Discrete Comput Geom 5:263-288 (1990)

Discrete & Computational Geometry

Hyperplane Arrangements with a Lattice of Regions*

Anders Björner,[1] Paul H. Edelman,[2] and Günter M. Ziegler[3]

[1] Department of Mathematics, Royal Institute of Technology, 10044 Stockholm, Sweden

[2] School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

[3] Department of Mathematics, Augsburg University, 8900 Augsburg, Federal Republic of Germany

Abstract. A hyperplane arrangement is a finite set of hyperplanes through the origin in a finite-dimensional real vector space. Such an arrangement divides the vector space into a finite set of regions. Every such region determines a partial order on the set of all regions in which these are ordered according to their combinatorial distance from the fixed base region.

We show that the base region is simplicial whenever the poset of regions is a lattice and that conversely this condition is sufficient for the lattice property for three-dimensional arrangements, but not in higher dimensions. For simplicial arrangements, the poset of regions is always a lattice.

In the case of supersolvable arrangements (arrangements for which the lattice of intersections of hyperplanes is supersolvable), the poset of regions is a lattice if the base region is suitably chosen. We describe the geometric structure of such arrangements and derive an expression for the rank-generating function similar to a known one for Coxeter arrangements. For arrangements with a lattice of regions we give a geometric interpretation of the lattice property in terms of a closure operator defined on the set of hyperplanes.

The results generalize to oriented matroids. We show that the adjacency graph (and poset of regions) of an arrangement determines the associated oriented matroid and hence in particular the lattice of intersections.

1. Introduction

Let $\mathcal{H}$ be a finite set of hyperplanes through the origin in a real vector space $\mathbb{R}^d$. We study the combinatorial structure of the set $\mathcal{R}$ of *regions*, that is, connected

* The work of Anders Björner was supported in part by a grant from the NSF. Paul Edelman's work was supported in part by NSF Grants DMS-8612446 and DMS-8700995. The work of Günter Ziegler was done while he held a Norman Levinson Graduate Fellowship at MIT.

components of

$$\mathbb{R}^d - \bigcup_{H \in \mathcal{H}} H,$$

the complement of the arrangement $\mathcal{H}$.

In particular, the *adjacency graph* of $\mathcal{H}$ has $\mathcal{R}$ as its set of vertices and connects two regions by an edge if they are adjacent (separated by exactly one hyperplane). If one of the regions B is chosen as a distinguished base region, the adjacency graph, directed away from B, gives rise to a finite, graded poset $P(\mathcal{H}, B)$, the *poset of regions* of $\mathcal{H}$ with respect to the base B as defined in [Ed3].

This paper has two main purposes. First we give conditions under which $P(\mathcal{H}, B)$ is a lattice, and discuss some related geometric and combinatorial questions. Secondly we prove that an oriented matroid can be reconstructed from its adjacency graph, which has interesting consequences for hyperplane arrangements.

If $P(\mathcal{H}, B)$ is a lattice, then B necessarily has to be a simplicial cone. We show that this condition is also sufficient for $d \leq 3$ but not for $d > 3$. However, for several highly structured arrangements more can be said.

If $\mathcal{H}$ is *simplicial* (every region of $\mathcal{H}$ is a simplicial cone), then $P(\mathcal{H}, B)$ is a lattice for every region B. This generalizes results by Björner [Bj1] on the weak ordering of finite Coxeter groups.

An arrangement $\mathcal{H}$ is called *supersolvable* if the lattice $L(\mathcal{H})$ of intersections of hyperplanes in $\mathcal{H}$ (ordered by reverse inclusion) is a supersolvable lattice as defined by Stanley [St2]. We give a geometric description of supersolvable arrangements. For such arrangements there is a natural choice of base region B such that $P(\mathcal{H}, B)$ is a lattice. Furthermore, for such choice of a base region the rank-generating function factors in a way which is similar to the expression known for the case of Coxeter arrangements.

Much of our discussion of the lattice property of the bounded posets $P(\mathcal{H}, B)$ depends on a simple "local" criterion for the lattice property in bounded posets of finite length (Lemma 2.1) that appears to be new.

We proceed to describe the lattice property and lattice operations geometrically in terms of a closure operator defined on the set $\mathcal{H}$ of hyperplanes. With this closure operator, the set $S(R)$ of hyperplanes in $\mathcal{H}$ separating a region R from the base region is both closed and co-closed. If $P(\mathcal{H}, B)$ is a lattice, the converse also holds and the lattice operations can be expressed in terms of the closure operator.

In the final section we show how our results generalize to oriented matroids, which can be thought of as combinatorial abstractions of hyperplane arrangements. The adjacency graph of a hyperplane arrangement determines the corresponding oriented matroid. Thus the lattice $L(\mathcal{H})$ of intersections can be constructed from the poset of regions $P(\mathcal{H}, B)$ for all B.

2. Preliminaries

In this section we give a brief review of the relevant known results concerning hyperplane arrangements, discuss some ideas in abstract convexity, and, most importantly, prove a lemma in pure lattice theory which is critical for the next section. We also take the opportunity to establish most of our notation for the rest of the paper.

For the most part our terminology and notation for lattices and posets is taken from Birkhoff [Bi]. A poset is called *bounded* if it has a maximum and a minimum element, denoted $\hat{1}$ and $\hat{0}$, respectively. If a poset P has a $\hat{0}$ then, for each $x \in P$, let $\mathrm{rk}(x)$, the *rank of* x, be the length of the longest chain in the interval $[\hat{0}, x]$, assuming that the length of such chains is bounded. By the *rank of* P, $\mathrm{rk}\, P$, we mean the length of the longest chain in P. The *rank generating function* of P is the polynomial $\zeta(P, q) = \sum_{x \in P} q^{\mathrm{rk}(x)}$. A poset is called *graded* if it is bounded and every maximal chain has the same finite length. A poset with $\hat{0}$ is called *ranked* if every interval $[\hat{0}, x]$ is graded. If x and y are in P then we say that y *covers* x, denoted $x \lessdot y$, if $x < y$ and $x < z \leq y$ implies $z = y$.

Critical for the next section is the following lemma. Surprisingly we were unable to find any previous reference to it in the literature.

Lemma 2.1. *Let P be a bounded poset of finite rank such that, for any x and y in P, if x and y both cover an element z then the join $x \vee y$ exists. Then P is a lattice.*

Proof. Since P is bounded, it is sufficient to show that P is a join semilattice, i.e., for any pair x and y in P, $x \vee y$ exists. It then easily follows that P is a lattice. We prove this by induction on $\mathrm{rk}\, P$. If $\mathrm{rk}\, P \leq 2$ then the lemma is obviously true.

Assume $\mathrm{rk}\, P = k$. Take $a, b \in P$ so that a and b do not cover a common element. We prove the existence of $a \vee b$. Let j_1 and j_2 be atoms of P so that $j_1 \leq a$ and $j_2 \leq b$. If $j_1 = j_2 = j$ then both a and b are in the interval $[j, \hat{1}]$ in P and so by induction $a \vee b$ exists.

If $j_1 \neq j_2$ then, since $j_1, j_2 \gtrdot \hat{0}$, $j_1 \vee j_2$ exists. Both a and $j_1 \vee j_2$ are in $[j_1, \hat{1}]$ so by induction $a \vee j_1 \vee j_2$ also exists. Finally, both $a \vee j_1 \vee j_2$ and b lie in $[j_2, \hat{1}]$ so $a \vee j_1 \vee j_2 \vee b = c$ exists and $a \vee b = c$ since $j_1 \leq a$ and $j_2 \leq b$. □

Let $\mathcal{H}$ be a collection (or *arrangement*) of hyperplanes in $\mathbb{R}^d$, $|\mathcal{H}| = n$. For the most part we are interested in arrangements such that every hyperplane in $\mathcal{H}$ contains the origin, i.e., $\mathcal{H}$ is a *central arrangement*. If $\mathcal{H}$ is not central then it is an *affine arrangement*. The *rank* of $\mathcal{H}$ is the dimension of the span of the normals to the hyperplanes. Note that for a central arrangement the rank is exactly the codimension of the intersection of all hyperplanes. Unless stated differently, we assume that $\mathcal{H}$ has rank d, that is the normals to the hyperplanes in $\mathcal{H}$ span all of $\mathbb{R}^d$.

The set $\mathbb{R}^d - \bigcup_{H \in \mathcal{H}} H$ is the disjoint union of open connected d-cells called *regions*. Let $\mathcal{R}$ denote the set of regions. If $R \in \mathcal{R}$ then the set of boundary hyperplanes $\mathcal{B}(R)$ is defined by

$$\mathcal{B}(R) = \{H \in \mathcal{H} \mid H \cap \bar{R} \text{ is of affine dimension } d-1\},$$

where $\bar{R}$ is the topological closure of R. A region R is called *simplicial* if

$|\mathscr{B}(R)| = d$, that is if R is a cone over an open simplex. A theorem of Shannon [Sh] shows that for any central arrangement $\mathscr{H}$ there are at least $d+1$ simplicial regions. The central arrangement $\mathscr{H}$ is called a *simplicial arrangement* if every region in $\mathscr{R}$ is simplicial.

The *adjacency graph* G of $\mathscr{H}$ has the regions as vertices and an edge between two vertices if the corresponding regions are *adjacent*, that is separated by exactly one hyperplane. The simplicial regions of $\mathscr{H}$ correspond to vertices of degree d and, by Shannon's theorem [Sh], G is a regular graph if and only if $\mathscr{H}$ is a simplicial arrangement.

Fix a region $B \in \mathscr{R}$, which we call the *base region*, and define for each $R \in \mathscr{R}$

$$S(R) = \{H \in \mathscr{H} \mid H \text{ separates } R \text{ from } B\}.$$

This allows us to define a partial order on $\mathscr{R}$ which we denote by $P(\mathscr{H}, B)$, or $P(\mathscr{H})$ if B is understood, by

$$R_1 \le R_2 \quad \Leftrightarrow \quad S(R_1) \subseteq S(R_2).$$

We call this poset the *poset of regions*. The Hasse diagram of this poset is the adjacency graph G of $\mathscr{H}$, directed away from B.

Given the fixed region B we choose a collection of vectors $\{z_H \in \mathbb{R}^d \mid H \in \mathscr{H}\}$ such that, for each $H \in \mathscr{H}$,

$$H = \{x \in \mathbb{R}^d \mid \langle z_H, x\rangle = 0\},$$

where $\langle\, ,\rangle$ is the standard Euclidean inner product, and for $b \in B$

$$\langle z_H, b\rangle > 0.$$

Let H^+ be the closed half-space bounded by H containing B and let H^- be the other closed half-space of H. If $z \in \mathbb{R}^d - \{0\}$ then let H_z be the hyperplane orthogonal to z.

We now collect a number of results concerning the structure of $P(\mathscr{H}, B)$ which are used in the subsequent sections:

Theorem 2.2 [Ed3].

(1) *For any arrangement $\mathscr{H}$ and base region B, $P(\mathscr{H}, B)$ is ranked and* $\operatorname{rk} R = |S(R)|$ *for every $R \in \mathscr{R}$.*
(2) *For $\mathscr{H}$ a central arrangement, $P(\mathscr{H}, B)$ is graded and self-dual under the map $R \mapsto -R$ where $-R = \{x \in \mathbb{R}^d \mid -x \in R\}$.*
(3) *If $\mathscr{H}$ is a central arrangement and B is a simplicial region, then each subset A of the atoms of $P(\mathscr{H}, B)$ has a join (least upper bound).* .

These facts appear in Proposition 1.1, Proposition 2.1, and Lemma 2.3, respectively, of [Ed3].

There is another partial order related to an arrangement $\mathcal{H}$. Let $L(\mathcal{H})$ be the set

$$L(\mathcal{H})=\left\{T\subseteq\mathbb{R}^d \mid T=\bigcap_{H\in\mathcal{I}} H \text{ for some subset } \mathcal{I}\subseteq\mathcal{H}\right\},$$

where we partially order $L(\mathcal{H})$ by reverse inclusion. We use the convention that $\mathbb{R}^d$ is the empty intersection. $L(\mathcal{H})$, the *lattice of intersections*, was introduced by Zaslavsky [Za] to study enumerative questions about the arrangement $\mathcal{H}$. As we see in Section 6, $L(\mathcal{H})$ can be recovered from the adjacency graph G, and hence from $P(\mathcal{H}, B)$ for any base region B.

The lattice of intersections $L(\mathcal{H})$ is a finite geometric lattice. Recall the definition

$$\chi(L, t)=\sum_{x\in L}\mu(\hat{0}, x)t^{\mathrm{rk}(\hat{1})-\mathrm{rk}(x)}$$

of the *characteristic polynomial* of a geometric lattice L, where μ is its Möbius function. In the sequel we will see that for some arrangements $\mathcal{H}$ there are intimate connections between the rank-generating function of the poset of regions, $\zeta(P(\mathcal{H}, B), q)$, and the characteristic polynomial of the lattice of intersections, $\chi(L(\mathcal{H}), t)$.

We continue this section of preliminaries by presenting three classes of central arrangements of hyperplanes which have special interest. The first, graphic arrangements, is very useful for creating examples in high dimensions. In this class the regions $\mathcal{R}$ have a nice combinatorial interpretation. The second class, polytopal arrangements, is of interest because the chains of $P(\mathcal{H}, B)$ have a useful interpretation in terms of a related convex polytope. The third class of examples, Coxeter arrangements, is of considerable importance in algebra and combinatorics.

Let G be a simple connected graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $E=\{e_{ij} \mid i \text{ is adjacent to } j\}$. From this graph we construct an arrangement of hyperplanes $\mathcal{H}(G)=\{H_{ij} \mid e_{ij}\in E\}$ in $\mathbb{R}^n$, where

$$H_{ij}=\{x\in\mathbb{R}^n \mid x_i-x_j=0\}.$$

This construction is due to Greene [Gre]. Notice that the rank of $\mathcal{H}(G)$ is $n-1$. An arrangement of type $\mathcal{H}(G)$ is called a *graphic arrangement.*

By an *acyclic orientation* of the graph G we mean an orientation of the edges of G so that there are no directed cycles. The edge e_{ij} is denoted (i, j) when the orientation is from i to j. Let Φ be the set of all acyclic orientations of G. The following is a fundamental observation of Greene [Gre]. For the proof see [GZ].

Theorem 2.3. *The regions of $\mathscr{H}(G)$ are in one-to-one correspondence with the elements of Φ. The correspondence is given by*

$$R(\alpha)=\{x\in\mathbb{R}^n | x_i<x_j \textit{ if } e_{ij} \textit{ is oriented } (i,j) \textit{ in } \alpha\}$$

for each acyclic orientation α and inversely

$$\alpha(R)=\{(i,j) | e_{ij}\in E \textit{ and } x_i<x_j \textit{ for } x\in R\}.$$

Fix an acyclic orientation $\beta\in\Phi$. For each $\alpha\in\Phi$ define

$$D(\alpha)=\{(i,j)\in\alpha | (j,i)\in\beta\}.$$

For each pair of acyclic orientations α_1 and α_2 of G we define $\alpha_1\leq\alpha_2$ if and only if $D(\alpha_1)\subseteq D(\alpha_2)$. It is easy to see that this partial ordering on Φ is the same as the one induced by the partial order $P(\mathscr{H}(G), B)$ where $B=R(\beta)$. In this ordering $\alpha_1>\alpha_2$ if and only if the orientations differ on exactly one edge e_{ij} and on that edge α_1 differs from β. This construction allows one to work easily with examples of arrangements of large rank.

The second construction is due to Carl Lee [Lee]. Let Q be a $(d-1)$-dimensional convex polytope with facets $\mathscr{F}=\{F_1, F_2, \ldots, F_k\}$. Embed Q into an affine $(d-1)$-dimensional subspace not containing 0 of $\mathbb{R}^d$. Let $\mathscr{H}=\{H_1, H_2, \ldots, H_k\}$ be the set of hyperplanes where $H_i=\operatorname{span}\{F_i\cup\{0\}\}$ for each $1\leq i\leq k$. We will call such an arrangement $\mathscr{H}$ a *polytopal arrangement.*

Let B be the region in $\mathscr{R}$ that contains Q. Notice that the set $\mathscr{B}(B)$ is the same as $\mathscr{H}$. (In fact this characterizes polytopal arrangements: an arrangement $\mathscr{H}$ is polytopal if and only if $\mathscr{B}(R)=\mathscr{H}$ for some region R.) Every maximal chain of $P(\mathscr{H}, B)$ induces a linear ordering of the set $\mathscr{H}$, say $(H_{i_1}, H_{i_2}, \ldots, H_{i_k})$, which in turn gives a linear ordering of the facets of Q: $(F_{i_1}, F_{i_2}, \ldots, F_{i_k})$. It was pointed out by [Lee] that the work of Bruggesser and Mani [BM] implies that all these linear orderings of $\mathscr{F}$ are shellings of the boundary of Q. In fact, these "curve shellings" induced by maximal chains in $P(\mathscr{H}, B)$ generalize the "line shellings" used by Bruggesser and Mani. It would be interesting to know more about the combinatorial properties of such curve shellings. Such properties of line shellings were studied by Danaraj and Klee [DK].

Generalizing the preceding paragraph, we observe that if $\mathscr{H}$ is **any** central arrangement with base region B, then the maximal chains in $P(\mathscr{H}, B)$ induce linear orderings of $\mathscr{H}$ which when restricted to the boundary hyperplanes $\mathscr{B}(B)$ give shelling orders of the facets of B (or, more precisely, of the boundary of the polytope over which B is a cone).

The third class of examples arises as follows. Let $\mathscr{C}$ be a finite collection of central hyperplanes in $\mathbb{R}^d$ and let $G_{\mathscr{C}}$ be the subgroup of $\mathrm{GL}(\mathbb{R}^d)$ generated by orthogonal reflections through the hyperplanes in $\mathscr{C}$. The finite groups arising in this way are the finite *Coxeter groups.* See Bourbaki [Bo] for a discussion of finite Coxeter groups. If $G_{\mathscr{C}}$ is a finite Coxeter group, let $\mathscr{H}$ be the set of hyperplanes whose corresponding orthogonal reflections belong to $G_{\mathscr{C}}$ (this set is in general strictly larger than $\mathscr{C}$). Arrangements $\mathscr{H}$ which correspond to finite Coxeter groups in this way are called *Coxeter arrangements*.

If $R_0, R_1, \ldots, R_k$ is a path of regions in the adjacency graph of a Coxeter arrangement, then reflection through the unique hyperplane which separates R_{i-1} from R_i must map R_{i-1} isometrically onto R_i, $1 \le i \le k$. So the product of these reflections will map R_0 onto R_k. It follows that the Coxeter group acts transitively on the set of regions $\mathscr{R}$ of its arrangement. In fact, the action is simply transitive, so once a base region B is chosen there is an induced one-to-one correspondence between regions and group elements. Clearly, due to symmetry under the group, the poset of regions $P(\mathscr{H}) = P(\mathscr{H}, B)$ is the same (up to isomorphism) for every base region B. The corresponding ordering of the group elements is known as the *weak order* (or *weak Bruhat order*) of the Coxeter group, see [Bj1] and [Ed3].

Let us now collect a few known facts about Coxeter arrangements that are relevant for this paper. Note that the transitive action on the regions together with the existence of at least one simplicial region [Sh] implies the first part of the following theorem, which is also otherwise well known [Bo, p. 153].

Theorem 2.4. *Let $\mathscr{H}$ be a Coxeter arrangement. Then:*

(1) [Bo] *$\mathscr{H}$ is a simplicial arrangement.*
(2) [Bj1] *$P(\mathscr{H})$ is a lattice.*
(3) *There exist integers $1 = e_1 \le e_2 \le \cdots \le e_d$, called* exponents, *such that*

$$[\text{So}] \quad \zeta(P(\mathscr{H}, q) = \prod_{i=1}^{d} (1+q+q^2+\cdots+q^{e_i}),$$

$$[\text{Te2}] \quad \chi(L(\mathscr{H}), t) = \prod_{i=1}^{d} (t-e_i).$$

In Section 3 we show that $P(\mathscr{H})$ is a lattice for *any* simplicial arrangement, thus generalizing (2). In Section 4 we discuss another class of arrangements for which the rank-generating function and characteristic polynomial factor just like for Coxeter arrangements.

As an example, consider K_n, the complete graph on the vertex set $\{1, 2, \ldots, n\}$, and the graphic arrangement $\mathscr{H}(K_n)$. This is a Coxeter arrangement corresponding to the symmetric group S_n of permutations of $\{1, 2, \ldots, n\}$. It is an arrangement of rank $n-1$ (all hyperplanes contain the line $x_1 = x_2 = \cdots = x_n$) and the exponents are $e_i = i$, $1 \le i \le n-1$. The lattice of intersections of $\mathscr{H}(K_n)$ is isomorphic to the lattice of partitions of the set $\{1, 2, \ldots, n\}$. The regions are in one-to-one correspondence with the acyclic orientations of K_n and also with the elements of S_n. The acyclic orientation of K_n corresponding to a certain permutation is simply its inversion graph.

Finally we discuss some ideas from abstract convexity that are relevant for Section 5. Our terminology is taken from [EJ]. Let X be a finite set and $\mathscr{L}$ a collection of subsets of X satisfying the properties:

(A1) $\varnothing, X \in \mathscr{L}$,
(A2) $\mathscr{L}$ is closed under intersection.

We can alternatively think of $\mathscr{L}$ as a closure operator on X. That is, for any subset $A \subseteq X$ define *the closure of* A, $\mathscr{L}(A)$, or $\bar{A}$, by

$$\mathscr{L}(A) = \bar{A} = \bigcap_{\{K \in \mathscr{L} \mid K \supseteq A\}} K.$$

It is easy to check that $\mathscr{L}$ is a *closure operator on* X with the additional property that $\mathscr{L}(\varnothing) = \varnothing$, i.e., $\mathscr{L}$ is a map $\mathscr{L}: 2^X \to 2^X$ with the properties:

(C1) $\mathscr{L}(A) \supseteq A$,
(C2) $A \subseteq B \Rightarrow \mathscr{L}(A) \subseteq \mathscr{L}(B)$,
(C3) $\mathscr{L}(A) = \mathscr{L}(\mathscr{L}(A))$.

We move freely between the interpretation of $\mathscr{L}$ as a collection of subsets and as a closure operator. Those sets $K \in \mathscr{L}$, or equivalently those sets $K \subseteq X$ such that $K = \mathscr{L}(K)$, are called *convex sets.*

A closure operator $\mathscr{L}$ is said to be an *antiexchange closure* if, for every pair $x, y \in X$, $x \neq y$, and a convex set K, $x, y \notin K$, we have that $x \in \mathscr{L}(K \cup y)$ implies $y \notin \mathscr{L}(K \cup x)$. A pair $(X, \mathscr{L})$ where $\mathscr{L}$ is an antiexchange closure is called a *convex geometry on* X. If X is understood, we just refer to $\mathscr{L}$ as the convex geometry. This is only one of numerous equivalent ways of defining a convex geometry. For other definitions see Theorem 2.1 of [EJ].

The collection $\mathscr{L}$ has a natural partial order by containment. Under this partial order $\mathscr{L}$ is a lattice where the meet operation is intersection and the join operation is closure of the union. We refer to this lattice as $L(\mathscr{L})$.

A finite lattice L is said to be *meet-distributive* if, for every $y \in L$ and x the meet of all elements covered by y, the interval $[x, y]$ is a Boolean algebra. For a survey of results on meet-distributive lattices see [Ed4].

Theorem 2.5 [EJ, Theorem 4.1]. *A lattice L is meet-distributive if and only if $L = L(\mathscr{L})$ for some convex geometry $(X, \mathscr{L})$.*

3. Simplicial Regions and Arrangements

In this section we focus on the question of how the geometry of the arrangement $\mathscr{H}$ and the base region B affects the combinatorial structure of the poset $P(\mathscr{H}, B)$. Our main concern is under what conditions is $P(\mathscr{H}, B)$ a lattice.

Theorem 3.1. *If $P(\mathscr{H}, B)$ is a lattice, then B is a simplicial region.*

Proof. Suppose that B is not simplicial. Then $|\mathscr{B}(B)| \geq d+1$ and B is covered by the set of regions

$$\{R \in \mathscr{R} \mid |S(R)| = 1 \text{ and } S(R) \in \mathscr{B}(B)\}.$$

If $Z=\{z_H | H \in \mathscr{B}(B)\}$ then Z is a dependent set and we let $D \subseteq Z$ be a minimal dependent subset of Z.

If $z \in D$ then

$$z=-\sum_{H \in U} \alpha_H z_H+\sum_{G \in V} \beta_G z_G, \qquad (*)$$

where $D=\{z\} \sqcup \{z_H | H \in U\} \sqcup \{z_G | G \in V\}$ (here $\sqcup$ means disjoint union) and the coefficients of α_H and β_G are all positive.

We observe that $V \neq \varnothing$ because otherwise $\langle z, b\rangle<0$ for all $b \in B$ which contradicts the choice of Z. Also $U \neq \varnothing$ because otherwise $\langle z_G, b\rangle>0$ for all $b \in B$ would imply that $\langle z, b\rangle>0$ and this means that $H_z \notin \mathscr{B}(B)$.

Consider the set of regions $\mathscr{R}'=\{R>B | S(R) \in V\}$. We will find an upper bound for $\mathscr{R}'$. Since $D-\{z\}$ is an independent set of vectors, the corresponding hyperplanes of $U \cup V$ can be thought of as coordinate hyperplanes. In particular, $(\bigcap_{H \in U} H^+) \cup (\bigcap_{G \in V} G^-)$ contains an open set of $\mathbb{R}^d$ and thus some region T. In other words, we can find a $T \in \mathscr{R}$ such that $V \subseteq S(T)$ and $S(T) \cap U=\varnothing$. Choose such a T minimal in $P(\mathscr{H}, B)$.

Now $V \subseteq S(T)$ means that T is an upper bound for the set $\mathscr{R}'$. On the other hand, from $(*)$ we see that $H_z \in S(T)$ as well. If we let R be the region so that $S(R)=\{H_z\}$ then from Theorem 2.2(2) we see that $-R \in \mathscr{R}$ and $S(-R)=\mathscr{H}-\{H_z\}$. Hence, both T and $-R$ are upper bounds for $\mathscr{R}'$ and $T \not\leq -R$.

Since T was chosen minimally there is therefore no least upper bound for $\mathscr{R}'$. □

For small dimensions, we have a converse to Theorem 3.1.

Theorem 3.2. *Let $\mathscr{H}$ be a central arrangement in $\mathbb{R}^d$ for $d \leq 3$, and B be a simplicial region. Then $P(\mathscr{H}, B)$ is a lattice.*

Proof. The cases $d=1$ and $d=2$ are trivial. Suppose then that $d=3$. Since B is simplicial, we know from Theorem 2.2(3) that the join of any set of atoms exists. It suffices to show that, for any atom R_0 of $P(\mathscr{H}, B)$, the interval $[R_0, -B]$ is a lattice. That this is sufficient follows from Lemma 2.1.

Let H be the hyperplane so that $S(R_0)=H$. Consider the affine arrangement $\mathscr{H}_1$ induced on the hyperplane

$$H_1=\{v \in \mathbb{R}^3 | \langle v, z_H\rangle=-1\}.$$

If we use $R_0 \cap H_1$ as the base region, then we get that

$$P(\mathscr{H}_1, R_0 \cap H_1) \approx [R_0, -B]$$

are isomorphic posets with isomorphism

$$R \mapsto R \cap H_1.$$

From the construction $P(\mathscr{H}_1, R_0 \cap H_1)$ is a bounded planar poset, i.e., the Hasse diagram can be drawn in the plane without intersecting edges. It is known that all bounded planar posets are lattices (see Example 7(a), p. 32 of [Bi]) and hence the proof is complete. □

Example 3.3. For $d=4$, Theorem 3.2 fails to hold as we illustrate with this example. Consider the arrangement $\mathscr{H}=\{H_1, H_2, \ldots, H_6\}$ in $\mathbb{R}^4$ given by

$$\begin{aligned}
H_1 &= \{x_1=0\},\\
H_2 &= \{x_2=0\},\\
H_3 &= \{x_1+2x_2+x_3=0\},\\
H_4 &= \{2x_1+x_2+x_3=0\},\\
H_5 &= \{x_3-x_4=0\},\\
H_6 &= \{x_4=0\}.
\end{aligned}$$

Pick as a simplicial base region

$$B=\{x\in\mathbb{R}^4 \mid x_1>0, x_2>0, x_3>x_4>0\}$$

and let R be the atom of $P(\mathscr{H}, B)$ such that $S(R)=\{H_6\}$, i.e.,

$$R=\{x\in\mathbb{R}^4 \mid x_1>0, x_2>0, x_3>x_4, x_4<0, x_1+2x_2+x_3>0, 2x_1+x_2+x_3>0\}.$$

Let $\tilde{H}=\{x_4=-1\}$ and $\tilde{\mathscr{H}}=\{H_i\cap\tilde{H} \mid 1\le i\le 5\}$ so $\tilde{\mathscr{H}}=\{\tilde{H}_1, \tilde{H}_2, \ldots, \tilde{H}_5\}$ is a three-dimensional affine arrangement given by

$$\begin{aligned}
\tilde{H}_1 &= \{x_1=0\},\\
\tilde{H}_2 &= \{x_2=0\},\\
\tilde{H}_3 &= \{x_1+2x_2+x_3=0\},\\
\tilde{H}_4 &= \{2x_1+x_2+x_3=0\},\\
\tilde{H}_5 &= \{x_3=-1\}.
\end{aligned}$$

Then as seen in the proof of Theorem 3.2 the interval $[R, -B]$ in $P(\mathscr{H}, B)$ is isomorphic to $P(\tilde{\mathscr{H}}, \tilde{R})$ where $\tilde{R}=R\cap\tilde{H}$.

The region $-\tilde{B}=-B\cap\tilde{H}$ is the maximum element of $P(\tilde{\mathscr{H}}, \tilde{R})$,

$$-\tilde{B}=\{x\in\mathbb{R}^3 \mid x_1<0, x_2<0, x_3<-1\}$$

and hence $-\tilde{B}$ covers a region $\tilde{T}$ in $P(\tilde{\mathscr{H}}, \tilde{R})$,

$$\tilde{T}=\{x\in\mathbb{R}^3 \mid x_1<0, x_2<0, x_3>-1, x_1+2x_2+x_3<0, 2x_1+x_2+x_3<0\}.$$

Notice that $S(\tilde{T}) = \tilde{\mathscr{H}} - \tilde{H}_5$.

The interval $[\tilde{R}, \tilde{T}]$ consists of exactly those regions in $P(\tilde{\mathscr{H}}, \tilde{R})$ which lie in the half-space $\tilde{H}_5^+$ so this interval is canonically isomorphic to the poset $P(\tilde{\mathscr{H}} - \tilde{H}_5, \tilde{R}')$ where $\tilde{R}'$ is the region containing $\tilde{R}$. We note that $\tilde{\mathscr{H}} - \tilde{H}_5$ is a central arrangement in $\mathbb{R}^3$ and $\tilde{R}'$ is not simplicial, so neither the interval $[\tilde{R}, \tilde{T}]$ in $P(\tilde{\mathscr{H}}, \tilde{R})$ nor the interval $[R, T]$ in $P(\mathscr{H}, B)$ is a lattice and hence $P(\mathscr{H}, B)$ is not a lattice.

So although it is necessary that B be simplicial in order for $P(\mathscr{H}, B)$ to be a lattice it is not sufficient. The following theorem presents a sufficient condition.

Theorem 3.4. *If $\mathscr{H}$ is a simplicial arrangement then $P(\mathscr{H}, B)$ is a lattice for all choices of base region B.*

Proof. As noted in Theorem 2.2(3) if B is a simplicial region then the join exists for every subset of regions covering B, hence in particular for every pair of regions covering B.

Assume R_1 and R_2 cover a common region R. We wish to show that $R_1 \vee R_2$ exists in $P(\mathscr{H}, B)$. Since maximal chains in $[R, -B]$ correspond to minimal paths from R to $-B$ in the adjacency graph, it is clear that the interval $[R, -B]$ in $P(\mathscr{H}, B)$ is isomorphic to the interval $[R, -B]$ in $P(\mathscr{H}, R)$. The regions R_1 and R_2 still cover R in $P(\mathscr{H}, R)$ and by Theorem 2.2(3), since R is simplicial, $R_1 \vee R_2$ exists in $P(\mathscr{H}, R)$. Since R_1 and R_2 are both less than $-B$ we also have that $R_1 \vee R_2 \in [R, -B]$ in $P(\mathscr{H}, R)$. Hence $R_1 \vee R_2$ exists in $P(\mathscr{H}, B)$ as well.

Now, applying Lemma 2.1 we see that $P(\mathscr{H}, B)$ is a lattice.

By Lemma 2.1 there must be some local criterion for deciding in general when $P(\mathscr{H}, B)$ has the lattice property. We do not have a good guess for such a criterion and thus we leave open the question of finding one.

4. Supersolvable Arrangements

Now we proceed to study "supersolvable" arrangements:

Definition 4.1. Let L be a finite geometric lattice of rank d. An element V of L is *modular* if

$$\mathrm{rk}(V \vee W) + \mathrm{rk}(V \wedge W) = \mathrm{rk}\, V + \mathrm{rk}\, W$$

for every $W \in L$. L is *supersolvable* if it has a maximal chain

$$\hat{0} = V_0 < V_1 < \cdots < V_d = \hat{1}$$

of modular elements [St2].

From now on let L be a supersolvable geometric lattice and $\hat{0} = V_0 < V_1 < \cdots < V_d = \hat{1}$ be a fixed maximal chain of modular elements in L. For $1 \le i \le d$, let e_i be the number of atoms of L that lie below V_i, but not below V_{i-1}, i.e., the number of atoms in $[\hat{0}, V_i] - [\hat{0}, V_{i-1}]$. Trivially we get $e_1 = 1$, and $\sum_{i=1}^{d} e_i$ is the number of atoms in L.

A main result of [St2] states that the characteristic polynomial of L factors as

$$\chi(L, t) = \prod_{i=1}^{d} (t - e_i).$$

In particular this shows that the multiset $\{e_1, \ldots, e_d\}$ does not depend on the maximal chain of modular elements in L.

Definition 4.2. A central arrangement $\mathscr{H}$ is *supersolvable* if its lattice $L(\mathscr{H})$ of intersections is a supersolvable lattice. The integers $e_1, \ldots, e_d$ associated with $L(\mathscr{H})$ are called the *exponents* of $\mathscr{H}$.

Supersolvable arrangements were first considered in [St3] and [JT] in the context of *free* arrangements of hyperplanes as defined by Terao [Te1].

The following result, generalizing Theorem 5.4 of [JT] to arbitrary dimension d, describes the geometric structure and construction of supersolvable arrangements inductively. An equivalent theorem for supersolvable arrangements in complex vector spaces was independently proved in Corollary 2.17 of [Te3].

Theorem 4.3. *Every arrangement $\mathscr{H}$ of rank at most 2 is supersolvable. An arrangement $\mathscr{H}$ of rank $d \ge 3$ is supersolvable if and only if it can be written as $\mathscr{H} = \mathscr{H}_0 \sqcup \mathscr{H}_1$ (disjoint union, $\mathscr{H}_1 \ne \varnothing$), where $\mathscr{H}_0$ is a supersolvable arrangement of rank $d-1$, and, for any $H', H'' \in \mathscr{H}_1$ ($H' \ne H''$), there is an $H \in \mathscr{H}_0$ such that $H' \cap H'' \subseteq H$.*

Proof. Every geometric lattice of rank 2 is supersolvable, from which the first statement follows.

If $\mathscr{H}$ is supersolvable, let V_{d-1} be the coatom in a maximal modular chain of $L(\mathscr{H})$ as above, and define $\mathscr{H}_0 = \{H \in \mathscr{H} | H \supseteq V_{d-1}\}$, $\mathscr{H}_1 = \mathscr{H} - \mathscr{H}_0$. Then $\mathscr{H}_0$ is obviously supersolvable of rank $d-1$, and, for $H', H'' \in \mathscr{H}$, $H' \ne H''$, we find that H' and H'' are both complements of V_{d-1} in $L(\mathscr{H})$. Thus $V_{d-1} \vee (H' \vee H'') = \hat{1}$, hence by modularity $H = V_{d-1} \wedge (H' \vee H'')$ is an atom in $L(\mathscr{H})$, that is a hyperplane in $\mathscr{H}$. But $H \le V_{d-1}$ means $H \in \mathscr{H}_0$, and $H \le H' \vee H''$ means $H \supseteq H' \cap H''$.

To prove the converse, let $\mathscr{H}_0$ and $\mathscr{H}_1$ be given as above. Any maximal chain of modular elements of $L(\mathscr{H}_0)$ together with $\hat{1}$ defines a maximal chain in $L(\mathscr{H})$. By the argument on p. 217 of [St1] it is sufficient to show that V_{d-1} (the maximal element of $L(\mathscr{H}_0)$) is modular in $L(\mathscr{H})$. For this let $Y \in L(\mathscr{H})$. For $Y \in L(\mathscr{H}_0)$ there is nothing to show. For $Y \in L(\mathscr{H}) - L(\mathscr{H}_0)$, write Y as a join of $\mathrm{rk}(Y)$ atoms such that the number of atoms (hyperplanes) from $\mathscr{H}_1$ in this expression is minimal. But for $H', H'' \in \mathscr{H}_1$, $H' \ne H''$, there is an $H \in \mathscr{H}_0$ such that $H \vee H' = H' \vee$

H''. Thus Y can be written as $Y = Y_0 \vee H'$, where $Y_0 \in L(\mathcal{H}_0)$, $H' \in \mathcal{H}_1$, and $\operatorname{rk} Y_0 = \operatorname{rk} Y - 1$ from semimodularity. This shows that

$$\begin{aligned}\operatorname{rk}(V_{d-1} \vee Y) + \operatorname{rk}(V_{d-1} \wedge Y) &= \operatorname{rk} \hat{1} + \operatorname{rk} Y_0 \\ &= (\operatorname{rk} V_{d-1} + 1) + (\operatorname{rk} Y - 1) \\ &= \operatorname{rk} V_{d-1} + \operatorname{rk} Y.\end{aligned}$$ □

We remark that the exponents $\{e_1, \ldots, e_d\}$ of $\mathcal{H}$ are recursively given by Theorem 4.3 as the multiset $\{e_1, \ldots, e_{d-1}\}$ of exponents of $\mathcal{H}_0$ together with $e_d = |\mathcal{H}_1|$.

To study the poset of regions of a supersolvable arrangement $\mathcal{H}$, we observe that Theorem 4.3 describes a canonical, order-preserving, surjective map

$$\pi \colon P(\mathcal{H}, B) \to P(\mathcal{H}_0, \pi(B))$$

which is inclusion of the regions of $\mathcal{H}$ into larger regions of $\mathcal{H}_0$, where $\mathcal{H}_0$ is a supersolvable arrangement of lower rank. This allows constructions and proofs by induction. For every region R of the arrangement $\mathcal{H}$, let $\mathcal{F}(R)$ be its *fiber* under π, that is $\mathcal{F}(R) = \pi^{-1}(\pi(R))$.

Inductively define a base region B for $\mathcal{H}$ to be *canonical* if it is chosen such that $\pi(B)$ is canonical for $\mathcal{H}_0$, and that $\mathcal{F}(B)$ is linearly ordered in $P(\mathcal{H}, B)$. For $n \leq 2$, every base region is canonical.

Note that, given any base region B_0 for H_0, the regions of the arrangement $\mathcal{H}$ contained in B_0 are linearly ordered by adjacency such that for every canonical B_0 for $\mathcal{H}_0$ there are exactly two canonical B for $\mathcal{H}$ such that $\pi(B) = B_0$. Thus every supersolvable arrangement of rank d has at least 2^d canonical base regions.

Theorem 4.4. *Let $\mathcal{H}$ be a supersolvable arrangement and B be a canonical base region for $\mathcal{H}$, then the rank-generating function for $P(\mathcal{H}, B)$ is*

$$\zeta(P(\mathcal{H}, B), q) = \prod_{i=1}^{d} (1 + q + q^2 + \cdots + q^{e_i}),$$

where the e_i are the exponents of $L(\mathcal{H})$.

Proof. If B is canonical, then all the fibers of $P(\mathcal{H}, B)$ under the mapping π are chains of length $|\mathcal{H}_1| = e_d$. Recall from Theorem 2.2(1) that $P(\mathcal{H}, B)$ is graded with rank being given by $\operatorname{rk} R = |S(R)|$. Thus, if R is a region and $h(R)$ denotes its rank in the fiber $\mathcal{F}(R)$, then $\operatorname{rk} R = \operatorname{rk}(\pi(R)) + h(R)$. Hence we can compute the rank-generating function by induction as the product of the rank-generating functions $\prod_{i=1}^{d-1} (1 + q + \cdots + q^{e_i})$ for $\mathcal{H}_0$ and $1 + q + \cdots + q^{e_d}$ for the fibers. □

It is easy to see that the conclusion of Theorem 4.4 can hold only if B is simplicial. It is not true, however, that the hypothesis that B is canonical can be

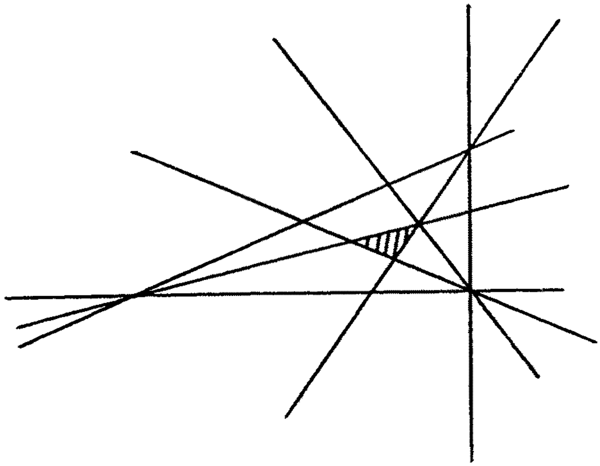

Fig. 4.1

replaced by B being simplicial. The central, supersolvable, arrangement in $\mathbb{R}^3$ (given by an affine section) shown in Fig. 4.1, with the base region B shaded, provides a counterexample. The rank-generating function for $P(\mathcal{H}, B)$ is $1+3q+6q^2+6q^3+6q^4+6q^5+3q^6+q^7$ instead of $(1+q)(1+q+q^2+q^3)^2$.

Corollary 4.5 [Te2], [JT]. *The number of regions of a supersolvable arrangement is*

$$\prod_{i=1}^{d} (1+e_i).$$

Theorem 4.6. *Let $\mathcal{H}$ be a supersolvable arrangement and B be a canonical base region, then $P(\mathcal{H}, B)$ is a lattice.*

Proof. As noted above, for every region R of the arrangement $\mathcal{H}$, $\mathcal{F}(R)$ is a chain of length $|\mathcal{H}_1|$ and forms an interval of $P(\mathcal{H}, B)$.

Let R_1 and R_2 be two regions of $\mathcal{H}$, and let $T \in \pi^{-1}(\pi(R_1) \vee \pi(R_2))$ be minimal in $\mathcal{F}(\pi(R_1) \vee \pi(R_2))$ such that $T \geq R_1$ and $T \geq R_2$. This is well-defined because we can assume that $P(\mathcal{H}_0, \pi(B))$ is a lattice by induction on the rank, and the maximal element of $\mathcal{F}(\pi(R_1) \vee \pi(R_2))$ is an upper bound for R_1 and R_2. Note that T is a minimal upper bound for R_1 and R_2 by construction.

Let T' be a different minimal upper bound. This implies that $T \not\leq T'$ and $\pi(T) < \pi(T')$.

Now let H be the hyperplane in $\mathcal{H}$ that separates T from the region it covers in $\mathcal{F}(T)$. Choose $H' \in S(T) - S(T')$: this is possible because $T \not\leq T'$. Note that $H' \in \mathcal{H}_1$ because $\pi(T) \leq \pi(T')$.

We now use that for every fiber $\mathcal{F}$ (i.e., for every region of $\mathcal{H}_0$) there is a unique linear order on $\mathcal{H}_1$ defined by $H_1 \leq H_2$ if and only if $H_1 \in S(R)$ for $R \in \mathcal{F}$, that is "H_2 is higher than H_1" in the corresponding region of $\mathcal{H}_0$.

By construction we have that $H' < H$ in $\mathcal{F}(T)$.

The construction of T implies $H \in S(R_1)$. On the other hand, $H' \notin S(T')$, thus $H' \notin S(R_1) \cup S(R_2)$ and $H' \notin S(R_1)$, and therefore $H < H'$ in $\mathcal{F}(R_1)$.

Finally $H \in S(T')$, but $H' \notin S(T')$, hence $H < H'$ in $\mathcal{F}(T')$.

Now if H_0 is the (unique) hyperplane in $\mathcal{H}_0$ that contains $H \cap H'$, then these data imply that $H_0 \in S(R_1) \Delta S(T)$ and $H_0 \in S(T) \Delta S(T')$ (here Δ means symmetric difference), which contradicts $\pi(R_1) \leq \pi(T) \leq \pi(T')$. Thus T as constructed above is the unique minimal upper bound for R_1 and R_2. □

Since the conclusions of Theorems 4.4 and 4.6 are also known to hold if $\mathcal{H}$ is a Coxeter arrangement (Theorem 2.4), it is natural to ask whether they generalize to the case where $\mathcal{H}$ is a "free" arrangement in the sense of Terao [Te1]. In this general case, Corollary 4.5 is known to hold [Te2], with respect to the generalized exponents as described by Terao, see [Te1] or [St3]. It has been conjectured by Björner [Bj2] and Terao and Wagreich [Wa, p. 137] that Theorem 4.4 holds for free arrangements, that is, for a suitable choice of a base chamber B, the rank-generating function of $P(\mathcal{H}, B)$ factors as $\prod_{i=1}^{d} (1+q+\cdots+q^{e_i})$ for the generalized exponents e_i. However, Terao [Te4] has found that the arrangement "$A_4(17)$" from Grünbaum's lists [Gr2, p. 89], [GS, p. 53] of simplicial arrangements forms a counterexample to this conjecture. The generalization of Theorem 4.6 to free arrangements is true for $d \leq 3$ by Theorem 3.2 and the existence of simplicial base regions [Sh]. However, for large enough dimension it is probably false, too.

If $\mathcal{H}$ is a Coxeter arrangement, then the posets of regions $P(\mathcal{H}, B)$ obtained for the different bases are canonically isomorphic via the transitive action of the corresponding Coxeter group on the regions. However, an analogous statement for supersolvable arrangements is false as we show in the next example.

Example 4.7. Let $\mathcal{H}$ be the supersolvable arrangement in $\mathbb{R}^3$ with exponents 1, 2, 2 (Fig. 4.2(a)). Its adjacency graph is given by Fig. 4.2(b). The base regions

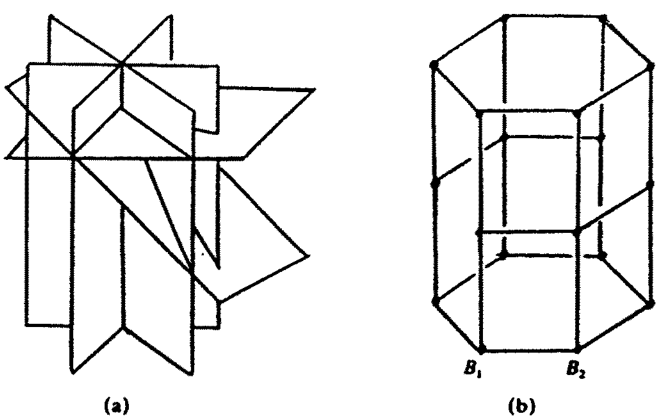

Fig. 4.2

B_1 and B_2 are both canonical, but the posets $P(\mathscr{H}, B_1)$ and $P(\mathscr{H}, B_2)$ (obtained by directing the adjacency graph from the respective base) are not isomorphic, though they share the lattice property and the same rank-generating function $(1+q)(1+q+q^2)^2$. □

In connection with Theorem 3.2, it seems interesting that the regions of supersolvable arrangements, though not simplicial in general, are of a very restricted combinatorial type. In particular, the regions of a supersolvable arrangement in $\mathbb{R}^d$ have at most $2(d-1)$ bounding hyperplanes. In fact, they are cones over $(d-1)$-dimensional polytopes that can be described inductively via the construction of Theorem 4.3. We omit the details.

5. Geometric Interpretation of the Lattice Property

When the poset of regions $P(\mathscr{H}, B)$ is a lattice the set $\mathscr{H}$ of hyperplanes admits the structure of a convex geometry which encodes much of the combinatorial information. In this section we describe this convex-geometric view on arrangements with a lattice of regions. The same analysis was previously carried out for Coxeter arrangements on p. 187 of [Bj1].

Let $\mathscr{H}$ be a finite, central arrangement of hyperplanes in $\mathbb{R}^d$ defining a set of regions $\mathscr{R}$. Choose a base region $B \in \mathscr{R}$, and for each $H \in \mathscr{H}$ define the normal vector z_H and the closed half-spaces H^+ and H^- as in Section 2. This means that $z_H \in H^+$ and $B \subseteq H^+$ for all $H \in \mathscr{H}$.

For each $H \in \mathscr{H}$, let $\mathscr{R}_H = \{R \in \mathscr{R} \mid H \notin S(R)\}$. Equivalently, $\mathscr{R}_H$ is the set of regions that are contained in the half-space H^+. Clearly, $\mathscr{R}_H$ is an order ideal in the poset of regions $P(\mathscr{H}, B)$.

Now define a closure operator on the subsets of $\mathscr{H}$ by

$$\bar{A} = \left\{ H \in \mathscr{H} \mid \mathscr{R}_H \supseteq \bigcap_{K \in A} \mathscr{R}_K \right\} \tag{5.1}$$

for $A \subseteq \mathscr{H}$. The formulation $\bar{A} = \{H \in \mathscr{H} \mid H^+ \supseteq \bigcap_{K \in A} K^+\}$ is equivalent.

Proposition 5.1. *The operator $A \mapsto \bar{A}$ is an antiexchange closure on the set of hyperplanes.*

Proof. It is clear that $A \mapsto \bar{A}$ is a closure operator, i.e., increasing, monotone, and idempotent. To check the antiexchange property assume that for some $H, G \in \mathscr{H} - A$ we have $A = \bar{A}$, $H \in \overline{A \cup \{G\}}$, and $H \neq G$. Then $\bigcap_{K \in A} \mathscr{R}_K$ is an order ideal that is not contained in $\mathscr{R}_G$. Let the region R be minimal in the set $(\bigcap_{K \in A} \mathscr{R}_K) - \mathscr{R}_G$, and pick a region R' which is covered by R. Then $G \in S(R) - S(R')$, and since $|S(R)| = |S(R')| + 1$ we have that $S(R) = S(R') \cup \{G\}$.

Now, $R' \in (\bigcap_{K\in A} \mathcal{R}_K) \cap \mathcal{R}_G$, so the assumption $H \in \overline{A \cup \{G\}}$ implies that $R' \in \mathcal{R}_H$, i.e., $H \notin S(R')$. But then also $H \notin S(R)$, i.e., $R \in \mathcal{R}_H$. So $R \in (\bigcap_{K\in A} \mathcal{R}_K) \cap \mathcal{R}_H$, which shows that $G \notin \overline{A \cup \{H\}}$. □

Remark 5.2. Assume that P is any finite poset and $(\mathcal{O}_i)_{i\in I}$ is a system of order ideals that satisfies the following separation property: if x covers y then there exists at most one ideal $\mathcal{O}_i$ such that $y \in \mathcal{O}_i$, $x \notin \mathcal{O}_i$. Then the closure operator

$$A \mapsto \bar{A} = \left\{ i \in I \,\middle|\, \mathcal{O}_i \supseteq \bigcap_{j\in A} \mathcal{O}_j \right\},$$

for $A \subseteq I$, has the antiexchange property. This is shown by a straightforward generalization of the preceding proof.

Remark 5.3. In the geometric setting of Proposition 5.1 the closure of a set $A \subseteq \mathcal{H}$ can also be characterized as

$$\bar{A} = \{H \in \mathcal{H} \mid z_H \in \operatorname{cone}\{z_K \mid K \in A\}\},$$

where by $\operatorname{cone}\{z_K\}$ we mean the convex cone generated by the rays $\{z_K\}$. This illuminates the connection with ordinary convexity.

A set of hyperplanes $A \subseteq \mathcal{H}$ will be called *convex* if $\bar{A} = A$ and *biconvex* if both A and its complement $A^c = \mathcal{H} - A$ are convex.

Proposition 5.4. *For each region $R \in \mathcal{R}$ the set $S(R)$ of separating hyperplanes is biconvex.*

Proof. Since $S(R) = S(-R)^c$, it suffices to show that

$$S(R)^c = \{H \in \mathcal{H} \mid H \notin S(R)\} = \{H \in \mathcal{H} \mid R \in \mathcal{R}_H\}$$

is convex. But this is immediately clear from the definition of closure in (5.1). □

Theorem 5.5. *Suppose that the poset of regions $P(\mathcal{H}, B)$ is a lattice. Then*:

(1) *$A = S(R)$ for some region R if and only if A is biconvex, for $A \subseteq \mathcal{H}$.*
(2) *$\overline{S(R_1) \cup S(R_2)} = S(T)$ if and only if $R_1 \vee R_2 = T$, for regions R_1, R_2, and T.*

Proof. Let R be any region and $[B, R]$ the closed interval below R in $P(\mathcal{H}, B)$. Clearly, $[B, R] \subseteq \mathcal{R}_H$ for all $H \in S(R)^c$, since $\mathcal{R}_H$ is an order ideal containing R. In fact,

$$[B, R] = \bigcap_{H\in S(R)^c} \mathcal{R}_H, \tag{5.2}$$

since a region R' in the intersection satisfies $H \notin S(R')$ for all $H \in S(R)^c$, i.e., $S(R') \subseteq S(R)$.

Part (2): Suppose that $\overline{S(R_1) \cup S(R_2)} = S(T)$. Then T is an upper bound to the regions R_1 and R_2 in $P(\mathscr{H})$. Also, if T' is another upper bound then $S(R_1) \cup S(R_2) \subseteq S(T')$, and, by Proposition 5.4, $\overline{S(R_1) \cup S(R_2)} \subseteq \overline{S(T')} = S(T')$. Hence, T is a least upper bound.

Conversely, assume that $R_1 \vee R_2 = T$. Write $\mathscr{R}(A)$ for the order ideal $\bigcap_{K \in A} \mathscr{R}_K$ when $A \subseteq \mathscr{H}$. Then, using (5.2) we have

$$\begin{aligned}\mathscr{R}(S(R_1) \cup S(R_2)) &= \mathscr{R}(S(-R_1)^c) \cap \mathscr{R}(S(-R_2)^c)\\ &= [B, -R_1] \cap [B, -R_2] = [B, (-R_1) \wedge (-R_2)] = [B, -T].\end{aligned}$$

Hence, $H \in \overline{S(R_1) \cup S(R_2)}$ if and only if $-T \in \mathscr{R}_H$, i.e., if and only if $H \in S(-T)^c = S(T)$.

Part (1): Necessity was proven in Proposition 5.4. Assume now that A and A^c are convex. Look at the order ideal $\mathscr{R}(A^c) = \bigcap_{K \in A^c} \mathscr{R}_K$. Clearly, by definition

$$R \in \mathscr{R}(A^c) \quad \Leftrightarrow \quad S(R) \subseteq A. \tag{5.3}$$

We have

$$\begin{aligned} R_1, R_2 \in \mathscr{R}(A^c) \quad &\Rightarrow \quad S(R_1) \cup S(R_2) \subseteq A\\ &\Rightarrow \quad \overline{S(R_1) \cup S(R_2)} \subseteq A \qquad \text{(since } A \text{ is convex)}\\ &\Rightarrow \quad S(R_1 \vee R_2) \subseteq A \qquad \text{(using part (2))}\\ &\Rightarrow \quad R_1 \vee R_2 \in \mathscr{R}(A^c).\end{aligned}$$

So, $\mathscr{R}(A^c)$ is closed under taking joins. Being an order ideal this means that $\mathscr{R}(A^c)$ has a unique maximal element, say R_0, and that $\mathscr{R}(A^c)$ is in fact a lower interval: $\mathscr{R}(A^c) = [B, R_0]$.

Now, $S(R_0) \subseteq A$ by (5.3). Conversely, if $H \in A$ then $\mathscr{R}_H \not\supseteq [B, R_0]$ since A^c is convex, which means that $H \in S(R_0)$. Hence, $A = S(R_0)$. □

Corollary 5.6. *If A and B are biconvex sets of hyperplanes then $\overline{A \cup B}$ is also biconvex.*

Easily available counterexamples show that the geometric property expressed by Corollary 5.6 is quite special; it seems to fail in most other classes of convex geometries.

Remark 5.7. The proof for part (2) of the theorem extends to show the following strengthening. If R_1 and R_2 are regions of an arbitrary arrangement $\mathscr{H}$ (the lattice property not assumed) and $\mathscr{M}$ denotes the set of minimal upper bounds to R_1 and R_2 in $P(\mathscr{H}, B)$, then

$$\overline{S(R_1) \cup S(R_2)} = \bigcap_{T \in \mathscr{M}} S(T). \tag{5.4}$$

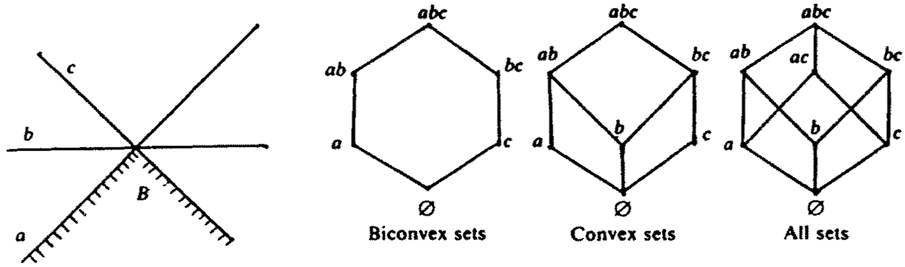

Fig. 5.1

Let $\mathscr{H}$ be an arrangement of hyperplanes with a lattice of regions $P(\mathscr{H}, B)$. The convex geometry on the set $\mathscr{H}$ defined in Proposition 5.1 contains surprisingly much information about the combinatorial-geometric structure of $\mathscr{H}$. Theorem 5.5 shows that $P(\mathscr{H}, B)$ is isomorphic to the lattice of biconvex subsets of $\mathscr{H}$ ordered by inclusion. In the next section we prove that the combinatorial type of the arrangement $\mathscr{H}$, up to isomorphism as an oriented matroid, is determined by $P(\mathscr{H}, B)$. It then follows that the same information is also carried by the convex geometry on $\mathscr{H}$.

It follows from the general properties reviewed in Section 2 that the mapping $R \mapsto S(R)$ embeds the poset of regions $P(\mathscr{H}, B)$ into the Boolean lattice $2^{\mathscr{H}}$ so that order and rank is preserved, for any arrangement $\mathscr{H}$. If $P(\mathscr{H}, B)$ is a lattice, then this embedding factors through the meet-distributive lattice of convex subsets of $\mathscr{H}$:

$$P(\mathscr{H}, B) \approx \{\text{biconvex sets}\} \overset{\varphi}{\to} \{\text{convex sets}\} \overset{\psi}{\to} \{\text{sets}\}.$$

Clearly, both φ and ψ preserve order and rank. Also, φ preserves joins (by Theorem 5.5) and ψ preserves meets (since the intersection of convex sets is convex). This is illustrated for an arrangement of three lines in $\mathbb{R}^2$ in Fig. 5.1.

6. Oriented Matroids

In this section we discuss how the results from the preceding sections can be generalized to oriented matroids. For the most part the details are not difficult and so we frequently leave the details of proofs to the reader. The technical issues involved in generalizing the results for supersolvable arrangements are somewhat more complicated and they are discussed at greater length.

We use the axiomatization of Folkman and Lawrence [FL] for oriented matroids. An oriented matroid $\mathcal{O} = (E, \mathscr{C}, *)$ is a triple consisting of a ground set E, a fixed-point free involution $*$ on E, and a collection of subset $\mathscr{C}$ of E called *circuits* satisfying the axioms:

(C1) *$C, D \in \mathscr{C}$ implies that $C \not\subset D$ (and $D \not\subset C$).*
(C2) *$C \in \mathscr{C}$ implies that $C^* \in \mathscr{C}$ and $C \cap C^* = \varnothing$.*

(C3) $C_1, C_2 \in \mathscr{C}$, $C_1 \neq C_2^*$, *and* $x \in C_1 \cap C_2^*$ *implies that there is a* $D \in \mathscr{C}$ *so that* $D \subseteq C_1 \cup C_2 - \{x, x^*\}$.

Here we use the convention that for a subset $S \subseteq E$, $S^* = \{x^* | x \in S\}$.

From an oriented matroid $\mathcal{O} = (E, \mathscr{C}, *)$ we define the *underlying matroid* $\bar{\mathcal{O}} = (\bar{E}, \bar{\mathscr{C}})$ to be the matroid with point set $\bar{E} = \{\bar{x} = \{x, x^*\} | x \in E\}$ and circuits $\bar{\mathscr{C}} = \{\bar{C} | C \in \mathscr{C}\}$ where $\bar{C} = \{\bar{x} | x \in \mathscr{C}\}$. By the rank of an oriented matroid rk $\mathcal{O}$ we mean the rank of the underlying matroid $\bar{\mathcal{O}}$. We call $\mathcal{O}$ an *oriented geometry* if $\bar{\mathcal{O}}$ is a geometry, i.e., $\bar{\mathcal{O}}$ has no loops and all the points are closed.

A subset $A \subseteq E$ is called an *acyclic set* if there is no circuit $C \in \mathscr{C}$ so that $C \subseteq A$ and $A \cap A^* = \varnothing$. Let $\mathscr{A}$ be the collection of all acyclic sets. The collection $\mathscr{A}$ forms a simplicial complex. All the maximal acyclic sets have size $|E|/2$ [Ed2, Corollary 2.2]. We call these the *acyclic orientations* of $\mathcal{O}$. The collection of all acyclic orientations of $\mathcal{O}$ we call $\Phi(\mathcal{O})$.

Associated with an oriented matroid $\mathcal{O}$ is a closure operator h on the set E defined by

$$h(B) = B \cup \{x \in E \,|\, \textit{there exists } C \in \mathscr{C} \textit{ such that } x^* \in C \subseteq B \cup \{x^*\}\}$$

for each $B \subseteq E$. This operator was studied in this form on p. 204 of [FL] and is equivalent to one studied by Las Vergnas [La2, Section 3]. The reader should observe that if A is an acyclic orientation then $h(A) = A$. If A is an acyclic orientation of an oriented geometry then by ex(A), the *extreme points of* A, we mean the unique minimal subset of A such that $h(\text{ex}(A)) = A$. That there is such a unique minimal subset follows from Theorem 2.2 of [La2] and Corollary 2.6 of [Ed1].

In what sense is an oriented matroid a generalization of an arrangement of hyperplanes? If $\mathscr{H}$ is a hyperplane arrangement then let $E = \{\pm z_H | H \in \mathscr{H}\}$, let $*$ be the change-of-sign map, and let $\mathscr{C}$ be the minimal subsets of E, not of the form $\{\pm z_H\}$, that are linearly dependent using only positive coefficients. Then $\mathcal{O} = (E, \mathscr{C}, *)$ is an oriented matroid. For $R \in \mathscr{R}$ let $A(R) = \{z \in E | \langle z, r\rangle > 0, r \in R\}$. Then $\Phi = \{A(R) | R \in \mathscr{R}\}$ is the collection of acyclic orientations of $\mathcal{O}$ [La1]. Moreover, $\text{ex}(A(R)) = \{z \in A(R) | H_z \in \mathscr{B}(R)\}$. Thus all the important geometric information in the arrangement $\mathscr{H}$ is easily interpreted in the context of oriented matroids. An oriented matroid that arises in the above way from a set of hyperplanes is called *coordinatizable.*

There is a natural way to generalize the construction of $P(\mathscr{H}, B)$ to a partial order on the acyclic orientations of $\mathcal{O}$. If we fix $B \in \Phi(\mathcal{O})$ then partially order Φ by $A_1 \leq A_2$ if and only if $S(A_1) \subseteq S(A_2)$ where $S(A) = A - B$. Call this partial order $P(\mathcal{O}, B)$. It has been shown by Cordovil [Co2] that many of the properties of posets of regions carry through for $P(\mathcal{O}, B)$.

Theorem 6.1 [Co1, Lemma 3.7], [Ed2, Theorem 2.3]. *For* $\mathcal{O}$ *an oriented geometry, and fixed acyclic orientation* B, $P(\mathcal{O}, B)$ *is ranked and* $\text{rk}\, A = |S(A)|$ *for every* $A \in \Phi(\mathcal{O})$. *Moreover,* $P(\mathcal{O}, B)$ *is self-dual under the map* $A \mapsto A^*$.

Theorem 6.2 [La2, Theorem 2.2]. *For $\mathcal{O}$ an oriented geometry and B a fixed acyclic orientation, an element $x \in E$ is in* $\mathrm{ex}(B)$ *if and only if there is an acyclic orientation $A \in \Phi(\mathcal{O})$ so that $S(A) = \{x^*\}$.*

We call an acyclic orientation B *simplicial* if $|\mathrm{ex}(B)| = \mathrm{rk}\, \mathcal{O}$. If every acyclic orientation of $\mathcal{O}$ is simplicial then we call $\mathcal{O}$ a *simplicial oriented matroid.* There are numerous examples of simplicial oriented matroids that are not coordinatizable, e.g., the "nonstretchable" simplicial arrangement of lines in Fig. 3.5 of [Gr3] gives rise to one. We should remark that (unlike the coordinatizable case) it is not known whether all oriented geometries possess a simplicial acyclic orientation. It has been conjectured that this is the case by Las Vergnas [La2].

Theorem 6.3. *For $\mathcal{O}$ an oriented geometry, if $P(\mathcal{O}, B)$ is a lattice then B is simplicial.*

Proof. The proof is a direct translation of the proof of Theorem 3.1 into oriented matroid terminology. We leave it to the reader. □

Theorem 6.4 [Co2, Theorem 2.13]. *For $\mathcal{O}$ an oriented geometry and B a simplicial acyclic orientation, each subset S of the atoms of $P(\mathcal{O}, B)$ has a unique least upper bound.*

Theorem 6.5. *If $\mathcal{O}$ is a simplicial oriented geometry, then $P(\mathcal{O}, B)$ is a lattice for all acyclic orientations B.*

Proof. The proof follows the lines of that of Theorem 3.4 only substituting the use of Theorem 6.4 for Theorem 2.2(3). □

If we fix an oriented matroid $\mathcal{O}$ and an acyclic orientation B, the closure operator h restricted to the subsets of B is an antiexchange closure [Ed1, Theorem 2.1]. This closure is just the oriented matroid generalization of the closure defined in Section 5 and in fact all the techniques from that section go through. Thus using these ideas we have

Theorem 6.6. *Suppose that for an oriented matroid $\mathcal{O}$ and fixed acyclic orientation B the poset $P(\mathcal{O}, B)$ is a lattice. Then for every biconvex set $S \subseteq B$, $B - S \cup S^*$ is an acyclic orientation of $\mathcal{O}$. Given two acyclic orientations A_1 and A_2 of $\mathcal{O}$, $A_1 \vee A_2$ is the acyclic orientation*

$$B - h(S(A_1) \cup S(A_2)) \cup [h(S(A_1) \cup S(A_2))]^*.$$

Finally we discuss the extensions of results in Section 4 on supersolvable arrangements to oriented matroids. An oriented matroid $\mathcal{O}$ is called *supersolvable* if the lattice of flats $L(\mathcal{O})$ of the underlying matroid $\bar{\mathcal{O}}$ is supersolvable. Note that there exist supersolvable oriented matroids which are not coordinatizable. These can be produced by adding some lines to the nonstretchable configuration of pseudolines in Fig. 3.6 of [Gr3]. We assume from now on that $\mathcal{O}$ is a geometry.

Suppose that $\mathcal{O}$ is supersolvable and $\overline{E_0}$, $\overline{E_0} \subseteq \bar{E}$, is the modular coatom in the modular chain in $L(\mathcal{O})$. Then the matroid $\bar{\mathcal{O}}$ restricted to $\overline{E_0}$, $\mathcal{O}(\overline{E_0})$, is also supersolvable. Let $\mathcal{O}(E_0)$ be the oriented matroid $\mathcal{O}$ restricted to the set $E_0 = \{e \in E \mid \bar{e} \in \overline{E_0}\}$. By E_1 we mean the set $E_1 = \{e \in E \mid e \notin E_0\}$. Thus E_0 and E_1 form a partition of E. There is a natural projection map $\pi: \Phi(\mathcal{O}) \to \Phi(\mathcal{O}(E_0))$ defined by $\pi(A) = A \cap E_0$ for each acyclic orientation A of $\mathcal{O}$.

Central to the results of Section 4 was the idea of a canonical region for a supersolvable arrangement. We now show how to define this for any supersolvable oriented geometry. First we require a lemma similar to Theorem 4.3. We leave the proof for the reader.

Lemma 6.7. *Let $\mathcal{O}$ be a supersolvable oriented geometry with partition E_0 and E_1 described above. For every pair $e_1, e_2 \in E_1$ there is a circuit $C \in \mathscr{C}$ so that $\bar{C} = \{\overline{e_1}, \overline{e_2}, \bar{y}\}$ for some $\bar{y} \in \overline{E_0}$.*

Theorem 6.8. *Let $\mathcal{O}$ be a supersolvable oriented geometry with partition E_0 and E_1 described as above. There is exactly one partition of $E_1 = S \sqcup S^*$ with the property that for every acyclic orientation $A \in \Phi(\mathcal{O}(E_0))$ both $A \cup S$ and $A \cup S^*$ are acyclic orientations of $\mathcal{O}$.*

Proof. We build the set S inductively by consecutively choosing among pairs $\{x, x^*\} \subseteq E_1$. Assume

$$E_1 = \{x_1, x_2, \ldots, x_t\} \cup \{x_1^*, x_2^*, \ldots, x_t^*\}$$

and we begin by defining $\{x_1\} \in T_1$. We assume that for $1 \leq k-1 < t$ we have chosen a subset $T_k \subseteq E_1$ where exactly one of each pair $\{x_i, x_i^*\}$ is in T_{k-1} for $1 \leq i \leq k-1$, and, for every $A \in \Phi(\mathcal{O}(E_0))$, $A \cup T_{k-1}$ and $A \cup T_{k-1}^*$ are acyclic sets in $\mathcal{O}$.

From Lemma 6.7 we deduce that there is a $y \in E_0$ so that exactly one of the following sets is in $\mathscr{C}$:

$$\begin{aligned} C_1 &= \{x_1, x_k, y\}, \\ C_2 &= \{x_1, x_k, y^*\}, \\ C_3 &= \{x_1, x_k^*, y\}, \\ C_4 &= \{x_1, x_k^*, y^*\}. \end{aligned}$$

In the cases where C_1 or $C_2 \in \mathscr{C}$, let $T_k = T_{k-1} \cup \{x_k^*\}$ and in the cases where C_3 or $C_4 \in \mathscr{C}$ let $T_k = T_{k-1} \cup \{x_k\}$. It is easy to show that these choices are independent of y and that T_k has the necessary properties. Moreover, by the way in which it was constructed, it must be unique up to the choice of $x_1 \in S$. □

By a *canonical acyclic orientation* of a supersolvable oriented matroid $\mathcal{O}$ we mean an acyclic orientation B of $\mathcal{O}$ so that $\pi(B)$ is a canonical acyclic orientation of $\mathcal{O}(E_0)$ and that $B \cap E_1$ is one of the sets described in Theorem 6.8. As in Section 4, any acyclic orientation of a rank 2 oriented geometry is canonical.

The key to the results in Section 4 is the fact that the fiber $\pi^{-1}(\pi(R))$ is a chain of length $|\mathscr{H}_1|$ for every region R. The oriented matroid analogue of this fact is

Theorem 6.9. *Let B be a canonical acyclic orientation of a supersolvable oriented geometry $\mathcal{O}$ with partition E_0 and E_1, and let $S = B \cap E_1$. Each acyclic orientation $A \in \Phi(\mathcal{O}(E_0))$ induces a unique linear ordering on $S = \{s_1, s_2, \ldots, s_k\}$ so that the orientations*

$$A \cup \{s_t, s_{t+1}, \ldots, s_k\} \cup \{s_1^*, s_2^*, \ldots, s_{t-1}^*\}$$

are acyclic for all $1 \leq t \leq k$.

Proof. For $x_1, x_2 \in S$ we say that $x_1 \leq x_2$ if there exists a $y \in A$ so that $x_1 \in h(\{x_2, y\})$. It is clear that $\leq$ is a partial order on S. If we show that $\leq$ is a total order then the lemma follows immediately. The proof of this fact follows from a case analysis similar to that used in the proof of Theorem 6.8. □

Corollary 6.10. *Let $\mathcal{O}$ be a supersolvable oriented geometry with canonical acyclic orientation B. Then, for each acyclic orientation $A \in \Phi(\mathcal{O})$, $\pi^{-1}(\pi(A))$ is a chain in $P(\mathcal{O}, B)$ of length $|E_1|$.*

Armed with Theorem 6.9 and Corollary 6.10 the reader will be able to fill in the proofs for the following theorems, following the proofs of the related theorems from Section 4.

Theorem 6.11. *Let $\mathcal{O}$ be a supersolvable oriented geometry and B be a canonical acyclic orientation. Then the rank generating function for $P(\mathcal{O}, B)$ is*

$$\prod_{i=1}^{d} (1+q+q^2+\cdots+q^{e_i}),$$

where $\{e_i\}$ are the exponents of $L(\bar{\mathcal{O}})$ and $\operatorname{rk} \mathcal{O} = d$.

Theorem 6.12. *Let $\mathcal{O}$ be a supersolvable oriented geometry and B be a canonical acyclic orientation. Then $P(\mathcal{O}, B)$ is a lattice.*

Finally in this section we show how the structure of the set $\Phi(\mathcal{O})$ uniquely determines the oriented geometry $\mathcal{O}$. Define the graph $G(\Phi)$ to be the one with vertex set Φ and for each pair of acyclic orientations $A, B \in \Phi$, A is adjacent to B in $G(\Phi)$ if there exists an $x \in A$ so that $A - x \cup x^* = B$. If $\mathcal{O}$ is coordinatizable, then $G(\Phi)$ is just the adjacency graph previously defined. As before, the Hasse diagram of $P(\mathcal{O}, B)$ is $G(\Phi)$ oriented away from B.

Let Q_n be the graph of the n-dimensional cube, i.e., $V(Q_n) = \{0, 1\}^n$ and two vertices of Q_n are adjacent if they differ in exactly one component. An *isometric embedding* of a graph G into a graph H is a map α: $V(G) \to V(H)$ which preserves distance, i.e., $d_G(x, y) = d_H(\alpha(x), \alpha(y))$ for each pair $x, y \in G$ where $d(x, y)$ is the length of the shortest path connecting x and y.

Lemma 6.13. *For an oriented geometry $\mathcal{O}$, $G(\Phi(\mathcal{O}))$ can be isometrically embedded in the cube graph Q_n where $n = |E|/2$, and Q_n is the smallest cube for which this can be done.*

Proof. Fix an acyclic orientation $A \in \Phi$, $A = \{x_1, x_2, \ldots, x_n\}$, and consider the map $\chi_A\colon G(\Phi) \to Q_n$ defined by

$$\chi_A(B) = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$$

for each $B \in \Phi$ where

$$\varepsilon_i = \begin{cases} 0 & \text{if } x_i \in B, \\ 1 & \text{if } x_i^* \in B \end{cases}$$

for all $1 \le i \le n$. That χ_n is an isometric embedding follows from Theorem 6.1. Since $d_{G(\Phi)}(A, A^*) = n$, Q_n is the smallest cube isometrically containing $G(\Phi)$. □

Theorem 6.14. *Assume $\mathcal{O} = (E, \mathscr{C}, *)$ and $\mathcal{O}' = (E', \mathscr{C}', \dagger)$ are oriented geometries with acyclic orientations Φ and Φ', respectively. If $G(\Phi)$ is isomorphic to $G(\Phi')$, then $\mathcal{O}$ is isomorphic to $\mathcal{O}'$ and $\bar{\mathcal{O}}$ is isomorphic to $\overline{\mathcal{O}'}$.*

Proof. Assume $G = G(\Phi)$ is isomorphic to $G' = G(\Phi')$ and let $A \in \Phi$ and $A' \in \Phi'$. By Lemma 6.13 the maps χ_A and $\chi_{A'}$ are isometric embeddings of G and G', respectively, into the cube graph Q_n where $n = d_G(A, A^*) = d_{G'}(A', A'^{\dagger})$. Hence $|E| = |E'|$.

By a result of Winkler [Wi, Theorem 1], isometric embedding into Q_n is unique up to an automorphism of Q_n. Let φ be the automorphism of Q_n such that $\varphi(\chi_A(G)) = \chi_{A'}(G')$. Then φ induces a bijection $\bar{\varphi}\colon E \to E'$ which respects $*$ and $\dagger$ and so that $\bar{\varphi}(\Phi) = \Phi'$, i.e., $\bar{\varphi}(e^*) = [\bar{\varphi}(e)]^{\dagger}$ and for each $B \in \Phi$, $\bar{\varphi}(B) \in \Phi'$.

As shown by Mandel, related in Theorem 1.1 of [Co2], we can recover the set of circuits $\mathscr{C}$ from Φ uniquely by

$$\mathscr{C} = \{B \subseteq E \mid B \text{ is minimal such that } B \cap B^* = \varnothing \text{ and } B \not\subseteq A \in \Phi\}.$$

Thus, since the map $\bar{\varphi}$ is a bijection, $\bar{\varphi}\colon E \to E'$, which preserves $*$ and $\dagger$ and $\bar{\varphi}(\Phi) = \Phi'$ we have that $\bar{\varphi}(\mathscr{C}) = \mathscr{C}'$ and thus $\bar{\varphi}$ is an isomorphism between $\mathcal{O}$ and $\mathcal{O}'$. It then follows immediately that $\bar{\mathcal{O}}$ is isomorphic to $\overline{\mathcal{O}'}$ and we are done. □

Theorem 6.14 was conjectured in the case of hyperplane arrangements (coordinatizable oriented matroids) by Grünbaum [Gr1, p. 397]. Using the duality between hyperplane arrangements and zonotopes (see Section 3 of [Ed3]) we see that this is equivalent to the fact that the 1-skeleton of a zonotope completely determines its combinatorial structure. A recent result of similar flavor [BlM], [Ka] states that simple polytopes are determined by their 1-skeletons.

References

[Bi] G. Birkhoff, *Lattice Theory*, AMS Colloquium publications, Vol. 25, 3rd edn., American Mathematical Society, Providence, RI, 1967.

[Bj1] A. Björner, Orderings of Coxeter groups, in *Proc. AMS-NSF Conference on Combinatorics and Algebra*, Boulder, 1983 (ed. C. Greene), Contemporary Mathematics, Vol. 34, American Mathematical Society, Providence, RI, 1984, pp. 175-195.

[Bj2] A. Björner, Supersolvable arrangements of hyperplanes, *Abstracts Amer. Math. Soc.* **6** (1985), 182.

[BlM] R. Blind and P. Mani, On puzzles and polytope isomorphisms, *Aequationes Math.* (to appear).

[Bo] N. Bourbaki, *Groupes et algèbres de Lie*, Hermann, Paris, 1968, Chapters 4-6.

[BM] H. Bruggesser and P. Mani, Shellable decompositions of cells and spheres, *Math. Scand.* **29** (1971), 197-205.

[Co1] R. Cordovil, Sur les matroides orientés de rang trois et les arrangements de pseudodroites dans le plan projectif réel, *European J. Combin.* **3** (1982), 307-318.

[Co2] R. Cordovil, A combinatorial perspective on the non-Radon partitions, *J. Combin. Theory, Ser. A* **38** (1985), 38-47.

[DK] G. Danaraj and V. Klee, Shellings of spheres and polytopes, *Duke Math. J.* **41** (1974), 443-451.

[Ed1] P. H. Edelman, The lattice of convex sets of an oriented matroid, *J. Combin. Theory Ser. B* **33** (1982), 239-244.

[Ed2] P. H. Edelman, The acyclic sets of an oriented matroid, *J. Combin. Theory Ser. B* **36** (1984), 26-31.

[Ed3] P. H. Edelman, A partial order on the regions of $\mathbb{R}^n$ dissected by hyperplanes, *Trans. Amer. Math. Soc.* **283** (1984), 617-631.

[Ed4] P. H. Edelman, Abstract convexity and meet-distributive lattices, *Proc. AMS-IMS-SIAM Conference on Combinatorics and Ordered Sets*, Arcata, 1985 (ed. I. Rival), Contemporary Mathematics, Vol. 57, American Mathematical Society, Providence, RI, 1986, pp. 127-150.

[EJ] P. H. Edelman and R. Jamison, The theory of convex geometries, *Geom. Dedicata* **19** (1985), 247-270.

[FL] J. Folkman and J. Lawrence, Oriented matroids, *J. Combin. Theory Ser. B* **25** (1978), 199-236.

[Gre] C. Greene, Acyclic Orientations (Notes), in *Higher Combinatorics* (ed. M. Aigner), Reidel, Dordrecht, 1977, pp. 65-68.

[GZ] C. Greene and T. Zaslavsky, On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs, *Trans. Amer. Math. Soc.* **280** (1983), 97-126.

[Gr1] B. Grünbaum, Convex polytopes, Interscience, London, 1967.

[Gr2] B. Grünbaum, Arrangements of hyperplanes, in *Proc. Second Louisiana Conference of Combinatorics and Graph Theory*, Baton Rouge (ed. Mullin, *et al.*), 1971.

[Gr3] B. Grünbaum, *Arrangements and Spreads*, AMS Regional Conference Series in Mathematics, Vol. 10, American Mathematical Society, Providence, RI, 1972.

[GS] B. Grünbaum and G. C. Shephard, Simplicial arrangements in projective 3-space, *Mitt. Math. Sem. Giessen* (Coxeter Festschrift, Teil IV) **16** (1984), 49-101.

[JT] M. Jambu and H. Terao, Free arrangements of hyperplanes and supersolvable lattices, *Adv. in Math.* **52** (1984), 248-258.

[Ka] G. Kalai, A simple way to tell a simple polytope from its graph, *J. Combin. Theory Ser. A* (submitted).

[La1] M. Las Vergnas, Sur les activités des orientations d'une géometrie combinatoire (Proc. Colloq. Mathématique Discrètes: Codes et Hypergraphes, Bruxelles, 1978), *Cahiers Centre Études Rech. Opér.* **20** (1978), 283-300.

[La2] M. Las Vergnas, Convexity in oriented matroids, *J. Combin. Theory Ser. B.* **29** (1980), 231-243.

[Lee] C. Lee, Personal communication.

[Sh] R. Shannon, Simplicial cells in arrangements of hyperplanes, *Geom. Dedicata* **8** (1979), 179–187.

[So] L. Solomon, The orders of the finite Chevalley groups, *J. Algebra* **3** (1966), 376–393.

[St1] R. Stanley, Modular elements in geometric lattices, *Algebra Universalis* **1** (1971), 214–217.

[St2] R. Stanley, Supersolvable lattices, *Algebra Universalis* **2** (1972), 197–217.

[St3] R. Stanley, T-free arrangements of hyperplanes, in *Progress in Graph Theory* (eds. J. A. Bondy and U. S. R. Murty), Academic Press, New York, 1984, p. 539.

[Te1] H. Terao, Arrangements of hyperplanes and their freeness, I, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 293–312.

[Te2] H. Terao, Generalized exponents of a free arrangement of hyperplanes and Shephard–Todd–Brieskorn formula, *Invent. Math.* **63** (1981), 159–179.

[Te3] H. Terao, Modular elements of lattices and topological fibration, *Adv. in Math.* **62** (1986), 135–154.

[Te4] H. Terao, Personal communication.

[Wa] P. Wagreich, *The Growth Function of a Discrete Group*, Lecture Notes in Mathematics, Vol. 959, Springer-Verlag, Berlin, 1982, pp. 125–144.

[Wi] P. Winkler, Isometric embedding in products of complete graphs, *Discrete Appl. Math.* **7** (1984), 221–225.

[Za] T. Zaslavsky, *Facing up to Arrangements: Face-Count Formulas for Partitions of Space by Hyperplanes*, Memoirs of the American Mathematical Society, Vol. 154, American Mathematical Society, Providence, RI, 1975.

Received October 5, 1987, and in revised form February 17, 1988.