Z. Wahrscheinlichkeitstheorie verw. Geb. 9, 112—130 (1968)

A Note on Time Sharing with Preferred Customers

ALAN G. KONHEIM*

Received April 27; in revised form May 12, 1967

Summary. The setting for this problem is a (single-server) service facility which is "time-shared" by m customers. *A processing schedule,* $\mathscr{S}$, is introduced to prescribe the times at which the facility is available to each customer. The processing schedule determines the random order with which customers exit from the facility. *The waiting time of the* j^{th} *customer,* $W^j_{\mathscr{S}}$, is defined as the difference between his exit time and service time; *the total waiting time,* $W_{\mathscr{S}}$, is then defined by

$$W_{\mathscr{S}} = \sum_{i=1}^{m} \beta^i W_{\mathscr{S}}$$

where the $\{\beta^i\}$ are positive real numbers. The weights $\{\beta^i\}$ reflect the cost per unit time of delay and indicate an a priori customer preferrence. In this paper we shall characterize the processing schedules $\mathscr{S}^*$ which realize

$$(*) \qquad \min_{\mathscr{S}} E(W_{\mathscr{S}})\,.$$

Our main result is that the schedules which minimize $E(W_{\mathscr{S}})$ in the m customer problem can be "put together" from the corresponding schedules in the 2 customer problems.

§ 1. Introduction

In this paper we continue the investigation of time-sharing (or queueing) disciplines initiated in [1]. The setting for this problem is a (single-server) service facility which is "time-shared" by m customers. A *processing schedule,* $\mathscr{S}$, is introduced to prescribe the times at which the facility is available to each customer. The processing schedule determines the (random) order with which the customers exit from the facility. The *waiting time of the j-th customer,* $W^j_{\mathscr{S}}$, is defined as the difference between his exit time and service time; the *total waiting time,* $W_{\mathscr{S}}$, is then defined by

$$W_{\mathscr{S}} = \sum_{i=1}^{m} \beta^i W^i_{\mathscr{S}},$$

where the $\{\beta^i\}$ are positive real numbers. The weights $\{\beta^i\}$ reflect the cost per unit time of delay. We shall characterize the schedules $\mathscr{S}^*$ which realize

$$(*) \qquad \min_{\mathscr{S}} E(W_{\mathscr{S}})\,.$$

Our main result is that the schedules which minimize $E(W_{\mathscr{S}})$ in the m customer problem can be "put together" from the corresponding schedules in the 2 customer problems. In § 2 we introduce the processing schedule and give a precise formula-

---

* This research was partially supported by the United States Air Force under Contract No. AF 49 (638)-1682.

tion of the problem (*). The case $m = 2$ is treated in § 3. We will show that (*) (for $m = 2$) is equivalent to finding a "shortest path" and will determine this path. In § 4 we show how the paths may be "put together" and prove in § 5 that this leads to the solution of (*).

§ 2. The Processing Schedule

We shall study a (single-server) service facility which is to serve m customers. The facility may serve only one customer at a time. The i-th customer requires service from the facility for a period of time of length T^i. We shall assume that the *(service times)* $\{T^i\}$ are independent random variables (on some probability space $(\Omega, \mathscr{B}, \Pr)$) and that T^i assumes (with probability one) values in the set $\{1, 2, \ldots, N^i\}$. We set

$$Q^i(k) = \Pr\{\omega\colon\ T^i(\omega) > k\}$$
$$p^i(k) = \Pr\{\omega\colon\ T^i(\omega) = k\}.$$

We define a processing schedule inductively;

Definition 2.1. $(m = 1)$

$\mathscr{S}$ *is a* $\begin{bmatrix} i \\ A \end{bmatrix}_1$ *processing schedule* will mean that

$$\mathscr{S} = \begin{cases} \emptyset & \text{if } A = 0 \\ (i, i, \ldots, i)\ (A \text{ times}) & \text{if } A > 0. \end{cases}$$

(We use the symbol $\emptyset$ to denote the *empty schedule.*)

$(m > 1).$

$\mathscr{S}$ *is a* $\begin{bmatrix} i_1 & i_2 \ldots & i_m \\ A_1 & A_2 \ldots & A_m \end{bmatrix}_m$ *-processing schedule* $(A = A_1 + A_2 + \cdots + A_m)$ will mean that

$$\mathscr{S} = \begin{cases} \emptyset & \text{if } A = 0 \\ ((\eta_1, \mathscr{S}_1), \ldots, (\eta_A, \mathscr{S}_A)) & \text{if } A > 0, \end{cases}$$

where

(1) $\eta_j \in \{i_1, i_2, \ldots, i_m\}$.

(2) If x_k^j denotes the number of times the symbol "i_j" appears in the list $\eta_1, \eta_2, \ldots, \eta_k$, then $x_A^j = A_j$ $(1 \leq j \leq m)$.

(3) If $\eta_j = i_\beta$, then $\mathscr{S}_\beta$ is a

$$\begin{bmatrix} i_1 & i_2 & \ldots\ i_{\beta-1} & i_{\beta+1} & \ldots\ i_m \\ A_1 - x_j^1 & A_2 - x_j^2 & \ldots A_{\beta-1} - x_j^{\beta-1} & A_{\beta+1} - x_j^{\beta+1} & \ldots A_m - x_j^m \end{bmatrix}_{m-1}$$

processing schedule.

Let $\mathscr{S} = ((\eta_1, \mathscr{S}_1), \ldots, (\eta_t, \mathscr{S}_t))$ $(t = N^1 + N^2 + \cdots + N^m)$ be a

$$\begin{bmatrix} 1 & 2 & \ldots & m \\ N^1 & N^2 & \ldots & N^m \end{bmatrix}_m$$

processing schedule. Define

$$\tau_{\mathscr{S}}^1(j, \cdot) = 1 + \sum_{k=1}^{t} \chi_{(0,t]}(T^j(\cdot) - x_k^j)\ ^1 \quad (1 \leq j \leq m) \tag{2.1}$$

$$\tau_{\mathscr{S}}^1(\cdot) = \min_{1 \leq j \leq m} \tau_{\mathscr{S}}^1(j, \cdot). \tag{2.2}$$

[1] χ_B denotes the characteristic function of the set B.

Note first that

$$\tau^1_{\mathscr{S}}(j, \omega) = k \text{ if and only if } x^j_{k-1} < T^j(\omega) = x^j_k .$$

The processing schedule $\mathscr{S}$ imposes the following time-sharing discipline. If $i \leqq \tau^1_{\mathscr{S}}$, the service facility serves the customer whose "name" is "η_i" in the interval $i - 1 \leqq t < i$. At time $\tau^1_{\mathscr{S}}$ the facility has just completed the service of *one* of the m customers [2]. Thereafter the facility serves the remaining $m - 1$ customers according to the processing schedule $\mathscr{S}_{\tau^1_{\mathscr{S}}}$. We call $\tau^1_{\mathscr{S}}$ the *first exit time according to* $\mathscr{S}$.

The processing schedule $\mathscr{S}$ thus determines a random order of exit $i_1, i_2, \ldots, i_m$ from the facility. Let $\tau^j_{\mathscr{S}}$ denote the exit time of "i_j"

$$0 = \tau^0_{\mathscr{S}} < \tau^1_{\mathscr{S}} < \cdots < \tau^m_{\mathscr{S}} = T^1 + T^2 + \cdots + T^m$$

We measure the effectiveness of the processing schedule in terms of the waiting times or delays it imposes on each customer. The *waiting time of customer "i_j"* is defined by

$$W^{i_j}_{\mathscr{S}} = \tau^j_{\mathscr{S}} - T^{i_j}$$

and the *total waiting time* by

$$W_{\mathscr{S}} = \sum_{i=1}^{m} \beta^i W^i_{\mathscr{S}} .$$

Our objective is to characterize the processing schedules which minimize

$$E(W_{\mathscr{S}}) = \int_{\Omega} W_{\mathscr{S}}(\omega) \Pr(d\omega) .$$

§ 3. The Case $m = 2$

We start with the formulae

$$W^1_{\mathscr{S}}(\omega) = \sum_{k=1}^{t} \xi^1_k \chi_{(0,t]}(T^1(\omega) - x^1_k) \chi_{[0,t]}(T^2(\omega) - x^2_k) \tag{3.1}$$

$$W^2_{\mathscr{S}}(\omega) = \sum_{k=1}^{t} \xi^2_k \chi_{(0,t]}(T^2(\omega) - x^2_k) \chi_{[0,t]}(T^1(\omega) - x^1_k) , \tag{3.2}$$

where

$$\xi^i_k = \begin{cases} 1 & \text{if} \quad \eta_k \neq i \\ 0 & \text{if} \quad \eta_k = i \end{cases} \quad (1 \leqq k \leqq t;\ i = 1, 2) .$$

To establish (3.1) to (3.2) we need only note that

$$\xi^i_k \chi_{(0,t]}(T^i(\omega) - x^i_k) \chi_{[0,t]}(T^{3-i}(\omega) - x^{3-i}_k)$$

is equal to 1 if

(1) $\eta_k \neq i$,

(2) $T^i(\omega) > x^i_k$, and

(3) $T^{3-i}(\omega) \geqq x^{3-i}_k$

---

[2] The "name" of the customer who first exits from the facility is "i_1" where

$$\tau^1_{\mathscr{S}}(\omega) = \tau^1_{\mathscr{S}}(i_1, \omega) = \min_{1 \leqq j \leqq m} \tau^1_{\mathscr{S}}(j, \omega) .$$

and zero otherwise. Thus (3.1) (resp. (3.2)) counts the number of intervals up to $\tau^1_{\mathscr{S}}$ during which the server is occupied with customer "2" (resp. "1"). The independence of T^1 and T^2 thus yields

Lemma 3.1.

$$E(W_{\mathscr{S}}) = \sum_{i=1}^{2} \beta^i \sum_{k=1}^{t} \xi^i_k \, Q^i(x^i_k) \, [Q^{3-1}(x^{3-i}_k) + p^{3-i}(x^{3-i}_k)]. \tag{3.3}$$

We may view (3.3) as defining a path integral. Let

$$\mathscr{X} = \{\boldsymbol{x} = (x^1, x^2) \colon \; 0 \leqq x^i \leqq N^i \;\; (i = 1, 2)\}$$

and

$$\boldsymbol{u}^i = (u^{i,1}, u^{i,2}) \left(u^{i,j} = \begin{cases} 1 & \text{if} \quad j = 1 \\ 0 & \text{otherwise} \end{cases} \right).$$

We take the usual partial ordering on $\mathscr{X}$.

Definition 3.1. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{X}$ with $\boldsymbol{x} \leqq \boldsymbol{y}$. By a *path* π *from* $\boldsymbol{x}$ *to* $\boldsymbol{y}$ we shall mean a sequence

$$\pi \colon \; \{\boldsymbol{x}_k\}_{k=0}^{t},$$

which satisfies the conditions:

(3.1.1) $\boldsymbol{x}_0 = \boldsymbol{x}$,

(3.1.2) $\boldsymbol{x}_t = \boldsymbol{y}$, and

(3.1.3) for $1 \leqq k \leqq t$, $\boldsymbol{x}_k - \boldsymbol{x}_{k-1} \in \{\boldsymbol{u}^1, \boldsymbol{u}^2\}$.

A point $\boldsymbol{x}_k$ is a *vertex* of π provided

(3.1.4) $k = 0$, or

(3.1.5) $k = t$, or

(3.1.6) $1 \leqq k < t$ and $\boldsymbol{x}_k - \boldsymbol{x}_{k-1} \neq \boldsymbol{x}_{k+1} - \boldsymbol{x}_k$.

The vertex $\boldsymbol{x}_k$ is an *r-vertex of* π provided

(3.1.7) $k = 0$, or

(3.1.8) $k = t$, or

(3.1.9) $1 \leqq k < t$ and $\boldsymbol{u}^r = \boldsymbol{x}_k - \boldsymbol{x}_{k-1} \neq \boldsymbol{x}_{k+1} - \boldsymbol{x}_k$.

$\Pi(\boldsymbol{x}, \boldsymbol{y})$ will denote the family of paths joining $\boldsymbol{x}$ to $\boldsymbol{y}$. If $\boldsymbol{x}_1 \leqq \boldsymbol{x}_2 \leqq \cdots \leqq \boldsymbol{x}_n$ and $\pi_i \in \Pi(\boldsymbol{x}_i, \boldsymbol{x}_{i+1})$ $(1 \leqq i < n)$ then we shall denote by

$$\pi = \pi_1 * \pi_2 * \cdots * \pi_{n-1}$$

the path joining $\boldsymbol{x}_1$ to $\boldsymbol{x}_n$ formed by the juxtaposition of the paths $\{\pi_i\}$.

Definition 3.2. If $\pi \colon \{\boldsymbol{x}_k\}_{k=0}^{t} \in \Pi(\boldsymbol{x}, \boldsymbol{y})$ we define the *integral along the path* π by

$$\int_{\pi} = \sum_{k=0}^{t-1} Q^1(x^1_k) \, Q^2(x^2_k) \, [\beta^2 \, \Theta^1_k + \beta^1 \, \Theta^2_k]$$

where

$$\Theta^i_k = \begin{cases} 1 & \text{if} \quad \boldsymbol{x}_{k+1} - \boldsymbol{x}_k = \boldsymbol{u}^i \\ 0 & \text{otherwise}. \end{cases}$$

The total waiting time for the $\begin{bmatrix} 1 & 2 \\ N^1 & N^2 \end{bmatrix}_2$-processing schedule

$$\mathscr{S} = ((\eta_1, \mathscr{S}_1), \ldots, (\eta_N, \mathscr{S}_N)) \qquad (N = N^1 + N^2)$$

is equal to $\int\limits_{\pi(\mathscr{S})}$ where $\pi^{(\mathscr{S})} \in \Pi(\boldsymbol{O}, \boldsymbol{N})$ $(\boldsymbol{N} = (N^1, N^2))$

$$\pi^{(\mathscr{S})}\colon \ \{\boldsymbol{x}_k^{(\mathscr{S})}\}_{k=0}^{N}$$

$$\boldsymbol{x}_k^{(\mathscr{S})} = \begin{cases} 0 & \text{if } \ k = 0 \\ \sum\limits_{r=1}^{k} [\eta_r (1 - \xi_r^1)\, \boldsymbol{u}^1 + (\eta_r - 1)(1 - \xi_r^2)\, \boldsymbol{u}^2] & \text{if } \ k \geqq 1 \end{cases}$$

We thus have the equivalence of the problems

$$\min_{\mathscr{S}} E(W_{\mathscr{S}})$$

and

$$\min \Big\{ \int\limits_{\pi} \colon \pi \in \Pi(\boldsymbol{O}, \boldsymbol{N}) \Big\}.$$

To compare $\int\limits_{\pi}$ along two paths it will be convenient to introduce the notion of the "area" of a rectangle.

Definition 3.3. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{X}$ with $\boldsymbol{x} \leqq \boldsymbol{y}$. The *rectangle spanned by* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ is the set

$$R(\boldsymbol{x}, \boldsymbol{y}) = \{\boldsymbol{z} = (z^1, z^2) \in \mathscr{X}\colon \ x^i \leqq z^i < y^i \ \ (i = 1, 2)\}.$$

The *area of* $R(\boldsymbol{x}, \boldsymbol{y})$ is defined by

$$\mu\{R(\boldsymbol{x}, \boldsymbol{y})\} = \sum_{\boldsymbol{z} \in R(\boldsymbol{x}, \boldsymbol{y})} [\beta^1 p^1(z^1 + 1)\, Q^2(z^2) - \beta^2 p^2(z^2 + 1)\, Q^1(z^1)].$$

This definition of "area" leads to the following discrete analogue of STOKES' Theorem.

Lemma 3.2. *Let* $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{X}$ *with* $\boldsymbol{x} \leqq \boldsymbol{y}$ *and* $\boldsymbol{y} = \boldsymbol{x} + \varrho^1 \boldsymbol{u}^1 + \varrho^2 \boldsymbol{u}^2$. *Define the paths* $\pi_i \in \Pi(\boldsymbol{x}, \boldsymbol{y})$ $(i = 1, 2)$ *by*

$$\pi_1\colon \ \{\boldsymbol{x}_k^{(1)}\}_{k=0}^{\varrho^1+\varrho^2} \ \ \boldsymbol{x}_k^{(1)} = \begin{cases} \boldsymbol{x} + k\, \boldsymbol{u}^1 & \text{if } \ 0 \leqq k \leqq \varrho^1 \\ \boldsymbol{x} + \varrho^1 \boldsymbol{u}^1 + (k - \varrho^1)\, \boldsymbol{u}^2 & \text{if } \ \varrho^1 < k \leqq \varrho^1 + \varrho^2, \end{cases}$$

$$\pi_2\colon \ \{\boldsymbol{x}_k^{(2)}\}_{k=0}^{\varrho^1+\varrho^2} \ \ \boldsymbol{x}_k^{(2)} = \begin{cases} \boldsymbol{x} + k\, \boldsymbol{u}^2 & \text{if } \ 0 \leqq k \leqq \varrho^2 \\ \boldsymbol{x} + \varrho^2 \boldsymbol{u}^2 + (k - \varrho^2)\, \boldsymbol{u}^1 & \text{if } \ \varrho^2 < k \leqq \varrho^1 + \varrho^2. \end{cases}$$

Then, $\int\limits_{\pi_2} - \int\limits_{\pi_1} = \mu\{R(\boldsymbol{x}, \boldsymbol{y})\}$.

Proof.

$$\begin{aligned} \int\limits_{\pi_2} - \int\limits_{\pi_1} = {} & \\ & \beta^1 \Big[Q^1(x^1 + \varrho^1) + \sum_{x^1 \leqq z^1 < x^1 + \varrho^1} p^1(z^1 + 1)\Big] \sum_{x^2 \leqq z^2 < x^2 + \varrho^2} Q^2(z^2) + {} \\ & \beta^2 Q^2(x^2 + \varrho^2) \sum_{x^1 \leqq z^1 < x^1 + \varrho^1} Q^1(z^1) - {} \\ & \beta^2 \Big[Q^2(x^2 + \varrho^2) + \sum_{x^2 \leqq z^2 < x^2 + \varrho^2} p^2(z^2 + 1)\Big] \sum_{x^1 \leqq z^1 < x^1 + \varrho^1} Q^1(z^1) - {} \\ & \beta^1 Q^1(x^1 + \varrho^1) \sum_{x^2 \leqq z^2 < x^2 + \varrho^2} Q^2(z^2) = \mu\{R(\boldsymbol{x}, \boldsymbol{y})\}. \end{aligned}$$

The following lemma plays the central role in characterizing the shortest paths.

Lemma 3.3. *Let* $R(\boldsymbol{x}.\boldsymbol{y})$ *and* $R(\boldsymbol{v},\boldsymbol{w})$ *be two rectangles satisfying the conditions:*

$$\begin{array}{ll} (3.2.1) & \boldsymbol{x},\boldsymbol{y}\in\mathscr{X},\ \boldsymbol{x}<\boldsymbol{y},\ \boldsymbol{y}=\boldsymbol{x}+\varrho^1\boldsymbol{u}^1+\varrho^2\boldsymbol{u}^2 \\ (3.2.2) & \boldsymbol{v},\boldsymbol{w}\in\mathscr{X},\ \boldsymbol{v}<\boldsymbol{w},\ \boldsymbol{w}=\boldsymbol{v}+\sigma^1\boldsymbol{u}^1+\sigma^2\boldsymbol{u}^2 \\ (3.2.3) & \varrho^1=\sigma^1=\varrho\,(\text{say}) \\ (3.2.4) & x^1=v^1=\delta\,(\text{say}) \end{array}$$

Given ζ^1,ζ^2 *such that* $-\delta\leqq\zeta^1<\zeta^2+\varrho\leqq N^1$ *there exist* $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ *all positive real numbers such that*

$$\begin{aligned} &\lambda_1\mu\{R(\boldsymbol{x},\boldsymbol{y})\}+\lambda_3\mu\{R(\boldsymbol{v}+\zeta^1\boldsymbol{u}^1,\boldsymbol{w}+\zeta^2\boldsymbol{u}^1)\}=\\ &\lambda_2\mu\{R(\boldsymbol{v},\boldsymbol{w})\}+\lambda_4\mu\{R(\boldsymbol{x}+\zeta^1\boldsymbol{u}^1,\boldsymbol{y}+\zeta^2\boldsymbol{u}^1)\}. \end{aligned}\tag{3.4}$$

Proof. A straightforward calculation verifies that the choice

$$\lambda_1=A'B',\quad \lambda_2=A'B,\quad \lambda_3=AB,\quad \lambda_4=AB'$$

where

$$A=\sum_{\delta\leqq z^1<\delta+\varrho}Q^1(z^1),\quad A'=\sum_{\delta+\zeta^1\leqq z^1<\delta+\varrho+\zeta^2}Q^1(z^1)$$
$$B=\sum_{x^2\leqq z^2<x^2+\varrho^2}Q^2(z^2),\quad B'=\sum_{v^2\leqq z^2<v^2+\sigma^2}Q^2(z^2)$$

satisfies (3.4).

Lemma 3.3 will be used in the sequel to determine the sign of the area of certain rectangles. Define $\{\Omega_i\}$ by writing (3.4) in the form

$$\lambda_1\Omega_1+\lambda_3\Omega_3=\lambda_2\Omega_2+\lambda_4\Omega_4$$

and put $\omega_i=\operatorname{sgn}\Omega_i$. In the Table we give the value of ω_4 determined according to (3.4) for certain values of $(\omega_1,\omega_3,\omega_2)$.

Table

$(\omega_1\ \omega_3,\ \omega_2)$	ω_4
(0, 0, 0)	0
(0, 0, 1)	-1
(0, 0, -1)	1
(0, 1, 0)	1
(0, 1, -1)	1
(0, -1, 0)	-1
(0, -1, 1)	-1
(1, 1, 0)	1
(-1, -1, 0)	-1
(-1, -1, 1)	-1
(1, 1, -1)	1

Definition 3.4. The path $\pi^*\in\Pi(\boldsymbol{O},\boldsymbol{N})$ represents an *optimum schedule* if

$$\int_{\pi^*}=\min\left\{\int_{\pi}:\pi\in\Pi(\boldsymbol{O},\boldsymbol{N})\right\}$$

Two paths $\pi_1,\pi_2\in\Pi(\boldsymbol{O},\boldsymbol{N})$ are *equivalent* if $\int_{\pi_1}=\int_{\pi_2}$.

Henceforth we assume that π: $\{\boldsymbol{x}_k\}_{k=0}^N$ represents an optimum schedule. Let

$$\boldsymbol{O}=\boldsymbol{x}_{v_0}<\boldsymbol{x}_{v_1}<\boldsymbol{x}_{v_2}<\cdots<\boldsymbol{x}_{v_t}=\boldsymbol{N}$$

denote the 1-vertices of π.

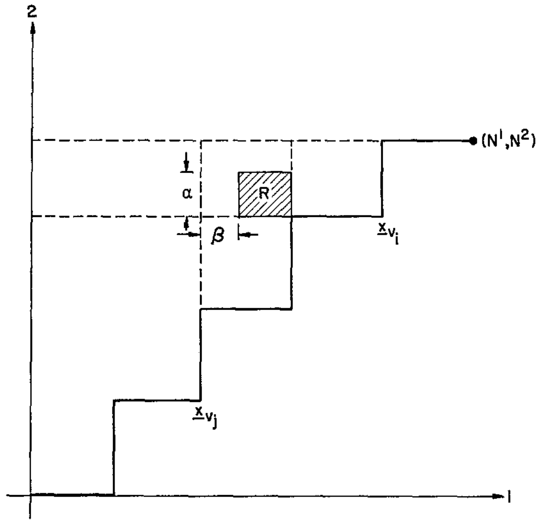
Theorem 3.4. *If* $0\leqq i\leqq j<t$, $0\leqq\alpha<(x^2_{v_{i+1}}-x^2_{v_i})$, $0<\beta\leqq(x^1_{v_{j+1}}-x^1_{v_j})$ *then*

$$\mu\{R(\boldsymbol{x}_{v_i}+\alpha\boldsymbol{u}^2+(x^1_{v_j}-x^1_{v_i})\boldsymbol{u}^1,\ \boldsymbol{x}_{v_{i+1}}+(x^1_{v_j}-x^1_{v_{i+1}}+\beta)\boldsymbol{u}^1)\}\leqq 0.\tag{3.5}$$

If $0\leqq j<i<t$, $0<\alpha\leqq(x^2_{v_{i+1}}-x^2_{v_i})$, $0\leqq\beta<(x^1_{v_{j+1}}-x^1_{v_j})$ *then*

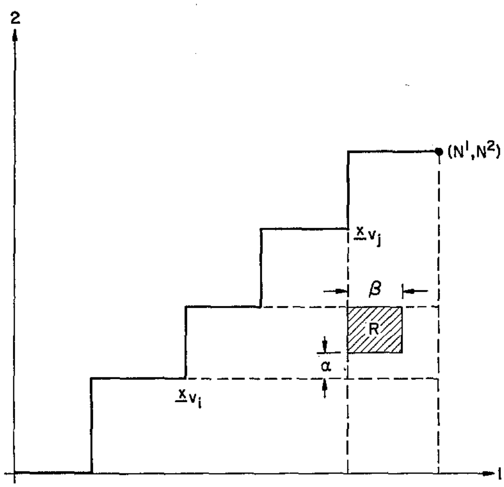
$$\mu\{R(\boldsymbol{x}_{v_i}+(x^1_{v_j}-x^1_{v_i}+\beta)\boldsymbol{u}^1,\ \boldsymbol{x}_{v_i}+\alpha\boldsymbol{u}^2+(x^1_{v_{j+1}}-x^1_{v_i})\boldsymbol{u}^1)\}\geqq 0\tag{3.6}$$

(see Figs. 1 and 2).



$R = R(x_{v_i} + (x^1_{v_j} - x^1_{v_i} + \beta)u^1, x_{v_i} + \alpha u^2 + (x^1_{v_{j+1}} - x^1_{v_i})u^1)$

Fig. 1



$R = R(x_{v_i} + \alpha u^2 + (x^1_{v_j} - x^1_{v_i})u^1, x_{v_{i+1}} + (x^1_{v_j} - x^1_{v_{i+1}} + \beta)u^1)$

Fig. 2

Proof. Suppose $i = j$ and on the contrary

$$\mu\{R(\boldsymbol{x}_{v_i} + \alpha \boldsymbol{u}^2, \boldsymbol{x}_{v_{i+1}} + (x^1_{v_i} - x^1_{v_{i+1}} + \beta)\boldsymbol{u}^1)\} > 0$$

for some pair (α, β) with $0 \leqq \alpha < (x^2_{v_{i+1}} - x^2_{v_i})$ and $0 < \beta \leqq (x^1_{v_{i+1}} - x^1_{v_i})$. Write

$$\pi = \pi_1 * \pi_2 * \pi_3$$

where

π_1 is that part of π joining $\boldsymbol{O}$ to $\boldsymbol{x}_{v_i} + \alpha \boldsymbol{u}^2$,
π_2 is that part of π joining $\boldsymbol{x}_{v_i} + \alpha \boldsymbol{u}^2$ to $\boldsymbol{x}_{v_{i+1}} + (-x^1_{v_{i+1}} + x^1_{v_i} + \beta)\boldsymbol{u}^1$,
π_3 is that part of π joining $\boldsymbol{x}_{v_{i+1}} + (-x^1_{v_{i+1}} + x^1_{v_i} + \beta)\boldsymbol{u}^1$ to $\boldsymbol{N}$.

Define the path $\hat{\pi}$ by $\hat{\pi} = \pi_1 * \hat{\pi}_2 * \pi_3$

$$\hat{\pi}_2 \colon \{\hat{\boldsymbol{x}}_k\}_{k=0}^{\beta + (x^2_{v_{i+1}} - x^2_{v_i}) - \alpha}$$

$$\hat{\boldsymbol{x}}_k = \begin{cases} \boldsymbol{x}_{v_i} + \alpha \boldsymbol{u}^2 + k \boldsymbol{u}^1 & \text{if} \quad 0 \leqq k \leqq \beta \\ \boldsymbol{x}_{v_i} + \beta \boldsymbol{u}^1 + (\alpha + k - \beta) \boldsymbol{u}^2 & \text{if} \quad \beta < k \leqq \beta - \alpha + (x^2_{v_{i+1}} - x^2_{v_i}) \end{cases}$$

Lemma 3.1 now shows that

$$\int_{\pi} - \int_{\hat{\pi}} = \int_{\pi_2} - \int_{\hat{\pi}_2} = \mu\{R(\boldsymbol{x}_{v_i} + \alpha \boldsymbol{u}^2, \boldsymbol{x}_{v_{i+1}} + (x^1_{v_i} - x^1_{v_{i+1}} + \beta) \boldsymbol{u}^1)\} > 0$$

which contradicts the minimality of $\int_{\pi}$.

A similar argument establishes (3.6) for $j = i - 1$.

To prove (3.5) for general (i, j) $(1 \leqq i \leqq j < t)$ we proceed by induction. Suppose therefore that (3.5) has been established for $j - i < \Delta$. We use Lemma 3.3 setting

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{x}_{v_i} + \alpha \boldsymbol{u}^2 \\ \boldsymbol{y} &= \boldsymbol{x}_{v_{i+1}} \\ \boldsymbol{v} &= \boldsymbol{x}_{v_i} + (x^2_{v_{i+1}} - x^2_{v_i}) \boldsymbol{u}^2 \\ \boldsymbol{w} &= \boldsymbol{x}_{v_{i+1}} + (x^2_{v_{i+2}} - x^2_{v_{i+1}}) \boldsymbol{u}^2 \\ \zeta^1 &= x^1_{v_j} - x^1_{v_i} \\ \zeta^2 &= x^1_{v_j} - x^1_{v_{i+1}} + \beta . \end{aligned}$$

Then

$\Omega_1 = \mu\{R(\boldsymbol{x}_{v_i} + \alpha \boldsymbol{u}^2, \boldsymbol{x}_{v_{i+1}})\} \leqq 0$ (by (3.5) with $j = i$ and $\beta = x^1_{v_{i+1}} - x^1_{v_i}$)

$\Omega_2 = \mu\{R(\boldsymbol{x}_{v_i} + (x^2_{v_{i+1}} - x^2_{v_i}) \boldsymbol{u}^2, \boldsymbol{x}_{v_{i+1}} + (x^2_{v_{i+2}} - x^2_{v_{i+1}})\} \geqq 0$
(by (3.6) with i replaced by $i + 1$, $j = i$, $\beta = 0$ and $\alpha = (x^2_{v_{i+2}} - x^2_{v_i})$).

$\Omega_3 = \mu\{R(\boldsymbol{x}_{v_{i+1}} + (x^1_{v_j} - x^1_{v_{i+1}}) \boldsymbol{u}^1, \boldsymbol{x}_{v_{i+2}} + (x^1_{v_j} - x^1_{v_{i+2}} + \beta) \boldsymbol{u}^1\} \leqq 0$
(by the induction hypothesis on (3.5)).

We may therefore conclude from the Table that

$$\Omega_4 = \mu\{R(\boldsymbol{x}_{v_i} + \alpha \boldsymbol{u}^2 + (x^1_{v_j} - x^1_{v_i}) \boldsymbol{u}^1, \boldsymbol{x}_{v_{i+1}} + (x^1_{v_j} - x^1_{v_{i+1}} + \beta) \boldsymbol{u}^1\} \leqq 0 \; .$$

completing the induction.

A similar induction proves (3.6) and we shall omit the details.

Equations (3.5) and (3.6) provide, a priori, only sufficient conditions for the minimality of $\int_{\pi}$. The next theorem provides a sharper characterization of the "shortest path".

Theorem 3.5. *If π represents an optimum schedule then there is an equivalent path which satisfies* (3.5), (3.6) *and the following two conditions:*
If $0 \leqq i < t$, $x^2_{v_{i+1}} - x^2_{v_i} > 0$, and $0 < \beta \leqq (x^1_{v_{i+1}} - x^1_{v_i})$ *then*

$$\mu\{R(\boldsymbol{x}_{v_i}, \boldsymbol{x}_{v_{i+1}} + (x^1_{v_i} - x^1_{v_{i+1}} + \beta) \boldsymbol{u}^1)\} < 0 . \tag{3.7}$$

If $0 \leqq i < t - 1$, $x^2_{v_{i+1}} - x^2_{v_i} > 0$, *and* $0 < \alpha \leqq (x^2_{v_{i+2}} - x^2_{v_{i+1}})$ *then*

$$\mu\{R(\boldsymbol{x}_{v_{i+1}} + (x^1_{v_i} - x^1_{v_{i+1}}) \boldsymbol{u}^1, \boldsymbol{x}_{v_{i+1}} + \alpha \boldsymbol{u}^2)\} > 0 . \tag{3.8}$$

Proof. Let ν_j ($j = 1, 2$) denote the following paths:

$$\nu_j\colon \{\Upsilon_k^{(j)}\}_{k=0}^{N^1+N^2}, \quad \Upsilon_k^{(j)} = \begin{cases} k\,\boldsymbol{u}^j & \text{if} \quad 0 \leqq k \leqq N^j \\ N^j \boldsymbol{u}^j + (k - N^{3-j})\,\boldsymbol{u}^{3-j} & \text{if} \quad N^j < k \leqq N^1 + N^2 \end{cases}$$

The proof is by induction on the number, $t + 1$, of 1-vertices of π. Note that $t \geqq 1$ and if $t = 1$ then $\pi = \nu_1$ or ν_2.

Case 1: $\pi = \nu_1$.

In this case both (3.7) and (3.8) are trivially satisfied.

Case 2: $\pi = \nu_2$.

If ν_1 and ν_2 are equivalent we take ν_1 as the path equivalent to π which satifies (3.7) and (3.8). Thus we may assume that

$$\int_{\nu_2} - \int_{\nu_1} = \mu\{R(\boldsymbol{O}, \boldsymbol{N})\} < 0.$$

If $\mu\{R(\boldsymbol{O}, (x, N^2))\} < 0$ for every x, $1 \leqq x \leqq N^1$, the path π itself satisfies (3.7) and (3.8). Otherwise define x^* to be the largest value of x, $1 \leqq x < N^1$, for which $\mu\{R(\boldsymbol{O}, (x, N^2))\} = 0$. It follows that

$$\mu\{R((x^*, 0), (x, N^2))\} < 0$$
$$x^* < x \leqq N^1.$$

The path $\pi^* = \pi_1^* * \pi_2^* * \pi_3^*$ where

$$\begin{array}{llll} \pi_1^* & \text{joins} & (0, 0) & \text{to} \quad (x^*, 0) \\ \pi_2^* & \text{joins} & (x^*, 0) & \text{to} \quad (x^*, N^2) \\ \pi_3^* & \text{joins} & (x^*, N^2) & \text{to} \quad (N^1, N^2) \end{array}$$

is equivalent to π and satisfies (3.7) and (3.8).

Next we assume that the theorem has been proved for optimal paths with fewer than $t + 1$ 1-vertices. We assert that we may assume $\mu\{R(\boldsymbol{x}_{v_i}, \boldsymbol{x}_{v_{i+1}})\} < 0$ if $0 \leqq i < t$ and $x_{v_{i+1}}^2 - x_{v_i}^2 > 0$. For if

$$\mu\{R(\boldsymbol{x}_{v_{i_0}}, \boldsymbol{x}_{v_{i_0}+1})\} = 0$$

for some i_0, $0 \leqq i_0 < t$ with $x_{v_{i_0+1}}^2 - x_{v_{i_0}}^2 > 0$ there is an equivalent path with fewer than $t + 1$ 1-vertices. Thus for each i, $0 \leqq i < t$ with $x_{v_{i+1}}^2 - x_{v_i}^2 > 0$ there exists a β_i, $0 \leqq \beta_i < (x_{v_{i+1}}^1 - x_{v_i}^1)$ such that

$$\mu\{R(\boldsymbol{x}_{v_i}, \boldsymbol{x}_{v_{i+1}} + (x_{v_{i+1}}^1 - x_{v_i}^1 + \beta_i)\,\boldsymbol{u}^1\} = 0$$
$$\mu\{R(\boldsymbol{x}_{v_i} + (x_{v_{i+1}}^1 - x_{v_i}^1 + \beta_i)\,\boldsymbol{u}^1, \boldsymbol{x}_{v_{i+1}} + (x_{v_{i+1}}^1 - x_{v_i}^1 + \beta)\,\boldsymbol{u}^1)\} < 0$$
$$\beta_i < \beta \leqq (x_{v_{i+1}}^1 - x_{v_i}^1).$$

It follows, as in Case 2, that we may find an equivalent path satisfying (3.7). Thus we may assume $\beta_i = 0$.

We assert that we may assume

$$\mu\{R(\boldsymbol{x}_{v_{i+1}} + (x_{v_i}^1 - x_{v_{i+1}}^1)\,\boldsymbol{u}^1, \boldsymbol{x}_{v_{i+1}} + (x_{v_{i+2}}^2 - x_{v_{i+1}}^2)\,\boldsymbol{u}^2)\} > 0$$

for all i, $0 \leqq i < t - 1$ for which $x_{v_{i+1}}^2 - x_{v_i}^2 > 0$. For if we assume the contrary then there is a path equivalent to π with fewer than $t + 1$ 1-vertices to which we may apply the induction hypothesis. Thus there exists for each i, $0 \leqq i < t-1$ for which $(x^2_{v_{i+1}} - x^2_{v_i}) > 0$ an α_i, $0 \leqq \alpha_i < (x^2_{v_{i+2}} - x^2_{v_{i+1}})$ such that

$$\mu\{R(\boldsymbol{x}_{v_i} + (x^1_{v_i} - x^1_{v_{i+1}})\boldsymbol{u}^1, \boldsymbol{x}_{v_{i+1}} + \alpha_i \boldsymbol{u}^2)\} = 0$$
$$\mu\{R(\boldsymbol{x}_{v_{i+1}} + (x^1_{v_i} - x^1_{v_{i+1}})\boldsymbol{u}^1 + \alpha_i \boldsymbol{u}^2, \boldsymbol{x}_{v_{i+1}} + \alpha \boldsymbol{u}^2)\} > 0$$
$$\alpha_i < \alpha \leqq (x^2_{v_{i+2}} - x^2_{v_{i+1}}).$$

Again there is an equivalent path for which (3.8) holds. (It is necessary to verify that (3.7) still is valid for this equivalent path. This presents no problems and we omit the details.)

A closer examination of the proof of Theorem 3.4 reveals that we have proved a somewhat stronger statement; if π satisfies conditions (3.5) and (3.6) then there is an equivalent path which satisfies conditions (3.5)—(3.8). Since, as we shall shortly prove, this latter path is uniquely determined, we will have proved.

Theorem 3.6. *The necessary and sufficient conditions that the path* π: $\{\boldsymbol{x}_k\}_{k=0}^N$ *(with* 1*-vertices* $\boldsymbol{O} = \boldsymbol{x}_{v_0} < \boldsymbol{x}_{v_1} < \cdots < \boldsymbol{x}_{v_t} = \boldsymbol{N}$*) represent an optimum schedule are* (3.5) *and* (3.6).

We now turn to the uniqueness question.

Lemma 3.7. *A necessary and sufficient condition that* ν_1 *represent an optimum schedule is*

$$\min_{0 \leqq x \leqq N^1,\, 0 \leqq y \leqq N^2} \mu\{R((x, 0), (N^1, y))\} \geqq 0. \tag{3.9}$$

Proof. (Necessity) Suppose on the contrary that

$$\mu\{R((x, 0), (N^1, y))\} < 0.$$

Define the path $\pi = \pi_1 * \pi_2 * \pi_3$ by

$$\begin{array}{llll} \pi_1 & \text{joins} \quad (0, 0) & \text{to} & (x, 0) \\ \pi_2 & \text{joins} \quad (x, 0) & \text{to} & (x, y) \\ \pi_3 & \text{joins} \quad (x, y) & \text{to} & (N^1, y). \end{array}$$

By Lemma 3.2

$$\int_\pi - \int_{\nu_1} = \mu\{R((x, 0), (N^1, y))\} < 0$$

which provides a contradiction.

(Sufficiency)

Suppose that π represents an optimum schedule and let

$$\boldsymbol{O} = \boldsymbol{x}_{v_0} < \boldsymbol{x}_{v_1} < \cdots < \boldsymbol{x}_{v_t} = \boldsymbol{N}$$

denote its 1-vertices. Equation (3.9) implies that

$$\mu\{R(\boldsymbol{x}_{v_i}, \boldsymbol{x}_{v_{i+1}})\} = 0 \qquad (0 \leqq i < t)$$

from which we conclude that $\int_{\nu_1} = \int_\pi$.

Similarly

Lemma 3.8. *A necessary and sufficient condition that* ν_2 *represent an optimum schedule is*

$$\max_{0 \leqq x \leqq N^1,\, 0 \leqq y \leqq N^2} \mu\{R((0, y), (x, N^2))\} \leqq 0. \tag{3.10}$$

Theorem 3.9. *There is a unique path π in $\mathscr{X}$ joing $\boldsymbol{O}$ to $\boldsymbol{N}$ which satisfies conditions* (3.5)—(3.8).

Proof. It evidently suffices to show that the first vertex (different from $\boldsymbol{O}$) is uniquely determined. Define

$$\sum_1 = \{\boldsymbol{z} = (z^1, z^2) \in \mathscr{X} : \mu\{R((z^1, 0), (N^1, z^2))\} < 0\}$$

$$\sum_2 = \{\boldsymbol{z} = (z^1, z^2) \in \mathscr{X} : \mu\{R((0, z^2), (z^1, N^2)\} > 0\}$$

$$\overset{*}{\sum_1} = \begin{cases} \left\{\boldsymbol{z} \in \sum_1 : \min\limits_{\substack{0 \leqq x \leqq z^1 \\ 0 \leqq y \leqq z^2}} \mu\{R((x, 0), (z^1, y))\} \geqq 0\right\} & \text{if } \sum_1 \neq \emptyset \\ \boldsymbol{N} & \text{if } \sum_1 = \emptyset \end{cases}$$

$$\overset{*}{\sum_2} = \begin{cases} \left\{\boldsymbol{z} \in \sum_2 : \max\limits_{\substack{0 \leqq x \leqq z^1 \\ 0 \leqq y \leqq z^2}} \mu\{R((0, y), (x, z^2))\} \leqq 0\right\} & \text{if } \sum_2 \neq \emptyset \\ \boldsymbol{N} & \text{if } \sum_2 = \emptyset \end{cases}$$

Let $\leqq_1$ and $\leqq_2$ denote the (lexicographic) orderings on $\mathscr{X}$

$(\xi_1, \xi_2) \leqq_1 (\eta_1, \eta_2)$ if either
(i) $\xi_2 < \eta_2$ or
(ii) $\xi_2 = \eta_2$ and $\xi_1 \leqq \eta_1$.
$(\xi_1, \xi_2) \leqq_2 (\eta_1, \eta_2)$ if either
(i) $\xi_1 < \eta_1$ or
(ii) $\xi_1 = \eta_1$ and $\xi_2 \leqq \eta_2$.

By Lemmata 3.7 and 3.8 we have

$$\sum_1 = \emptyset \quad \text{if and only if } \nu_1 \text{ represents an optimum schedule;}$$

$$\sum_2 = \emptyset \quad \text{if and only if } \nu_2 \text{ represents an optimum schedule.}$$

Let (ξ^*, η^*) denote the maximum element of $\overset{*}{\sum_1}$ (under $\leqq_1$) and (ξ^{**}, η^{**}) the maximum element of $\overset{*}{\sum_2}$ (under $\leqq_2$). We assert that the first non zero vertex of π, $\zeta = (\zeta^1, \zeta^2)$ is given by

$$\zeta = \begin{cases} (\xi^*, 0) & \text{if } \xi^* > 0 \\ (0, \eta^{**}) & \text{if } \xi^* = 0 \end{cases}$$

We must examine several cases.

Case 1: The first non-zero vertex of π is a 1-vertex. The second non-zero vertex is $(x^1_{v_1}, x^2_{v_2})$. By (3.5)—(3.8) we have $(x^1_{v_1}, x^2_{v_2}) \in \overset{*}{\sum_1}$ so that $(x^1_{v_1}, x^2_{v_2}) \leqq_1 (\xi^*, \eta^*)$.

Case 1.1: $x^2_{v_2} < \eta^*$, $x^1_{v_1} > \xi^*$. We have

$$\begin{aligned} &\mu\{R((\xi^*, 0), (x^1_{v_1}, \eta^*))\} \geqq 0 && \text{(by (3.6))} \\ &\mu\{R((\xi^*, 0), (N^1, \eta^*))\} < 0 && \text{(by hypothesis)} \end{aligned}$$

so that

$$\mu\{R((x^1_{v_1}, 0), (N^1, \eta^*))\} < 0$$

showing that $(x^1_{v_1}, \eta^*) \in \sum_1$. Moreover since

$$\min_{\substack{0 \leqq u \leqq x^1_{v_1} \\ 0 \leqq v \leqq \eta^*}} \mu\{R((u, 0), (x^1_{v_1}, v))\} \geqq 0$$

(by (3.6)) we have $(x^1_{v_1}, \eta^*) \in \overset{*}{\sum_1}$. This contradicts the maximality of (ξ^*, η^*).

Case 1.2: $x^2_{v_2} = \eta^*, \quad x^1_{v_1} < \xi^*$.

Condition (3.7) yields

$$\mu\{R((x^1_{v_1}, 0),\ (\xi^*, \eta^*))\} < 0$$

which contradicts the membership of (ξ^*, η^*) in $\overset{*}{\sum_1}$.

Case 1.3: $x^2_{v_2} < \eta^*, \quad x^1_{v_1} < \xi^*$.

Condition (3.7) yields

$$\mu\{R((x^1_{v_1}, 0),\ (\xi^*, x^2_{v_2}))\} < 0$$

which contradicts the membership of (ξ^*, η^*) in $\overset{*}{\sum_1}$. Thus in Case 1 we may conclude that $\boldsymbol{x}_{v_1} = (\xi^*, 0)$.

A similar treatment of the second case (the first non-zero vertex of π is a 2-vertex) yields $x^2_{v_1} = \eta^*$.

One final remark; let $|\pi|$ denote the number of vertices of π. The proof of Theorem 3.5 implies.

Theorem 3.10. *If* π^* *is the (unique) path satisfying* (3.5)—(3.8) *then*

$$|\pi|^* \leqq 1 + \min\{|\pi|:\ \pi \text{ represents an optimum schedule}\}.$$

This fact is of practical interest since we desire schedules which minimize the number of "switchings". Theorem 3.9 displays a curious asymmetry. This is due to our choice of a preferred coordinate (the "1" coordinate) which is evidenced in conditions (3.7)—(3.8). There is a dual set of conditions (favoring the "2" coordinate) which yields a unique path π^{**}. It is easy to show that

$$\min|\pi^*|, |\pi^{**}| = \min\{|\pi|:\ \pi \text{ represents an optimum schedule}\}.$$

Definition 3.5. Let $d_2 \begin{Bmatrix} 1 & 2 \\ x^1 & x^2 \\ \beta^1 & \beta^2 \end{Bmatrix}$ denote the minimum value of

$$E(\beta^1 W^1_{\mathscr{S}} + \beta^2 W^2_{\mathscr{S}})$$

where the service times T^1, T^2 have distributions

$$\Pr\{\omega:\ T^1(\omega) > k\} = \frac{Q^1(k + x^1)}{Q^1(x^1)}$$

$$\Pr\{\omega:\ T^2(\omega) > k\} = \frac{Q^2(k + x^2)}{Q^2(x^2)}$$

(d_2 is defined for $x^i < N^i$ as above and is equal to 0 for $x^1 \geqq N^1$ or $x^2 \geqq N^2$.)

Theorem 3.11. $d_2 \begin{Bmatrix} 1 & 2 \\ x^1 & x^2 \\ \beta^1 & \beta^2 \end{Bmatrix}$ *satisfies the recurrence*

$$d_2 \begin{Bmatrix} 1 & 2 \\ x^1 & x^2 \\ \beta^1 & \beta^2 \end{Bmatrix} = \min \left\{ \beta^1 + \frac{Q^2(x^2+1)}{Q^2(x^2)} d_2 \begin{Bmatrix} 1 & 2 \\ x^1 & x^2+1 \\ \beta^1 & \beta^2 \end{Bmatrix} \right. \\ \left. \beta^2 + \frac{Q^1(x^1+1)}{Q^1(x^1)} d_2 \begin{Bmatrix} 1 & 2 \\ x^1+1 & x^2 \\ \beta^1 & \beta^2 \end{Bmatrix} \right\}^* \tag{3.11}$$

Proof. We must find a shortest path from (x^1, x^2) to $\boldsymbol{N}$. The first vertex (different from (x^1, x^2)) must either be a 1-vertex or a 2-vertex. If the first vertex (different from (x^1, x^2)) is a 1-vertex then the average total waiting time is

$$\beta^2 + \frac{Q^1(x^1+1)}{Q^1(x^1)} d_2 \begin{Bmatrix} 1 & 2 \\ x^1+1 & x^2 \\ \beta^1 & \beta^2 \end{Bmatrix}$$

while if the first vertex is a 2-vertex the average total waiting time is

$$\beta^1 + \frac{Q^2(x^2+1)}{Q^2(x^2)} d_2 \begin{Bmatrix} 1 & 2 \\ x^1 & x^2+1 \\ \beta^1 & \beta^2 \end{Bmatrix}$$

This proves (3.11).

The techniques of this section may be applied to obtain a shortest path joining $\boldsymbol{x}$ to $\boldsymbol{N}$. Among all such shortest paths there is a uniquely determined path satisfying the analogue of conditions (3.7)—(3.8). We shall say that such a path is a *minimum path.*

§ 4. Consistency Conditions

We begin in this section by establishing certain consistency conditions relating the minimum paths.

Let $1 \leqq \xi < \eta \leqq m$ and

$$\mathscr{X}^{\xi,\eta} = \{(z^1, z^2) \colon 0 \leqq z^1 \leqq N^\xi,\ 0 \leqq z^2 \leqq N^\eta\}.$$

We shall use superscripts to denote the space ($\mathscr{X}^{\xi,\eta}$) in which a rectangle lies and its area. Thus $R^{\xi,\eta}(\boldsymbol{x}, \boldsymbol{y})$ denotes the rectangle in $\mathscr{X}^{\xi,\eta}$ spanned by $\boldsymbol{x}$ und $\boldsymbol{y}$ (in $\mathscr{X}^{\xi,\eta}$) and $\mu^{\xi,\eta}(R^{\xi,\eta}(\boldsymbol{x}, \boldsymbol{y}))$ will denote its area

$$\mu^{\xi,\eta}(R^{\xi,\eta}(\boldsymbol{x}, \boldsymbol{y})) = \sum_{z \in R^{\xi,\varepsilon}(\boldsymbol{x},\boldsymbol{y})} [\beta^\xi p^\xi(z^1+1)\, Q^\eta(z^2) - \beta^\eta p^\eta(z^2+1)\, Q^\xi(z^1)].$$

Let $\pi^{\xi,\eta}$ denote the minimum path (in $\mathscr{X}^{\xi,\eta}$) joining $\boldsymbol{O}$ to (N^ξ, N^η).

---

* We assume here that $x^i < N^i$ $(i = 1, 2)$.

Lemma 4.1. *If* $1 \leqq \xi < \eta < \zeta \leqq m$, $0 \leqq x^1 < y^1 \leqq N^\xi$, $0 \leqq x^2 < y^2 \leqq N^\eta$ *and* $0 \leqq x^3 < y^3 \leqq N^\zeta$ *then*

$$\begin{aligned} &\beta^\zeta [Q^\zeta(x^3) - Q^\zeta(y^3)]\, \mu^{\xi,\eta}\{R^{\xi,\eta}((x^1, x^2), (y^1, y^2))\} \\ &+ \beta^\xi [Q^\xi(x^1) - Q^\xi(y^1)]\, \mu^{\eta,\zeta}\{R^{\eta,\zeta}((x^2, x^3), (y^2, y^3))\} \\ &= \beta^\eta [Q^\eta(x^2) - Q^\eta(y^2)]\, \mu^{\xi,\zeta}\{R^{\xi,\zeta}((x^1, x^3), (y^1, y^3))\}. \end{aligned} \tag{4.1}$$

Proof. The proof of (4.1) is by direct calculation and we shall omit the details. Let

$$\mathscr{V}(\xi, \eta) = \begin{cases} \xi \text{ if the first non-zero vertex of } \pi^{\xi,\eta} \text{ is a } \xi\text{-vertex}, \\ \eta \text{ if the first non-zero vertex of } \pi^{\xi,\eta} \text{ is a } \eta\text{-vertex}. \end{cases}$$

Theorem 4.2.

$$\textit{If} \quad \mathscr{V}(\xi,\eta) = \xi \quad \textit{and} \quad \mathscr{V}(\eta,\zeta) = \eta \quad \textit{then} \quad \mathscr{V}(\xi,\zeta) = \xi \tag{4.2}$$

$$\textit{If} \quad \mathscr{V}(\xi,\eta) = \eta \quad \textit{and} \quad \mathscr{V}(\eta,\zeta) = \zeta \quad \textit{then} \quad \mathscr{V}(\xi,\zeta) = \zeta \tag{4.3}$$

$$\textit{If} \quad \mathscr{V}(\xi,\eta) = \xi \quad \textit{and} \quad \mathscr{V}(\xi,\zeta) = \zeta \quad \textit{then} \quad \mathscr{V}(\eta,\zeta) = \zeta \tag{4.4}$$

$$\textit{If} \quad \mathscr{V}(\eta,\zeta) = \eta \quad \textit{and} \quad \mathscr{V}(\xi,\zeta) = \zeta \quad \textit{then} \quad \mathscr{V}(\xi,\eta) = \eta \tag{4.5}$$

$$\textit{If} \quad \mathscr{V}(\xi,\zeta) = \xi \quad \textit{and} \quad \mathscr{V}(\xi,\eta) = \eta \quad \textit{then} \quad \mathscr{V}(\eta,\zeta) = \eta \tag{4.6}$$

$$\textit{If} \quad \mathscr{V}(\xi,\zeta) = \xi \quad \textit{and} \quad \mathscr{V}(\eta,\zeta) = \zeta \quad \textit{then} \quad \mathscr{V}(\xi,\eta) = \xi. \tag{4.7}$$

Proof. We shall only prove (4.2) since the proofs of (4.3) to (4.7) are similar.

Let the first vertex of $\pi^{\xi,\eta}$ be $(A, 0)$, the first vertex of $\pi^{\eta,\zeta}$ be $(B, 0)$ and suppose on the contrary that the first vertex of $\pi^{\xi,\zeta}$ is $(0, C)$ (with $A > 0$, $B > 0$, $C > 0$). In Lemma 4.1 set

$$x^1 = x^2 = x^3 = 0 \qquad y^1 = A \qquad y^2 = B \qquad y^3 = C.$$

Then

$$\begin{aligned} &\beta^\zeta [1 - Q^\zeta(C)]\, \mu^{\xi,\eta}\{R^{\xi,\eta}((0,0), (A, B))\} \\ &+ \beta^\xi [1 - Q^\xi(A)]\, \mu^{\eta,\zeta}\{R^{\eta,\zeta}((0,0), (B, C))\} \\ &= \beta^\eta [1 - Q^\eta(B)]\, \mu^{\xi,\zeta}\{R^{\xi,\zeta}((0,0), (A, C))\}. \end{aligned} \tag{4.8}$$

The left-hand side of (4.8) is non-negative. Suppose for the moment that we know $Q^\eta(B) < 1$. Then, the right-hand side of (4.8) must be non-negative and this is a contradiction to (3.7). But if $Q^\eta(B) = 1$ then $Q^\zeta(C) = 1$ and this contradicts the fact that $\pi^{\xi,\zeta}$ is a minimum path.

Let

$$\mathscr{Z}^{1,2,\ldots,m} = \{\boldsymbol{z} = (z^1, \ldots, z^m):\ 0 \leqq z^i \leqq N^i\ (1 \leqq i \leqq m)\}$$

and

$$\boldsymbol{u}^i = (u^{i,1}, u^{i,2}, \ldots, u^{i,m}) \quad u^{i,j} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

By a *path from* $\boldsymbol{x}$ *to* $\boldsymbol{y}$ we shall mean a sequence

$$\pi:\ \{\boldsymbol{x}_k\}_{k=0}^{t}$$

satisfying the conditions;

(1) $\boldsymbol{x}_0 = \boldsymbol{x}$

(2) $\boldsymbol{x}_t = \boldsymbol{y}$

(3) if $1 \leqq k$ then $\boldsymbol{x}_k - \boldsymbol{x}_{k-1} \in \{\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots, \boldsymbol{u}^m\}$.

Let $E^{\mu,\nu}$ be the projection of $\mathscr{X}^{1,2,\ldots,m}$ onto $\mathscr{X}^{\mu,\nu}$. If π is a path in $\mathscr{X}^{1,2,\ldots,m}$ joining $\boldsymbol{x}$ to $\boldsymbol{y}$ then $E^{\mu,\nu}\pi$ is the path in $\mathscr{X}^{\mu,\nu}$ joining (x^μ, x^ν) to (y^μ, y^ν).

Lemma 4.3. *There exists a unique path π in $\mathscr{X}^{1,2,\ldots,m}$ joining 0 to $(N^1, N^2, \ldots, N^m)$ such that*

$$E^{\xi,\eta}\pi = \pi^{\xi,\eta} \qquad 1 \leqq \xi < \eta \leqq m.$$

Proof. Lemma 4.3 is an immediate consequence of

Lemma 4.4. *There exists a unique i^*, $1 \leqq i^* \leqq i$, such that*

$$\begin{aligned} \mathscr{V}(i, i^*) &= i^* \quad 1 \leqq i < i^* \\ \mathscr{V}(i^*, j) &= i^* \quad i^* < j \leqq m. \end{aligned} \tag{4.9}$$

Proof. The proof is by induction on m; for $m = 3$ we use Theorem 4.2. Suppose therefore that Lemma 4.4 has been proved for $m < n$. Thus there exists an j^*, $1 \leqq j^* < n$ such that

$$\begin{aligned} \mathscr{V}(i, j^*) &= j^* \quad 1 \leqq i < j^* \\ \mathscr{V}(j^*, j) &= j^* \quad j^* < j < n. \end{aligned}$$

We must consider two cases;

Case 1: $\mathscr{V}(j^*, n) = j^*$.

We then set $i^* = j^*$ and note that (4.9) holds for $m = n$.

Case 2: $\mathscr{V}(j^*, n) = n$.

For $1 \leqq i < j^*$ we have

$$\mathscr{V}(i, j^*) = j^* \qquad \mathscr{V}(j^*, n) = n$$

and hence by (4.3) $\mathscr{V}(i, n) = n$.

For $j^* < j < n$ we have

$$\mathscr{V}(j^*, j) = j^* \qquad \mathscr{V}(j^*, n) = n$$

and hence by (4.4) $\mathscr{V}(j, n) = n$. We set $i^* = n$ and (4.9) has been established for $m = n$.

§ 5. General m

Let $\boldsymbol{x} = (x^1, x^2, \ldots, x^m)$ $(0 \leqq x^i < N^i;\ 1 \leqq i \leqq m)$. We begin by defining a class of processing schedules $\mathscr{S}(x^{i_1}, x^{i_2}, \ldots, x^{i}p)$ $(1 \leqq i_1 < i_2 < \cdots < i_p \leqq m)$; $\mathscr{S}(x^{i_1}, x^{i_2}, \ldots, x^{i_p})$ is a processing schedule for the p-customer time-sharing problem in which

(1) The names of the customers are

$$\text{“}i_1\text{” “}i_2\text{” } \ldots \text{ “}i_p\text{”}. \tag{5.1}$$

(2) The service time distribution of "i_j" is

$$\Pr\{\omega\colon\ T^{i_j}(\omega) > k\} = \frac{Q^{i_j}(k + x^{i_j})}{Q^{i_j}(x^{i_j})}\ (1 \leqq j \leqq p). \tag{5.2}$$

Let $\pi^{\mu,\nu}(x^\mu, x^\nu)$ be the unique minimum path (of Theorem 3.9) joining (x^μ, x^ν) to (N^μ, N^ν). We note that $\pi^{\mu,\nu}(x^\mu, x^\nu)$ determines a processing schedule, say $\mathscr{S}$, for the time-sharing problem of (5.1) to (5.2) [3]; we shall write $\pi^{\mu,\nu}(x^\mu, x^\nu) \leftrightarrow \mathscr{S}$ to indicate this correspondence. We define $\mathscr{S}(x^{i_1}, x^{i_2}, \ldots, x^{i_p})$ inductively;

Definition 5.1. $(p = 2)$

$$\pi^{i_1, i_2}(x^{i_1}, x^{i_2}) \leftrightarrow \mathscr{S}(x^{i_1}, x^{i_2})$$

$(p > 2)$

$$\mathscr{S}(x^{i_1}, \ldots, x^{i_p}) = ((\eta_1, \mathscr{S}_1), \ldots, (\eta_t, \mathscr{S}_t))$$

$$t = \sum_{j=1}^{p} (N^{i_j} - x^{i_j}),$$

where

(1) $\eta_i \in \{i_1, i_2, \ldots, i_p\} \quad (1 \leqq i \leqq t)$,

(2) If $x_k^{i_j}$ denotes the number of times the symbol "i_j" appears in the list $\eta_1, \eta_2, \ldots, \eta_k$, then $x_{t}^{i_j} = N^{i_j} - x^{i_j}$ $(1 \leqq j \leqq p)$,

(3) If $\eta_\nu = i_\beta$ then

$$\mathscr{S}_\nu = \mathscr{S}(x^{i1} - x_\nu^{i1}, \ldots, x^{i_{\beta-1}} - x_\nu^{i_{\beta-1}}, x^{i_{\beta+1}} - x_\nu^{i_{\beta+1}}, \ldots, x^{i_p} - x_\nu^{i_p}),$$

(4) The path π: $\{\xi_k\}_{k=0}^t$

$$\begin{aligned} \xi_k &= \left\{\sum_{j=1}^{p} x_k^{i_j} \boldsymbol{u}^{i_j}\right\} + (x^{i_1}, x^{i_2}, \ldots, x^{i_p}) && \text{if} \quad 1 \leqq k \leqq t \\ &= (x^{i_1}, x^{i_2}, \ldots, x^{i_p}) && \text{if} \quad k = 0 \end{aligned}$$

in

$$\mathscr{Z}^{i_1,\ldots,i_p} = \{\boldsymbol{z} = (z^1, z^2, \ldots, z^p) \colon 0 \leqq z^j \leqq N^{i_j} \ (1 \leqq j \leqq p)\}$$

$\boldsymbol{u}^{i_i} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in j-th coordinate) is the unique path of Lemma 4.3 joining $(x^{i_1}, x^{i_2}, \ldots, x^{i_p})$ to $(N^{i_1}, N^{i_2}, \ldots, N^{i_p})$ whose projection on $\mathscr{Z}^{i_\mu, i_\nu}$ is $\pi^{i_\mu, i_\nu}(x^{i_\mu}, x^{i_\nu})$ $(1 \leqq \mu < \nu \leqq p)$.

Theorem 5.1. *Let*

$$d_m \begin{Bmatrix} 1 & 2 & 3 & \ldots & m \\ x^1 & x^2 & x^3 & \ldots & x^m \\ \beta^1 & \beta^2 & \beta^3 & \ldots & \beta^m \end{Bmatrix}$$

be the expected total waiting time for the processing schedule

$$\mathscr{S}(x^1, x^2, \ldots, x^m) \text{ [4]}.$$

[3] For $p = 2$, $i_1 = \mu$, $i_2 = \nu$.

[4] d_m is defined as above for $0 \leqq x^i < N^i$ $(1 \leqq i \leqq m)$. We extend the definition of d_m for $0 \leqq x^i \leqq N^i$ as follows; if $0 \leqq x^i < N^i$ for $i \in \{j_1, j_2, \ldots, j_p\}$ $(1 \leqq j_1 < \ldots < j_p \leqq m)$ and $x^i = N^i$ for $i \notin \{j_1, j_2, \ldots, j_p\}$ then

$$d_m \begin{Bmatrix} 1 & 2 & & m \\ x^1 & x^2 & \ldots & x^m \\ \beta^1 & \beta^2 & & \beta^m \end{Bmatrix} = d_p \begin{Bmatrix} j_1 & j_2 & & j_p \\ x^{j1} & x^{j2} & \ldots & x^{jp} \\ \beta^{j1} & \beta^{j2} & & \beta^{jp} \end{Bmatrix}$$

Then,

$$d_m \begin{Bmatrix} 1 & 2 & \dots & m \\ x^1 & x^2 & \dots & x^m \\ \beta^1 & \beta^2 & \dots & \beta^m \end{Bmatrix} = \sum_{\substack{(\mu,\nu)\\ 1\leqq\mu<\nu\leqq m}} d_2 \begin{Bmatrix} \mu & \nu \\ x^\mu & x^\nu \\ \beta^\mu & \beta^\nu \end{Bmatrix} \tag{5.3}$$

Proof. The proof of (5.3) is by induction on m; for $m = 2$ (5.3) certainly holds. We now assume that (5.3) has been proved for $m - 1$. We begin by noting that

$$\begin{aligned} d_m \begin{Bmatrix} 1 & 2 & \dots & m \\ x^1 & x^2 & \dots & x^m \\ \beta^1 & \beta^2 & \dots & \beta^m \end{Bmatrix} &= \sum_{\substack{i=1\\ i\to k}} \beta^i + \\ &+ \frac{Q^k(x^k+1)}{Q^k(x^k)} d_m \begin{Bmatrix} 1 & 2 & \dots & k-1 & k & k+1 & \dots & m \\ x^1 & x^2 & \dots & x^{k-1} & 1+x^k & x^{k+1} & \dots & x^m \\ \beta^1 & \beta^2 & \dots & \beta^{k-1} & \beta^k & \beta^{k+1} & \dots & \beta^m \end{Bmatrix} \\ &+ \frac{Q^k(x^k) - Q^k(x^k+1)}{Q^k(x^k)} d_{m-1} \begin{Bmatrix} 1 & 2 & \dots & k-1 & k+1 & \dots & m \\ x^1 & x^2 & \dots & x^{k-1} & x^{k+1} & \dots & x^m \\ \beta^1 & \beta^2 & \dots & \beta^{k-1} & \beta^{k+1} & \dots & \beta^m \end{Bmatrix} \end{aligned} \tag{5.4}$$

where $0 \leqq x^i < N^i$ $(1 \leqq i \leqq m)$ and

(1) the first vertex (different from (x^μ, x^k)) of the minimum path $\pi^{\mu,k}(x^\mu, x^k)$ is a 2-vertex for $1 \leqq \mu < k$,

(2) the first vertex (different from (x^k, x^ν)) of the minimum path $\pi^{k,\nu}(x^k, x^\nu)$ is a 1-vertex for $k < \nu \leqq m$.

Certainly (5.3) holds for $x^1 + x^2 + \dots + x^m > K$ for some sufficiently large K. (We need only take K so that at least one $x^i = N^i$.) We assume now that $x^1 + x^2 + \dots + x^m = K$. We have

$$\begin{aligned} d_m \begin{Bmatrix} 1 & 2 & \dots & k-1 & k & k+1 & \dots & m \\ x^1 & x^2 & \dots & x^{k-1} & 1+x^k & x^{k+1} & \dots & x^m \\ \beta^1 & \beta^2 & \dots & \beta^{k-1} & \beta^k & \beta^{k+1} & \dots & \beta^m \end{Bmatrix} &= \\ \sum_{\substack{(\mu,\nu)\\ 1\leqq\mu<\nu\leqq m\\ \mu,\nu\neq k}} d_2 \begin{Bmatrix} \mu & \nu \\ x^\mu & x^\nu \\ \beta^\mu & \beta^\nu \end{Bmatrix} + \sum_{\substack{\mu\\ 1\leqq\mu<k}} d_2 \begin{Bmatrix} \mu & k \\ x^\mu & 1+x^k \\ \beta^\mu & \beta^k \end{Bmatrix} &+ \sum_{\substack{\nu\\ k<\nu\leqq m}} d_2 \begin{Bmatrix} k & \nu \\ 1+x^k & x^\nu \\ \beta^k & \beta^\nu \end{Bmatrix} \end{aligned} \tag{5.5}$$

(by the induction on K).

$$d_{m-1} \begin{Bmatrix} 1 & 2 & \dots & k-1 & k+1 & \dots & m \\ x^1 & x^2 & \dots & x^{k-1} & x^{k+1} & \dots & x^m \\ \beta^1 & \beta^2 & \dots & \beta^{k-1} & \beta^{k+1} & \dots & \beta^m \end{Bmatrix} = \sum_{\substack{(\mu,\nu)\\ 1\leqq\mu\leqq\nu\leqq m\\ \mu,\nu\neq k}} d_2 \begin{Bmatrix} \mu & \nu \\ x^\mu & x^\nu \\ \beta^\mu & \beta^\nu \end{Bmatrix} \tag{5.6}$$

(by the induction on m). Finally we observe that

$$d_2 \begin{Bmatrix} \mu & k \\ x^\mu & x^k \\ \beta^\mu & \beta^k \end{Bmatrix} = \beta^\mu + \frac{Q^k(x^k+1)}{Q^k(x^k)} d_2 \begin{Bmatrix} \mu & k \\ x^\mu & 1+x^k \\ \beta^\mu & \beta^k \end{Bmatrix} \tag{5.7}$$

$$(1 \leqq \mu < k)$$

$$d_2 \left\{\begin{matrix} k & \nu \\ x^k & x^\nu \\ \beta^k & \beta^\nu \end{matrix}\right\} = \beta^\nu + \frac{Q^k(1+x^k)}{Q^k(x^k)}\, d_2 \left\{\begin{matrix} k & \nu \\ x^k+1 & x^\nu \\ \beta^k & \beta^\nu \end{matrix}\right\} \tag{5.8}$$

$$(k < \nu \leqq m)\,.$$

Substituting (5.5) and (5.6) into (5.4) and using (5.7) and (5.8) proves (5.3) for $x^1 + x^2 + \cdots + x^m = K$ and thus completes the proof of Theorem 5.1.

Let

$$\mathscr{P} \left\{\begin{matrix} 1 & 2 & \ldots & m \\ x^1 & x^2 & \ldots & x^m \\ \beta^1 & \beta^2 & \ldots & \beta^m \end{matrix}\right\}$$

denote the class of all processing schedules for the m-person time-sharing problem (with customers "1", "2", ..., "m") in which the service time distributions are

$$\Pr\{\omega\colon\ T^i(\omega) > k\} = \frac{Q^i(x^i+k)}{Q^i(x^i)}$$

$$(1 \leqq i \leqq m)\,.$$

Theorem 5.2.

$$\min_{\mathscr{S} \in \mathscr{P} \left\{\begin{smallmatrix} 1 & 2 & \ldots & m \\ x^1 & x^2 & \ldots & x^m \\ \beta^1 & \beta^2 & & \beta^m \end{smallmatrix}\right\}} E(W_{\mathscr{S}}) = d_m \left\{\begin{matrix} 1 & 2 & & m \\ x^1 & x^2 & \ldots & x^m \\ \beta^1 & \beta^2 & & \beta^m \end{matrix}\right\} \tag{5.9}$$

Proof. Let

$$e_m \left\{\begin{matrix} 1 & 2 & & m \\ x^1 & x^2 & \ldots & x^m \\ \beta^1 & \beta^2 & \ldots & \beta^m \end{matrix}\right\} \tag{5.10}$$

denote the left-hand side of (5.9). Clearly $e_m = d_m$ for $m = 2$. We suppose that (5.9) has been proved for time-sharing problems with fewer than m customers. Next we observe that (5.9) holds for $x^1 + x^2 + \cdots + x^m$ sufficiently large. We assume that (5.9) holds for $x^1 + x^2 + \cdots + x^m$ larger than K and attempt to verify (5.9) for $x^1 + x^2 + \cdots + x^m = K$.

But (5.10) satisfies the recurrence relation

$$\begin{aligned} e_m \left\{\begin{matrix} 1 & 2 & . & m \\ x^1 & x^2 & \ldots & x^m \\ \beta^1 & \beta^2 & & \beta^m \end{matrix}\right\} = \min_{\substack{k \\ 1 \leqq k \leqq m}} \Bigg\{ & \sum_{\substack{i=1 \\ i \neq k}}^{m} \beta^i \\ & + \frac{Q^k(x^k+1)}{Q^k(x^k)}\, e_m \left\{\begin{matrix} 1 & 2 & & k-1 & k & k+1 & & m \\ x^1 & x^2 & \ldots & x^{k-1} & 1+x^k & x^{k+1} & \ldots & x^m \\ \beta^1 & \beta^2 & & \beta^{k-1} & \beta^k & \beta^{k+1} & & \beta^m \end{matrix}\right\} \\ & + \frac{Q^k(x^k) - Q^k(x^k+1)}{Q^k(x^k)}\, e_{m-1} \left\{\begin{matrix} 1 & 2 & & k-1 & k+1 & & m \\ x^1 & x^2 & \ldots & x^{k-1} & x^{k+1} & \ldots & x^m \\ \beta^1 & \beta^2 & & \beta^{k-1} & \beta^{k+1} & & \beta^m \end{matrix}\right\} \Bigg\}\,. \end{aligned} \tag{5.11}$$

We apply the induction hypothesis in (5.11);

$$e_m \begin{Bmatrix} 1 & 2 & & k-1 & k & k+1 & & m \\ x^1 & x^2 & \dots & x^{k-1} & 1+x^k & x^{k+1} & \dots & x^m \\ \beta^1 & \beta^2 & & \beta^{k-1} & \beta^k & \beta^{k+1} & & \beta^m \end{Bmatrix}$$

$$= \sum_{\substack{(\mu,\nu) \\ 1 \leqq \mu < \nu \leqq m \\ \mu,\nu \neq k}} d_2 \begin{Bmatrix} \mu & \nu \\ x^\mu & x^\nu \\ \beta^\mu & \beta^\nu \end{Bmatrix} + \sum_{\substack{\mu \\ 1 \leqq \mu < k}} d_2 \begin{Bmatrix} \mu & k \\ x^\mu & x^k + 1 \\ \beta^\mu & \beta^k \end{Bmatrix} + \sum_{\substack{\nu \\ k < \nu \leqq m}} d_2 \begin{Bmatrix} k & \nu \\ x^k + 1 & x^\nu \\ \beta^k & \beta^\nu \end{Bmatrix} \tag{5.12}$$

$$e_{m-1} \begin{Bmatrix} 1 & 2 & & k-1 & k+1 & & m \\ x^1 & x^2 & \dots & x^{k-1} & x^{k+1} & \dots & x^m \\ \beta^1 & \beta^2 & & \beta^{k-1} & \beta^{k+1} & & \beta^m \end{Bmatrix} = \sum_{\substack{(\mu,\nu) \\ 1 \leqq \mu < \nu \leqq m \\ \mu,\nu \neq k}} d_2 \begin{Bmatrix} \mu & \nu \\ x^\mu & x^\nu \\ \beta^\mu & \beta^\nu \end{Bmatrix} \tag{5.13}$$

Finally we note that for each k, $1 \leqq k \leqq m$

$$d_2 \begin{Bmatrix} \mu & k \\ x^\mu & x^k \\ \beta^\mu & \beta^k \end{Bmatrix} \leqq \beta^\mu + \frac{Q^k(x^k+1)}{Q^k(x^k)} d_2 \begin{Bmatrix} \mu & k \\ x^\mu & 1+x^k \\ \beta^\mu & \beta^k \end{Bmatrix} \tag{5.14}$$

$$(1 \leqq \mu < k)$$

$$d_2 \begin{Bmatrix} k & \nu \\ x^k & x^\nu \\ \beta^k & \beta^\nu \end{Bmatrix} \leqq \beta^\nu + \frac{Q^k(x^k+1)}{Q^k(x^k)} d_2 \begin{Bmatrix} k & \nu \\ 1+x^k & x^\nu \\ \beta^k & \beta^\nu \end{Bmatrix} \tag{5.15}$$

$$(k < \nu \leqq m).$$

by Theorem 3.11. Substituting (5.12) and (5.13) into (5.11) and using (5.14) and (5.15) we obtain

$$e_m \begin{Bmatrix} 1 & 2 & & m \\ x^1 & x^2 & \dots & x^m \\ \beta^1 & \beta^2 & & \beta^m \end{Bmatrix} \geqq d_m \begin{Bmatrix} 1 & 2 & & m \\ x^1 & x^2 & \dots & x^m \\ \beta^1 & \beta^2 & & \beta^m \end{Bmatrix}$$

from which we immediately deduce equality.

Reference

1. Chazan, D., A. G. Konheim, and B. Weiss: A note on time sharing. IBM Research Paper RC 1656 (submitted to the Journal on Combinatorial Theory) July 29, 1966.

Prof. A. G. Konheim
Thomas J. Watson Research Center
International Business Machines Corporation
Yorktown Heights, N. Y. 10598
USA

Institut f. Angewandte Mathematik
Universität Heidelberg
69 Heidelberg
Berliner Straße 17