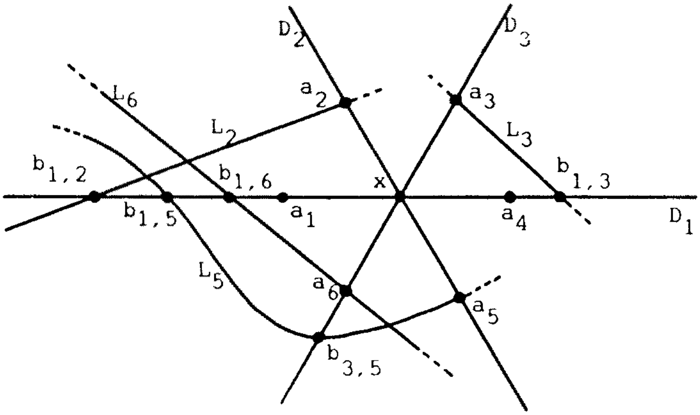

Fig. 6

Remarks. R. J. Canham (see [1]) has already observed that the arrangement $\mathcal{A}'$ of nine pseudolines that we have obtained in a somewhat natural way in Fig. 8, satisfies $p_3 = 9$. Thus, Conjecture 1 cannot be extended to arrangements of pseudolines. (See [1] for a study of this notion.)

This seems to be the first combinatorial relation that is satisfied by arrangements of lines but not by arrangements of pseudolines.

We finally notice that the truth of Conjecture 1 easily implies that the following conjecture of Grünbaum is also true [4]:

Conjecture 2 [1, p. 29]. *If $\mathcal{A}$ is an arrangement of $n \geq 5$ lines with exactly $\frac{1}{2}n(n-3)$ four-sided faces, then $\mathcal{A}$ is a simple arrangement.*

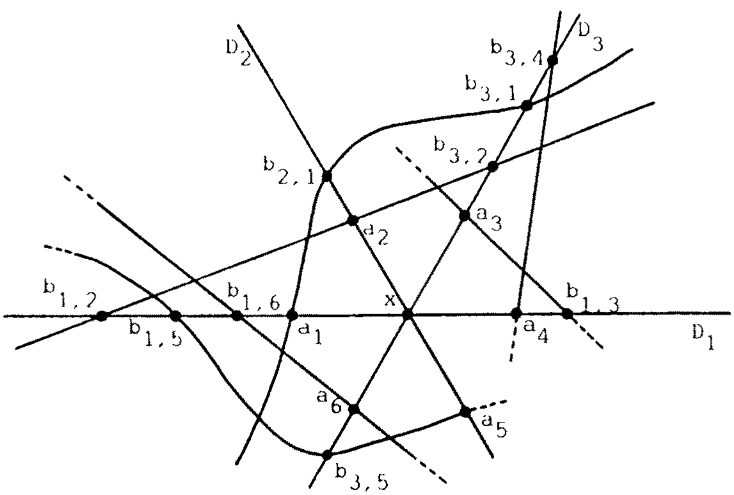

Fig. 7

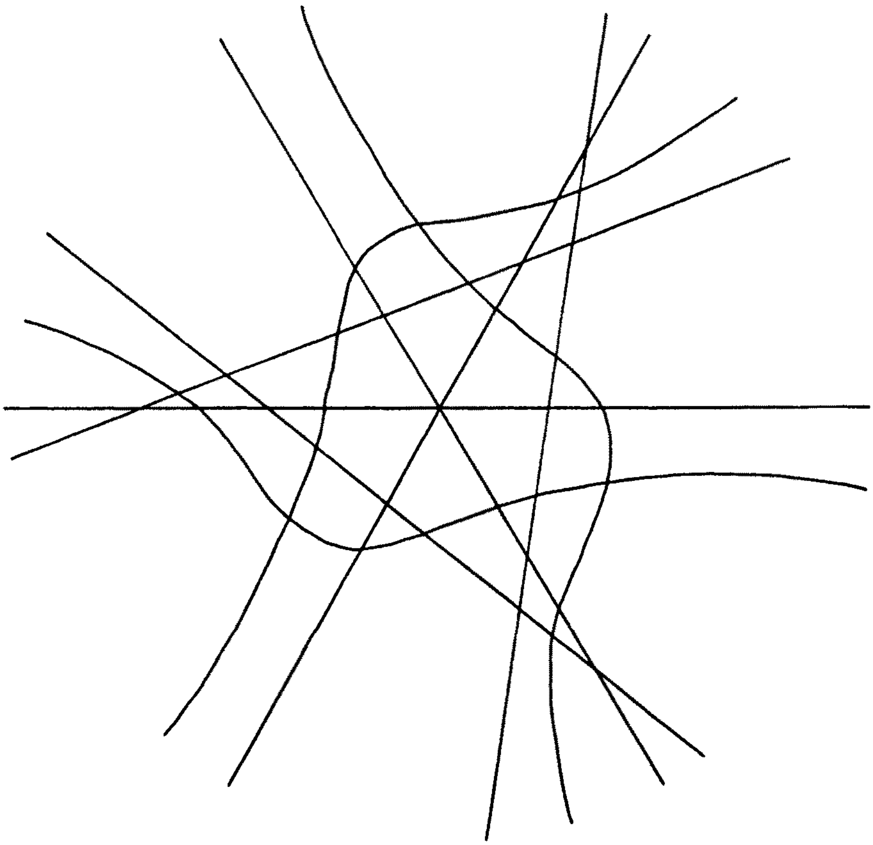

Fig. 8. A nonstretchable arrangement of nine pseudolines.

However, this method does not apply if "arrangement of lines" is replaced by "arrangement of pseudolines" in Conjecture 2. In [4] we have shown with other arguments that Conjecture 2 is true in the more general context of arrangements of pseudolines.

References

1. B. Grünbaum, *Arrangements and Spreads*, Regional Conference Series in Mathematics 10, American Mathematical Society, Providence, RI, 1972.
2. F. Levi, Die Teilung der projektiven Ebene durch Gerade oder Pseudogerade, *Ber. Math.-Phys. Kl. Sächs. Akad. Wiss. Leipzig* **78** (1926), 256–267.
3. J.-P. Roudneff, The maximum number of triangles in arrangements of (pseudo)-lines, submitted.
4. J.-P. Roudneff, Quadrilaterals and pentagons in arrangements of lines, *Geom. Dedicata*, to appear.

Received May 5, 1986.