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Ergodic Theorems for Subadditive Spatial Processes

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Summary. Let $\{X_G, G$ bounded Borel subset of $\mathbb{R}^\nu\}$ be a subadditive spatial process with finite constant γ. It will be proved that as $G \to \infty$ (in some sense), the average $(1/|G|)$. X_G converges in L^1, and if in addition the process is strongly subadditive, it converges almost surely towards an invariant random variable with expectation γ.

0. Introduction

In [7] general multidimensional spatial processes were investigated; under reasonable conditions mean and individual ergodic theorems were obtained, many applications were given. This paper is concerned with multidimensional subadditive spatial processes of the type recently considered by other authors.

Our work provides a full generalization of the results of Kingman. The main results are Theorem (3.10) and (4.6). For almost sure convergence we assume strong subadditivity. The L^1-convergence theorem is proved in an analoguous way to that of Kingman; with the strong subadditivity assumption it can also be proved in another way. The almost sure convergence is proved by a different method; this turns out to be a special case of a more general theorem in our joint paper [7] (Theorem (4.10)). The decomposition theorem is obtained, in contrast to Kingman, as a by-product of the almost sure convergence theorem.

A structure which is specific to spatial processes is the covariation. This means that, if we translate the realisation (=configuration) ω and the domain G at the same time, then the process does not change its value: $X_{G+u}(T_u\omega) = X_G(\omega)$. This property, which until now has not been taken into account, will be exploited to overcome certain difficulties (see, for example [10]). Note that the ordinary one-dimensional stochastic processes have in the canonic representation the same property: $x_{s+1,t+1}(\omega) = x_{s,t}(T\omega)$, where T is the shift translation.

The underlying space is $\mathbb{R}^\nu$, but the results remain valid for the case of the integer lattice $\mathbb{Z}^\nu$. In this case the subsets G have the usual form, being parallelepipeds, and $|G|$ denotes the cardinality of G.

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At the end of the paper we discuss some subadditive but not strongly subadditive processes, namely the cluster processes in discret and continuous cases, and prove the ergodic theorems.

1. Preliminaries

To avoid unnecessary repetition, we use the same definitions, notations and assume knowledge of the fundamental facts of Sects. 2 and 3 in [7].

A spatial process is a family $X=\{X_G, G\in\mathscr{C}\}$ of integrable random variables. X is said to be *covariant* if

(1.1) (Covariation) $X_{G+u}(T_u)=X_G(.)$ a.s.

It is called *subadditive* is, for any disjoint sets $A, G\in\mathscr{C}$

(1.2) $X_{A\cup G}\leqq X_A+X_G$ a.s.,

and *strongly subadditive* if it is subadditive and if

(1.3) $X_{A\cup G}+X_{A\cap G}\leqq X_A+X_G$ a.s.

for any $A, G\in\mathscr{C}$ with $A\cap G\neq\emptyset$.

We define the random variables $X_{G,A}$ as follows

(1.4) $X_{G,A}=X_{A\cup G}-X_A$

when $A\cap G=\emptyset$ and $A, G\in\mathscr{C}$ (with the convention $X_{G,\emptyset}=X_G$).

Then one can easily check that the strong subadditivity is equivalent with the property that:

(1.5) for any $G\in\mathscr{C}$, the family $\{X_{G,A}, A\in\mathscr{C}, A\cap G=\emptyset\}$ is decreasing: $X_{G,A'}\leqq X_{G,A}\leqq X_G$ a.s. when $A'\supset A$, $A\cap G=\emptyset$.

(1.5) yields a practical criterion to find out quickly whether the process is strongly subadditive or not.

We can define the strong superadditivity analogously by chainging, in (1.2) and (1.3), "$\leqq$" to "$\geqq$", and in (1.5) "decreasing" to "increasing".

2. Subadditive Set Functions

A set function is a real valued function D defined on $\mathscr{C}$. We write $D=\{D_G, G\in\mathscr{C}\}$. We can analogously define subadditivity or strong subadditivity for set functions as above.

If we set for any spatial process X

$$D_G=\mathbb{E}X_G \quad (G\in\mathscr{C})$$

then D has the same property of subadditivity or strong subadditivity as X. If X

is covariant, D is translation invariant, i.e.

(2.1) $D_{G+u}=D_G$ for any $u\in S_1$ and $G\in\mathscr{C}$.

(2.2) **Lemma.** *If D is a translation invariant subadditive set function, then*

(2.3) $$\lim_{\substack{G\in\mathscr{Q}\\ d(G)\to\infty}}\frac{D_G}{|G|}=\lim_n\frac{D_{G_n}}{|G_n|}=\inf_n\frac{D_{G_n}}{|G_n|}$$

for any sequence $\{G_n\}$ in $\mathscr{Q}$ with $d(G_n)\to\infty$. The limit may be $-\infty$.

Proof. It is easy. See for example Ruelle [9], Proposition (7.2.4). It is enough to require that D is defined on $\mathscr{Q}$. □

We define $D_{G,\Lambda}$ for $\Lambda, G\in\mathscr{C}$, $\Lambda\cap G=\emptyset$, by

(2.4) $D_{G,\Lambda}=D_{\Lambda\cup G}-D_\Lambda$,

with the convention $D_{G,\emptyset}=D_G$. The following lemma is helpful and will be used later in the proof of the almost sure convergence theorem. We first remark that because of the strong subadditivity the limit

(2.5) $$l=\lim_{\substack{\Lambda\in F_+\cap\mathscr{C}\\ \Lambda\to F_+}} D_{F_0,\Lambda}$$

always exists (but may be $-\infty$).

(2.6) **Lemma.** *Let $\{D_G, G\in\mathscr{C}\}$ be a translation invariant, strongly subadditive set function, with spatial constant γ, defined by*

(2.7) $$\gamma=\inf\frac{1}{|F_n|}\cdot D_{F_n}.$$

Then γ is finite if and only if l is finite, and we have

(2.8) $l=\gamma\cdot|F_0|$.

Proof. We suppose that l is finite; the case when l is infinite can be proved in an analoguous manner.

Then, given $\varepsilon>0$, there exists an integer $m=m(\varepsilon)$ such that

(2.9) $D_{F_0,\Lambda}\leqq l+\varepsilon$ for all $\Lambda\supset F_m\cap F_+$, $\Lambda\in F_+\cap\mathscr{C}$.

Let $F\in\mathscr{Q}$ and define F^1 to be the subset of $F\cap S_1$ consisting of those points x whose distance to the boundary of F is at leat equal to $m+1$, and $F^2=F\cap S_1\setminus F^1$. Then, if $x\in F^1$, we have $F\cap(F_+ +u)\supset F\cap(F_m\cap F_+ +u)$; hence by (2.9)

(2.10) $$D_{(F-u)\cap F_0,\,(F-u)\cap F_+}=D_{F\cap(F_0+u),\,F\cap(F_m\cap F_+ +u)}\leqq l+\varepsilon;$$

otherwise it holds from the strong subadditivity property

(2.11) $D_{F_0, \Lambda} \leqq D_{F_0}$ for all $\Lambda \in F_+ \cap \mathscr{C}$.

The decomposition, for $F \in \mathscr{Q}$,

(2.12) $D_F = \sum_{u \in F \cap S_1} D_{(F-u) \cap F_0, (F-u) \cap F_+}$,

which is similar to (2.8) in [1], leads, by means of (2.10) and (2.11) to the inequality

$$\frac{1}{|F|} \cdot D_F \leqq \frac{1}{|F|} \cdot (\# F^1) \cdot (l+\varepsilon) + \frac{1}{|F|} \cdot (\# F^2) \cdot D_{F_0}.$$

By Lemma (3.1) in [7], the first term on the *rhs* tends to $\frac{1}{|F_0|} \cdot (l+\varepsilon)$ and the second to zero when $d(F) \to \infty$. Hence

$$\gamma = \lim_{d(F) \to \infty} \frac{1}{|F|} \cdot D_F \leqq \frac{1}{|F_0|} \cdot (l+\varepsilon),$$

which implies $\gamma \leqq l/|F_0|$.

The direction "$\geqq$" is simple: the decomposition (2.12) and the definition of l imply

$$\frac{1}{|F|} \cdot D_F \geqq l \cdot \frac{\#(F \cap S_1)}{|F|},$$

hence, by Lemma (3.1) in [7], $\gamma \geqq l/|F_0|$. The lemma is proved. □

Note that if the basis in $\mathbb{R}^\nu$ is orthonormal or if $S = \mathbb{Z}^\nu$, we have $|F_0| = 1$ and the limit in (2.5) is just equal to γ. Recall that the limit in (2.5), supposed to be finite, with (2.8) means that

(2.13) given $\varepsilon > 0$ there exists an integer $m = m(\varepsilon)$ such that

$|D_{F_0, \Lambda} - \gamma \cdot |F_0|| \leqq \varepsilon$ for any $\Lambda \supset F_m \cap F_+$, $\Lambda \in F_+ \cap \mathscr{C}$.

With the help of Lemma (2.6) we now prove a limit theorem for strongly subadditive set functions.

(2.14) **Proposition.** *Let D be a translation invariant, strongly subadditive set function with finite constant γ. Suppose that*

(2.15) $\sup_{G \in F_0 \cap \mathscr{C}} |D_G| = K < \infty$.

Then

(2.16) $\lim \frac{D_G}{|G|} = \gamma$

as $d(G) \to \infty$, $G \in \mathscr{K}$ (i.e. *convex*).

Proof. Firstly, because of the strong subadditivity we have for $G \in F_0 \cap \mathscr{C}$ and $\Lambda \in F_+ \cap \mathscr{C}$

$$D_{G,\Lambda} \leqq D_G$$

and

$$D_{G,\Lambda} = D_{\Lambda \cup G} - D_\Lambda = D_{\Lambda \cup F_0} - D_\Lambda - (D_{\Lambda \cup F_0} - D_{\Lambda \cup G}) \geqq D_{F_0,\Lambda} - D_{F_0 \setminus G}$$

hence by (2.8) and (2.15)

$$(2.17) \quad |D_{G,\Lambda}| \leqq |\gamma| \cdot |F_0| + K = K' < +\infty.$$

Secondly, by virtue of Lemma (3.1) in [7],

$$(2.18) \quad \gamma \cdot |F_0| \cdot \frac{\text{number}\{u: u \in S_1 \cap G\}}{|G|} \to \gamma$$

when $d(G) \to \infty$, $G \in \mathscr{K}$.

Now let $\varepsilon > 0$ and m be as in (2.13) and let h be a number greater than the diameter of F_m. Using the decomposition

$$(2.19) \quad D_G = \sum_{u \in S_1 \cap (G \setminus G^h)} D_{(G-u) \cap F_0,\, (G-u) \cap F_+} + \sum_{\substack{u \in S_1 \cap G^h \\ (F_0 + u) \cap G \neq \emptyset}} D_{(G-u) \cap F_0,\, (G-u) \cap F_+}$$

we obtain for $G \in \mathscr{K}$ the estimate

$$(2.20) \quad \left| \frac{1}{|G|} \cdot D_G - \gamma \cdot |F_0| \cdot \frac{1}{|G|} \cdot \text{number}\{u: u \in S_1 \cap G\} \right| \leqq \varepsilon \cdot \frac{1}{|G|} \cdot \text{number}\{u: u \in S_1 \cap (G \setminus G^h)\} + (K' + \varepsilon) \cdot \frac{1}{|G|} \cdot \text{number}\{u: u \in S \cap G^h\},$$

due to (2.17) and (2.13).

As $d(G) \to \infty$, the first term on the *rhs* tends to $\varepsilon \cdot \frac{1}{|F_0|}$ while the second tends to zero. Thus, by (2.18), the therem is proved. □

(2.21) *Remarks.* The statement (2.16) is in fact true when the convergence "$d(G) \to \infty$" and the convexity of G are replaced by van Hove's convergence. The proof is the same. What we need is the negligibility of the boundary effect, which is already assured by van Hove's convergence.

Proposition (2.14) generalizes slightly a result of Robinson and Ruelle ([8], Proposition 2) in connection with mean entropy in statistical mechanics. In [8] D was assumed to be nonpositive.

3. L^1-Convergence

Let us now consider a subadditive and covariant spatial process $X=\{X_G, G\in\mathcal{Q}\}$. We limit ourselves in this section on $\mathcal{Q}$, instead of $\mathcal{C}$. We call the number

(3.1) $$\gamma=\inf_n \frac{\mathbb{E}X_{F_n}}{|F_n|}$$

the (spatial) constant of the process. By Lemma (2.2) we have

(3.2) $$\gamma=\lim_{\substack{d(G)\to\infty\\ G\in\mathcal{Q}}} \frac{\mathbb{E}X_G}{|G|}=\lim \frac{\mathbb{E}X_{F_n}}{|F_n|}.$$

For special processes we obtain the following.

(3.3) **Proposition.** *Let X be a subadditive spatial prozess with finite constant γ. Suppose*

(3.4) $X_{G+u}=X_G$ *a.s. for any* $G\in\mathcal{Q}$.

Then there exists an invariant random variable ξ, $\xi\in L^1(P)$, such that

(3.5) $$\xi=\lim_{\substack{d(G)\to\infty\\ G\in\mathcal{Q}}} \frac{1}{|G|}\cdot X_G=\lim \frac{1}{|G_n|}\cdot X_{G_n} \quad \text{in } L^1$$

for any sequence $\{G_n, n\in\mathbb{N}\}$ of subsets in $\mathcal{Q}$ with $d(G_n)\to\infty$. ξ has the form

(3.6) $$\xi=\lim_{\substack{d(G)\to\infty\\ G\in\mathcal{Q}}} \frac{1}{|G|}\cdot X_G=\inf_{\mathcal{Q}} \frac{1}{|G|}\cdot X_G \quad \text{a.s.}$$
$$=\lim \frac{1}{|G_n|}\cdot X_{G_n}=\inf\cdot\frac{1}{|G_n|}\cdot X_{G_n} \quad \text{a.s.}$$

for any sequence $\{G_n, n\in\mathbb{N}\}$, $G_n\in\mathcal{Q}$, and $d(G_n)\to\infty$.

Proof. Note that (3.4) is stronger than the covariation, and that if we have

(3.7) $$\xi=\lim \frac{1}{|G_n|}\cdot X_{G_n} \quad \text{in } L^1$$

for any sequence $\{G_n\}_n$, $G_n\in\mathcal{Q}$, $d(G_n)\to\infty$, and ξ independent (up to null set) on $\{G_n\}_n$, then

(3.8) $$\xi=\lim_{\substack{d(G)\to\infty\\ G\in\mathcal{Q}}} \frac{1}{|G|}\cdot X_G \quad \text{in } L^1$$

in the sense (2.14) in [7], that is, given $\varepsilon>0$, there exists a positive number $d=d(\varepsilon)$, such that

$$\left\| \xi - \frac{1}{|G|} \cdot X_G \right\|_1 \leqq \varepsilon$$

holds for all $G \in \mathcal{Q}$ with $d(G) \geqq d$.

Now, since $\mathcal{Q}$ is countable, it follows from Lemma (2.2) that a random variable ξ is well defined by (3.6).

As will be proved in the theorem below (in the first step of the proof), for any regular sequence $\{G_n, n \in \mathbb{N}\}$, $G_n \in \mathcal{Q}$, $d(G_n) \to \infty$, the limit

$$(3.9) \quad \lim \frac{1}{|G_n|} \cdot X_{G_n} = s$$

exists in L^1 and

$$s = \limsup \frac{1}{|G_n|} \cdot X_{G_n} \quad \text{a.s.,}$$

hence, by (3.6).

$s = \xi$ almost surely.

To complete the proof it remains to prove (3.7) for any sequence $\{G_n, n \in \mathbb{N}\}$, $G_n \in \mathcal{Q}$, and $d(G_n) \to \infty$. This will be done in the last step of the proof below. We have only to use the fact that the limit in (3.9) does not depend on the choice of the sequence $\{G_n\}_n$. □

We now state the mean ergodic theorem.

(3.10) **Theorem.** *Let $X = \{X_G, G \in \mathcal{Q}\}$ be a covariant subadditive spatial process with finite constant γ.*

Then there exists an invariant random variable ξ such that

$$(3.11) \quad \xi = \lim_{\substack{d(G) \to \infty \\ G \in \mathcal{Q}}} \frac{1}{|G|} \cdot X_G \quad \text{in } L^1$$

$$\mathbb{E}\xi = \gamma,$$

and moreover,

$$(3.12) \quad \xi = \limsup \frac{1}{|G_n|} X_{G_n} \quad \text{almost surely for any regular sequence } \{G_n\}_n \text{ in } \mathcal{Q}.$$

Proof. I. Without loss of generality we can assume that X non-positive, since if not we consider the auxiliary process X' defined by

$$X'_G = X_G - \sum_{u \in S_1 \cap G} X_{F_0}(T_{-u}),$$

which is non positive and subadditive, with finite constant, the asymptotic behavior of the second term on the *rhs* is well-known by Theorem (3.7) in [7].

For any fixed $k \in \mathbb{N}$, we define the lattice.

$S_k = \{(x^1, \ldots, x^\nu) \in \mathbb{R}^\nu : x^i = n^i \cdot k, n^i \in \mathbb{Z}, i = 1, \ldots, \nu\}$.

Then for any $k \in \mathbb{N}$ the limit

$$(3.13) \quad Y_k = \lim \frac{1}{|G_n|} \sum_{\substack{u \in S_k \\ (F_k + u) \subset G_n}} X_{F_k}(T_{-u})$$

exists almost surely for any regular sequence $\{G_n\}_n$ in $\mathcal{Q}$, and

$$(3.14) \quad \mathbb{E}\, Y_k = \frac{1}{|F_k|} \mathbb{E} X_{F_k}.$$

(One needs to replace in Theorem (3.7), [7], S_1 by S_k and F_0 by F_k). Let $\{G_n\}_n$ be now a regular sequence in $\mathcal{Q}$ and define

$$(3.15) \quad s = \limsup \frac{1}{|G_n|} \cdot X_{G_n}.$$

We prove that $\left\| s - \frac{1}{|G_n|} \cdot X_{G_n} \right\|_1 \to 0$.

Using the subadditivity, covariation and non-positivity of X, we have for any fixed $k \in \mathbb{N}$

$$(3.16) \quad \frac{1}{|G_n|} \cdot X_{G_n} \leqq \frac{1}{|G_n|} \cdot \sum_{\substack{u \in S_k \\ (F_k + u) \subset G_n}} X_{F_k}(T_{-u}).$$

It follows from (3.13) and (3.14) that

$$\mathbb{E} s \leqq \frac{1}{|F_k|} \cdot \mathbb{E} X_{F_k}.$$

Hence

$$(3.17) \quad \mathbb{E} s \leqq \gamma.$$

On the other hand, applying Fatou's lemma for the sequence $\left\{ -\frac{1}{|G_n|} \cdot X_{G_n} \right\}$ we obtain with the help of (2.3)

$$\mathbb{E} s \geqq \mathbb{E} \left(\limsup \frac{X_{G_n}}{|G_n|} \right) \geqq \limsup \frac{\mathbb{E} X_{G_n}}{|G_n|} = \gamma.$$

Thus, together with (3.17),

$$(3.18) \quad \mathbb{E} s = \gamma.$$

The random variable s is integrable. This allows us to apply Lebesgue's dominated convergence theorem to the sequence $\{-Z_n\}_n$, with

$$Z_n = \sup_{m \geqq n} \left\{ \frac{1}{|G_m|} X_{G_m} \right\}$$

and this leads to

$$\text{(3.19)} \quad \begin{aligned} &\|s - Z_n\|_1 \to 0 \\ &\mathbb{E} Z_n \to \gamma \end{aligned}$$

Now, since $Z_n \geqq \frac{1}{|G_n|} \cdot X_{G_n}$ and

$$\left\| Z_n - \frac{1}{|G_n|} \cdot X_{G_n} \right\|_1 = \mathbb{E} Z_n - \frac{1}{|G_n|} \mathbb{E} X_{G_n} \to \gamma - \gamma = 0,$$

it follows from (3.19) that

$$\text{(3.20)} \quad \left\| s - \frac{1}{|G_n|} \cdot X_{G_n} \right\|_1 \to 0.$$

It is trivial that s is invariant.

II. We prove that s as defined in (3.15) is independent (up to a null set) of the sequence $\{G_n\}_n$. This is simple. Proposition (3.3), applied to the process $\{X'_G, G \in \mathcal{Q}\}$ with

$$X'_G = \mathbb{E}\left(\frac{1}{|G|} X_G \,|\, \mathcal{J} \right),$$

yields an invariant random variable ξ, $\xi \in L^1(P)$, and, for any regular sequence $\{G_n\}_n$,

$$\text{(3.21)} \quad \xi = \lim \frac{1}{|G_n|} X'_{G_n} \quad \text{in } L^1,$$

by (3.5).

Thus, in view of (3.20) and (3.21), we obtain.

$$s = \xi \quad \text{a.s.}$$

III. Finally we prove the convergence in (3.11). This convergence does not depend on the position of the sets G in $\mathcal{Q}$, but only on the inner radius $d(G)$. This is plausible, since X_G and X_{G+u} have the same distribution and for the mean convergence this fact is decisive. Suppose that the convergence in (3.11) does not hold. Then we can find a sequence $\{G_n\}_n$ in $\mathcal{Q}$ with $d(G_n) \to \infty$ and

$$\text{(3.22)} \quad \left\| \xi - \frac{1}{|G_n|} \cdot X_{G_n} \right\|_1 \geqq \delta > 0$$

for some δ. We may clearly assume that the sets G_n are choosen in such a way that they can, in a suitable manner, be translated into sets G'_n with

$$G'_1 \subset G'_2 \subset \dots.$$

Then the sequence $\{G'_n\}$ is regular, however we have

$$\left\| \xi - \frac{1}{|G'_n|} X_{G'_n} \right\|_1 = \left\| \xi - \frac{1}{|G_n|} X_{G_n} \right\|_1 \geqq \delta.$$

This is imposible, by steps I and II.

The theorem is completely proved. □

(3.23) *Remark.* It results from step II and Proposition (3.3) that the limit ξ in theorem above has the form

$$(3.24) \quad \xi = \inf_{G \in \mathcal{Q}} \mathbb{E}\left(\frac{1}{|G|} \cdot X_G | \mathcal{I}\right) = \lim_{\substack{d(G) \to \infty \\ G \in \mathcal{Q}}} \mathbb{E}\left(\frac{1}{|G|} \cdot X_G | \mathcal{I}\right)$$

almost surely.

In the next part we will give another form of ξ if X is strongly subadditive. In that case, the L^1-convergence theorem can be proved in another way.

4. The Almost Sure Convergence Theorem

We now go on to the individual ergodic theorem for strongly subadditive spatial proceses. Before stating and proving it, we recall a fundamental result in [7], Theorem (4.10).

(4.1) **Theorem.** I. *Let $\{X_G, G \in \mathscr{C}\}$ be a covariant spatial process. Suppose there exist random variables Y, $Z \in L^1(P)$ with $Y \geqq 0$ such that*

$$(4.2) \quad |X_{G,\Lambda}| \leqq Y \quad \text{a.s.} \quad \text{for any } \Lambda \in F_+ \cap \mathscr{C} \text{ and } G \in F_0 \cap \mathscr{K};$$

$$(4.3) \quad \lim_{\substack{\Lambda \to F_+ \\ \Lambda \in F_+ \cap \mathscr{C}}} X_{F_0, \Lambda} = Z$$

almost surely (in the sense of (2.12), [7]).

Then

$$(4.4) \quad \lim \frac{1}{|G_n|}, X_{Gn} = \frac{1}{|F_0|} \mathbb{E}(Z | \mathcal{I}) \quad \text{a.s.}$$

for any regular sequence $\{G_n\}_n$ of subsets in $\mathscr{K}$.

II. *If* (4.2) *is replaced by the weaker form*

$$(4.2') \quad |X_{F_0,\Lambda}| \leqq Y \quad \text{a.s.} \quad \text{for any } \Lambda \in F_+ \cap \mathscr{C},$$

then (4.4) *holds for any regular sequence $\{G_n\}_n$ in $\mathcal{Q}$.*

The random variable $X_{G,\Lambda}$ is defined in (1.4).

A consequence of (4.4) is the decomposition of the process. Let X be a spatial process satisfying (4.2′) and (4.3). If we write

$$(4.5)\quad X_G = \sum_{u \in S_1 \cap G} Z(T_{-u}) + W_G \qquad (G \in \mathscr{Q})$$

then, by (4.4), $\frac{W_G}{|G|}$ will tend to zero almost surely when G becomes large. Hence the essential part of X is an additive process.

We will use Theorem (4.1) to prove the almost sure convergence theorem for strongly subadditive processes.

To avoid inconvenience we assume that, concerning the covariation and strong subadditivity, (1.1), (1.2) and (1.3) are true, independently of $u \in S_1$, Λ and G in $\mathscr{C}$, pointwise on a set of measure 1 and therefore without loss of generality on the whole propability space, although the result is true without this assumption. We now formulate the result.

(4.6) **Theorem.** *Let X be a covariant, strongly subadditive spatial process with finite constant γ.*

I. *Then there exists an invariant random variable ξ such that*

$$\mathbb{E}\xi = \gamma$$

$$(4.7)\quad \lim \frac{1}{|G_n|} \cdot X_{G_n} = \xi \qquad \textit{almost surely}$$

for any regular sequence $\{G_n\}_n$ in $\mathscr{Q}$.

II. *If moreover the process is non-positive, or if, more generally, there exists a random variable Y_0, $Y_0 \in L^1$ and $Y_0 \geqq 0$ a.s. such that*

$$(4.8)\quad |X_G| \leqq Y_0 \quad \text{a.s.} \quad \textit{for any} \;\; G \in F_0 \cap \mathscr{C},$$

then the convergence in (4.7) is valid for any regular sequence $\{G_n\}$ in $\mathscr{K}$.

(4.9) *Remark.* Condition (4.8) just says that the essential supremum of $\{X_G, G \in F_0 \cap \mathscr{C}\}$ is integrable.

Proof of Theorem. The strong subadditivity implies the following decreasing property

$$(4.10)\quad X_{F_0,\Lambda} \leqq X_{F_0,G} \leqq X_{F_0} \qquad \text{for } \Lambda \supset G,\ \Lambda, G \in F_+ \cap \mathscr{C}.$$

Therefore the limit

$$(4.11)\quad Z = \lim_{\substack{\Lambda \to F_+ \\ \Lambda \in F_+ \cap \mathscr{C}}} X_{F_0,\Lambda} = \lim_{m \to \infty} X_{F_m \cap F_0,\, F_m \cap F_+}$$

exists almost surely and defines a random variable. It holds for any $\Lambda \in F_+ \cap \mathscr{C}$ that

$$(4.12)\quad |X_{F_0,\Lambda}| \leqq \max\{|X_{F_0}|, |Z|\} \quad \text{a.s.}$$

Thus conditions (4.2′) and (4.3) of Theorem (4.1) will be satisfied if $Z \in L^1(P)$, and in view of the monotone convergence theorem, this is the same as having

(4.13) $$\inf_{\Lambda \in F_+ \cap \mathscr{C}} \mathbb{E} X_{F_0, \Lambda} < \infty.$$

But Lemma (2.6) applied to the set function

$$D_G = \mathbb{E} X_G \quad (G \in \mathscr{C})$$

just yields

(4.14) $$\lim_{\substack{\Lambda \to F_+ \\ \Lambda \in F_+ \cap \mathscr{C}}} \mathbb{E} X_{F_0, \Lambda} = \inf_{\Lambda \in F_+ \cap \mathscr{C}} \mathbb{E} X_{F_0, \Lambda} = \gamma \cdot |F_0|,$$

hence, by the monotone convergence theorem,

(4.15) $$\mathbb{E} Z = \gamma \cdot |F_0|$$

and

(4.16) $$\|X_{F_0, \Lambda} - Z\|_1 \to 0 \quad \text{as } \Lambda \to F_+$$

(in the sense of (2.13) in [7]).

Thus, statement I of the theorem follows from Theorem (4.1), part II. We have

(4.17) $$\xi = \frac{1}{|F_0|} \cdot \mathbb{E}(Z \mid \mathscr{J}) \quad \text{a.s.}$$

$$\mathbb{E}\xi = \frac{1}{|F_0|} \cdot \gamma \cdot |F_0| = \gamma.$$

To prove part II we have to verify condition (4.2). The strong subadditivity implies for $\Lambda \in F_+ \cap \mathscr{C}$ and $G \in F_0 \cap \mathscr{C}$

$$X_{G, \Lambda} \leqq X_G,$$
$$X_{G, \Lambda} = X_{F_0, \Lambda} - (X_{\Lambda \cup F_0} - X_{\Lambda \cup G})$$
$$= Z - X_{F_0 \setminus G},$$

hence

(4.18) $$|X_{G, \Lambda}| \leqq |Z| + Y_0 \quad \text{a.s.}$$

Thus condition (4.2) is fulfilled, and the theorem is proved. □

(4.19) *Decomposition.* As mentioned above, any spatial process, under assumptions of Theorem (4.1), can be decomposed into an additive process plus a process $W = \{W_G, G \in \mathscr{Q}\}$ whose asymptotic behavior is negligible. What more do we know about W if X is strongly subadditive? According to (4.5) W is strongly subadditive. Moreover all terms in the sum on the *rhs* of

$$X_G = \sum_{u \in S_1 \cap G} X_{(G-u) \cap F_0, (G-u) \cap F_+}(T_{-u}), \qquad G \in \mathscr{Q},$$

have the form $X_{F_0, \Lambda}$ or X_{F_0}, hence by (4.10)

$$X_G \geqq \sum_{u \in S_1 \cap G} X_{F_0}(T_{-u}),$$

so that $W_G \geqq 0$.

Those are exactly the properties which Kingman (1968) obtained in the one-dimensional case. Here the decomposition theorem is a by-product of the almost sure convergence theorem, while Kingman used it to prove the latter.

We see, thanks the covariation property, that the decomposition of X has a nice form and the individual ergodic theorem for X is practically reduced to Birkhoff's ergodic theorem. The random variable Z as defined in (4.11) plays the same role as f_0 in Kingman (1968), but here, due to the strong subadditivity, it is simply obtained.

(4.20) *Processes of Finite Range.* Each random variable $X_{G, \Lambda}$, for $\Lambda \cap G = \emptyset$, can in general be interpreted as the value of the process in G, given the external condition in Λ. A wide class of processes has the property that, given any area $G \in \mathscr{C}$, there exists a "boundary" domain, say ∂G, bounded, disjoint from G, such that

(4.21) $$X_{G, \Lambda} = X_{G, \partial G} \quad \forall \Lambda \in \mathscr{C} \quad \text{with} \quad \Lambda \supset \partial G, \ \Lambda \cap G = \emptyset.$$

This means that the influence on G remains unchanged if the external domain increases over the boundary ∂G. Such processes are called of *finite range.* In these cases Z has the following form

(4.22) $$Z = X_{\partial F_0 \cap F_0, \, \partial F_0 \cap F_+}.$$

Hammersley's example (1974, Sect. 10) about the specific volume covered by spheres or figure processes in Minlos (1967) fall under this category.

If X is strongly subadditive and of finite range, then the spatial constant is necessarily finite, by Lemma (2.6). In fact, it follows from (4.12) and (4.22) that

$$\inf_{\Lambda \in F_+ \cap \mathscr{C}} \mathbb{E} X_{F_0, \Lambda} > -\infty.$$

We can also apply Theorem (4.1) directly. We resume this in

(4.23) **Corollary.** *Theorem* (4.6) *is true for strongly subadditive covariant spatial processes of finite range.*

(4.24) *Remarks.* In the situation of Theorem (4.6), say part II, the L^1-convergence theorem can be proved as follows. It results from (4.18) that

(4.25) $$\sup \{\mathbb{E} |X_{G, \Lambda}| : \Lambda \in F_+ \cap \mathscr{C}, F \in F_0 \cap \mathscr{C}\} < +\infty.$$

But this and (4.16) are exactly the conditions of Theorem (4.1) in [7], thus

(4.26) $$\frac{1}{|G|} \cdot X_G \xrightarrow{1} \xi, \qquad \xi = \frac{1}{|F_0|} \cdot \mathbb{E}(Z \mid \partial),$$

as $d(G) \to \infty$, $G \in \mathscr{K}$.

If in (4.26) the set G is limited to be a parallelepiped in $\mathcal{Q}$, we don't need (4.8), since from (4.12) we have

$$\sup\{\mathbb{E}|X_{F_0, A}|: A \in F_+ \cap \mathscr{C}\} < \infty,$$

and this, together with (4.16), is sufficient for (4.26) (see Remark (4.28) in [7]). Theorem (4.1) in [7] is much earlier to prove than theorem (3.10).

(4.27) Results valid for parallelepipeds in $\mathcal{Q}$ are also valid for their translates by $(1/2, \ldots, 1/2)$; we need to translate F_+ and F_0 by the same vector and repeat all the proofs. They are also true for open or closed parallelepipeds. We have only to use Lemma (3.1) and Theorem (3.7) in [7] to remove the boundary effects.

Conclusion. Our approach to the multidimensional subadditive processes is different from that in the one-dimensional case of Kingman. It shows that subadditive processes in fact fall under a wider class of processes, for which the ergodic theorems, under weaker conditions, are valid.

However, Kingman's idea to prove the almost sure convergence theorem in fact works if one takes the property of covariation into account. One first proves the decomposition theorem, as done by Smythe,

(4.28) $X_G = Y_G + W_G \quad (G \in \mathcal{Q})$

where $\{Y_G, G \in \mathcal{Q}\}$ is an additive process with the same constant and $\{W_G, G \in \mathcal{Q}\}$ is a non-negative strongly subadditive process with constant zero. But by [7] (Corollary (4.20) or statement (3.9) of Theorem (3.7)),

(4.29) $\dfrac{1}{|G_n|} \cdot Y_{G_n} \to \xi \quad$ a.s.

for any regular sequence $\{G_n\}$ in $\mathcal{Q}$. Moreover (3.11) and (3.12) of Theorem (3.10) imply

(4.30) $\dfrac{1}{|G_n|} \cdot W_{G_n} \to 0 \quad$ a.s.

Hence

$$\frac{1}{|G_n|} \cdot X_{G_n} \to \xi \quad \text{a.s.}$$

Note that the condition of *regularity* of sequence $\{G_n\}_n$ in the almost sure convergence theorem is very important and can not be omitted. If this condition is not satisfied, the almost sure convergence theorem does not hold. Counterexamples can be found in Tempel'man (1972), Chap. 9. Our regularity condition is, for the sake of simplicity, stronger than the original by Tempel'man.

All the above considerations and results remain valid without change when S is the ν-dimensional integer lattice $\mathbb{Z}^\nu$, instead of $\mathbb{R}^\nu$. In this case $S = S_1$ and $|G|$ denotes the counting measure of G. The sets G_n in all limit passages are the usual parallelepipeds

(4.31) $G_n = \{x \in S: l_n^i \leqq x^i \leqq l_n^i + a_n^i, = 1, \ldots, \nu\},$

with l_n^i, $a_n^i \in \mathbb{Z}$ and $a_n^i \geqq 0$. All sequence $\{G_n\}$ are countable. A sequence $\{G_n\}$ is regular if $\min_i a_n^i \to \infty$ and $G_n \subset G_{n+1}$ for any n; or if $\min_i a_n^i \to \infty$ and there exists a positive constant c such that

(4.32) $|G_n| \geqq c \cdot [l(G_n)]^\nu$ for any n,

where $l(G_n) = \max_i a_n^i$.

5. Cluster Processes

The Lattice Case. Grimmett (1976) has considered independent processes on the two-dimensional integer lattice which are constructed by randomly and independently coloring each site of the lattice with a given probability $p (0 \leqq p \leqq 1)$. He proved the mean and almost sure convergence theorems for the cluster number per site. We now describe the situation in arbitrary multidimensional integer lattices and prove his theorem in a more general form.

Let $S = \mathbb{Z}^\nu$, $\Omega = \{0, 1\}^S$ be the configuration space (=phase space), equipped with the product σ-field $\mathscr{F}$, and let $\mathscr{C}$ be the collection of all finite subsets of S. A point $x \in S$ is occupied in ω iff $\omega_x = 1$. An $\omega \in \Omega$ is a configuration with occupied (black) and vacant (white) sites. We join two occupied nearest neighbors by a bond. In this way each ω induces a graph in S. A cluster of ω is a connected component of this graph.

Let $T_u (u \in S)$ be the usual translation operator in Ω, and P be an invariant probability measure (=state) on $\mathscr{F}$. We now define, for any finite subset G of S and $\omega \in \Omega$,

(5.1) $X_G(\omega)$ = number of clusters in G, induced by the restriction of ω to G.

Then $X = \{X_G, G \in \mathscr{C}\}$ is a subadditive covariant process, but not strongly subadditive[1]. Indeed we have for any disjoints sets $\Lambda, G \in \mathscr{C}$

(5.2) $X_{G,\Lambda}(\omega) = \#$ {clusters of $\omega_{\Lambda \cup G}$ in $\Lambda \cup G$ having occupied vertices in G} $- \#$ {cluster of ω_Λ in Λ which can be extended to some occupied sites in G}.

Both terms on the *rhs* are naturally equal to zero if each point of Λ has no neighboors in G, or if none of the sites in G is occupied.

The function $\Lambda \mapsto X_{G,\Lambda}(\omega)$, defined for $\Lambda \in \mathscr{C}$ and $\Lambda \cap G = \emptyset$, is not monotone, as one can easily check (see Fig. 1).

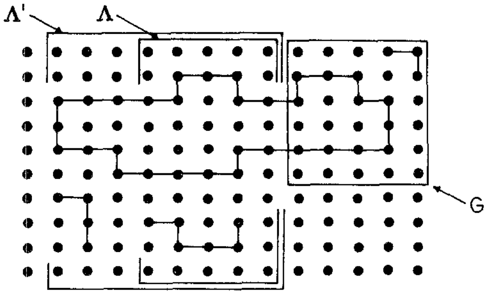

Fig. 1

[1] The author thanks G.R. Grimmett for his indication to this point

However the process X satisfies the hypotheses of Theorem (4.1). We have

(5.3) **Theorem.** *Let P be an invariant probability measure on $\mathscr{F}$. Then there exists an invariant random variable ξ, $\xi \in L^1$, such that, firstly*

$$(5.4) \quad L^1-\lim \frac{1}{|G_n|} \cdot X_{G_n} = \xi, \qquad \mathbb{E}\xi = \inf \frac{1}{|F_n|} \cdot \mathbb{E} X_{F_n}$$

for any sequence of parallelepipeds with $\min_i a_n^i \to \infty$; and secondly

$$(5.5) \quad \lim \frac{1}{|G_n|} \cdot X_{G_n} = \xi$$

almost surely for any regular sequence $\{G_n\}$ of parallelepipeds.

Proof. The statements (5.4) follows from Theorem (3.10), since the process is subadditive and the spatial constant is finite. In the expression (5.2), the first term on the right hand side is at most equal to the number of occupied sites in G. Furthermore, any occupied site in G can be linked with the other ones in at most 2ν ways, therefore

$$(5.6) \quad |X_{G,\Lambda}(\omega)| \leqq n_G(\omega) + 2\nu\, n_G(\omega),$$

where n_G denotes the number of occupied sites in G. In particular, if $G = F_0$ (in this case, $= \{0\}$ and hence $|F_0| = 1$), (5.6) gives

$$(5.7) \quad |X_{F_0,\Lambda}(\omega)| \leqq 2\nu + 1, \qquad \text{for any } \Lambda \in F_+ \cap \mathscr{C}.$$

On the other hand, it is easy to check that the limit

$$(5.8) \quad \lim_{\Lambda \to F_+} X_{F_0,\Lambda} = Z$$

exists everywhere. In fact

(5.9) $Z(\omega) = \#\{$clusters of $\omega_{F_+ \cup F_0}$ in $F_+ \cup F_0$ which contain the point origin$\}$ $- \#\{$clusters of ω_{F_+} in F_+ which can be extended to the origin$\}$.

For $\nu = 2$, Z has the form

$$(5.10) \quad Z(\omega) = \begin{cases} 0 & \text{if } \omega_0 = 0 \\ 1 & \text{if } \omega_0 = 1 \text{ and } A(\omega) = 0 \\ 0 & \text{if } \omega_0 = 1 \text{ and } A(\omega) = 1 \\ -1 & \text{if } \omega_0 = 1 \text{ and } A(\omega) = 2, \end{cases}$$

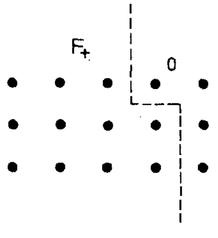

Fig. 2

where $A(\omega)$ denotes the number of clusters of ω_{F_+} in F_+ which can be linked to the origin.

With (5.7) and (5.8), conditions of Theorem (4.1), part II, are satisfied, hence statement (5.5) is proved and we have

$$\xi = \mathbb{E}(Z \mid \mathscr{I}) \quad \text{a.s.}$$
$$\gamma = \mathbb{E}Z = (\text{for } v=2)\, P\{\omega: \omega_0 = 1, A(\omega)=0\} - P\{\omega: \omega_0=1, A(\omega)=2\}.$$

The proof is completed. □

The probability measure P in the theorem is not necessarily independent. It can be, for example, a Gibbsian state with non-zero interaction. For an ergodic Gibbsian state the limit ξ is almost surely constant, and equal to γ. In particular processes arising from independent colorings are of these type; this is the case which Grimmett considered.

The Continuous Case.[2] Now let $S = \mathbb{R}^v$ and Ω be the space of all countable point sets $\{x_i, x_i \in S\}$ which have only finitely many points in bounded subsets of S; Ω is equipped with the usual vague σ-field. The automorphismes $\{T_u, u \in S_1\}$ are the translations. $n_G(\omega)$ denotes the number of points of ω in G. Now let δ be a fixed positive number. Then we join any two points of a configuration ω by a bond if their distance is not greater than δ. In this way ω induces a graph in $\mathbb{R}^v$. Any connected component of this graph is called a δ-cluster. We define for $G \in \mathscr{C}$

(5.11) $X_G(\omega) =$ number of clusters in G induced by the restriction ω_G,

and will investigate the asymptotic behaviour of the average $\frac{1}{|G|} \cdot X_G$ as G becomes large.

As in the lattice case, $X = \{X_G, G \in \mathscr{C}\}$ is subadditive but not strongly subadditive. For disjoint sets $\Lambda, G \in \mathscr{C}$ we have

(5.12) $X_{G,\Lambda}(\omega) = \#\{$clusters of $\omega_{\Lambda \cup G}$ in $\Lambda \cup G$ which have points in $G\}$ $- \#\{$clusters of ω_Λ in Λ which can be extended into $G\}$,

and

(5.13) $|X_{G,\Lambda}(\omega)| \leqq n_G(\omega) + k_\delta \cdot n_G(\omega)$,

where k_δ is the maximal number of points with pairwise distance not greater than δ which a sphere of radius δ in S can contain. In particular

(5.14) $|X_{G,\Lambda}(\omega)| \leqq (1+k_\delta)\, n_{F_0}(\omega) \equiv Y(\omega)$
for any $G \in F_0 \cap \mathscr{C}$ and $\Lambda \in F_+ \cap \mathscr{C}$.

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[2] The author thanks R. Lang for his suggestion about this example.

The limit

$$(5.15)\quad \lim_{\Lambda \nearrow F_+} X_{F_0, \Lambda}(\omega) = Z(\omega)$$

again exists everywhere and has a form analogous to (5.9).

If the probability measure P on Ω is supposed to be of first order, i.e. the first moment measure is a radon measure on S, then the random variable Y in (5.14), and therefore Z, is integrable, so that the conditions of Theorem (4.1), part I, are satisfied, as well as the conditions of the mean convergence theorem in [7] (see Remark (4.28)). Hence we obtain

(5.16) **Theorem.** *Let P be an invariant (w.r.t. T_u, $u \in S_1$) probability measure on Ω (=point process) of first order. Then there exists an invariant random variable ξ such that*

$$(5.17)\quad L^1 - \lim_{\substack{d(G) \to \infty \\ G \in \mathcal{K}}} \frac{1}{|G|} \cdot X_G = \xi, \qquad \mathbb{E}\xi = \inf \mathbb{E} X_{F_n}/|F_n|,$$

and

$$(5.18)\quad \lim \frac{1}{|G|} \cdot X_G = \xi$$

almost surely for any countable regular sequence of convex subsets $\{G\}$ of S.

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