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A Characterization of Semimartingales on Nuclear Spaces

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Summary. This work is devoted to prove the following fact: Suppose that Φ is a nuclear space whose dual Φ' is nuclear under the strong topology. If X' is a weakly adapted mapping with values in Φ' such that for any $\phi \in \Phi$, $X'(\phi)$ has a modification which is a semimartingale then there exists a unique projective system of Hilbert space-valued semimartingales indexed by the Hilbert-Schmidt neighbourhood base of the dual space whose 'projective limit' is X'.

In the last part we study in detail a semimartingale defined as the convolution of a distribution by a random Dirac measure whose support is determined by the trajectories of a real-valued semimartingale.

Introduction

Suppose that Φ is a nuclear space whose dual Φ' is also nuclear under the strong topology. In [12] we have defined the semimartingales in Φ' as the projective systems of Hilbert space-valued semimartingales indexed by the Hilbert-Schmidt neighbourhood base of Φ', constructed the stochastic integrals of the previsible processes with respect to these semimartingales, studied some subclasses of them and given some applications to the stochastic flows and partial differential equations. Any semimartingale in Φ' induces a linear mapping from Φ into some vector subspace of the real-valued semimartingales. If this subspace can be equipped with a topology under which it is a Banach space then any linear mapping on Φ with values in this Banach space induces also a semimartingale in Φ' and this statement can be proved using the classical tools of functional analysis (cf. [12]). However, the space in which the linear mapping takes its values may be a non-locally convex vector space, namely S^0 (cf. the notations) and in this case the methods that we have mentioned above do not work. In this work we prove that any such mapping induces also a semimartingale in Φ' and the second section is completely devoted to the proof of this fact. In the first section we give the basic

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definitions and some results obtained in [12]. The third section deals in detail with a special example of a semimartingale in the space of the distributions on $\mathbb{R}$ defined as the convolution of a distribution with the Dirac measure whose support is a real-valued semimartingale. Using Ito's formula we calculate the explicit forms of the elements of the corresponding projective systems. In particular, the representation theorems which are the generalizations of Ito's formula in some certain sense, suggest a rigorous definition for the so-called stochastic evolution equations or partial stochastic differential equations. Further applications to non-linear flows and Physics will be given in the forthcoming papers.

I. Notations and Preliminaries

Φ denotes a locally convex, nuclear space whose topological dual Φ' is nuclear under its strong topology $\beta(\Phi', \Phi)$ denoted by Φ'_β. If U is an absolutely convex (i.e. convex and balanced) neighbourhood (of zero) in Φ, we denote by $\Phi(U)$ the quotient set $\Phi/p_U^{-1}(0)$ completed with respect to the norm p_U where p_U denotes the gauge function of U and $k(U)$ denotes the canonical mapping from Φ onto $\Phi(U)$. Let us recall that Φ is called nuclear if there exists a neighbourhood base (of zero) $\mathscr{U}$ such that for any $U\in\mathscr{U}$, there exists $V\subset U$, $V\in\mathscr{U}$, for which the canonical mapping $k(U, V)$: $\Phi(V)\to\Phi(U)$, defined by $k(U) = k(U, V)\circ k(V)$, is nuclear (cf. [6], [8]). If B is a bounded, absolutely convex subset of Φ, we note by $\Phi[B]$ the completion of the subspace of Φ, spanned by B, with respect to the norm p_B (i.e. gauge function of B). It is well known that, in each nuclear space, there exists a base of neighbourhoods $\mathscr{U}_h(\Phi)$ such that $\Phi(U)$ is a separable Hilbert space whose dual can be identified by $\Phi'[U^0]$, U^0 being the polar of U, and Φ is (a subspace of) the projective limit of

$$\{(\Phi(U), k(U)) \colon U\in\mathscr{U}_h(\Phi)\}$$

(cf. [8], p. 102). We shall denote by $\mathscr{K}_h(\Phi')$ the set

$$\{U^0 \colon U\in\mathscr{U}_h(\Phi)\}$$

and $\mathscr{K}_h(\Phi)$ is defined by interchanging Φ and Φ'_β.

$(\Omega, \mathscr{F}, P)$ denotes a complete probability space and $(\mathscr{F}_t; t\geqq 0\}$ will represent a right continuous, increasing family of the sub-σ-algebras of $\mathscr{F}$. We suppose as usual that $\mathscr{F}_0$ contains all the P-negligeable subsets of Ω. S^0 denotes the space of the equivalent classes (with respect to evanescent processes) of real-valued semimartingales. If $x\in S^0$, we define $|x|_k$, $k\in\mathbb{N}$, as

$$|x|_k = \sup\left\{E\left[\left|\int_0^k h_s\, dx_s\right| \wedge 1\right] \colon h\in\varepsilon_1\right\} \tag{I.1}$$

where $E(.)$ denotes the mathematical expectation with respect to the probability P, ε_1 is the set of the previsible processes h such that

$$h=\sum_{i=1}^{n-1} a_i 1_{]\!]t_i,t_{i+1}]\!]}, \qquad 0\leqq t_1<\ldots<t_n<+\infty,$$

a_i being an $\mathscr{F}_{t_i}$-measurable random variable bounded by one. $]\!]t_i, t_{i+1}]\!]$ is the stochastic interval $]t_i, t_{i+1}]\times\Omega$, and the integral at the right hand side of (I.1) is the stochastic integral. Define $|x|_0$ as

$$|x|_0=\sum_{k=1}^{\infty} 2^{-k}|x|_k$$

then $d(x, y)=|x-y|_0$ is a metric on S^0 and (S^0, d) is a Fréchet space (non-locally convex) (cf. [3] and [4]). Moreover, the set of local martingale and the set of previsible processes of finite variation, having uniformly bounded jumps (both) are the closed subsets of (S^0, d) (cf. [4], Remark IV.3 and Theorem IV.4). If x is a special semimartingale with its canonical decomposition

$$x=m+a$$

i.e. a is a previsible process of finite variation with $a_0=0$ and m is a local martingale, we denote by $\|x\|_p$ the following quantity

$$\|x\|_p=\left\|[m, m]_\infty^{1/2}+\int_0^\infty |da_s|\right\|_{L^p}, \qquad p\geqq 1,$$

where L^p denotes $L^p(\Omega, \mathscr{F}, P)$. The set of real-valued semimartingales for which the above quantity is finite is denoted by S^p and this is a Banach space under $\|\cdot\|_p$ (cf. [3]).

Now let us give the definitions of 'the stochastic process' with values in Φ':

Definition I.1. Let X be the following set:

$$X=\{X^U\colon U\in\mathscr{U}_h(\Phi'_\beta)\},$$

where each X^U is a stochastic process with values in the separable Hilbert space $\Phi'(U)$. X will be called a projective system on Φ' (of stochastic processes) if for any $U\in\mathscr{U}_h(\Phi'_\beta)$ and $V\subset U$, $V\in\mathscr{U}_h(\Phi'_\beta)$, $k(U, V)\circ X^V$ and X^U are undistinguishable.

Definition I.2. Let X be a projective system of stochastic processes on Φ', denoted by

$$X=\{X^U\colon U\in\mathscr{U}_h(\Phi'_\beta)\}.$$

We say that X has a limit in Φ' if, for any $t\geqq 0$, there exists a measurable mapping X'_t from $(\Omega, \mathscr{F})$ into $(\Phi', \mathscr{C}(\Phi'))$ where $\mathscr{C}(\Phi')$ denotes the cylindrical σ-algebra of Φ' such that

$$k(U)\circ X'_t=X_t^U \qquad \text{a.e.}$$

for any $U\in\mathscr{U}_h(\Phi'_\beta)$.

In the following, each time that the index set of a projective system is not specified, it will be understood that it is either $\mathscr{U}_h(\Phi)$ or $\mathscr{U}_h(\Phi'_\beta)$.

Definition I.3. Let X be a projective system of stochastic processes on Φ' with a limit X' in Φ'. The pair (X, X) is called a g-process on Φ'. We say that the g-process (X, X') or the projective system X possesses the property π if any element X^U of X possesses the property π in $\Phi'(U)$.

Remark. In the sequel, if there is no confusion the pair (X, X') will be denoted by a single letter.

Definition I.4. Suppose that X and Y are two projective systems of stochastic processes on Φ'. We say that X and Y are undistinguishable if X^U and Y^U are undistinguishable stochastic processes, for any $U \in \mathscr{U}_h(\Phi'_\beta)$.

Definition I.5. Let X' be a mapping on $\mathbb{R}_+ \times \Omega$ with values in Φ' such that, for any $t \geqq 0$ and $\phi \in \Phi$, $\omega \mapsto \langle \phi, X'_t(\omega) \rangle$ is $\mathscr{F}$-measurable. We say that X' generates a projective system of stochastic processes (on Φ') if there exists a projective system whose limit is X'.

If χ is a separable Hilbert space and Z is a measurable mapping on $\mathbb{R}_+ \times \Omega$ with values in χ, Z will be called a semimartingale if it is right continuous with left limits, adapted and it has a decomposition

$$Z = M + A, \qquad A_0 = 0,$$

where M is a local martingale and A is a process of finite variation (cf. [5]). It is well known that Z has a uniquely defined continuous local martingale part. We denote by $[\![Z, Z]\!]_t$ the following process which is of finite variation:

$$[\![Z, Z]\!]_t = \langle\!\langle Z^c, Z^c \rangle\!\rangle_t + \sum_{s \leqq t} \|\Delta Z_s\|_\chi^2$$

where $\langle\!\langle Z^c, Z^c \rangle\!\rangle$ is the unique previsible process such that $\|Z_t^c\|_\chi^2 - \langle\!\langle Z^c, Z^c \rangle\!\rangle_t$ is a local martingale.

Definition I.6. A g-process X on Φ' is called a semimartingale on Φ' if X^U is a semimartingale in $\Phi'(U)$ for any $U \in \mathscr{U}_h(\Phi'_\beta)$. The set of semimartingale on Φ' will be denoted by $S^0(\Phi')$.

If Z is a semimartingale with values in a separable Hilbert space χ, we denote by $\|Z\|_p$, $p \geqq 1$, the following quantity:

$$\|Z\|_p = \left\| [\![M, M]\!]_\infty^{1/2} + \int_0^\infty \|dA_s\| \right\|_{L^p(\Omega, \mathscr{F}, P)}.$$

The set (of equivalence classes) of semimartingale for which this quantity is finite is denoted by $S^p(\chi)$ and $S^0(\chi)$ denotes simply all the semimartingales with values in χ. In fact, for $\|\cdot\|_p$ to be a norm we should choose the canonical decomposition of Z i.e.

$$Z = M + A$$

where A is also previsible with $A_0=0$. Hence, in the sequel, when we write $\|Z\|_p$ we will understand that $\|Z\|_p$ is calculated for the canonical decomposition of Z.

Definition I.7. Let X be a g-process on Φ'. X is called an S^0-semimartingale or we say $X\in S^p(\Phi')$ if, for any $U\in\mathscr{U}_h(\Phi')$, X^U is in $S^p(\Phi'(U))$.

In [12] we have shown the following result:

Theorem 1. *Suppose that X' is a weakly adapted mapping $\mathbb{R}_+\times\Omega$ with values in Φ' such that, for any $\phi\in\Phi$, the mapping on $\mathbb{R}_+\times\Omega$*

$$(t,\omega)\to\langle\phi, X'_t(\omega)\rangle$$

has a modification which is an S^1-semimartingale. Then there exists a unique projective system (up to a projective system of evanescent process) X such that (X, X') is an element of $S^1(\Phi')$.

If Φ is bornological, then there exists a mapping Y: $\mathbb{R}_+\times\Phi\to\Phi'$ with right continuous trajectories having left limits in Φ', such that $k(U)\circ Y$ and X^U are undistinguishable for any $U\in\mathscr{U}_h(\Phi'_\beta)$.

For the sake of completeness we will sketch the proof: X' induces a linear mapping on Φ with values in S^1, which is sequentially continuous. Hence $k(U)\circ X'$ induces a linear continuous mapping on $\Phi[U^0]$ with values in S^1. Decomposing $k(U)$ as $k(U,V)\circ k(V)$, for some $V\in\mathscr{U}_h(\Phi'_\beta)$, $V\subset U$, we see that $k(U)\circ X'$ is a nuclear mapping on $\Phi[U^0]$ with values in S^1. Take any representation of it (cf. [8])

$$k(U)\circ X'=\sum_{i=1}^{\infty}\lambda_i F_i(U)\otimes x^i$$

where $(\lambda_i)\in l^1$ (i.e. the space of the summable sequences), $\{F_i(U)\}$ is equicontinuous in $\Phi'(U)$ and $(x^i)\subset S^1$ is bounded. Define

$$X_t^U(\omega)=\sum_{i=1}^{\infty}\lambda_i F_i(U)\,x_t^i(\omega)$$

then X^U has right continuous trajectories with left limits in $\Phi'(U)$, it is in $S^1(\Phi'(U))$ and $k(U,V)(X_t^V)=X_t^U$ a.e. for any $V\in\mathscr{U}_h(\Phi'_\beta)$, $V\subset U$. Since both of them are right continuous they are undistinguishable. Of course the limit of $\{X^U\colon U\in\mathscr{U}_h(\Phi'_\beta)\}$ in Φ' is X'. If Φ is bornological, then the mapping induced by X' itself is continuous, hence we can do these things globally and show that there exists some $K\in\mathscr{K}_h(\Phi')$ and a semimartingale $\tilde{Y}$ with values in $\Phi'[K]$. Then it is sufficient to inject $\tilde{Y}$ into Φ'. ||QED

In the following section we shall show that the first part of this theorem is true for S^0.

II. The Characterization of the Semimartingales

In the following we shall denote by ε the space of the simple previsible, uniformly bounded, real-valued stochastic processes whose unit ball is denoted by ε_1. $\hat{\varepsilon}$ denotes the space of the uniformly bounded previsible processes completed under the supremum topology, whose unit ball will be noted by $\hat{\varepsilon}_1$. If x is a real valued semimartingale, then it is well known that the mapping $h \to (h.x)_t$ is continuous on $\hat{\varepsilon}$ with values in L^0 where $h.x$ denotes the stochastic integral of h with respect to x and L^0 is the space of the equivalence classes of random variables under the convergence in probability (cf. [3], Theorem 3, p. 329).

By the help of these results we shall prove the following theorem:

Theorem II.1. *Suppose that X' is a weakly adapted mapping on $\mathbb{R}_+ \times \Omega$ with values in Φ' such that, for any $\phi \in \Phi$, the mapping*

$$(t, \omega) \to \langle \phi, X'_t(\omega) \rangle$$

has a modification which is a semimartingale. Then there exists a projective system of semimartingales on Φ' having X' as its limit.

The proof of the theorem will be made in several steps. We prove first the following fact:

Lemma II.1. *X' defines a linear mapping $\tilde{X}$ on Φ with values in S^0. If $U \in \mathscr{U}_h(\Phi'_\beta)$, then denote by $J(h, \phi)$ the mapping on $\hat{\varepsilon} \times \Phi[U^0]$ with values in S^0 defined by*

$$J(h, \phi) = h . \tilde{X}(\phi).$$

Then J is continuous on $\hat{\varepsilon} \times \Phi[U^0]$ with values in (S^0, d).

Proof. If $\phi \in \Phi$, denote by $\tilde{X}(\phi)$ a modification of $(t, \omega) \to \langle \phi, X'_t(\omega) \rangle$ which is a semimartingale (more exactly an element of (S^0, d)). If $\tilde{Y}(\phi)$ is another such modification, $\tilde{X}(\phi)$ and $\tilde{Y}(\phi)$ are undistinguishable, hence $\phi \to \tilde{X}(\phi)$ is well defined. If $U \in \mathscr{U}_h(\Phi'_\beta)$, $\tilde{X} \circ i_{U^0}$ is a linear mapping on $\Phi[U^0]$ with values in S^0 where i_{U^0} denotes the injection $\Phi[U^0] \hookrightarrow \Phi$. Suppose that (ϕ_n) converges to ϕ in $\Phi[U^0]$ and $(\tilde{X}(\phi_n))$ converges to y in (S^0, d). Since i_{U^0} is continuous, $\langle \phi_n, X'_t(\omega) \rangle$ converges pointwise to $\langle \phi, X'_t(\omega) \rangle$ for any $(t, \omega) \in \mathbb{R}_+ \times \Omega$. Hence

$$y_t(\omega) = \langle \phi, X'_t(\omega) \rangle = \tilde{X}(\phi)_t(\omega) \qquad \text{a.e.}$$

for any $t \geqq 0$ and this implies that $y = \tilde{X}(\phi)$. Consequently the mapping $\phi \to \tilde{X}(\phi)$ is continuous on $\Phi[U^0]$. Moreover, by what we have said above $h \to h.\tilde{X}(\phi)$ is continuous on $\hat{\varepsilon}$ with values in (S^0, d) (as one can show using the closed graph theorem) and a theorem of functional analysis (cf. [8], p. 88, Theorem 5.1) says that the bilinear mapping $(h, \phi) \to h.\tilde{X}(\phi)$ is continuous on $\hat{\varepsilon} \times \Phi[U^0]$ with values in (S^0, d). ||QED

Now we prove the following

Lemma II.2. *Let (ϕ_i) be a countable dense subset of U^0 in $\Phi[U^0]$ and $k \in \mathbb{N}$. Then we have*

$$P\{\omega : \sup_{t \leqq k} \sup_i |\tilde{X}(\phi_i)_t(\omega)| < +\infty\} = 1.$$

Proof. To prove this result we take $[0, k]$ instead of $\mathbb{R}_+$ and modify $\hat{\varepsilon}$, $\hat{\varepsilon}_1$ by $h_t = 0$ for $t > k$ (cf. the remark at p. 401 of [3]). Moreover it is sufficient to take the supremum with respect to t on a countable dense subset D_k of $[0, k]$. Let us suppose now that the contrary holds. Then there exists $a \in]0, 1]$ such that

$$P\{\sup_{t\in D_k} \sup_i |\tilde{X}(\phi_i)_t| = +\infty\} > a > 0.$$

Choose $V \in \mathscr{U}_h(\Phi'_\beta)$ with $V \subset U$ such that the injection $i(V^0, U^0)$: $\Phi[U^0] \hookrightarrow \Phi[V^0]$ is nuclear. Since any nuclear mapping is universally 0-decomposable (cf. [1], p. 203) and

$$\tilde{X} \circ i_{U^0} = (\tilde{X} \circ i_{V^0}) \circ i(V^0, U^0)$$

the mapping $\phi \to \tilde{X} \circ i_{U^0}(\phi)$ is 0-decomposable at any $t \geqq 0$, i.e. there exists a random variable $\tilde{X}^U_t$ with values in $\Phi'(U)$ such that

$$\tilde{X}(\phi)_t = (\phi \mid \tilde{X}^U_t) \quad \text{a.e.} \tag{II.1}$$

for any $\phi \in \Phi[U^0]$. Similarly, for any $h \in \hat{\varepsilon}$, by Lemma II.1, the mapping $\phi \to (h.\tilde{X}(\phi))_t$ is 0-decomposable, hence there exists a $\Phi'(U)$-valued random variable $(h.\tilde{X})^U_t$ such that

$$(h.\tilde{X}(\phi))_t = (\phi \mid (h.\tilde{X})^U_t) \quad \text{a.e.} \tag{II.2}$$

for any $\phi \in \Phi[U^0]$. Denote by $L^0(\Phi'(U))$ the space of equivalence classes of $\Phi'(U)$-valued random variables under the topology of convergence in probability. Then (II.2) defines a linear mapping $h \to (h.\tilde{X})^U_k$ on $\hat{\varepsilon}$ with values in $L^0(\Phi'(U))$. Suppose that (h^n) converges to h in $\hat{\varepsilon}$ and $(h^n.\tilde{X})^U_k$ converges to Y in $L^0(\Phi'(U))$. By Lemma II.1, $(h^n.\tilde{X}(\phi))$ converges to $h.\tilde{X}(\phi)$ in (S^0, d), for any $\phi \in \Phi[U^0]$, hence

$$(\phi \mid Y) = (h.\tilde{X}(\phi))_k = (\phi \mid (h.\tilde{X})^U_k) \quad \text{a.e.}$$

Consequently the mapping $h \to (h.\tilde{X})^U_k$ is continuous on $\hat{\varepsilon}$ with values in $L^0(\Phi'(U))$ and the image of $\hat{\varepsilon}_1$ under this mapping is bounded in $L^0(\Phi'(U))$. One can show, as in the one-dimensional case (cf. [3], p. 402), that the boundedness of this set is equivalent to

$$\sup_{h\in\hat{\varepsilon}_1} P\{\|(h.\tilde{X})^U_k\| > c\} \xrightarrow[c\to\infty]{} 0.$$

We have

$$\|\tilde{X}^U_t\| = \sup_i |(\phi_i \mid \tilde{X}^U_t)| = \sup_i |\tilde{X}(\phi_i)_t| \quad \text{a.e.}$$

hence $(t, \omega) \to \|\tilde{X}^U_t(\omega)\|$ has an optional right lower semi-continuous modification which will be denoted by $f_t(\omega)$. Define T_n by

$$T_n = \inf\{t \in D_k : f_t > n\} \wedge k,$$

then T_n is a stopping time and the lower semi-continuity of f implies that

$$P\{f_{T_n} \geqq n\} > a.$$

Since the mapping $\phi \to \tilde{X}(\phi)_{T_n}$ is 0-decomposable, there exists $Z_n \in L^0(\Phi'(U))$ such that

$$(\phi \mid Z_n) = \tilde{X}(\phi)_{T_n} \quad \text{a.e.}$$

for any $\phi \in \Phi[U^0]$. Since $1_{]\!]0, T_n]\!]}$ belongs to $\hat{\varepsilon}_1$, Z_n belongs to the image of $\hat{\varepsilon}_1$ under the mapping $h \to (h.\tilde{X})^U_k$. Moreover

$$\|Z_n\|_{\Phi'(U)} = \sup_i |(\phi_i \mid Z_n)| = \sup_i |\tilde{X}(\phi_i)_{T_n}| = f_{T_n} \quad \text{a.e.}$$

consequently, for any $n \in \mathbb{N}$, there exists Z_n in the image of $\hat{\varepsilon}_1$ with

$$P\{\|Z_n\| \geqq n\} > a$$

but this contradicts the boundedness of the set $\{(h.\tilde{X})^U_t : h \in \hat{\varepsilon}_1\}$ in $L^0(\Phi'(U))$. ‖QED

Lemma II.3. *Under the hypotheses of Theorem* II.1, *there exists a right continuous projective system with left limits whose projective limit is* X'.

Proof. Let $U \in \mathscr{U}_h(\Phi'_\beta)$, by the dual characterization of the nuclear spaces (cf. [8]), we may suppose that the injection i_{U^0}: $\Phi[U^0] \hookrightarrow \Phi$ is nuclear. Choose any representation of it as

$$i_{U^0} = \sum_{i=1}^{\infty} \lambda_i F_i(U) \otimes \phi_i$$

where $(\lambda_i) \in l^1$, $(F_i(U)) \subset \Phi'(U)$ is equicontinuous and $(\phi_i) \subset \Phi$ is bounded. Define $X^U_t(\omega)$ by

$$X^U_t(\omega) = \sum_{i=1}^{\infty} \lambda_i \tilde{X}(\phi_i)_t(\omega) F_i(U).$$

By Lemma II.2,

$$\sum_{i=1}^{\infty} |\lambda_i| \sup_{t \leqq k} |\tilde{X}(\phi_i)_t(\omega)| < +\infty \quad \text{a.e.}$$

for any $k \in N$. Therefore $t \to X^U_t(\omega)$ is right continuous with left limits in $\Phi'(U)$, for almost all $\omega \in \Omega$, in the norm topology of $\Phi'(U)$. If $\phi \in \Phi[U^0]$, we have

$$\tilde{X}(\phi)_t = (\phi \mid X^U_t) \quad \text{a.e.}$$

by the continuity of $\phi \to \tilde{X}(\phi)$ from $\Phi[U^0]$ into (S^0, d). If $V \in \mathscr{U}_h(\Phi'_\beta)$ with $V \subset U$ and if $\phi \in \Phi[U^0]$, then

$$(\phi \mid k(U, V) \circ X^V_t) = (i(V^0, U^0)(\phi) \mid X^V_t) = \tilde{X}(\phi)_t = (\phi \mid X^U_t) \quad \text{a.e.}$$

since $\Phi[U^0]$ is separable

$$k(U, V) \circ X^V_t = X^U_t \quad \text{a.e.}$$

and both sides are right continuous, hence the stochastic processes $k(U, V) \circ X^V$ and X^U are undistinguishable. ||QED

Remark. The uniqueness of the projective system (up to a projective system of evanescent processes) is obvious because of its right continuity.

To complete the proof of Theorem II.1, we should show that the elements of $\{X^U: U \in \mathscr{U}_h(\Phi'_\beta)\}$ are the semimartingales. For the notational convenience, in this part, we shall denote X^U by $X(U)$.

Let 0 be any element of $\mathscr{U}_h(\Phi'_\beta)$, choose V, W and U in $\mathscr{U}_h(\Phi'_\beta)$ such that $W \subset V \subset U \subset 0$ and that $k(V, W)$, $k(U, V)$ and $k(0, U)$ are nuclear (cf. the notations). Define

$$T_n = \inf\{t \geqq 0: \|X_t(W)\| > n\}, \quad n \in \mathbb{N}.$$

T_n is a stopping time and it increases to infinity with $n \in \mathbb{N}$. Denote by $X^n(W)$ the following process:

$$X_t^n(W) = X_t(W)\, 1_{\{t < T_n\}} + X_{T_n -}(W)\, 1_{\{t \geqq T_n\}}$$

where $X_{T_n -}(W)$ denotes the left hand side limit of $t \to X_t(W)$ at $t = T_n$. $X^n(W)$ is a uniformly bounded right continuous stochastic process having left limits. Moreover, for any $\phi \in \Phi[W^0]$, $(\phi \mid X^n(W))$ is a semimartingale. Let

$$Q_t^n(W) = \sum_{0 < s \leqq t} \Delta X_s^n(W)\, 1_{\{\|\Delta X_s^n(W)\| > n\}},$$

$Q^n(W)$ is an adapted process of finite variation. Let

$$Y^n(W) = X^n(W) - Q^n(W)$$

then $Y^n(W)$ has uniformly bounded jumps and $(\phi \mid Y^n(W))$ is a semimartingale for any $\phi \in \Phi[W^0]$. If we can show that the image of $Y^n(W)$ under $k(0, W)$ is undistinguishable from a $\Phi'(0)$-valued semimartingale, the proof of Theorem II.1 will be completed. Since the jumps of $Y^n(W)$ are uniformly bounded, for any $\phi \in \Phi[W^0]$, $(\phi \mid Y^n(W)) = Y^n(W)(\phi)$ is a special semimartingale. Hence it has a unique decomposition

$$M^n(W)(\phi) + A^n(W)(\phi)$$

where $M^n(W)(\phi)$ is a local martingale and $A^n(W)(\phi)$ is a previsible process, of finite variation. For notational simplicity, let us write (omitting n and W)

$$Y(\phi) = M(\phi) + A(\phi).$$

Lemma II.4. *$\phi \to M(\phi)$ and $\phi \to A(\phi)$ are linear, continuous mappings on $\Phi[W^0]$ with values in (S^0, d).*

Proof. The fact that $\phi \to M(\phi)$ and $\phi \to A(\phi)$ are well defined and linear follows from the uniqueness of the canonical decomposition of a special semimartingale (cf. [3]). Let $\phi \in \Phi[W^0]$, then there exists a sequence of stopping times (S_k) increasing to infinity such that $(M(\phi)_{t \wedge S_k};\ t \geqq 0)$ is a uniformly integrable martingale and $(A(\phi)_{t \wedge S_k}; t \geqq 0)$ is of integrable variation. We have then

$$E[\Delta M(\phi)_{t\wedge S_k}+\Delta A(\phi)_{t\wedge S_k}|\mathscr{F}_{t-}]=\Delta A(\phi)_{t\wedge S_k}$$

since $A(\phi)$ is previsible. Hence

$$|\Delta A(\phi)_{t\wedge S_k}|\leqq E[|\Delta Y(\phi)_{t\wedge S_k}||\mathscr{F}_{t-}]\leqq n\|\phi\|_{\Phi[W^0]},$$

since n is independent of S_k, this estimation implies that

$$|\Delta A(\phi)_t|\leqq n\|\phi\|_{\Phi[W^0]},\quad |\Delta M(\phi)_t|\leqq 2n\|\phi\|_{\Phi[W^0]}\quad \text{a.e.}$$

for any $t\geqq 0$ and $\phi\in\Phi[W^0]$. Suppose that (ϕ_k) converges to ϕ in $\Phi[W^0]$ and $(M(\phi_k))$ to m in (S^0, d). Then $A(\phi_k)$ converges also to some $a\in S^0$ in (S^0, d). However

$$|\Delta M(\phi_k)_t|=|M(\phi_k)_t-M(\phi_k)_{t}|\leqq 2n\sup_k\|\phi_k\|_{\Phi[W^0]}<+\infty$$

and

$$|\Delta A(\phi_k)_t|\leqq n.\sup_k\|\phi_k\|_{\Phi[W^0]}<+\infty.$$

Since the set of local martingales and the set of previsible processes of finite variation having uniformly bounded jumps are closed in (S^0, d) (cf. [3], [4]), m is a local martingale and a is a previsible process of finite variation. Since Y is a stochastic process with values in $\Phi'(W)$; $(Y(\phi_k))$ converges to $Y(\phi)$ in (S^0, d) as one can see applying the closed graph-theorem. Hence

$$Y(\phi)=m+a,\qquad a_0=0,$$

and the uniqueness of the canonical decomposition of $Y(\phi)$ implies that $m=M(\phi)$ and $a=A(\phi)$ i.e. the mappings $\phi\to A(\phi)$ and $\phi\to M(\phi)$ are continuous. ||QED

We have:

Lemma II.5. *Let $\{\phi_j\}$ be a countable, dense subset of V^0 in $\Phi[V^0]$. Then for any $k\in\mathbb{N}$, we have*

$$P\{\omega:\sup_{t\leqq k}\sup_j|M(i(W^0, V^0)(\phi_j))_t|<+\infty\}=1$$

and

$$P\{\omega:\sup_{t\leqq k}\sup_j|A(i(W^0, V^0)(\phi_j))_t|<+\infty\}=1.$$

Proof. By the closed graph theorem the mappings $(\phi,h)\to h.A(\phi)$ and $(\phi,h)\to h.M(\phi)$ are continuous on $\Phi[W^0]\times\hat{\varepsilon}$ with values in (S^0,d). Then the proof follows from Lemma II.2. ||QED

Since $i(V^0, U^0)$ is nuclear it can be represented as

$$\sum_{j=1}^{\infty} \lambda_j F_j(U) \otimes \phi_j$$

where $(F_j(U)) \subset \Phi'(U)$ is equicontinuous $(\lambda_j) \in l^1$ and $(\phi_j) \subset \Phi[V^0]$ is bounded. Define

$$\tilde{A}_t(\omega) = \sum_{j=1}^{\infty} \lambda_j A(i(W^0, V^0)(\phi_j))_t(\omega) F_j(U)$$

$$\tilde{M}_t(\omega) = \sum_{j=1}^{\infty} \lambda_j M(i(W^0, V^0)(\phi_j))_t(\omega) F_j(U).$$

By Lemma II.5, the sums are uniformly almost surely convergent hence $\tilde{A}$ and $\tilde{M}$ are right continuous with left limits in the norm topology of $\Phi'(U)$. Moreover, for any $\phi \in \Phi[U^0]$, we have

$$\begin{aligned}(\phi|\tilde{A}_t) + (\phi|\tilde{M}_t) &= (\phi|k(U, V) \circ k(V, W)(Y_t)) \\ &= (\phi|k(U, W)(Y_t)) \quad \text{a.e.}\end{aligned}$$

Since $\tilde{A}, \tilde{M}$ and $k(U, W) \circ Y$ are with values in $\Phi'(U)$ and all of them are right continuous, $\tilde{A} + \tilde{M}$ and $k(U, W) \circ Y$ are undistinguishable. Moreover $\tilde{M}$ and $\tilde{A}$ have uniformly bounded jumps. Since $(\phi|\tilde{M})$ is a local martingale, for any $\varphi \in \Phi[U^0]$, a stopping time argument shows that $\tilde{M}$ is in fact a local martingale with values in $\Phi'(U)$. Hence $k(0, U) \circ \tilde{M}$ is also a local martingale with values in $\Phi'(0)$. $\tilde{A}$ is a right continuous previsible process with values in $\Phi'(U)$, without loss of generality we can suppose it bounded. The continuity of the mapping

$$(h, \varphi) \mapsto \int_0^t h_s \, d\tilde{A}_s(\varphi)$$

from $\varepsilon \times \Phi[U^0]$ into L^0 implies

i) the set $T = \left\{ \int_0^t h_s \, d\tilde{A}_s(\varphi) \colon h \in \varepsilon_1, \varphi \in U^0 \right\}$ is bounded in L^0,

ii) $T \subset L^1$ and T is convexe.

Consequently, there exists a probability measure Q equivalent to P (c.f. [3], p. 402) such that, for any $\varphi \in \Phi[U^0]$, $\tilde{A}(\varphi)$ is a previsible Q-quasimartingale and it can be decomposed as

$$\tilde{A}(\varphi) = N(\varphi) + B(\varphi)$$

where $N(\varphi)$ is a continuous local martingale and $B(\varphi)$ is of integrable variation (on $[0, t]$), previsible. By the methods which we have already used, N can be lifted as a continuous local martingale with values in $\Phi'(0)$ and B as a process of integrable variation in $\Phi'(0)$ (c.f. [12], Theorem II.I and II.2), hence $\tilde{A}$ can be lifted as a Q-semimartingale with values in $\Phi'(0)$ so also as a P-semimartingale. Moreover $\tilde{A}$ is previsible in $\Phi'(0)$ so its continuous local martingale part is null i.e. $\tilde{A}$ is of finite variation in $\Phi'(0)$.

Now let us complete the proof of Theorem II.1:
We have

$$X_{t\wedge T_n}(W)=X_t^n(W)+\Delta X_{T_n}\,1_{\{t\geqq T_n\}}=Y_t^n(W)+Q_t^n(W)+\Delta X_{T_n}(W)\,1_{\{t\geqq T_n\}}$$

Hence

$$X_{t\wedge T_n}(0)=k(0,W)(Y_t^n(W))+k(0,W)(Q_t^n(W))$$
$$+k(0,W)(\Delta X_{T_n}(W))\,1_{\{t\geqq T_n\}}$$

and $k(0,W)(Q_t^n(W))$ and $k(0,W)(\Delta X_{T_n})\,1_{\{t\geqq T_n\}}$ are the semimartingales in $\Phi'(0)$. Moreover

$$k(0,W)(Y_t^n(W))=k(0,U)(\tilde{M}_t)+k(0,U)(\tilde{A}_t)$$

where $(\tilde{M}_t)$ is a local martingale with values in $\Phi'(U)$ and $k(0,U)(\tilde{A}_t)$ is a process of finite variation (it is even previsible and with uniformly bounded jumps). Hence the stochastic process $\{X_{t\wedge T_n}(0); t\geqq 0\}$ is a semimartingale with values in $\Phi'(0)$ and T_n increases to infinity, so $\{X_t(0); t\geqq 0\}$ is a semimartingale with values in $\Phi'(0)$. Since $0\in\mathscr{U}_h(\Phi'_\beta)$ is arbitrary, the proof of Theorem II.1 is completed. ||QED

We state some important consequences of Theorem II.1:

Corollary II.1. i) *Suppose that g is a continuous linear mapping on Φ' and (X', X) is a semimartingale on Φ'. There exists a unique projective system of semimartingales X^g such that $(g\circ X', X^g)$ is a semimartingale.*

ii) *Suppose that (X', X) is a semimartingale on Φ' and Q is another probability measure equivalent to P. Then (X', X) is also a semimartingale under Q.*

As a consequence of the Theorem of Meyer-Dellacherie-Mokobodski (cf. [3], [5]) and Theorem II.1 we have the following

Corollary II.2. *Let X' be a weakly measurable mapping on $\mathbb{R}_+\times\Omega$ with values in Φ' such that, for any $\phi\in\Phi$, $(t,\omega)\to\langle\phi, X'_t(\omega)\rangle$ has a right continuous modification $\tilde{X}(\phi)$. Denote by $\mathscr{I}$ the algebra generated by the previsible rectangles. Then there exists a projective system of semimartingales whose projective limit is X' if and only if the set*

$$D_\phi=\left\{\int_0^\infty 1_H(s)\,d\tilde{X}(\phi)_s;\ H\in\mathscr{I}\right\}$$

is bounded in L^0, for any $\phi\in\Phi$.

If Φ'_β is a nuclear Fréchet space one can define also the semimartingales in the ordinary sense:

Definition II.1. Suppose that Φ'_β is a nuclear Fréchet space and Y a mapping on $\mathbb{R}_+\times\Omega$ with values in Φ'. Y is called an *s*-semimartingale if for any $U\in\mathscr{U}_h(\Phi'_\beta)$, $k(U)\circ Y$ is a semimartingale with values in $\Phi'(U)$.

Remark. Since $\mathscr{U}_h(\Phi'_\beta)$ is countable, Y is necessarily right continuous with left limits.

Definition II.2. Suppose that Φ'_β is a nuclear Fréchet space and Y is a right continuous stochastic process with values in Φ'_β. Y is called a w-semimartingale if, for any $\phi\in\Phi$, the stochastic process $(t,\omega)\to\langle\phi, Y_t(\omega)\rangle$ is a semimartingale.

By the argument that we have used to prove the fact that the projective system of Theorem II.1 is a projective system of semimartingales, one can show the following:

Corollary II.3. *Suppose that Φ'_β is a nuclear Fréchet space. Then any w-semimartingale in Φ' is an s-semimartingale.*

The following result is an interpretation of some results of Functional Analysis from a probabilistic point of view:

Corollary II.4. i) *Suppose that F is a Banach space. Then F is finite dimensional if and only if any w-semimartingale with values in F is an s-semimartingale.*

ii) *Suppose that E is a locally convex Fréchet space. Then E is nuclear if and only if every w-semimartingale with values in E is an s-semimartingale.*

Proof. i) 'If' part is trivial. Conversely suppose that (x_n) is a weakly summable sequence. It is a (discrete) w-semimartingale, by the hypothesis it is a semimartingale hence it is absolutely summable, Dvoretzky-Rogers Theorem implies that F is finite dimensional (cf. [8], p. 184, Corollary 3).

ii) The same method shows that the space of the weakly summable sequences in E is algebraically isomorphic to the space of the absolutely summable sequences in E. Since E is a Fréchet space, this isomorphism is also a topological one and this is a sufficient condition for the nuclearity of E (*cf.* [8]). ‖QED

III. Some Applications

Suppose that z is a real-valued semimartingale and define δ_z on $\mathscr{D}$ (i.e. the space of C^∞-functions on $\mathbb{R}$ with compact support) as

$$\delta_z(\phi)_t=\phi(z_t).$$

Then by Theorem II.1, δ_z generates a projective system of semimartingales such that the corresponding g-process is a semimartingale on $\mathscr{D}'$. If $\phi\in\mathscr{D}$, for any $x\in\mathbb{R}$, by Ito's formula we have

$$\begin{aligned}\phi(x+z_t)=\phi(x+z_0)&+\int_0^t\phi'(x+z_{s-})\,dz_s\\&+\frac{1}{2}\int_0^t\phi''(x+z_{s-})\,d\langle z^c,z^c\rangle_s\\&+\sum_{0<s\leq t}[\phi(x+z_s)-\phi(x+z_{s-})-\Delta z_s\,\phi'(x+z_{s-})]\qquad \text{P-a.e.}\qquad \text{(III.1)}\end{aligned}$$

For any $(t,\omega)\in\mathbb{R}_+\times\Omega$, the mapping $x\to\phi(x+z_t(\omega))$ is again an element of $\mathscr{D}$.

The mappings

$$x \to \int_0^t \phi''(x+z_{s-})\, d\langle z^c, z^c\rangle_s$$

$$x \to \sum_{0<s\leqq t} [\phi(x+z_s)-\phi(x+z_{s-})-\Delta z_s\, \phi'(x+z_{s-})] = A_t(\phi(x))$$

are continuous for any $t \geqq 0$ and for almost all $\omega \in \Omega$. Define T_n as

$$T_n = \inf\{t \geqq 0 : |z_t| > n\} \wedge n, \qquad n \in \mathbb{N}.$$

T_n is a stopping time and it increases to infinity by n. Moreover the set

$$\{\phi'(\,.\,+z_{(t\wedge T_n)-}(\omega)) : (t,\omega) \in \mathbb{R}_+ \times \Omega\}$$

is a bounded subset of $\mathscr{D}$ for almost all $\omega \in \Omega$ since it is a subset of

$$\{\phi'(\,.\,+y) : |y| \leqq n\}$$

for almost all $\omega \in \Omega$ and the right translation is a continuous mapping on $\mathscr{D}$. Let L_n be any element of $\mathscr{K}_h(\mathscr{D})$ absorbing $\{\phi'(\,.\,+z_{(t\wedge T_n)-});\ (t,\omega) \in \mathbb{R}_+ \times \Omega\}$. Denote by z^n the process stopped at T_n. Then $\phi(\,.\,+z^n)$ is a bounded, previsible stochastic process in the separable Hilbert space $\mathscr{D}[L_n]$ so the stochastic integral

$$\int_0^t \phi'(\,.\,+z^n_{s-})\, dz_s$$

defines a semimartingale with values in $\mathscr{D}[L_n]$. Consequently the negligeable set on which (III.1) does not hold is independent of $x \in \mathbb{R}$ id we modify the stochastic integral as above. Moreover, the same method shows that the mapping

$$x \to \int_0^{t\wedge T_n} \phi''(x+z_{s-})\, d\langle z^c, z^c\rangle_s$$

is C^∞ and of compact support for any $t \geqq 0$ and for almost all $\omega \in \Omega$. We have

$$|A_t(\phi(x))| \leqq \sup_{s\leqq t} |\phi''(x+z_{s-}+\theta\Delta z_s)| \sum_{s\leqq t} (\Delta z_s)^2, \qquad \theta \in (0,1),$$

$\overline{\mathrm{co}}(\{z_s(\omega); s \leqq t\} + \{z_{s-}(\omega); s \leqq t\})$ is a compact set hence the mapping

$$x \to A_t(\phi(x))$$

is of compact support for any $t \geqq 0$ and for almost all $\omega \in \Omega$. For fixed (t,ω), denote by $K_t(\omega)$ the support of $x \to A_t(\phi(x))$. If R is in $\mathscr{D}'$, by the local characterization of the distributions, R is equal to $D_x^\beta g$ on $K_t(\omega)$ where $\beta \in \mathbb{N}$, g is a continuous function of compact support whose support includes $K_t(\omega)$ (cf. [9]). Consequently, we have

$$\sum_{s\leqq t}[R(\phi(.+z_s)-\phi(.+z_{s-})-\phi'(.+z_{s-})\Delta z_s)]$$
$$\leqq \sum_{s\leqq t}|\int g(x)(D_x^\beta \phi(x+z_s)-D_x^\beta \phi(x+z_{s-})-\Delta z_s D_x^{\beta+1}\phi(x+z_{s-}))\,dx|$$
$$\leqq \tfrac{1}{2}\sup_{x\in\mathbb{R}}|D_x^{\beta+2}\phi(x)|\int|g(x)|\,dx.\sum_{s\leqq t}(\Delta z_s)^2<+\infty \quad \text{a.e.}$$

hence the sum converges in $\mathscr{D}$ for any $t\geqq 0$ and for any $\omega\in\Omega$ such that $t\to z_t(\omega)$ is right continuous with left limits. Therefore, for any $R\in\mathscr{D}'$, we have

$$R(\phi(.+z_t^n))=R(\phi(.+z_0))+R\left(i_{K_n}\left(\int_0^{t\wedge T_n}\phi'(.+z_{s-})\,dz_s\right)\right)$$
$$+\frac{1}{2}R\left(\int_0^{t\wedge T_n}\phi''(.+z_{s-})\,d\langle z^c,z^c\rangle_s\right)$$
$$+\sum_{0<s\leqq t}[R(\phi(.+z_s)-\phi(.+z_{s-})-\phi'(.+z_{s-})\Delta z_s)] \quad \text{a.e.}$$

where i_{K_n} is the injection from $\mathscr{D}[K_n]$ into $\mathscr{D}$. Denoting the adjoint of i_{K_n} by $k(K_n^0)$, we have

$$R\left(i_{K_n}\left(\int_0^{t\wedge T_n}\phi'(.+z_{s-})\,dz_s\right)\right)=\left(k(K_n^0)(R)\,\Big|\int_0^{t\wedge T_n}\phi'(.+z_{s-})\,dz_s\right)$$
$$=\int_0^{t\wedge T_n}(k(K_n^0)(R)\,|\,\phi'(.+z_{s-}))\,dz_s$$
$$=\int_0^{t\wedge T_n}R(\phi'(.+z_{s-}))\,dz_s \quad \text{a.e.}$$

since we have modified the stochastic integral to obtain a $\mathscr{D}[K_n]$-valued stochastic integral. As one can show easily R commutes also with the Stieltjes integral and we obtain

$$R(\phi(.+z_t))=R(\phi(.+z_0))+\int_0^t R(\phi'(.+z_{s-}))\,dz_s$$
$$+\frac{1}{2}\int_0^t R(\phi''(.+z_{s-}))\,d\langle z^c,z^c\rangle_s$$
$$+\sum_{0<s\leqq t}[R(\phi(.+z_s))-R(\phi(.+z_{s-}))$$
$$-R(\phi'(.+z_{s-}))\Delta z_s] \qquad \text{(III.2)}$$

for any $t\geqq 0$ and for almost all $\omega\in\Omega$. Since both sides are right continuous they are undistinguishable and we have the following

Theorem III.1. *Let z be a real valued semimartingale. If R is an element of $\mathscr{D}'$, define $X_t^r(\omega)$ as*

$$X_t^r(\omega)=R*\delta_{z_t(\omega)},$$

where '$*$' denotes the convolution. Then X' generates a unique projective system of semimartingales X such that the g-process (X', X) is a semimartingale on $\mathscr{D}'$.

Proof. By (III.2), for any $\phi \in \mathscr{D}$, $\langle \phi, X'_t \rangle$ has a modification which is a semimartingale. Then the theorem follows from Theorem II.1. ||QED

Theorem III.1 and the relation III.2 are equivalent to

Theorem III.2. *Denote by $\tilde{X}$ the mapping on $\mathscr{D}$ with values in S^0 defined by (up to an evanescent process)*

$$\tilde{X}(\phi)_t = R(\phi(\,.\,+z_t)).$$

Then $\tilde{X}$ is a linear, sequentially continuous with values in (S^0, d) such that

$$\begin{aligned}\tilde{X}(\phi)_t = \tilde{X}(\phi)_0 &+ \int_0^t (\tilde{X} \circ D_x)(\phi)_{s-}\, dz_s \\ &+ \frac{1}{2} \int_0^t (\tilde{X} \circ D_x^2)(\phi)_{s-}\, d\langle z^c, z^c \rangle_s \\ &+ \sum_{0<s\leqq t} [\tilde{X}(\phi)_s - \tilde{X}(\phi)_{s-} - (\tilde{X} \circ D_x)(\phi)_{s-}\, \Delta z_s] \quad \text{a.e.} \qquad \text{(III.3)}\end{aligned}$$

for any $t \geqq 0$.

Proof. The relation (III.3) is obvious. The sequential continuity of $\tilde{X}$ follows from the closed graph theorem and from the fact that the random variable $X'_t(.)$ takes its values in $\mathscr{D}'$, for any $t \in \mathbb{R}_+$. ||QED

Remark. The representation (III.3) gives a rigorous definition of the partial stochastic differential equations.

By Theorem II.1 and Corollary II.1, the preceeding theorem is also equivalent to saying that

Theorem III.3. *Denote respectively by $\{X^U; U \in \mathscr{U}_h(\mathscr{D}'_\beta)\}$, $\{X^{1,U}; U \in \mathscr{U}_h(\mathscr{D}'_\beta)\}$ and $\{X^{2,U}; U \in \mathscr{U}_h(\mathscr{D}'_\beta)\}$ the projective systems of semimartingales corresponding to X', $D_x X'$ and $D_x^2 X'$, where D_x^i denotes the derivation in $\mathscr{D}'$ of order $i=1,2$. Then one has the following relation:*

$$\begin{aligned}X_t^U = X_0^U &- \int_0^t X_{s-}^{1,U}\, dz_s + \frac{1}{2} \int_0^t X_{s-}^{2,U}\, d\langle z^c, z^c \rangle_s \\ &+ \sum_{0<s\leqq t} [X_s^U - X_{s-}^U + X_{s-}^{1,U} . \Delta z_s] \quad \textit{a.e.} \qquad \text{(III.4)}\end{aligned}$$

Moreover, as stochastic processes, both sides of (III.4) are undistinguishable for any $U \in \mathscr{U}_h(\mathscr{D}'_\beta)$.

Proof. Let $U \in \mathscr{U}_h(\mathscr{D}'_\beta)$ and $\phi \in \mathscr{D}[U^0]$. We have

$$\begin{aligned}&\tilde{X}(\phi)_t = (\phi \mid X_t^U) \quad \text{a.e.} \\ &\tilde{X}(D_x \phi)_t = \langle D_x \phi, X'_t \rangle = -\langle \phi, D_x X'_t \rangle = -(\phi \mid X_t^{1,U}) \quad \text{a.e.} \\ &\tilde{X}(D_x^2 \phi)_t = (\phi \mid X_t^{2,U}) \quad \text{a.e.,}\end{aligned}$$

hence the stochastic processes at each side of the equalities are undistinguishable. Then, by (III.3), we have

$$(\phi|X_t^U)=(\phi|X_0^U)-\int_0^t(\phi|X_{s-}^{1,U})\,dz_s+\frac{1}{2}\int_0^t(\phi|X_{s-}^{2,U})\,d\langle z^c,z^c\rangle_s$$
$$+\sum_{0<s\leq t}[(\phi|X_s^U-X_{s-}^U)+\Delta z_s(\phi|X_{s-}^{1,U})] \quad \text{a.e.} \qquad \text{(III.5)}$$

The integrals $\int X_{s-}^{1,U}\,dz_s$ and $\int X_{s-}^{2,U}\,d\langle z^c,z^c\rangle_s$ converge in $\mathscr{D}'(U)$. Using the local structure of the distributions and Taylor's formula, as we have already done to prove (III.2), we obtain, for any $s\leq t$ and for any $t\geq 0$,

$$|[(\phi|X_s^U-X_{s-}^U)+\Delta z_s(\phi|X_{s-}^{1,U})]|\leq\tfrac{1}{2}\mathscr{C}_t(\omega)\sup_x|\phi^\beta(x)|(\Delta z_s)^2$$

where $\mathscr{C}_t(\omega)>0$ depends only on t, ω and R. Hence

$$\sum_{0<s\leq t}\|X_s^U-X_{s-}^U+\Delta z_s X_{s-}^{1,U}\|<+\infty \quad \text{a.e.}$$

consequently both sides of (III.4) are right continuous and the theorem follows. ‖QED

By Theorem I.1, since $\mathscr{D}$ is bornological, if (X,X'), corresponding to $R\in\mathscr{D}'$ and $z\in S^0$, belongs to $S^1(\mathscr{D}')$, then there exists right continuous stochastic processes having left limits with values in $\mathscr{D}'$, say $\hat{X},\hat{X}^1$ and $\hat{X}^2$ such that

$$k(U)\circ\hat{X}=X^U,$$
$$k(U)\circ\hat{X}^1X^{1,U}$$
$$k(U)\circ\hat{X}^2=X^{2,U}$$

up to an evanescent process, for any $U\in\mathscr{U}_h(\mathscr{D}'_\beta)$. Moreover, there exists some K in $\mathscr{K}_h(\mathscr{D}')$ and semimartingales Y, Y^1 and Y^2 with values in $\mathscr{D}'[K]$ such that

$$\hat{X}=i_K(Y), \qquad \hat{X}^1=i_K(Y^1), \qquad \hat{X}^2=i_K(Y^2)$$

where i_K is the injection $\mathscr{D}'[K]\hookrightarrow\mathscr{D}'$. One can now prove, by the method that we have used for the proof of Theorem III.3, the following

Theorem III.4. *Suppose that the semimartingale (X,X') corresponding to $R\in\mathscr{D}'$, $z\in S^0$, be in $S^1(\mathscr{D}')$. Then, using the above notations, one has*

$$Y_t=Y_0-\int_0^t Y_{s-}^1\,dz_s+\frac{1}{2}\int_0^t Y_{s-}^2\,d\langle z^c,z^c\rangle_s$$
$$+\sum_{0<s\leq t}[Y_s-Y_{s-}+\Delta z_s\,Y_{s-}^1] \quad \text{a.e.} \qquad \text{(III.6)}$$

and the corresponding stochastic processes are undistinguishable in $\mathscr{D}'[K]$.

Example. If $z\in S^2$ then $X'_t(\omega)$ defined by

$$X'_t(\phi)=\phi(z_t), \qquad \phi\in\mathscr{D},$$

generates an element of $S^1(\mathscr{D}')$ as one can see by Ito's formula.

Remark. For notational simplicity we have treated only the one-dimensional case, however all the results of this section extend trivially to higher dimensions.

By definition $\hat{X}^1(\phi)$ is a modification of $D_x X'_t(\phi)$ for fixed $\phi\in\mathscr{D}$. Hence, for fixed $t\in\mathbb{R}_+$, we have

$$\begin{aligned}\hat{X}^1_t(\phi)&=\langle D_x X'_t, \phi\rangle=-\langle X'_t, D_x\phi\rangle=-\langle \hat{X}_t, D_x\phi\rangle\\ &=\langle D_x\hat{X}_t,\phi\rangle=D_x\hat{X}_t(\phi) \qquad \text{a.e.}\end{aligned}$$

Since $\hat{X}^1_\cdot(\phi)$ and $D_x\hat{X}_\cdot(\phi)$ are right continuous stochastic processes, they are undistinguishable. Similarly, for any $\phi\in\mathscr{D}$, the stochastic processes $\hat{X}^2_\cdot(\phi)$ and $D_x^2\hat{X}_\cdot(\phi)$ are also undistinguishable. Consequently we have

Theorem III.5. *For any $\phi\in\mathscr{D}$ one has the following equality*

$$\begin{aligned}\hat{X}_t(\phi)=\hat{X}_0(\phi)&-\int_0^t D_x\hat{X}_{s-}(\phi)\,dz_s+\frac{1}{2}\int_0^t D_s^2\hat{X}_{s-}(\phi)\,d\langle z^c,z^c\rangle_s\\ &+\sum_{0<s\leq t}[\hat{X}_s(\phi)-\hat{X}_{s-}(\phi)+D_x\hat{X}_{s-}(\phi)\,\Delta z_s].\end{aligned}$$

Remark. i) Any semimartingale indexed by a compact interval of the positive real numbers can be regarded as an element of S^2 under a convenient change of the probability P (cf. [3]).

ii) If $z_t=B_t$ i.e. the Standard Wiener process, then stopping B_t on the increasing, compact subsets of $\mathbb{R}$ by a bounded sequence of stopping times, we see that $(\hat{X}_t)$ corresponding to $R*\delta_{B_t}$ satisfies the following equation

$$d\hat{X}_t=-D_x\hat{X}_t\,dB_t+\tfrac{1}{2}D_x^2\hat{X}_t\,dt, \qquad X_0=R, \qquad R\in\mathscr{D}',$$

and same method works also for the following case:

$$\langle X'_t,\phi\rangle=\langle(\exp-\int_0^t V(\,.\,+B_s)\,ds).R,\phi(\,.\,+B_t)\rangle$$

and the corresponding stochastic process $(\hat{X}_t)$ satisfies the following equation:

$$\begin{aligned}&d\hat{X}_t=-D_x\hat{X}_t\,dB_t-V\hat{X}_t\,dt+\tfrac{1}{2}D_x^2\hat{X}_t\,dt,\\ &\hat{X}_0=R, \qquad R\in\mathscr{D}'\end{aligned}$$

where V is an infinitely differentiable function on $\mathbb{R}$. In [12] this relation has been called the stochastic form of Feynman-Kac formula. Of course all this relations should be interpreted in the sense of Theorem III.5.

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