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# A Boundary Property of Semimartingale Reflecting Brownian Motions*

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**Summary.** We consider a class of reflecting Brownian motions on the non-negative orthant in $\mathbf{R}^K$. In the interior of the orthant, such a process behaves like Brownian motion with a constant covariance matrix and drift vector. At each of the $(K-1)$-dimensional faces that form the boundary of the orthant, the process reflects instantaneously in a direction that is constant over the face. We give a necessary condition for the process to have a certain semimartingale decomposition, and then show that the boundary processes appearing in this decomposition do not charge the set of times that the process is at the intersection of two or more faces. This boundary property plays an essential role in the derivation (performed in a separate work) of an analytical characterization of the stationary distributions of such semimartingale reflecting Brownian motions.

## 1. Introduction

Let $K$ be a positive integer, $\Gamma=(\Gamma_{ij})$ be a $K \times K$ non-degenerate covariance matrix (symmetric and positive definite), and $\theta=(\theta_i)$ be a $K$-vector. (All vectors should be envisioned as column vectors.) Let $R$ be a $K \times K$ matrix, and let $S$ denote the non-negative orthant $\mathbf{R}_+^K$.

In this paper, we are concerned with certain semimartingale reflecting Brownian motions, called SRBM's, associated with the data $(S, \theta, \Gamma, R)$. By an SRBM we mean a continuous, adapted $K$-dimensional process $Z$ together with a family of probability measures $\{P_z, z \in S\}$ (one for each starting point in $S$), defined on some filtered space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\})$, such that for each $z \in S$, under $P_z$,

$$Z(t)=X(t)+RY(t) \in S, \qquad t \geqq 0 \tag{1}$$

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where

(i) $X$ is an $\{\mathscr{F}_t\}$-adapted, $K$-dimensional Brownian motion with covariance matrix $\Gamma$ and drift vector $\theta$, and $P_z$- a.s., $X$ starts from $z$,

(ii) $Y$ is a continuous, $\{\mathscr{F}_t\}$-adapted, $\mathbf{R}_+^K$-valued process that $P_z$-a.s. satisfies:

(a) $Y(0)=0$

(b) $Y$ is non-decreasing

(c) for each $k\in\{1, \ldots, K\}$, the $k$-th component $Y_k$ of $Y$ increases only when $Z$ is on the face $F_k \equiv \{x\in\mathbf{R}_+^K : x_k=0\}$, i.e.,

$$\int_0^\infty 1_{S\setminus F_k}(Z(s))\, dY_k(s)=0.$$

For brevity, we shall use $Z$ to denote the SRBM determined by $Z$ and the probability measures $\{P_z, z\in S\}$. In the language of stochastic differential equations (s.d.e.'s), $Z$ is a *weak* solution of the s.d.e.:

$$dZ(t)=dX(t)+R\,dY(t), \tag{2}$$

together with the auxiliary conditions (i)–(ii) and $Z(t)\in S$ for all $t\geqq 0$. Loosely speaking, $Z$ behaves like the Brownian motion $X$ in the interior of the orthant and is reflected at the boundary, the direction of reflection on the face $F_k$ being given by the $k$-th column of the matrix $R$. Although we shall not need it here, it can be shown that the amount of the time that $Z$ spends on the boundary has zero Lebesgue measure and hence reflection at the boundary is instantaneous.

A necessary and sufficient condition on the data for existence and/or uniqueness of an associated SRBM is not known. However, in studying diffusion processes arising as the heavy traffic limits of open queueing networks, Harrison and Reiman [3] have shown that when $R$ has ones on the diagonal and $R-I$ is non-positive and has spectral radius strictly less than one, then there is a unique path-to-path mapping from any $(\theta, \Gamma)$ Brownian motion $X$ starting at $z\in S$ to a pair of continuous processes $(Y, Z)$, adapted to $X$, such that (i) and (ii) hold. This yields a sufficient condition for a unique *strong* solution to the problem defined by (1) and (i)–(ii). Other sufficient conditions for existence of a solution to this problem can be gleaned from [6]. In contemporary work, Mandelbaum and Van der Heyden [8] are investigating conditions for the existence/uniqueness of pathwise solutions to this problem. In Sect. 2, we prove that the following natural feasibility condition is necessary for the existence of a (weak) solution to (1) and (i)–(ii) for each $z\in S$. Here a principal submatrix of a square $(k\times k)$ matrix $A$ is a matrix obtained by deleting all rows and columns of $A$ with indices in some (possibly empty) subset of $\{1, \ldots, k\}$. Inequalities are to be interpreted componentwise.

**Definition.** A square matrix $A$ is called *completely*$-\mathscr{S}$ if for each principal submatrix $\tilde{A}$ of $A$, there is $\tilde{y}\geqq 0$ such that $\tilde{A}\tilde{y}>0$.

*Condition* (SF). We say the matrix $R$ satisfies the semimartingale feasibility condition (SF) if $R$ is a completely-$\mathscr{S}$ matrix.

*Remark 1.* It is easy to see that in the definition of completely-$\mathscr{S}$, $\tilde{y}$ may be chosen such that $\tilde{y}>0$ and $\tilde{A}\tilde{y}\geqq 1$.

*Remark 2.* In the literature [2], a matrix for which there is a $y\geqq 0$ such that $Ay>0$ is called an $\mathscr{S}$ (after Stiemke) matrix. Following Cottle [1], we have used the prefix "completely" to indicate when this property is inherited by all principal submatrices.

The geometric interpretation of condition (SF) is as follows. For an SRBM $Z$ starting from the smooth part $F_j\setminus\bigcup_{k\neq j} F_k$ of some face $F_j$, the $j$-th component $Y_j$ of $Y$ is the only one to increase until the random positive time $\tau$ at which $Z$ first reaches one of the other faces $F_k$, $k\neq j$. Thus, for $0\leqq t<\tau$, the projection of $Z(t)$ onto the direction $e_j$ normal to $F_j$ is given by

$$Z_j(t)=e_j\cdot(X(t)+R^j Y_j(t))=X_j(t)+R_j^j Y_j(t),$$

where $R^j$ is the $j$-th column of $R$ and $R_j^j$ is the element in the $j$-th row and column of $R$. It follows that a necessary condition for $Z$ to remain in $S$ (and hence for $Z_j$ to remain non-negative) is that $R_j^j$ be strictly positive, i.e., the inward normal component of the direction of reflection $R^j$ on the $j$-th face is strictly positive. But this is precisely the condition that each $1\times 1$ principal submatrix (each diagonal element) of $R$ is an $\mathscr{S}$ matrix. (Indeed, one can normalize the directions of reflection $R^j$ such that all of the diagonal elements of $R$ are 1.) Similarly, by considering $Z$ starting from any $(K-j)$-dimensional edge formed by the intersection of $j$ distinct faces $(2\leqq j\leqq K)$, we see that a necessary condition for $Z$ to remain in $S$ is that there is a non-negative linear combination of the directions of reflection for those faces such that its projection onto each of the inward normals to those faces is positive. This is precisely the condition that each $j\times j$ principal submatrix of $R$ is an $\mathscr{S}$ matrix. Combining these necessary conditions, one obtains condition (SF). The preceding informal argument is formalized in Theorem 2 of Sect. 2.

As a preliminary to the proof of the main result of this paper, it is shown in Sect. 3 that the completely-$\mathscr{S}$ matrices are the same as Cottle's [1] completely-$\mathscr{Q}$ matrices which are the same as the strictly semi-monotone matrices, and that this class is closed under transposition. This result (Lemma 3), which seems to be new, has also been discovered independently by Mandelbaum and Van der Heyden [7].

In Sect. 4, we establish the main result of this paper. Assuming there is an SRBM $Z$ associated with $(S, \theta, \Gamma, R)$, and hence that the transpose of $R$ is completely-$\mathscr{S}$, it is shown that the "boundary processes" $Y_k$ (also sometimes called boundary local times or control processes) do not "charge" the set of times that $Z$ is at the intersection of two or more faces. More precisely, if for any $\mathbf{J}\subset\{1, \ldots, K\}$, $Z_{\mathbf{J}}$ denotes the process whose components are those of $Z$ with indices in $\mathbf{J}$, then we prove the following.

**Theorem 1.** *For each* $\mathbf{J}\subset\{1, \ldots, K\}$ *satisfying* $|\mathbf{J}|\geqq 2$, *and any* $k\in\mathbf{J}$, *we have* $P_z$-a.s. *for each* $z\in S$:

$$\int_0^\infty 1_{\{Z_{\mathbf{J}}(s)=0\}}\, dY_k(s)=0. \tag{3}$$

*Remark.* Since $Y_k$ can increase only when $Z_k$ is zero, the case $k \in \mathbf{J}$ is the only important one.

The above result is not only of intrinsic interest for the understanding of semimartingale reflecting Brownian motions, but also plays an essential role in the derivation of an analytical characterization of the stationary distributions for such processes [4]. This is of relevance to applications in queueing and storage theory where SRBM's arise naturally in the heavy traffic limit, as shown in Reiman [9].

## 2. A Necessary Condition

The main result of this section is the following.

**Theorem 2.** *Suppose $Z$ is an* SRBM *associated with the data $(S, \theta, \Gamma, R)$. Then the semimartingale feasibility condition* (SF) *must hold, i.e., $R$ is a completely-$\mathcal{S}$ matrix.*

*Proof.* For a proof by contradiction, suppose $R$ is not completely-$\mathcal{S}$. Then there is a principal submatrix $\tilde{R}$ of $R$ such that for all $\tilde{y} \geqq 0$, $\tilde{R}\tilde{y} \ngtr 0$. Let $\mathbf{J} \subset \mathbf{K} \equiv \{1, \ldots, K\}$ be the indices of the rows and columns of $R$ that are *retained* in forming $\tilde{R}$ from $R$. Let $z \in S$ such that $z_j = 0$ for $j \in \mathbf{J}$ and $z_j > 0$ for $j \in \mathbf{K} \setminus \mathbf{J}$. Define

$$\tau = \inf\{t \geqq 0 \colon Z_j(t) = 0 \text{ for some } j \in \mathbf{K} \setminus \mathbf{J}\}. \tag{4}$$

Then since $z \notin \bigcup_{j \in \mathbf{K} \setminus \mathbf{J}} F_j$ and the paths of $Z$ are continuous, we have

$$P_z(t < \tau) \uparrow 1 \quad \text{as} \quad t \downarrow 0. \tag{5}$$

Let $\tilde{Z}$, $\tilde{X}$, and $\tilde{Y}$ denote the processes obtained from $Z$, $X$, and $Y$ by retaining only those components with indices in $\mathbf{J}$. Then by (1) and (ii) of Sect. 1, we have $P_z$-a.s.:

$$\tilde{Z}(t) = \tilde{X}(t) + \tilde{R}\tilde{Y}(t) \qquad \text{for } 0 \leqq t < \tau, \tag{6}$$

since the components of $Y$ indexed by $\mathbf{K} \setminus \mathbf{J}$ do not increase before the time $\tau$. Under $P_z$, $\tilde{X}$ is a $|\mathbf{J}|$-dimensional Brownian motion with non-degenerate covariance matrix and drift $\tilde{\theta}$, and $\tilde{X}$ starts from the origin. Thus, there are $\delta_0 > 0$ and $\varepsilon > 0$ such that $P_z(\tilde{X}(t) < 0) > \varepsilon$ for all $0 < t \leqq \delta_0$. Combining this with (5), we see that there is $t_0 > 0$ such that

$$P_z(\tilde{X}(t_0) < 0, t_0 < \tau) > \varepsilon/2. \tag{7}$$

Thus, by (6), (7) and the fact that $P_z$-a.s., $\tilde{Z}(t) \geqq 0$ for all $t$, we have

$$P_z(\tilde{R}\tilde{Y}(t_0) = \tilde{Z}(t_0) - \tilde{X}(t_0) > 0) \geqq P_z(\tilde{X}(t_0) < 0, t_0 < \tau) > \varepsilon/2.$$

But this contradicts the facts that $\tilde{R}$ is not an $\mathcal{S}$ matrix and $\tilde{Y}(t) \geqq 0$ $P_z$-a.s. for all $t \geqq 0$. □

## 3. Completely-$\mathscr{S}$ Matrices

In this section we establish some preliminary results needed for the proof of Theorem 1. In particular, we equate the completely-$\mathscr{S}$ matrices to the completely-$\mathscr{Q}$ matrices associated with solvability of the linear complementarity problem. We also prove this class is closed under transposition.

**Definition.** A square $(k \times k)$ matrix $A$ is called *completely-$\mathscr{Q}$* if for each $j \in \{1, \ldots, k\}$, each $j \times j$ principal submatrix $\tilde{A}$ of $A$, and each vector $\tilde{x} \in \mathbf{R}^j$, there is at least one solution $(\tilde{y}, \tilde{z})$ of the linear complementarity problem:

$$(\mathrm{LCP}) \begin{cases} \tilde{z} = \tilde{x} + \tilde{A}\tilde{y} \\ \tilde{z} \geqq 0, \quad \tilde{y} \geqq 0 \\ \tilde{z} \cdot \tilde{y} = 0. \end{cases}$$

**Definition.** A square matrix $A$ is called *strictly semi-monotone* if for each principal submatrix $\tilde{A}$ of $A$, the system

$$\tilde{A}\tilde{w} \leqq 0, \qquad \tilde{w} \geqq 0$$

has the unique solution $\tilde{w} = 0$.

**Lemma 1.** *A square matrix is completely-$\mathscr{Q}$ if and only if it is strictly semimonotone.*

*Proof.* See Cottle [1, Theorem 1]. □

**Lemma 2.** *If a square matrix is completely-$\mathscr{Q}$, then it is completely-$\mathscr{S}$.*

*Proof.* This follows easily by taking solutions of the (LCP) corresponding to $\tilde{x}_i = -1$, $i = 1, \ldots, j$. □

**Lemma 3.** *A square matrix is completely-$\mathscr{S}$ if and only if it is completely-$\mathscr{Q}$, and this class is closed under transposition of matrices.*

*Proof.* Suppose $A$ is a square matrix that is completely-$\mathscr{S}$. Then, for any principal submatrix $\tilde{A}$ of $A$, there is $\tilde{y} > 0$ such that $\tilde{A}\tilde{y} > 0$. Suppose $\tilde{w} \geqq 0$ such that $\tilde{A}'\tilde{w} \leqq 0$, where the prime denotes transpose. Then $\tilde{w} \equiv 0$, for if not, we have the contradiction:

$$0 < \tilde{w} \cdot (\tilde{A}\tilde{y}) = (\tilde{A}'\tilde{w}) \cdot \tilde{y} \leqq 0.$$

Since $\tilde{A}'$ was an arbitrary principal submatrix of $A'$, it follows that $A'$ is strictly semi-monotone. But then by Lemma 1, $A'$ is completely-$\mathscr{Q}$ and hence completely-$\mathscr{S}$, by Lemma 2. Repeating the argument with $A'$ in place of $A$ completes the desired circle of implications:

$$A \in \bar{\mathscr{S}} \Rightarrow A' \in \bar{\mathscr{Q}} \Rightarrow A' \in \bar{\mathscr{S}} \Rightarrow A \in \bar{\mathscr{Q}} \Rightarrow A \in \bar{\mathscr{S}},$$

where $\bar{\mathscr{S}}$ denotes the set of completely-$\mathscr{S}$ matrices and $\bar{\mathscr{Q}}$ the set of completely-$\mathscr{Q}$ matrices. □

## 4. Proof of the Boundary Property

The following notation will be used in this section. If $\mathbf{J} \subset \mathbf{K} \equiv \{1, \ldots, K\}$, then for any $x \in \mathbf{R}^K$ and $K \times K$ matrix $A$, $x_\mathbf{J}$ will denote the vector whose components are those of $x$ with indices in $\mathbf{J}$, and $A_\mathbf{J}$ will denote the principal submatrix of $A$ obtained by deleting those rows and columns of $A$ *not* indexed by elements of $\mathbf{J}$.

The proof of Theorem 1 is given in several steps. First, the result is proved for the case of zero drift ($\theta = 0$) and $\mathbf{J} = \{1, \ldots, K\}$. Then, using backwards induction it is extended to all $\mathbf{J} \subset \{1, \ldots, K\}$ with $|\mathbf{J}| \geqq 2$. Finally, using a Girsanov transformation, the result is extended to all constant drifts $\theta$.

Henceforth it is assumed that $Z$ is an SRBM associated with $(S, \theta, \Gamma, R)$. Then, by Theorem 2, $R$ is a completely-$\mathscr{S}$ matrix, and so by Lemma 3, $R'$ is completely-$\mathscr{S}$.

**Lemma 4.** *Suppose* $\theta = 0$ *and* $\mathbf{J} = \{1, \ldots, K\}$ *with* $K \geqq 2$. *Then for each* $k \in \mathbf{J}$ *and* $z \in S$, (3) *holds* $P_z$-a.s.

*Proof.* Fix $z \in S$. Then by the semimartingale representation (1) of $Z$ and Itô's formula, for any function $f$ that is twice continuously differentiable in some domain containing $S$, we have $P_z$-a.s. for all $t \geqq 0$:

$$f(Z(t)) - f(Z(0)) = \int_0^t \nabla f(Z(s))\, dX(s) + \sum_{k=1}^K \int_0^t v_k \cdot \nabla f(Z(s))\, dY_k(s) + \int_0^t Lf(Z(s))\, ds, \tag{8}$$

where $X$ is a $(0, \Gamma)$-Brownian motion, $v_k$ is the $k$-th column of the reflection matrix $R$ and

$$L = \frac{1}{2} \sum_{i,j=1}^K \Gamma_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

We shall substitute functions into (8) that allow us to estimate the left member of (3). Each such function is $L$-harmonic in some domain containing $S$ and for each $k$, its directional derivative in the direction $v_k$ is bounded below on $S$ and is very large positive near the origin. These functions are defined below as integrals of Newtonian potentials associated with a well chosen line of point sources located outside of $S$.

Since $R'$ is completely-$\mathscr{S}$, there is $\gamma \in \mathbf{R}_+^K$ and $\delta \in [1, \infty)^K$ such that

$$R' \gamma = \delta. \tag{9}$$

Define

$$\alpha = \Gamma \gamma. \tag{10}$$

For each $x \in S$ and $r \in (0, 1)$,

$$\begin{aligned} d^2(x,r) &\equiv (x+r\alpha)' \Gamma^{-1} (x+r\alpha) \\ &= x' \Gamma^{-1} x + 2r\alpha' \Gamma^{-1} x + r^2 \alpha' \Gamma^{-1} \alpha \\ &= x' \Gamma^{-1} x + 2r\gamma' x + r^2 \alpha' \Gamma^{-1} \alpha \\ &\geqq r^2 \hat{\alpha} \end{aligned} \tag{11}$$

where $\hat{\alpha} \equiv \alpha' \Gamma^{-1} \alpha = \gamma' \Gamma \gamma > 0$, since $\Gamma$ is symmetric and positive definite and $\gamma \neq 0$. Then, for each $\varepsilon \in (0, 1)$,

$$\phi_\varepsilon(x) \equiv \begin{cases} \dfrac{1}{2-K} \int_\varepsilon^1 r^{K-2} (d^2(x,r))^{\frac{2-K}{2}} \, dr, & K \geqq 3 \\ \frac{1}{2} \int_\varepsilon^1 \ln(d^2(x,r)) \, dr, & K=2 \end{cases} \tag{12}$$

is twice continuously differentiable in some domain containing $S$, and on each compact subset of $S$, it is bounded, uniformly in $\varepsilon$. Moreover, since the integrands in (12) are $L$-harmonic as functions of $x \in \mathbf{R}^K \setminus \{-r\alpha\}$, it is readily verified that for each $\varepsilon \in (0, 1)$,

$$L\phi_\varepsilon = 0 \tag{13}$$

in some domain containing $S$.

For the verification of the directional derivative properties of $\phi_\varepsilon$, for each $k \in \{1, \ldots, K\}$, let $u_k = (\Gamma^{-1})' v_k$. Then

$$u_k \cdot \alpha = (R' \Gamma^{-1} \alpha)_k = \delta_k \geqq 1. \tag{14}$$

Combining this with

$$\nabla \phi_\varepsilon(x) = \int_\varepsilon^1 r^{K-2} \Gamma^{-1} (x+r\alpha) (d^2(x,r))^{-\frac{K}{2}} \, dr, \tag{15}$$

yields

$$v_k \cdot \nabla \phi_\varepsilon(x) = \int_\varepsilon^1 r^{K-2} (u_k \cdot x + r\delta_k) (d^2(x,r))^{-\frac{K}{2}} \, dr. \tag{16}$$

Let $\beta_k = \delta_k / \|u_k\|$, where $\|\cdot\|$ is the Euclidean norm in $\mathbf{R}^K$. Then for $\varepsilon \in (0, 1)$ and $x \in S$ satisfying $\|x\| < \varepsilon \beta_k$, we have $|u_k \cdot x| < \varepsilon \delta_k$ and for $r > \varepsilon$,

$$\begin{aligned} d^2(x,r) &\leqq \|\Gamma^{-1}\| \, \|x+r\alpha\|^2 \\ &\leqq \|\Gamma^{-1}\| (\|x\| + \|r\alpha\|)^2 \\ &\leqq \|\Gamma^{-1}\| (\beta_k + \|\alpha\|)^2 r^2 \end{aligned} \tag{17}$$

where $\|\Gamma^{-1}\|$ denotes the norm of $\Gamma^{-1}$ as an operator from $\mathbf{R}^K$ to $\mathbf{R}^K$ with the Euclidean norm. Setting

$$c_k = (\|\Gamma^{-1}\| (\beta_k + \|\alpha\|)^2)^{-K/2} \delta_k$$

and substituting the above in (16) yields:

$$\begin{aligned} v_k \cdot \nabla \phi_\varepsilon(x) &\geqq c_k \int_\varepsilon^1 r^{K-2}(r-\varepsilon) r^{-K} \, dr \\ &\geqq -c_k[\ln \varepsilon + 1] \end{aligned} \tag{18}$$

for all $x \in S$ satisfying $\|x\| < \varepsilon \beta_k$. Note that for $\varepsilon$ small, the term in the last line above is large and positive.

Now, for any $x \in S$,

$$v_k \cdot \nabla \phi_\varepsilon(x) = -\delta_k \int_\varepsilon^1 r^{K-2}(\rho_k(x)-r)(d^2(x,r))^{-K/2} \, dr \tag{19}$$

where $\rho_k(x) \equiv -(u_k \cdot x)/\delta_k$. If $\rho_k(x) \leqq \varepsilon$, then the right member of (19) is non-negative. Thus, to obtain a lower bound for $v_k \cdot \nabla \phi_\varepsilon$ on $S$, it suffices to consider $x \in S$ such that $\rho_k(x) > \varepsilon$. For such $x$,

$$\begin{aligned} &\int_\varepsilon^1 r^{K-2}(\rho_k(x)-r)(d^2(x,r))^{-K/2} \, dr \\ &\quad \leqq \int_\varepsilon^{\rho_k(x)} r^{K-2}(\rho_k(x)-r)(d^2(x,r))^{-K/2} \, dr \\ &\quad \leqq (\rho_k(x)-\varepsilon) \max_{r \in [\varepsilon, \rho_k(x)]} \left\{ \frac{\rho_k(x)-r}{d^2(x,r)} \right\} \\ &\qquad \cdot \max_{r \in [\varepsilon, \rho_k(x)]} \left\{ \frac{r^{K-2}}{(d^2(x,r))^{(K-2)/2}} \right\}. \end{aligned} \tag{20}$$

Since $d^2(x,r)$ is a quadratic in $r$ with positive coefficients, the first maximum above is achieved at $r=\varepsilon$, and by (11), the second maximum is crudely dominated by $(\hat{\alpha})^{(2-K)/2}$. Thus, the last line of (20) is bounded above by

$$\frac{(\rho_k(x)-\varepsilon)^2}{d^2(x,\varepsilon)} (\hat{\alpha})^{(2-K)/2}. \tag{21}$$

Since $\Gamma^{-1}$ is positive definite, there is $\lambda > 0$ such that $x' \Gamma^{-1} x \geqq \lambda \|x\|^2$ and so (cf. (11)),

$$d^2(x,\varepsilon) \geqq \lambda \|x\|^2 + \varepsilon^2 \hat{\alpha} \geqq (\lambda \wedge \hat{\alpha})(\|x\|^2 + \varepsilon^2). \tag{22}$$

On the other hand, by the definition of $\rho_k(x)$,

$$\begin{aligned} (\rho_k(x)-\varepsilon)^2 &\leqq 2((\rho_k(x))^2 + \varepsilon^2) \\ &\leqq 2(\|u_k\|^2 \|x\|^2 \delta_k^{-2} + \varepsilon^2) \\ &\leqq 2(\|u_k\|^2 \delta_k^{-2} \vee 1)(\|x\|^2 + \varepsilon^2). \end{aligned} \tag{23}$$

It follows from (22)–(23) that (21) is bounded above by a constant not depending on $x$ or $\varepsilon$. Hence, there is $\hat{c}_k \geqq 0$ such that for all $x \in S$ and $\varepsilon \in (0, 1)$,

$$v_k \cdot \nabla \phi_\varepsilon(x) \geqq -\hat{c}_k. \tag{24}$$

We are now ready to prove (3) holds $P_z$-a.s. for each $k$. For each positive integer $m$, define

$$T_m = \inf\{t \geqq 0 \colon \|Z(t)\| \geqq m \ \text{ or } \ Y_k(t) \geqq m \ \text{for some } k\} \wedge m.$$

Replacing $f$ by $\phi_\varepsilon$ and $t$ by $T_m$ in (8), we see from (13) that $P_z$-a.s.:

$$\begin{aligned} \phi_\varepsilon(Z(T_m)) - \phi_\varepsilon(Z(0)) = & \int_0^{T_m} \nabla \phi_\varepsilon(Z(s))\, dX(s) \\ & + \sum_{k=1}^{K} \int_0^{T_m} v_k \cdot \nabla \phi_\varepsilon(Z(s))\, dY_k(s). \end{aligned} \tag{25}$$

Since $\phi_\varepsilon$ and its first derivatives are bounded on each compact subset of $S$, by the definition of the stopping time $T_m$ and since $\theta = 0$, the stochastic integral with respect to $dX$ in (25) has zero expectation. Thus, taking expectations in (25) yields:

$$\begin{aligned} E^{P_z}[\phi_\varepsilon(Z(T_m)) - \phi_\varepsilon(Z(0))] &= \sum_{k=1}^{K} E^{P_z}\left[\int_0^{T_m} v_k \cdot \nabla \phi_\varepsilon(Z(s))\, dY_k(s)\right] \\ &\geqq -(\ln \varepsilon + 1) \sum_{k=1}^{K} c_k E^{P_z}\left[\int_0^{T_m} 1_{\{\|Z(s)\| < \varepsilon \beta_k\}}\, dY_k(s)\right] - \sum_{k=1}^{K} \hat{c}_k E^{P_z}[Y_k(T_m)], \end{aligned} \tag{26}$$

where the lower bounds (18) and (24) have been used to obtain the last inequality. Now, the left member of (26) is bounded as $\varepsilon \downarrow 0$, since $\phi_\varepsilon$ is uniformly bounded on compact subsets of $S$. Also, the last sum in (26) is independent of $\varepsilon$. Thus, dividing (26) by $-(\ln \varepsilon + 1)$ and letting $\varepsilon \downarrow 0$ yields:

$$\lim_{\varepsilon \downarrow 0} \sum_{k=1}^{K} c_k E^{P_z}\left[\int_0^{T_m} 1_{\{\|Z(s)\| < \varepsilon \beta_k\}}\, dY_k(s)\right] \leqq 0.$$

Since each term in the above sum is non-negative and $c_k > 0$, it follows by Fatou's lemma that for each $k \in \{1, \ldots, K\}$,

$$\int_0^{T_m} 1_{\{0\}}(Z(s))\, dY_k(s) = 0 \ P_z\text{-a.s.}$$

Letting $m \to \infty$ yields the desired result. □

**Lemma 5.** *Suppose $\theta = 0$. Then Theorem 1 holds.*

*Proof.* Our proof is by backward induction on $|\mathbf{J}|$. By Lemma 1, the result holds for $|\mathbf{J}| = K$. Now, suppose it holds for all $\mathbf{J}$ such that $j < |\mathbf{J}| \leqq K$, some

$j \geqq 2$. Fix $\mathbf{J} \subset \{1, \ldots, K\}$ such that $|\mathbf{J}|=j$, let $k \in \mathbf{J}$ and $z \in S$. Then, by the induction hypothesis, $P_z$-a.s.,

$$\int_0^\infty 1_{\{Z_{\mathbf{J}}(s)=0\}}\, dY_k(s) = \int_0^\infty 1_{\{Z_{\mathbf{J}}(s)=0,\, Z_{\mathbf{J}^c}(s)>0\}}\, dY_k(s)$$

where $\mathbf{J}^c \equiv \mathbf{K} \setminus \mathbf{J}$. Thus, by monotone convergence, it suffices to prove for each $\eta \in \mathbf{R}_+^{K-j}$ satisfying $\eta > 0$, that

$$\int_0^\infty 1_{\{Z_{\mathbf{J}}(s)=0,\, Z_{\mathbf{J}^c}(s)>\eta\}}\, dY_k(s) = 0 \quad P_z\text{-a.s.} \tag{27}$$

For this, fix an $\eta$ and define a sequence of stopping times $\{T_m\}$ as follows.

$$T_0 \equiv 0$$
$$T_1 = \inf\{s \geqq 0 \colon Z_i(s) < \eta_i/2 \ \text{ for some } i \in \mathbf{J}^c\}$$
$$T_2 = \inf\{s \geqq 0 \colon Z_{\mathbf{J}^c}(s) > \eta\}$$

and for $m \geqq 1$,

$$T_{2m+1} = T_{2m} + T_1 \circ \theta_{T_{2m}}$$
$$T_{2m+2} = T_{2m+1} + T_2 \circ \theta_{T_{2m+1}}$$

where $\theta_t$ is the usual shift operator, defined by

$$\theta_t(Z(\cdot)) = \begin{cases} Z(\cdot + t) & \text{for } t < \infty, \\ \partial & \text{for } t = \infty, \end{cases}$$

and $\partial$ is a cemetery point isolated from $S$. By the continuity of the paths of $Z$, $T_m \to \infty$ as $m \to \infty$, and we have $P_z$-a.s.:

$$\int_0^\infty 1_{\{Z_{\mathbf{J}}(s)=0,\, Z_{\mathbf{J}^c}(s)>\eta\}}\, dY_k(s) \leqq \sum_{m=1}^\infty \int_{T_{2m}}^{T_{2m+1}} 1_{\{Z_{\mathbf{J}}(s)=0\}}\, dY_k(s). \tag{28}$$

Although it is possible that $T_2 < T_1$, this does not affect the validity of (28). Consider $m \geqq 1$. Then for $i \in \mathbf{J}^c$, $Y_i$ increases only when $Z_i = 0$ and so $P_z$-a.s.: $(Y(t) - Y(T_{2m}))_{\mathbf{J}^c} = 0$ for all $t \in [T_{2m}, T_{2m+1}]$. Thus, on $\{T_{2m} < \infty\}$, we have $P_z$-a.s. for all $t \in [0, T_1 \circ \theta_{T_{2m}}]$

$$Z_{\mathbf{J}}(t + T_{2m}) - Z_{\mathbf{J}}(T_{2m}) = X_{\mathbf{J}}(t + T_{2m}) - X_{\mathbf{J}}(T_{2m}) + R_{\mathbf{J}}(Y_{\mathbf{J}}(t + T_{2m}) - Y_{\mathbf{J}}(T_{2m})).$$

Then Itô's formula (8) holds on $\{T_{2m} < \infty\}$ for $f \in C^2(\mathbf{R}_+^j)$ with $(X, Y, Z)$ replaced by $(X_{\mathbf{J}}, Y_{\mathbf{J}}, Z_{\mathbf{J}})\,((\cdot + T_{2m}) \wedge T_{2m+1})$ and with

$$L = \frac{1}{2} \sum_{i, l \in \mathbf{J}} \Gamma_{il} \frac{\partial^2}{\partial x_i \partial x_l}.$$

The same proof as in Lemma 4, but with the dimension reduced from $K$ to $j=|\mathbf{J}|$, shows that

$$1_{\{T_{2m}<\infty\}} \int_{T_{2m}}^{T_{2m+1}} 1_{\{Z_J(s)=0\}}\, dY_k(s)=0 \;\; P_z\text{-a.s.} \tag{29}$$

For this, one uses the strong Markov property of $X$ and the assumption that there is $\gamma\in\mathbf{R}^j_+$ and $\delta\in[1,\infty)^j$ such that $R'_{\mathbf{J}}\gamma=\delta$. Substituting (29) in (28) then yields the desired result. □

**Lemma 6.** *Theorem 1 holds for all $\theta\in\mathbf{R}^K$.*

*Proof.* Let $\mathbf{J}\subset\{1,\dots,K\}$ satisfying $|\mathbf{J}|\geqq 2$, $k\in\mathbf{J}$, and $\theta\in\mathbf{R}^K$. Suppose $Z$, with associated probability measures $\{P^\theta_z, z\in S\}$, is an SRBM with data $(S,\theta,\Gamma,R)$. Then for each $z\in S$, since $X$ is a $(\theta,\Gamma)$-Brownian motion on $(\Omega,\mathscr{F},\{\mathscr{F}_t\},P^\theta_z)$, by the Girsanov transformation [5, p. 176], there is a probability measure $P^0_z$ on $(\Omega,\mathscr{F})$ such that under $P^0_z$, $X$ is a $(0,\Gamma)$-Brownian motion starting from $z$ and for each positive integer $m$, $P^\theta_z$ and $P^0_z$ are mutually absolutely continuous on $\mathscr{F}_m$. It follows that $Z$ with the probability measures $\{P^0_z, z\in S\}$ on $(\Omega,\mathscr{F},\{\mathscr{F}_t\})$ is an SRBM for the data $(S,0,\Gamma,R)$. Then by Lemma 5, (3) holds $P^0_z$-a.s. for each $z\in S$. But since $P^\theta_z$ and $P^0_z$ are mutually absolutely continuous on $\mathscr{F}_m$, it follows that (3) holds $P^\theta_z$-a.s. with $m$ in place of the upper limit $\infty$ there. Letting $m\to\infty$ yields the desired result. □

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