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p-Sets for Random Walks

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Summary. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\{X_n(\omega)\}_{n=1}$ be a sequence of i.i.d. random vectors whose state space is Z^m for some positive integer m, where Z denotes the integers. For $n=1,2,\dots$ let $S_n(\omega)$ be the random walk defined by $S_n(\omega)=\sum_{j=1}^{n} X_j(\omega)$. For $x \in Z^m$ and $\alpha \in U^m$, the m-dimensional torus, let $\langle \alpha, x\rangle = e^{2\pi i} \sum_{j=1}^{m} \alpha_j x_j$. Finally let $\phi(\alpha)=E\{\langle \alpha, X_1(\omega)\rangle\}$ be the characteristic function of the X's.

In this paper we show that, under mild restrictions, there exists a set $\Omega_0 \subset \Omega$ with $P\{\Omega_0\}=1$ such that for $\omega \in \Omega_0$ we have

$$\lim_{n\to\infty} \left| \frac{1}{n} \sum_{j=1}^{n} \langle \alpha, S_j(\omega)\rangle \right| = 0 \quad \text{for all } \alpha \in U^m,\ \alpha \neq 0.$$

As a consequence of this theorem, we obtain two corollaries. One is concerned with occupancy sets for m-dimensional random walks, and the other is a mean ergodic theorem.

1. Preliminaries

Let G be a locally compact abelian group. Let $\hat{G}$ be the dual of G, that is $\hat{G}$ is the set of continuous homomorphisms of G to T, the unit circle in the complex plane. $\hat{G}$ is a group under pointwise multiplication and inherits a natural topology from G. Now put the discrete topology on $\hat{G}$, call it $\hat{G}_d$. Then $\hat{G}_d = \bar{G}$ is a compact (but very large) group such that G is dense in $\bar{G}$. $\bar{G}$ is called the Bohr compactification of G. If m is Haar measure on $\bar{G}$ then $m(G)=0$. For details of this, see e.g. Rudin [3, Ch. 1].

Now suppose $\{m_n\}_{n=1}^{\infty}$ is a sequence of probability measures on G. In an obvious way, we may consider them also as measures on $\bar{G}$. We shall say the sequence $\{m_n\}$ is ergodic provided m_n converges weakly to m. The reason for the

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name is that such sequences provide mean ergodic theorems for unitary representations of G on an arbitrarily Hilbert space, as shown in [2]. It follows from the Paul Lévy continuity theorem that $\{m_n\}$ is ergodic if and only if $\int_G \langle\gamma, g\rangle\, m_n(dg) \to 0$ for every $\gamma \in \hat{G}$, $\gamma \neq$ identity since the Fourier transform $\hat{m}$ of m is zero for every non-trivial $\gamma \in G$.

Now suppose $0 \leqq p \leqq 1$. We shall call a measurable subset I_p of G a p-set provided $\lim_n m_n(I_p) = p$ for every ergodic sequence $\{m_n\}$. There exist many p-sets and examples are not difficult to construct. Let H be a closed subset of $\bar{G}$ with $m(H) = p$, such that H is a continuity set for m, i.e. the boundary of H has Haar measure zero. Then it follows at once that $H \cap G$ is a p-set. For example, let I be a closed subinterval of the unit circle with Lebesgue measure p, and let $\gamma \in \hat{G}$ be of infinite order. Then $\{\bar{g} \in \bar{G} \mid \langle\gamma, \bar{g}\rangle \in I\}$ is a closed continuity set for m with measure p, and $\{g \in G \mid \langle\gamma, g\rangle \in I\}$ is a p-set. Now if $G = Z^m$ for some $m \geqq 1$, then $\hat{G} = U^m$, where U is the unit circle. Thus if $\alpha = (\alpha_1, \ldots, \alpha_m) \in U^m$ is not a root of unity then

$$\{l = (l_1, \ldots, l_m) \in Z^m \mid \langle\alpha, l\rangle \in I\}$$

is a p-set.

2. The Main Results

With the notation as defined in the abstract, we shall prove the

Theorem. *If $|\phi(\alpha)| < 1$ for $\alpha \in U^m$, $\alpha \neq 0$ and if $E|X_1(j, \omega)| < \infty$ for $j = 1, \ldots, m$ then there exists a set $\Omega_0 \subset \Omega$ with $P\{\Omega_0\} = 1$ such that for $\omega \in \Omega_0$*

$$\lim_{n\to\infty} \left| \frac{1}{n} \sum_{j=1}^{n} \langle\alpha, S_j(\omega)\rangle \right| = 0$$

for all $\alpha \in U^m$, $\alpha \neq 0$.

Remarks. i) By $X_n(j, \omega)$ or $S_n(j, \omega)$ we mean the j^{th} coordinate of $X_n(\omega)$ or $S_n(\omega)$ respectively.

ii) The hypothesis $|\phi(\alpha)| < 1$ for $\alpha \neq 0$ is inessential. If we consider the class of random walks for which $|\phi(\alpha_0)| = 1$ for some $\alpha_0 \neq 0$ then this is the class of random walks which lives on some subgroup of Z^m. We can then prove an analogous theorem for that subgroup.

3. Proof of the Theorem

For $m = 1$ the theorem was proved in [1]. However the case $m > 1$ is considerably more difficult and appears to be interesting in its own right.

Lemma 1. *Let $\delta_j = \pm 1$ for $j = 1, \ldots, 2m$ with $\sum_{j=1}^{2m} \delta_j = 0$. Define $k_j = -\sum_{l=1}^{j} \delta_l$, l*

$=1,\dots,2m-1$. *Then for arbitrary numbers* $x_1,\dots,x_{2m}$ *we have* $\sum_{j=1}^{2m}\delta_j x_j = \sum_{j=1}^{2m-1} k_j(x_{j+1}-x_j)$. *Moreover* $|k_j|\leqq m$ *for all* j, *and* $k_{2j-1}\neq 0$ *for* $j=1,\dots,m$.

The proof of the lemma is obvious and will be omitted. □

Lemma 2. *Let* m *be a positive integer and let* $r_j\in(0,1)$, $j=1,\dots,m$. *Let* $M\leqq N$ *be positive integers. Then*

$$\sum_{M\leqq j_1\leqq\dots\leqq j_{2m}\leqq N}\prod_{k=1}^{m} r_k^{j_{2k}-j_{2k-1}}\leqq\frac{2^m N^m}{\prod_{j=1}^{m}(1-r_j)}.$$

Proof. The proof is by induction on m. For $m=1$ we have

$$\sum_{M\leqq j_1\leqq j_2\leqq N} r^{j_2-j_1}=\sum_{j_1=M}^{N}\frac{1-r^{N-j_1+1}}{1-r}\leqq\frac{2N}{1-r}.$$

Now suppose the conclusion holds for m. Then

$$\begin{aligned}&\sum_{M\leqq j_1\leqq j_2\leqq\dots\leqq j_{2(m+1)}\leqq N}\prod_{k=1}^{m+1} r_k^{j_{2k}-j_{2k-1}}\\ &=\sum_{M\leqq j_1\leqq j_2\leqq N} r_1^{j_2-j_1}\sum_{j_2\leqq j_3\leqq\dots j_{2(m+1)}\leqq N}\prod_{k=2}^{m+1} r_k^{j_{2k}-j_{2k-1}}\\ &\leqq\sum_{M\leqq j_1\leqq j_2\leqq N} r_1^{j_2-j_1}\frac{2^m N^m}{\prod_{j=2}^{m+1}(1-r_j)}\leqq\frac{2^{m+1}N^{m+1}}{\prod_{k=1}^{m+1}(1-r_k)},\end{aligned}$$

by the induction hypothesis and the case $m=1$. □

Lemma 3. *Let* m *and* n *be positive integers and let* $\alpha\in U^m$ *such that* $\alpha\neq 0$, $2\alpha\neq 0,\dots,n\alpha\neq 0$. *Then for all positive integers* N *we have*

$$E\left|\sum_{j=1}^{N}\langle\alpha,S_j(\omega)\rangle\right|^{2n}\leqq\frac{N^n 2^n(2n)!}{(1-\max_{1\leqq j\leqq n}|\phi(j\alpha)|)^n}.$$

Proof. Define $\delta_k=\begin{cases}1 & k=1,\dots,n\\ -1 & k=n+1,\dots,2n\end{cases}$. We have

$$\begin{aligned}E\left|\sum_{j=1}^{N}\langle\alpha,S_j(\omega)\rangle\right|^{2n}&=E\sum_{j_1,\dots,j_{2n}=1}^{N}\prod_{k=1}^{n}\langle\alpha,S_{j_k}(\omega)\rangle\prod_{k=n+1}^{2n}\overline{\langle\alpha,S_{j_k}(\omega)\rangle}\\ &=E\sum_{j_1,\dots,j_{2n}=1}^{N}\left\langle\alpha,\sum_{k=1}^{2n}\delta_k S_{j_k}(\omega)\right\rangle\\ &\leqq\sum_{j_1,\dots,j_{2n}=1}^{N}\left|E\left\langle\alpha,\sum_{k=1}^{2n}\delta_k S_{j_k}(\omega)\right\rangle\right|.\end{aligned}\tag{3.1}$$

Now let P_{2n} be the family of permutations of $(1,\ldots,2n)$ and for $\sigma\in P_{2n}$ define

$$S_\sigma=\{\bar J=(j_1,\ldots,j_{2n})\,|\,1\leqq j_k\leqq N, j_{\sigma(1)}\leqq j_{\sigma(2)}\leqq\cdots\leqq j_{\sigma(2n)}\}.$$

Then clearly $\bigcup_{\sigma\in P_{2n}} S_\sigma$ is the collection of all sequences $(j_1,\ldots,j_{2n})$ with $1\leqq j_k\leqq N$. Therefore we obtain from (3.1) that

$$E\left|\sum_{j=1}^{N}\langle\alpha,S_j(\omega)\rangle\right|^{2n}\leqq\sum_{\sigma\in P_{2n}}\sum_{\bar J\in S_\sigma}\left|E\left\langle\alpha,\sum_{k=1}^{2n}\delta_k S_{j_k}(\omega)\right\rangle\right|. \tag{3.2}$$

Note that $\sum_{k=1}^{2n}\delta_k S_{j_k}(\omega)=\sum_{k=1}^{2n}\delta_{\sigma(k)}S_{j_{\sigma(k)}}(\omega)$ and that for each $\sigma\in P_{2n}$ the numbers $\delta_{\sigma(k)}$, $k=1,\ldots,2n$ satisfy the conditions of Lemma 1. Therefore there exist integers $k_r(\sigma)$, $r=1,\ldots,2n-1$ such that

$$\sum_{k=1}^{2n}\delta_{\sigma(k)}S_{j_{\sigma(k)}}(\omega)=\sum_{r=1}^{2n-1}k_r(\sigma)[S_{j_{\sigma(r+1)}}(\omega)-S_{j_{\sigma(r)}}(\omega)],$$

and we obtain

$$\begin{aligned}
&\sum_{\bar J\in S_\sigma}\left|E\left\langle\alpha,\sum_{k=1}^{2n}\sigma_k S_{j_k}(\omega)\right\rangle\right|\\
&\quad=\sum_{1\leqq j_{\sigma(1)}\leqq\cdots\leqq j_{\sigma(2n)}\leqq N}\left|E\left\langle\alpha,\sum_{r=1}^{2n-1}k_r(\sigma)[S_{j_{\sigma(r+1)}}(\omega)-S_{j_{\sigma(r)}}(\omega)]\right\rangle\right|\\
&\quad=\sum_{1\leqq j_{\sigma(1)}\leqq\cdots\leqq j_{\sigma(2n)}\leqq N}\left|E\prod_{r=1}^{2n-1}\langle k_r(\sigma)\,\alpha,S_{j_{\sigma(r+1)}}(\omega)-S_{j_{\sigma(r)}}(\omega)\rangle\right|\\
&\quad=\sum_{1\leqq j_{\sigma(1)}\leqq\cdots\leqq j_{\sigma(2n)}\leqq N}\prod_{r=1}^{2n-1}|\phi(k_r(\sigma)\,\alpha)|^{j_{\sigma(r+1)}-j_{\sigma(r)}}\\
&\quad\leqq\sum_{1\leqq j_{\sigma(1)}\leqq\cdots\leqq j_{\sigma(2n)}\leqq N}\prod_{r=1}^{n}|\phi(k_{2r-1}(\sigma)\,\alpha)|^{j_{\sigma(2r)}-j_{\sigma(2r-1)}}.
\end{aligned}\tag{3.3}$$

The first equality in (3.3) follows from the preceding remark, the second from properties of exponentials, and the third from the fact that the blocks $S_{j_{\sigma(r+1)}}(\omega)-S_{j_{\sigma(r)}}$ are pairwise independent. The final inequality is obvious.

Now apply Lemma 2 to the last expression in (3.3) to obtain (3.4)

$$\sum_{\bar J\in S_\sigma}\left|E\left\langle\alpha,\sum_{k=1}^{2n}\delta_k S_{j_k}(\omega)\right\rangle\right|\leqq\frac{2^nN^n}{\sum_{r=1}^{n}(1-|\phi(k_{2r-1}(\sigma)\,\alpha)|)}\leqq\frac{2^nN^n}{(1-\max_{j=1,\ldots,n}|\phi(j\alpha)|)^n}. \tag{3.4}$$

The proof of the Lemma is now completed by summing over the permutations on $(1,\ldots,2n)$. □

Lemma 4. *For $k=1,\ldots,m$ we have*

$$E\left|\frac{\partial}{\partial\alpha_k}\frac{1}{N}\sum_{j=1}^{N}\langle\alpha,S_j(\omega)\rangle\right|=O(N).$$

Proof.

$$E\left|\frac{\partial}{\partial\alpha_k}\frac{1}{N}\sum_{j=1}^{N}\langle\alpha, S_j(\omega)\rangle\right| = E\left|\frac{1}{N}\sum_{j=1}^{N} i\alpha_k S_j(k,\omega)\langle\alpha, S_j(\omega)\rangle\right|$$
$$\leqq \frac{1}{N}\sum_{j=1}^{N} E|S_j(k,\omega)| \leqq C\frac{N(N+1)}{2N} = O(N). \quad \square$$

Lemma 5. *Let f be a complex-valued function defined on U^m, and suppose f has continuous partial derivatives. Suppose $\left|\frac{\partial f}{\partial\alpha_j}\right| \leqq K$ for all $\alpha\in U^m$. Let $C = \prod_{j=1}^{m}[a_j, b_j]$ be a subcube of U^m. Then for all $\alpha, \beta\in C$ we have*

$$|f(\beta)| \leqq |f(\alpha)| + 2K\sum_{j=1}^{m}(b_j - a_j).$$

Proof. The proof is by induction on m. The case $m=1$ follows from the mean-value theorem, and the induction proof is straight forward. We omit the details. $\square$

Now fix an m-cube $C = \prod_{j=1}^{m}[a_j, b_j]$ such that C does not contain a root of unity of U^m of order $\leqq 2m$. We have

Lemma 6. $\lim_{N\to\infty}\sup_{\alpha\in C}\left|\frac{1}{N}\sum_{j=1}^{N}\langle\alpha, S_j(\omega)\rangle\right| = 0.$

Proof. Let $[x]$ denote the greatest integer $\leqq x$. For each positive integer N, partition C into $[N^{\frac{3}{2}}]^m$ subcubes by partitioning each interval $[a_j, b_j]$ into $[N^{\frac{3}{2}}]$ subintervals of equal length. Choose a point α_k from each subcube and define

$$A_N = \left\{\omega \,\middle|\, \left|\frac{1}{N}\sum_{j=1}^{N}\langle\alpha_k, S_j(\omega)\rangle\right| < \frac{1}{N^{\frac{1}{16}}},\ k=1,\dots,[N^{\frac{3}{2}}]^m\right\}.$$

Then we have

$$P(A_N^c) \leqq \sum_{k=1}^{[N^{\frac{3}{2}}]^m} P\left\{\left|\frac{1}{N}\sum_{j=1}^{N}\langle\alpha_k, S_j(\omega)\rangle\right| \geqq \frac{1}{N^{\frac{1}{16}}}\right\}$$
$$\leqq \sum_{k=1}^{[N^{\frac{3}{2}}]^m} N^{\frac{m}{4}} E\left(\left|\frac{1}{N}\sum_{j=1}^{N}\langle\alpha_k, S_j(\omega)\rangle\right|^{4m}\right) \tag{3.5}$$

by the Čebyšev inequality. Now let $|\phi(\alpha_0)| = \max_{\alpha\in C}\max_{j=1,\dots,2m}|\phi(j_\alpha)|$. Then by hypothesis $|\phi(\alpha_0)| < 1$. Now we may apply Lemma 3 and (3.5) to obtain

$$P(A_N^c) \leqq \frac{N^{\frac{3m}{2}} N^{\frac{m}{4}}}{N^{4m}}\,\frac{N^{2m}2^{2m}(4m)!}{(1-|\phi(\alpha_0)|)^{2m}}$$
$$= O\left(\frac{1}{N^{\frac{m}{8}}}\right). \tag{3.6}$$

Now define $B_N=\left\{\omega \mid \sup_{\alpha}\left|\frac{\partial}{\partial\alpha_k}\frac{1}{N}\sum_{j=1}^{N}\langle\alpha, S_j(\omega)\rangle\right|\leqq N^{\frac{5}{4}},\ k=1,\dots,m\right\}$.

Then it follows from Lemma 4 and the Čebyšev inequality that

$$P(B_N^c)\leqq\frac{mO(N)}{N^{\frac{5}{4}}}=O\left(\frac{1}{N^{\frac{1}{4}}}\right). \tag{3.7}$$

Combining (3.6) and (3.7) we have

$$P\{(A_N\cap B_N)^c\}=O\left(\frac{1}{N^{\frac{1}{8}}}\right). \tag{3.8}$$

If $\omega\in A_N\cap B_N$ and $\alpha\in C$ then α is in one of the $[N^{\frac{3}{2}}]^m$ subcubes of C. Suppose α is in the subcube corresponding to α_r. Then

$$\begin{aligned}\left|\frac{1}{N}\sum_{j=1}^{N}\langle\alpha, S_j(\omega)\rangle\right| &\leqq\left|\frac{1}{N}\sum_{j=1}^{N}\langle\alpha_r, S_j(\omega)\rangle\right|+N^{\frac{5}{4}}\sum_{j=1}^{m}\frac{b_j-a_j}{[N^{\frac{3}{2}}]}\\ &\leqq\frac{1}{N^{\frac{1}{16}}}+\frac{m}{[N^{\frac{1}{4}}]}=O\left(\frac{1}{N^{\frac{1}{4}}}\right).\end{aligned} \tag{3.9}$$

The first inequality follows from Lemma 5 and the second is obvious. We conclude that for $\omega\in A_N\cap B_N$ we have

$$\sup_{\alpha\in C}\left|\frac{1}{N}\sum_{j=1}^{N}\langle\alpha, S_j(\omega)\rangle\right|=O\left(\frac{1}{N^{\frac{1}{16}}}\right). \tag{3.10}$$

From (3.8) we have

$$\sum_{N=1}^{\infty}P\left\{(A_{N^{16}}\cap B_{N^{16}})^c\right\}<\infty \tag{3.11}$$

and we use the Borel-Cantelli lemma to conclude that $P\{\Omega_c\}=1$, where $\Omega_c=\{\omega\mid\omega$ is in all but a finite number of $A_{N^{16}}\cap B_{N^{16}}\}$, and clearly for $\omega\in\Omega_c$ we have

$$\lim_{N\to\infty}\sup_{\alpha\in C}\left|\frac{1}{N^{16}}\sum_{j=1}^{N^{16}}\langle\alpha, S_j(\omega)\rangle\right|=0.$$

Now a well-known argument shows that the same holds along the entire sequence proving the lemma. □

It is easily seen that we may write $U^m-\{\text{roots of unity of order }\leqq 2m\}=\bigcup_{M=1}^{\infty}C_M$, where each C_M is a cube of the type in Lemma 6. Consequently for $\omega\in\bigcap_{M=1}^{\infty}\Omega_{C_M}$ and $\alpha\in\bigcup_{M=0}^{\infty}C_M$ the averages converge to zero.

If α is a nonzero root of unity of order $\leqq 2m$ let

$$A_N=\left\{\omega \mid \left|\frac{1}{N}\sum_{j=1}^{N}\langle\alpha, S_j(\omega)\rangle\right|<\frac{1}{N^{16}}\right\}.$$

Using Lemma 3 with $n=1$ and the Čebyšev inequality we obtain $P\{A_N^c\} \leqq N^{\frac{1}{8}} O\left(\frac{1}{N}\right) = O\left(\frac{1}{N^{\frac{1}{8}}}\right)$. Using the same technique as in Lemma 6 we obtain the almost sure convergence for each such α. Combining this with the previous paragraph concludes the proof of the theorem.

4. Applications

In this section we give several applications of the main result. Let $\{S_n(\omega)\}_{n=1}^{\infty}$ be a random walk in Z^m satisfying the hypotheses of the theorem. Let $\{\mu_n\{\cdot, \omega\}\}_{n=1}^{\infty}$ be the sequence of random measures on Z^m obtained by placing mass $1/n$ on $S_1(\omega), \ldots, S_n(\omega)$ for each n. Then the Fourier transform of $\mu_n(\cdot, \omega)$ is given by $\hat{\mu}_n(\alpha, \omega) = \frac{1}{n} \sum_{j=1}^{n} \langle \alpha, S_j(\omega) \rangle$. The theorem tells us that for $\omega \in \Omega_0$ we have $\lim_{n\to\infty} \hat{\mu}_n(\alpha, \omega) = 0$ for $\alpha \in U^m$, $\alpha \neq 0$. Therefore the sequence $\{\hat{\mu}_n(\cdot, \omega)\}_{n=1}^{\infty}$ is an ergodic sequence of measures, as defined in Sect. 1. Now let I_p be an arbitrary p-set in Z^m as defined in Sect. 1. Then it follows that

$$\lim_{n\to\infty} \mu_n(I_p, \omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{I_p}(S_j(\omega)) = p,$$

i.e. the limiting proportion of time that the random walk $\{S_n(\omega)\}_{n=1}^{\infty}$ spends in I_p is p. We summarize this in

Corollary 1. *Let $\mathscr{I}$ be the class of all p-sets in Z^m with $0 \leqq p \leqq 1$. Let $\{S_n(\omega)\}_{n=1}^{\infty}$ be a random walk in Z^m satisfying the hypotheses of the theorem. Then there exists $\Omega_0 \in \Omega$ with $P\{\Omega_0\} = 1$ such that for $\omega \in \Omega_0$ and $I_p \in \mathscr{I}$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{I_p}(S_j(\omega)) = p.$$

For a second application let H be an arbitrary Hilbert space and let $\{U_J, J \in Z^m\}$ be a group of unitary operators indexed by Z^m, i.e. $U_{J_1} U_{J_2} = U_{J_1+J_2}$. Let P be the orthogonal projection on the subspace of H defined by $\{f \in H \mid U_J f = f,\ J \in Z^m\}$. Let $\{\mu_n(\cdot, \omega)\}_{n=1}^{\infty}$ be the random ergodic measures defined above. Then $\int_{Z^m} U_J \, d\mu_n(J, \omega) = \frac{1}{n} \sum_{j=1}^{n} U_{S_j(\omega)}$. By applying the results of [2] and the theorem we obtain

Corollary 2. *Let $\{S_n(\omega)\}_{n=1}^{\infty}$ be a random walk in Z^m satisfying the hypotheses of the theorem, and let $\{U_J, J \in U^m\}$ be a group of unitary operators on a Hilbert space H, indexed by U^m. Then there exists $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$, such that for $\omega \in \Omega_0$ we have strong*

$$\operatorname*{limit}_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} U_{S_j(\omega)} = P,$$

i.e. *for every* $f \in H$,

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=1}^{n} U_{S_j(\omega)} f - Pf \right\| = 0.$$

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