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Probability Theory and Related Fields

Stationary States and their Stability of the Stepping Stone Model Involving Mutation and Selection

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1. Introduction

The stepping stone model is a generalization of the Wright-Fisher model in population genetics, by taking account of geographical structure, [7], [8], [12]. For this we are particularly interested in the influence of geographical factors on stationary states in the genetical evolution. The model is originally formulated as a discrete time Markov process with values in an infinite product space which describes an evolution of a population consisting of a number of colonies, having non-overlapping generations. However we here treat a diffusion model which, obtained by taking a diffusion approximation, is an infinite dimensional diffusion process.

In the absence of mutation and selective force one of the authors obtained in [13] a complete characterization of stationary states in terms of migration rates as geographical factors. In the present paper we consider the stepping stone model involving mutation and selection, and investigate stationary states and their stability paying our attention to the mutual influences of mutation, selection and migration. Furthermore we also discuss regularity of finite dimensional marginal distributions of the stationary states.

For the discrete time model it is to be noted that analogues to some of our results are recently obtained by Itatsu in [6], but our method can essentially cover the discrete time case.

Before stating the present problem and our results we here give a description of the discrete time model and briefly survey the diffusion approximation of it.

Let S be a countable set. Each element i of S corresponds to a subpopulation, which is called a colony. Assuming that there are two alleles A_1 and A_2 in each colony, we denote by $0 \leq x_i \leq 1$ the gene frequency of the A_1-allele in the colony i. We assume that a genetical evolution is caused by migration among colonies, and mutation, selection and random sampling drift within each colony.

Let N be the population size of each colony, and let

$$\begin{aligned} X_N &= \{0, 1/N, 2/N, \dots, 1\}^S \\ &= \{x=\{x_i\}: x_i \in \{0, 1/N, 2/N, \dots, 1\} \quad \textit{for all } i \in S\} \end{aligned}$$

be the totality of configulations of gene frequencies of the population size N. Then the time evolution is defined as a discrete time Markov process $\{x^{(N)}(n), P_x\}$ taking values in X_N with the following transition probability:

$$(1.1) \qquad \begin{aligned} &P_x[x_i^{(N)}(n+1)=k_i/N, i\in A \mid x^{(N)}(n)] \\ &= \prod_{i\in A} \binom{N}{k_i} \rho_i^{(N)}(x^{(N)}(n))^{k_i}(1-\rho_i^{(N)}(x^{(N)}(n)))^{N-k_i} \end{aligned}$$

for any finite subset $A \subset S$ and $0 \leqq k_i \leqq N$ $(i \in A)$. Here $\rho^{(N)}(x)$ involves genetical factors, which is assumed to be determined in the following way:

(1) Let $u_i^{(N)} \geqq 0$ $(v_i^{(N)} \geqq 0)$ be a mutation rate from A_1 to A_2 (from A_2 to A_1) in the colony i, and set

$$\rho^1(x) = \{\rho_i^1(x) = (1-u_i^{(N)})x_i + v_i^{(N)}(1-x_i)\},$$

(2) let $q_{ij}^{(N)} \geqq 0$ be a migration rate from i to j satisfying $\sum_{j\in S} q_{ij}^{(N)} = 1$ for any $i \in S$, and set

$$\rho^2(x) = \{\rho_i^2(x) = \sum_{j\in S} q_{ji}^{(N)} x_j / \sum_{k\in S} q_{ki}^{(N)}\}, \quad \text{and}$$

(3) let $-1 < s_i^{(N)} < \infty$ be a relative fitness of A_1 in the colony i, and set

$$\rho^3(x) = \{\rho_i^3(x) = (1+s_i^{(N)})x_i / [(1+s_i^{(N)})x_i + (1-x_i)]\}.$$

Then $\rho^{(N)}(x)$ is defined by their composition, namely

$$\rho^{(N)}(x) = \rho^3(\rho^2(\rho^1(x))).$$

In order to get a diffusion approximation we assume that

$$(1.2) \qquad \begin{aligned} &u_i^{(N)} = u_i/N + o(1/N), \qquad v_i^{(N)} = v_i/N + o(1/N), \\ &q_{ij}^{(N)} = q_{ij}/N + o(1/N) \qquad (i \neq j), \\ &q_{ii}^{(N)} = 1 - \sum_{j \neq i} q_{ij}^{(N)}, \quad \text{and} \quad s_i^{(N)} = s_i/N + o(1/N). \end{aligned}$$

In addition, we impose the following technical assumption:

$$(1.3) \qquad \sup_{i\in S}\Big(\sum_{j\neq i} q_{ji} + u_i + v_i + |s_i|\Big) < +\infty.$$

Then it is not hard to show that if $x_N \to x \in X$ as $N \to \infty$, $\{x^{(N)}(t) = x^{(N)}([Nt]), P_{x_N}\}$ converges to a Feller diffusion process $\{x(t), P_x\}$ taking values in $X = [0,1]^S = \{x = \{x_i\}: 0 \leqq x_i \leqq 1 \textit{ for all } i \in S\}$ and governed by the generator L,

$$(1.4) \qquad L = \tfrac{1}{2}\sum_{i\in S} a_i(x)\frac{\partial^2}{\partial x_i^2} + \sum_{i\in S} b_i(x)\frac{\partial}{\partial x_i},$$

where

$$a_i(x) = x_i(1-x_i)$$

$$b_i(x) = \sum_{j \neq i} m_{ji}(x_j - x_i) + v_i - (u_i + v_i)\,x_i + s_i\,x_i(1-x_i)$$

with $m_{ji} = q_{ji} / \sum_{k \neq i} q_{ki}$.

Now we state our model and results about it. Let $C(X)$ be the Banach space of all continuous functions on X with the supremum norm $\| \ \|$ (X is equipped with the product topology), and let $C_f^m(X)$ be the totality of C^m-functions on X depending only on finitely many coordinates.

Let L be the operator formally given in (1.4) and defined on $C_f^2(X)$. Under the assumption (1.3) it is known that there exists a unique semi-group $\{T_t\}$ of contraction operators on $C(X)$ such that

(1.5) $$T_t 1 = 1,\ T_t f \geqq 0 \quad \text{for any } f \geqq 0 \quad \text{of} \quad C(X), \quad \text{and}$$

$$T_t f = \int_0^t T_s L f\, ds \quad \text{for any} \quad f \in C_f^2(X),$$

which is associated with the above diffusion process $\{x(t), P_x\}$ (cf. [14], [2]).

Let $\mathscr{P}(X)$ be the totality of probability measures on X, equipped with the topology of weak convergence. Denote by T_t^* the adjoint operator on $P(X)$ associated with T_t, and denote by $\mathscr{S}$ the totality of stationary states of $\{T_t\}$, that is

$$\mathscr{S} = \{\mu \in \mathscr{P}(X) \colon T_t^* \mu = \mu \quad \textit{for all } t \geqq 0\}.$$

Then $\mathscr{S}$ is a non-empty, convex and compact set. We denote by $\mathscr{S}_{\text{ext}}$ the totality of extremal elements of $\mathscr{S}$.

In this paper we restrict our consideration to the following class of parameters:

(1.6) $S = Z^d$ (d-dimensional integer lattice space); u_i, v_i, and s_i are independent of $i \in S$; and $m_{ji} = m_{j-i,0}$.

Accordingly, we write $u_i = u$, $v_i = v$, $s_i = s$, and $m_{i,0} = m_i$. Furthermore we assume that $\{m_i\}$ is irreducible, namely $\{i \in Z^d \colon m_i > 0\}$ generates Z^d as an additive group.

Let $\mathscr{I}^{Z^d}$ be the totality of Z^d-translation-invariant probability measures on $X = [0,1]^{Z^d}$. Then $\mu \in \mathscr{I}^{Z^d}$ is extremal in $\mathscr{I}^{Z^d}$ if and only if μ is ergodic with respect to Z^d-translations.

According to values of the parameters u, v and s, let us introduce the following classification, which exhausts all possibilities:

Case 1. $u = v = s = 0$,
Case 2. $u > 0$, $v > 0$, and s is arbitrary,
Case 3. $u > 0$, $v = 0$ (or $u = 0$, $v > 0$) and s is arbitrary,
Case 4. $u = v = 0$, and $s \neq 0$.

Let $M=\{m_{ij}\}$ with $m_{ii}=m_0=-\sum_{j\neq 0} m_j$, and set $P_t=\exp tM^*$, where M^* stands for the transposed matrix of M. Then P_t is a transition matrix of a continuous time random walk on Z^d. For $x\in X$ δ_x denotes the Dirac measure at x. In particular $\delta_{\mathbf{1}}$ and $\delta_{\mathbf{0}}$ are called genetical uniform states, where $\mathbf{c}=\{x_i\equiv c\}\in X$ for $0\leqq c\leqq 1$. Then the following theorem is known. (See [13].)

Theorem 1.1. *Assume Case 1. If $d=1$ or 2, then $\mathscr{S}_{\text{ext}}=\{\delta_{\mathbf{0}},\delta_{\mathbf{1}}\}$; moreover for $\mu\in\mathscr{P}(X)$, $\lim_{t\to\infty} T_t^*\mu=\lambda\delta_{\mathbf{1}}+(1-\lambda)\delta_{\mathbf{0}}$ holds if and only if*

$$\lim_{t\to\infty}\sum_{j\in Z^d} P_t(i,j)\int x_j\,\mu(dx)=\lambda \quad \textit{for all } i\in Z^d.$$

If $d\geqq 3$, then $\mathscr{S}_{\text{ext}}=\{\nu_c\colon 0\leqq c\leqq 1\}$ where $\nu_c=\lim_{t\to\infty} T_t^*\delta_{\mathbf{c}}$. Moreover for $\mu\in\mathscr{P}(X)$, $\lim_{t\to\infty} T_t^*\mu$ exists if and only if $\lim_{t\to\infty}\pi_t\mu(\equiv\mu_\infty)$ exists, where

$$\pi_t x=\{(\pi_t x)_i=\sum_{j\in Z^d} P_t(i,j)\,x_j\}\in X,$$

and $\pi_t\mu$ denotes the image measure of μ by π_t; in this case, $\mu_\infty[x=\mathbf{c}$ for some $0\leqq c\leqq 1]=1$ and

$$\lim_{t\to\infty} T_t^*\mu=\int_{[0,1]}\nu_c\,\mu_\infty(d\mathbf{c}).$$

Our main results of this paper are the following theorems:

Theorem 1.2. *Assume Case 2. For every s there exists a unique stationary state ν, and $\lim_{t\to\infty} T_t^*\mu=\nu$ holds for all $\mu\in\mathscr{P}(X)$.*

Theorem 1.3. *Assume Case 3. If $u>0$ and $v=0$ there exists a critical parameter $0<s_c<\infty$ such that if $s<s_c$, $\delta_{\mathbf{0}}$ is a unique stationary state and $\lim_{t\to\infty} T_t^*\mu=\delta_{\mathbf{0}}$ holds for every $\mu\in\mathscr{P}(X)$, while if $s>s_c$ there exists another extremal stationary state ν and $(\mathscr{S}\cap\mathscr{I}^{Z^d})_{\text{ext}}=\{\delta_{\mathbf{0}},\nu\}$. Furthermore, if $s>s_c$, ν is stable and $\delta_{\mathbf{0}}$ is unstable in the following sense:*

$$\lim_{t\to\infty} T_t^*\mu=\nu \textit{ holds for every } \mu\in\mathscr{I}^{Z^d} \textit{ satisfying } \mu[\{\mathbf{0}\}]=0.$$

If $u=0$ and $v>0$ the same statement is valid by replacing **0** *by* **1**.

Theorem 1.4. *Assume Case 4. Then $(\mathscr{S}\cap\mathscr{I}^{Z^d})_{\text{ext}}=\{\delta_{\mathbf{0}},\delta_{\mathbf{1}}\}$. Furthermore, under an additional assumption: $\sum_{i\in Z^d}|i|\,m_i<\infty$, if $s>0$, $\delta_{\mathbf{1}}$ is stable and $\delta_{\mathbf{0}}$ is unstable, while if $s<0$ $\delta_{\mathbf{0}}$ is stable and $\delta_{\mathbf{1}}$ is unstable in the sense of Theorem* 1.3.

Next, we will discuss regularity of finite dimensional marginal distributions of stationary states.

Let $\mu\in\mathscr{P}(X)$. We write $A\Subset Z^d$ if A is a finite subset of Z^d. Denote by μ_A the marginal distribution of μ on X_A on $X_A=[0,1]^A$. Then we obtain

Theorem 1.5. *Let ν be any extremal stationary state other than δ_0 and δ_1, appearing in Theorems 1.1 to 1.3. Then for every $A \Subset Z^d$, ν_A is absolutely continuous with respect to the Lebesgue measure on X_A, and admits a probability density $p_A(x_A)$ that is strictly positive and of C^∞-class in $\mathring{X}_A = (0,1)^A$.*

Our basic tool for the proof of these results is the dual process, which has been used very successfully in the theory of interacting particle systems. It would be worthwhile to note that our dual process indeed is a branching random walk with an interaction, of which extinction problem leads us to the proof of Theorem 1.3 and 1.4. Readers unfamiliar with such field we refer to Liggett's book [10].

2. Dual Process

In what follows we may assume that the selective fitness $s \leqq 0$, because the case $s > 0$ can be reduced to this case by interchanging the roles of A_1 and A_2. In fact changing all the variables x_i according to $x_i \to 1 - x_i$ transforms the operator L and this transformation simply amounts to replacing s, u and v by $-s$, v and u, respectively.

Let I be the totality of multi-indices on Z^d, that is

$$I = \{\alpha = \{\alpha_i\}_{i \in Z^d} \colon \alpha_i \in \{0, 1, 2, \ldots\}, \quad \text{and} \quad |\alpha| = \sum_{i \in Z^d} \alpha_i < \infty\}.$$

If $\alpha_i = 0$ for all $i \in Z^d$, α is denoted by 0 and if $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$, α is denoted by ε_i. We write $\alpha \geqq \beta$ if $\alpha_i \geqq \beta_i$ for all $i \in Z^d$. For α and β of I we define $\alpha + \beta$ and $\alpha - \beta$ (if $\alpha \geqq \beta$) by componentwise addition and subtraction. For $\alpha \in I$ set $f_\alpha(x) = \prod_{i \in Z^d} x_i^{\alpha_i}$ and $f_0(x) \equiv 1$. Then the linear hull of $\{f_\alpha(x)\}_{\alpha \in I}$ is dense in $C(X)$.

Let $(\alpha(t), \mathbf{P}_\alpha)_{\alpha \in I}$ be a continuous time Markov chain generated by the infinitesimal matrix $\{Q_{\alpha,\beta}\}$,

$$\begin{aligned}
(2.1) \qquad Q_{\alpha, \alpha - \varepsilon_i} &= \tfrac{1}{2} \alpha_i (\alpha_i - 1) + \alpha_i v, \\
Q_{\alpha, \alpha + \varepsilon_i} &= \alpha_i (-s), \\
Q_{\alpha, \alpha - \varepsilon_i + \varepsilon_j} &= \alpha_i m_{ji} = \alpha_i m_{j-i} \qquad (i \neq j), \\
Q_{\alpha, \beta} &= 0 \qquad \text{for all the other } \beta (\neq \alpha).
\end{aligned}$$

(Here and below $Q_{\alpha,\alpha}$ is understood to be $-\sum_{\beta \neq \alpha} Q_{\alpha,\beta}$.) Noting that $Lf_\alpha(x) = \sum_{\beta \in I} Q_{\alpha,\beta}(f_\beta(x) - f_\alpha(x)) - u|\alpha| f_\alpha(x)$, by the Feynman-Kac formula we have the following relation of duality between $\{T_t\}$ and $(\alpha(t), \mathbf{P}_\alpha)$.

Lemma 2.1.

$$\langle T_t^* \mu, f_\alpha \rangle = \mathbf{E}_\alpha \left[\langle \mu, f_{\alpha(t)} \rangle \exp\left(-u \int_0^t |\alpha(s)|\, ds \right) \right].$$

Thus we see that the dual process $(\alpha(t), \mathbf{P}_\alpha)$ has a complete information of $T_t^* \mu$. It also is interesting to understand the dual process $(\alpha(t), \mathbf{P}_\alpha)$ in the following way. Ignoring the quadratic term of $Q_{\alpha, \alpha-\varepsilon_i}$, it is nothing but a branching random walk with a binary branching rate $\gamma=-s$, a death rate v and a transition rate $\{m_i\}$. In addition, an interaction works on it so that multiple occupancies may make the death rate increase rapidly. Thus we may say that it is a branching random walk with an interaction.

Now the proof of Theorem 1.2 follows immediately from this lemma. Let σ_0 be the extinction time: $\sigma_0=\inf\{t \geqq 0: \alpha(t)=0\}$. Since $|\alpha(t)| \geqq 1$ for all $t<\sigma_0$, we see that

$$\begin{aligned}\lim_{t\to\infty} \langle T_t^* \mu, f_\alpha\rangle &= \lim_{t\to\infty} \mathbf{E}_\alpha\left[\langle \mu, f_{\alpha(t)}\rangle \exp\left(-u\int_0^t |\alpha(s)|\, ds\right); t \leqq \sigma_0\right] \\ &\quad + \lim_{t\to\infty} \mathbf{E}_\alpha\left[\langle \mu, f_{\alpha(t)}\rangle \exp\left(-u\int_0^t |\alpha(s)|\, ds\right); t > \sigma_0\right] \\ &= \mathbf{E}_\alpha\left[\exp\left(-u\int_0^{\sigma_0} |\alpha(s)|\, ds\right); \sigma_0 < \infty\right],\end{aligned}$$

which completes the proof of Theorem 1.2.

It is well noted that the positivity of v is not used in the proof given above. Since $\sigma_0=\infty$ a.s. if $v=0$, we actually proved that $\lim_{t\to\infty} T_t^* \mu=\delta_0$ (resp. δ_1) if $u>0$, $v=0$ and $s \leqq 0$ (resp. $u=0$, $v>0$ and $s \geqq 0$).

For the remaining cases it suffices to discuss under the assumption,

$$\gamma(=-s) \geqq 0, \quad v \geqq 0 \quad \text{and} \quad u=0. \tag{2.2}$$

Then the duality relation turns into

$$\langle T_t^* \mu, f_\alpha\rangle = \mathbf{E}_\alpha[\langle \mu, f_{\alpha(t)}\rangle] \quad \text{for } \alpha \in I. \tag{2.3}$$

For two stochastic processes $(\alpha(t), \mathbf{P}^1)$ and $(\alpha(t), \mathbf{P}^2)$ taking values in I we write $(\alpha(t), \mathbf{P}^1) \geqq (\alpha(t), \mathbf{P}^2)$, if

$$\mathbf{E}^1[f(\alpha(t))] \geqq \mathbf{E}^2[f(\alpha(t))] \quad \text{for all } t \geqq 0 \tag{2.4}$$

for all bounded and monotone non-decreasing function f defined on I. Here f is called to be monotone non-decreasing if $f(\alpha) \geqq f(\beta)$ holds whenever $\alpha \geqq \beta$.

The following lemma is intuitively clear from the form of the generator of the dual process (for the proof see Corollary A.1 and Corollary A.3 in Appendix.

Lemma 2.2.

(i) $(\alpha(t), \mathbf{P}_\alpha) \leqq (\alpha(t), \mathbf{P}_\beta)$ *if* $\alpha \leqq \beta$,

(ii) $(\alpha(t), \mathbf{P}_{\alpha+\beta}) \leqq (\alpha^1(t)+\alpha^2(t), \mathbf{P}_\alpha \otimes \mathbf{P}_\beta)$,

where $(\alpha^k(t), \mathbf{P}_\alpha)$, $k=1, 2$, *are copies of* $(\alpha(t), \mathbf{P}_\alpha)$ *and* $\mathbf{P}_\alpha \otimes \mathbf{P}_\beta$ *stands for the product measure of* $\mathbf{P}_\alpha$ and $\mathbf{P}_\beta$.

Let S_k be the translation operator by $k \in Z^d$, i.e. $S_k \alpha=\{\alpha_{i+k}\}$, $S_k x=\{x_{i+k}\}$, $S_k f(x)=f(S_k x), \ldots$, etc.

Lemma 2.3. *For $t>0$ and $m\geqq 2$ and for $f_1, f_2, \dots, f_m$ from $C(X)$*

$$\|T_t((S_{a_1}f_1)(S_{a_2}f_2)\dots(S_{a_m}f_m))-(S_{a_1}T_tf_1)(S_{a_2}T_tf_2)\dots(S_{a_m}T_tf_m)\|$$

vanishes as $\min_{1\leqq i\neq j\leqq m} |a_i-a_j|\to\infty$.

Proof. Since the proof is essentially the same for all $m\geqq 2$ we will prove the lemma for $m=2$ only. It is enough to show it for $f_1=f_\alpha$ and $f_2=f_\beta$ with every α and β from I. Let $\tau=\inf\{t>0\colon \alpha_i(t)>0$ *and* $\beta_i(t)>0$ *for some* $i\in Z^d\}$ be the collision time of two independent processes $\alpha(t)$ and $\beta(t)$. Notice that $S_kT_t=T_tS_k$, $S_kf_\alpha=f_{S_{-k}\alpha}$ and $(\alpha(t), \mathbf{P}_{S_k\alpha})=(S_k\alpha(t), \mathbf{P}_\alpha)$. Then for $t>0$ we have

$$\begin{aligned}
&|T_t(f_\alpha S_{-k}f_\beta)(x)-T_tf_\alpha(x)\,T_tS_{-k}f_\beta(x)|\\
&=|T_tf_{\alpha+S_k\beta}(x)-T_tf_\alpha(x)\,T_tf_{S_k\beta}(x)|\\
&=|\mathbf{E}_{\alpha+S_k\beta}[f_{\alpha(t)}(x)]-\mathbf{E}_\alpha\otimes\mathbf{E}_{S_k\beta}[f_{\alpha(t)+\beta(t)}(x)]|\\
&\leqq \mathbf{P}_\alpha\otimes\mathbf{P}_{S_k\beta}[\tau<t]\to 0 \quad\text{as}\quad |k|\to\infty,
\end{aligned}$$

because the sum of the two independent processes $\alpha(t)$ and $\beta(t)$ is identical in law to the single process starting from $\alpha(0)+\beta(0)$ until the collision time τ.

For $\alpha\in I$ we write $\operatorname{supp}\alpha=\{i\in Z^d\colon \alpha_i>0\}$, and denote by $|\operatorname{supp}\alpha|$ the cardinality of $\operatorname{supp}\alpha$.

Lemma 2.4. *For any* $N\geqq 1$

$$\lim_{t\downarrow 0}\lim_{n\to\infty}\mathbf{P}_{n\varepsilon_0}[|\operatorname{supp}\alpha(t)|\geqq N]=1.$$

Proof. For $a\in Z^d$ satisfying $m_a>0$, we introduce a simpler Markov chain $(\bar\alpha(t), \bar{\mathbf{P}}_\alpha)$ on I, governed by the following infinitesimal matrix $\{\bar Q_{\alpha,\beta}\}$:

$$\begin{aligned}
\bar Q_{\alpha,\alpha-\varepsilon_i}&=\tfrac12\alpha_i(\alpha_i-1)+\alpha_i(v+\sum_{j\neq 0,a} m_j),\\
\bar Q_{\alpha,\alpha-\varepsilon_i+\varepsilon_{i+a}}&=\alpha_i m_a,\\
\bar Q_{\alpha,\beta}&=0 \qquad\text{for all the other } \beta \text{ with } \beta\neq\alpha.
\end{aligned}$$

By Corollary A.2 in Appendix it is easily seen that $(\alpha(t), \mathbf{P}_\alpha)\geqq(\bar\alpha(t), \bar{\mathbf{P}}_\alpha)$ for all $\alpha\in I$. Let us introduce a sequence of stopping times $\{\tau_N\}_{N=0}^\infty$; $\tau_0=0$, $\tau_1=\inf\{t>0\colon \bar\alpha_a(t)=\bar\alpha_a(t-)+1$ and $\bar\alpha_0(t)=\bar\alpha_0(t-)-1\}, \dots, \tau_{N+1}=\tau_1\cdot\theta_{\tau_N}$, where θ_t stands for the shift operator.

We first claim that

$$\lim_{n\to\infty}\mathbf{P}_{n\varepsilon_0}[\tau_N<t]=1 \qquad \text{for all } t>0 \quad\text{and}\quad N\geqq 1. \tag{2.5}$$

Using the strong Markov property we have for $N\geqq 1$

$$\begin{aligned}
\bar{\mathbf{E}}_{n\varepsilon_0}[\exp(-\tau_N)]=&\frac{\delta(n)}{1+q(n)}\bar{\mathbf{E}}_{(n-1)\varepsilon_0}[\exp(-\tau_N)]\\
&+\frac{nm_a}{1+q(n)}\bar{\mathbf{E}}_{(n-1)\varepsilon_0+\varepsilon_a}[\exp(-\tau_{N-1})],
\end{aligned}$$

where

$$q(n)=\tfrac{1}{2}n(n-1)+n(v+\sum_{j\neq 0} m_j) \quad \text{and} \quad \delta(n)=q(n)-n\,m_a.$$

Set $u^N(n)=\bar{\mathbf{E}}_{n\varepsilon_0}[\exp(-\tau_N)]$. Since $u^N(n)=\bar{\mathbf{E}}_{n\varepsilon_0+\varepsilon_a}[\exp(-\tau_N)]$,

$$u^N(n)-u^N(n-1)=\frac{1+nm_a}{1+q(n)}(u^{N-1}(n-1)-u^N(n-1)) - \frac{1}{1+q(n)}u^{N-1}(n-1). \tag{2.6}$$

We observe that the sum of the left-hand sides and that of the second terms of the right-hand sides over n are bounded and that $u^{N-1}(n)\geqq u^N(n)$ to obtain

$$\sum_{n=1}^{\infty}\frac{1+nm_a}{1+q(n)}|u^{N-1}(n-1)-u^N(n-1)|<\infty,$$

which together with $\sum_{n=1}^{\infty}\frac{1+nm_a}{1+q(n)}=\infty$ implies that $\lim_{n\to\infty}(u^{N-1}(n)-u^N(n))=0$. Since $u^0(n)=1$, by induction $\lim_{n\to\infty}u^N(n)=1$; hence we have (2.5).

Next, we show that

$$\lim_{n\to\infty}\bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}_a(\tau_N)=N]=1 \qquad \text{for all } N\geqq 1. \tag{2.7}$$

It is trivial for $N=1$. Assume that (2.7) holds for some N. Using (2.5) and Lemma 2.2 we see that for all $m\geqq 1$ and $t>0$,

$$\begin{aligned}&\lim_{n\to\infty}\bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}_0(\tau_N)\geqq m]\\ &\quad\geqq \lim_{n\to\infty}\bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}_0(s)\geqq m \quad \textit{for all } s\in[0,t], \quad \textit{and} \quad \tau_N<t]\\ &\quad\geqq \bar{\mathbf{P}}_{m\varepsilon_0}[\bar{\alpha}_0(s)\geqq m \quad \textit{for all } s\in[0,t]].\end{aligned}$$

Thus, letting $t\to 0$, we have

$$\lim_{n\to\infty}\bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}_0(\tau_N)\geqq m]=1. \tag{2.8}$$

Using (2.5) again it follows that for all $k\geqq 1$

$$\begin{aligned}&\lim_{n\to\infty}\bar{\mathbf{P}}_{n\varepsilon_0+k\varepsilon_a}[\bar{\alpha}_a(\tau_1)=k+1]\\ &\quad\geqq \lim_{n\to\infty}\bar{\mathbf{P}}_{n\varepsilon_0}\otimes\bar{\mathbf{P}}_{k\varepsilon_a}[\bar{\alpha}^2(s)=k\varepsilon_a \quad \textit{for all } s\in[0,\tau_1^1)]\\ &\quad=1,\end{aligned} \tag{2.9}$$

where τ_1^1 stands for the corresponding one to τ_1 for the copy $(\bar{\alpha}^1(t),\bar{\mathbf{P}}_{n\varepsilon_0})$.

Combining these with the assumption of induction we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}_a(\tau_{N+1})=N+1] \\
&\quad =\lim_{n\to\infty} \sum_{m=0}^{\infty} \bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}_\alpha(\tau_N)=N, \bar{\alpha}_0(\tau_N)=m]\,\mathbf{P}_{m\varepsilon_0+N\varepsilon_a}[\bar{\alpha}_a(\tau_1)=N+1] \\
&\quad =1,
\end{aligned}$$

which proves (2.7).

Furthermore, using (2.5) and (2.7) we see that

$$\begin{aligned}
&\bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}_a(t)\geqq N] \\
&\quad \geqq \bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}_a(t)\geqq N, \tau_N<t \quad \textit{and} \quad \bar{\alpha}_\alpha(\tau_N)=N] \\
&\quad \geqq \bar{\mathbf{P}}_{n\varepsilon_0}[\tau_N<t, \bar{\alpha}_a(\tau_N)=N]\,\bar{\mathbf{P}}_{N\varepsilon_a}[\bar{\alpha}_a(s)\geqq N \quad \textit{for any } s\in[0,t]] \\
&\quad \geqq \exp(-q(N)\,t)\,\bar{\mathbf{P}}_{n\varepsilon_0}[\tau_N<t, \bar{\alpha}_a(\tau_N)=N].
\end{aligned}$$

Hence we have for all $N\geqq 1$ and $t>0$

$$\underline{\lim}_{n\to\infty} \bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}_a(t)\geqq N]\geqq \exp(-q(N)\,t). \tag{2.10}$$

Finally we claim that for all $k\geqq 1$ and all $N\geqq 1$

$$\lim_{t\downarrow 0}\,\underline{\lim}_{n\to\infty} \bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}_a(t)\geqq N, \bar{a}_{2a}(t)\geqq N, \ldots, \bar{\alpha}_{ka}(t)\geqq N]=1. \tag{2.11}$$

If $k=1$, it follows immediately from (2.10). Assume that (2.11) holds for some $k\geqq 1$. Then for $t>0$, $s>0$ and $n\geqq n_0$

$$\begin{aligned}
&\bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}_a(t+s)\geqq N, \bar{\alpha}_{2a}(t+s)\geqq N, \ldots, \bar{\alpha}_{(k+1)a}(t+s)\geqq N] \\
&\quad \geqq \bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}_a(t)\geqq n_0, \bar{\alpha}_{2a}(t+s)\geqq N, \ldots, \bar{\alpha}_{(k+1)a}(t+s)\geqq N] \\
&\qquad -\bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}_a(t+s)<N] \\
&\quad \geqq \bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}(t)\geqq n_0]\,\bar{\mathbf{P}}_{n_0\varepsilon_a}[\bar{\alpha}_{2a}(s)\geqq N, \ldots, \bar{\alpha}_{(k+1)a}(s)\geqq N] \\
&\qquad -\bar{\mathbf{P}}_{n\varepsilon_0}[\bar{\alpha}_0(t+s)<N].
\end{aligned}$$

Letting $n\to\infty$, $t\downarrow 0$, $n_0\to\infty$ and $s\downarrow 0$ in this order, and using (2.10) and the assumption of induction we obtain

$$\underline{\lim}_{s\downarrow 0}\,\underline{\lim}_{t\downarrow 0}\,\underline{\lim}_{n\to\infty} \bar{\mathbf{P}}_{n\varepsilon_0}[\bar{a}_a(t+s)\geqq N, \bar{\alpha}_{2a}(t+s)\geqq N, \ldots, \bar{\alpha}_{(k+1)a}(t+s)\geqq N]=1.$$

Therefore, (2.11) holds for all $k\geqq 1$, which completes the proof of Lemma 2.4.

Lemma 2.5. *Suppose that $\mu\in\mathscr{I}^{Z^d}$ and $\mu[\{\mathbf{1}\}]=0$. Then, for all $t>0$*

$$\lim_{n\to\infty}\ \sup_{|\operatorname{supp}\alpha|\geqq n} \langle T_t^*\mu, f_\alpha\rangle=0.$$

Proof. It follows from Lemma 2.3 that for every $t>0$, $\eta>0$ and $k\geqq 2$ there exists $L>0$ such that if $\min_{1\leqq i\neq j\leqq k}|a_i-a_j|>L$,

$$\|T_t f_{\varepsilon_{a_1}+\varepsilon_{a_2}+\cdots+\varepsilon_{a_k}}-T_t f_{\varepsilon_{a_1}} T_t f_{\varepsilon_{a_2}}\cdots T_t f_{\varepsilon_{a_k}}\|<\eta.$$

Using the Hölder inequality and the translation invariance of T_t we have

$$\langle T_t^*\mu, f_{\varepsilon_{a_1}+\varepsilon_{a_2}+\cdots+\varepsilon_{a_k}}\rangle \leqq \int (T_t f_{\varepsilon_0}(x))^k \mu(dx)+\eta.$$

Hence for all $k\geqq 1$

$$\overline{\lim_{n\to\infty}} \sup_{|\operatorname{supp}\alpha|\geqq n} \langle T_t^*\mu, f_\alpha\rangle \leqq \int (T_t f_{\varepsilon_0}(x))^k \mu(dx).$$

If $x\neq \mathbf{1}$ and $t>0$, then $T_t f_{\varepsilon_0}(x)=\mathbf{E}_{\varepsilon_0}[f_{\alpha(t)}(x)]<1$, since $\mathbf{P}_{\varepsilon_0}[\alpha(t)=\alpha]>0$ for all $\alpha\in I$. Therefore, letting $k\to\infty$, we obtain Lemma 2.5.

Theorem 2.6. *Suppose that* $\nu\in\mathscr{S}\cap\mathscr{I}^{Z^d}$ *and* $\nu[\{\mathbf{1}\}]=0$. *Then* $\nu[x_i<1$ *for all* $i\in Z^d]=1$.

Proof. For $m\geqq 1$, $t>0$ and $t_0>0$ we have

$$\begin{aligned}\nu[x_0=1]&=\lim_{n\to\infty}\langle \nu, f_{n\varepsilon_0}\rangle=\lim_{n\to\infty}\langle T_{t+t_0}^*\nu, f_{n\varepsilon_0}\rangle\\&=\lim_{n\to\infty}\mathbf{E}_{n\varepsilon_0}[\langle T_t^*\nu, f_{\alpha(t_0)}\rangle]\\&\leqq \sup_{|\operatorname{supp}\alpha|\geqq m}\langle T_t^*\nu, f_\alpha\rangle+\lim_{n\to\infty}\mathbf{P}_{n\varepsilon_0}[|\operatorname{supp}\alpha(t_0)|<m].\end{aligned}$$

By Lemma 2.4 and 2.5 the last line vanishes in the limit as $t_0\downarrow 0$ and $m\to\infty$ (in this order), so that $\nu[x_0=1]=0$. This proves Theorem 2.6 by virtue of the translation invariance of ν.

3. Proof of Theorem 1.3.

Let us consider the subcase: $u=0$ and $v>0$. The other subcase is reduced to it by exchanging the role of A_1 and A_2. Furthermore, we assume that $\gamma=-s\geqq 0$, for it has already been shown that $\lim_{t\to\infty}T_t^*\mu=\delta_{\mathbf{1}}$ for any $\mu\in\mathscr{P}(X)$ if $s>0$ (see a remark made after the proof of Theorem 2.2 given in the previous section). Recall that σ_0 is the extinction time of $\alpha(t)$. From $v>0$ it follows that $\mathbf{P}_\alpha[\sigma_0<\infty]>0$ for all $\alpha\in I$. Lemma 2.1 implies that $\lim_{t\to\infty}T_t^*\mu=\delta_{\mathbf{1}}$ for all $\mu\in\mathscr{P}(X)$ if and only if

(3.1) $$\mathbf{P}_\alpha[\sigma_0<\infty]=1 \quad \text{for any } \alpha\in I.$$

On the other hand we have the following:

Theorem 3.1. *Suppose that*

(3.2) $$\mathbf{P}_\alpha[\sigma_0<\infty]<1 \quad \textit{for some } \alpha\in I,$$

and define $\nu\in\mathscr{I}^{Z^d}$ *by the relation* $\langle \nu, f_\alpha\rangle=\mathbf{P}_\alpha[\sigma_0<\infty]$ $(=\lim_{t\to\infty}T_t f_\alpha(\mathbf{0}))$ *for all* $\alpha\in I$.

Then,

(i) $(\mathscr{S} \cap \mathscr{I}^{Z^d})_{\text{ext}} = \{\delta_{\mathbf{1}}, \nu\}$, *and*

(ii) $\lim_{t\to\infty} T_t^* \mu = \lambda \delta_{\mathbf{1}} + (1-\lambda)\nu$ *for all* $\mu \in \mathscr{I}^{Z^d}$ *with* $\lambda = \mu[\{\mathbf{1}\}]$.

Lemma 3.2. *If the condition* (3.2) *holds, then*

$$\mathbf{P}_\alpha[\lim_{t\to\infty} |\alpha(t)| = \infty \mid \sigma_0 = \infty] = 1 \qquad \textit{for all } \alpha \in I.$$

Proof. Lemma 2.2 implies that

$$\mathbf{P}_\alpha[\sigma_0 < \infty] \geqq \mathbf{P}_{\varepsilon_0}[\sigma_0 < \infty]^{|\alpha|}.$$

Since $\mathbf{P}_{\alpha(t)}[\sigma_0 < \infty]$ is a bounded martingale we see

$$\lim_{t\to\infty} \mathbf{P}_{\alpha(t)}[\sigma_0 < \infty] = 0 \quad \text{a.s. on} \quad [\sigma_0 = \infty].$$

Hence by the condition (3.2) we get the equality in the lemma.

Now, let us prove Theorem 3.1. Let $\mu \in \mathscr{I}^{Z^d}$ satisfy $\mu[\{\mathbf{1}\}] = 0$. It follows from Lemma 2.5 that for every $t_1 > 0$ and $\eta > 0$ we have some $m \geqq 1$ such that $\sup_{|\text{supp}\,\alpha| \geqq m} \langle T_{t_1}^* \mu, f_\alpha \rangle < \eta$. By Lemma 2.4 there are some $t_0 > 0$ and $n_0 \geqq 1$ satisfying

$$\mathbf{P}_{n\varepsilon_0}[|\text{supp}\,\alpha(t_0)| \geqq m] > 1 - \eta \qquad \text{for all } n \geqq n_0.$$

Hence,

$$\langle T_{t_0+t_1}^* \mu, f_{n\varepsilon_0} \rangle = \mathbf{E}_{n\varepsilon_0}[\langle T_{t_1}^* \mu, f_{\alpha(t_0)} \rangle] \leqq 2\eta$$

for all $n \geqq n_0$. Also,

$$\begin{aligned}
&\mathbf{E}_\alpha[\langle T_{t_0+t_1}^* \mu, f_{\alpha(t)} \rangle; t < \sigma_0] \\
&\quad \leqq \mathbf{E}_\alpha[\langle T_{t_0+t_1}^* \mu, f_{\alpha(t)} \rangle; |\text{supp}\,\alpha(t)| > M] \\
&\qquad + \mathbf{E}_\alpha[\langle T_{t_0+t_1}^* \mu, f_{\alpha(t)} \rangle; |\text{supp}\,\alpha(t)| \leqq M, |\alpha(t)| > nM] \\
&\qquad + \mathbf{E}_\alpha[\langle T_{t_0+t_1}^* \mu, f_{\alpha(t)} \rangle; |\alpha(t)| \leqq nM, t < \sigma_0] \\
&\quad \leqq \sup_{|\text{supp}\,\alpha| > M} \langle T_{t_0+t_1}^* \mu, f_\alpha \rangle + \langle T_{t_0+t_1}^* \mu, f_{n\varepsilon_0} \rangle \\
&\qquad + \mathbf{P}_\alpha[|\alpha(t)| \leqq nM, t < \sigma_0].
\end{aligned}$$

First letting $t \to \infty$, and then $M \to \infty$, we get by Lemma 3.2 and Lemma 2.5

$$\overline{\lim_{t\to\infty}}\, \mathbf{E}_\alpha[\langle T_{t_0+t_1}^* \mu, f_{\alpha(t)} \rangle; t < \sigma_0] \leqq \langle T_{t_0+t_1}^* \mu, f_{n\varepsilon_0} \rangle \leqq 2\eta,$$

which shows that the left-hand side equals zero. Hence,

$$\begin{aligned}
\lim_{t\to\infty} \langle T_t^* \mu, f_\alpha \rangle &= \lim_{t\to\infty} \langle T_{t+t_0+t_1}^* \mu, f_\alpha \rangle \\
&= \lim_{t\to\infty} (\mathbf{E}_\alpha[\langle T_{t_0+t_1}^* \mu, f_{\alpha(t)} \rangle; t < \sigma_0] + \mathbf{P}_\alpha[\sigma_0 \leqq t]) \\
&= \langle \nu, f_\alpha \rangle.
\end{aligned}$$

Thus the part (ii) has been proved, and (i) follows from (ii).

By virtue of Theorem 3.1 the proof of Theorem 1.3 is reduced to the extinction problem of the dual process. Since the extinction probability is monotone in γ, it suffices to prove that (3.2) holds for a sufficiently large $\gamma>0$. Note that if we neglect the quadratic term of $Q_{\alpha,\alpha-\varepsilon_i}$, the dual process is nothing but a branching random walk. This shows that (3.1) holds if $\gamma(=-s)\leqq v$.

Let us introduce a simpler Markov process $(\alpha(t),\tilde{\mathbf{P}}_\alpha)$, taking values in $\tilde{I}=\{\alpha\in I\colon \alpha_i\in\{0,1,2\}$ *for any* $i\in Z^d\}$, with the infinitesimal matrix $\{\tilde{Q}_{\alpha,\beta}\}$: Choose $a\in Z^d$ satisfying $a\neq 0$ and $m_a>0$, and set

$$\begin{aligned}\tilde{Q}_{\alpha,\alpha-\varepsilon_i}&=\delta I(\alpha_i>0),\\ \tilde{Q}_{\alpha,\alpha+\varepsilon_i}&=\gamma I(\alpha_i=1),\\ \tilde{Q}_{\alpha,\alpha-\varepsilon_i+\varepsilon_{i+a}}&=m_a I(\alpha_i=2,\ \alpha_{i+a}=0),\\ \tilde{Q}_{\alpha,\beta}&=0 \qquad \text{for any other } \beta(\neq\alpha),\end{aligned}$$

where $\delta=1+2v+2\sum_{j\neq 0} m_j$, and $I(A)$ stands for the indicator function of $A\subset\tilde{I}$.

Since, as easily seen from Corollary A.2, $(\alpha(t),\mathbf{P}_\alpha)\geqq(\alpha(t),\tilde{\mathbf{P}}_\alpha)$, our problem is reduced to showing that $\tilde{\mathbf{P}}_\alpha[\sigma_0=\infty]>0$ holds for some $\alpha\in\tilde{I}$ for a sufficiently large γ.

Theorem 3.3. *There exists* $\gamma_0>0$ *such that if* $\gamma>\gamma_0$, $\tilde{\mathbf{P}}_\alpha[\sigma_0=\infty]>0$ *for all* $\alpha\neq 0$ *from* $\tilde{I}$.

Proof. It is enough to show the theorem for $d=1$ and $a=1$. For each $0<q<1$ let us associate a set-valued Markov process $\{B(n)\}_{n=1}^{\infty}$, which is called a branching process with interference, to be compared with $(\alpha(t),\tilde{\mathbf{P}}_\alpha)$. It is defined by simply setting

$$B(0)=B, \quad \text{and} \quad B(n)=\bigcup_{i\in B(n-1)} B_i^n \qquad \text{for } n\geqq 1,$$

where $\{B_i^n\}_{i\in Z^d,n\geqq 1}$ is a system of set-valued random variables which are mutually independent and have a common probability law given by

$$\operatorname{Prob}[B_i^n=\{i,i+1\}]=q^2,\ \operatorname{Prob}[B_i^n=\phi]=(1-q)^2,$$

and

$$\operatorname{Prob}[B_i^n=\{i\}]=\operatorname{Prob}[B_i^n=\{i+1\}]=q(1-q).$$

We need the following lemma (see [1] for the proof).

Lemma 3.4. *There exists a critical value* $0<q_c<1$ *such that if* $q>q_c$

$$P_B^q[B(n)\neq\phi \quad \text{for all } n\geqq 1]>0 \qquad \text{for all } B\neq\phi,$$

and if $q<q_c$

$$P_B^q[B(n)=\phi \quad \text{for some } n\geqq 1]=1 \qquad \text{for all } B.$$

Therefore, the proof of Theorem 1.3 is reduced to the following.

Lemma 3.5. *Let $a=1$ and set $m=m_a$ in the definition of $\tilde{Q}_{\alpha,\beta}$. For every $m>0$, $\delta>0$, and $q>q_c$ there exists $t_0>0$ and $\gamma>0$ such that if $B\Subset Z^1$ and $\alpha\geqq\sum_{i\in B}\varepsilon_i$, then*

$$(B(n), P_B^q)\leqq(\operatorname{supp}\alpha(nt_0), \tilde{\mathbf{P}}_\alpha).$$

Here the order relation is defined according to the understanding that a set-valued process is considered as a process on I by identifying $B\Subset Z^1$ with $\alpha=\sum_{i\in B}\varepsilon_i$.

Proof. Fix $B\Subset Z^1$, $B\neq\phi$, and introduce a modified Markov chain $(\alpha(t), \tilde{\mathbf{P}}_\alpha^*)$ on $\tilde{I}$ associated with B, with the infinitesimal matrix $\{\tilde{Q}_{\alpha,\beta}^*\}$ as follows: decomposing B into intervals of Z^1 such that

$$B=[a_1,b_1]\cup[a_2,b_2]\cup\ldots\cup[a_k,b_k]$$

with

$$a_1\leqq b_1, b_1+1<a_2\leqq b_2,\ldots,b_{k-1}+1<a_k\leqq b_k,$$

we set

$$\begin{aligned}
\tilde{Q}_{\alpha,\alpha-\varepsilon_i}^* &=\delta I(\alpha_i>0) \quad \text{if } i\in\{b_1,\ldots,b_k\},\\
&=\delta I(\alpha_i>0)+mI(\alpha_i=2) \quad \text{otherwise},\\
\tilde{Q}_{\alpha,\alpha+\varepsilon_i}^* &=\gamma I(\alpha_i=1),\\
\tilde{Q}_{\alpha,\alpha-\varepsilon_i+\varepsilon_{i+1}}^* &=mI(\alpha_i=2, \alpha_{i+1}=0) \quad \text{if } i\in\{b_1,\ldots,b_k\},\\
&=0 \quad \text{otherwise},\\
\tilde{Q}_{\alpha,\beta}^* &=0 \quad \text{for any other } \beta(\neq\alpha),
\end{aligned}$$

where $m=m_1$. Then it holds that

$$(\alpha(t), \tilde{\mathbf{P}}_\alpha)\geqq(\alpha(t), \tilde{\mathbf{P}}_\alpha^*) \quad \text{for all } \alpha\in\tilde{I}, \tag{3.3}$$

$$(\alpha(t), \tilde{\mathbf{P}}_\alpha^*)=(\sum_{i\in B}\alpha^i(t), \bigotimes_{i\in B}\tilde{\mathbf{P}}_{\varepsilon_i}^*) \quad \text{for } \alpha=\sum_{i\in B}\varepsilon_i. \tag{3.4}$$

For $i\in B\backslash\{b_1,\ldots,b_k\}$ set $c(m,\delta,\gamma,t)=\tilde{\mathbf{P}}_{\varepsilon_i}^*[\alpha_i(t)\geqq 1]$ and for $i\in\{b_1,\ldots,b_k\}$ set $d(m,\delta,\gamma,t)=\tilde{\mathbf{P}}_{\varepsilon_i}^*[\alpha_i(t)\geqq 1, \alpha_{i+1}(t)\geqq 1]$. The both c and d are independent of B and of particular choice of i.

For a fixed $t>0$ we write $B^*(1)=\{i\in Z^1\colon \alpha_i(t)\geqq 1\}$ and $P_B^*=\tilde{\mathbf{P}}_\alpha^*$ with $\alpha=\sum_{i\in B}\varepsilon_i$. We note that $\{B_i^n\ni i\}$ and $\{B_i^n\ni i+1\}$ are independent events for each i and n, showing that the events $\{B(1)\ni i\}$, $i\in Z^1$, are mutually independent. Then, taking this fact together with (3.4) into account, it is easy to check that if

$$c(m,\delta,\gamma,t)\geqq 1-(1-q)^2 \quad \text{and} \quad d(m,\delta,\gamma,t)\geqq 1-(1-q)^3, \tag{3.5}$$

it holds that

$$(B(1), P_B^q)\leqq(B^*(1), P_B^*). \tag{3.6}$$

(The meaning of "$\leqq$" would be clear.)

Thus we obtain by (3.3) that if $\alpha=\sum_{i\in B}\varepsilon_i$, then $(B(1), P_B^q)\leqq(\alpha(t), \tilde{\mathbf{P}}_\alpha)$. By Corollary A.1 in Appendix we also have $(\alpha(t), \tilde{\mathbf{P}}_\alpha)\leqq(\alpha(t), \tilde{\mathbf{P}}_\beta)$ if $\alpha\leqq\beta$. Finally we apply the Markov property to conclude that $(B(n), P_B^q)\leqq(\alpha(nt), \tilde{\mathbf{P}}_\alpha)$.

To complete the proof of Lemma 3.5 we have to check (3.5) for some $t>0$, $\gamma>0$, and $q>q_c$. Let $m>0$ and $\delta>0$ be fixed. We claim that

$$\lim_{\gamma\to\infty}\tilde{\mathbf{P}}^*_{\varepsilon_i}[\alpha_i(t)\geqq 1]=1 \tag{3.8}$$

uniformly in t of any compact interval and in $i\in Z^d$. Let $i\in\{b_1,\dots,b_k\}$. $u_1(t)=\tilde{\mathbf{P}}^*_{\varepsilon_i}[\alpha_i(t)\geqq 1]$ and $u_2(t)=\tilde{\mathbf{P}}^*_{2\varepsilon_i}[\alpha_i(t)\geqq 1]$ satisfy the following system of differential equations:

$$\frac{du_1}{dt}=\gamma(u_2-u_1)-\delta u_1,$$

$$\frac{du_2}{dt}=\delta(u_1-u_2)+m(\tilde{\mathbf{P}}^*_{\varepsilon_i+\varepsilon_{i+1}}[\alpha_i\geqq 1]-u_2),$$

with $u_1(0)=u_2(0)=1$. From these equations and the inequality $\tilde{\mathbf{P}}^*_{\varepsilon_i+\varepsilon_{i+1}}[\alpha_i(t)\geqq 1]\geqq u_1(t)$, we can easily deduce an integral inequality for $u_1(t)$, and by passing to the limit as $\gamma\to\infty$ we get (3.8). (The details are omitted.) For $i\notin\{b_1,\dots,b_k\}$ the same proof is valid.

Let $i\in\{b_1,\dots,b_k\}$. We will next show that

$$\varliminf_{\gamma\to\infty} d(m,\delta,\gamma,t)\geqq 1-\exp(-(m+\delta)t). \tag{3.9}$$

Note that $v_1(t)=d(m,\delta,\gamma,t)$ and $v_2(t)=\tilde{\mathbf{P}}^*_{2\varepsilon_i}[\alpha_i(t)\geqq 1,\ \alpha_{i+1}(t)\geqq 1]$ satisfy

$$\frac{dv_1}{dt}=\gamma(v_2-v_1)-\delta v_1,$$

$$\frac{dv_2}{dt}=\delta(v_1-v_2)+m(\tilde{\mathbf{P}}^*_{\varepsilon_i+\varepsilon_{i+1}}[\alpha_i(t)\geqq 1, \alpha_{i+1}(t)\geqq 1]-v_2),$$

with $v_1(0)=v_2(0)=0$.

Noticing that by (3.8) $\lim_{\gamma\to\infty}\tilde{\mathbf{P}}^*_{\varepsilon_i+\varepsilon_{i+1}}[\alpha_i(t)\geqq 1,\ \alpha_{i+1}(t)\geqq 1]=1$ uniformly in t of any compact interval, (3.9) is obtained in the same process as taken for (3.8) and briefly sketched above.

Consequently, for every $m>0$, $\delta>0$, and $q>q_c$ there exist $t>0$ and $\gamma>0$ that satisfy the condition (3.5).

Finally Theorem 3.3 follows immediately from Lemma 3.4 and Lemma 3.5. Furthermore, we obtain Theorem 1.3 by combining this with Theorem 3.1.

4. Proof of Theorem 1.4

In this section we will prove Theorem 1.4. So we assume that $u=v=0$, $\gamma(=-s)>0$. Then the dual process $(\alpha(t), \mathbf{P}_\alpha)_{\alpha\in I}$ is a branching random walk with an interaction, which does not extinct.

We first show the following:

Lemma 4.1. $(\mathscr{S} \cap \mathscr{I}^{Z^d})_{\text{ext}} = \{\delta_{\mathbf{0}}, \delta_{\mathbf{1}}\}$.

Proof. Let $\nu \in \mathscr{S} \cap \mathscr{I}^{Z^d}$. For every $t > 0$ and $a \in Z^d$ it holds that

$$\mathbf{P}_{\varepsilon_0}[\alpha(t) = 2\varepsilon_0] > 0, \quad \text{and} \quad \mathbf{P}_{\varepsilon_0}[\alpha(t) = \varepsilon_0 + \varepsilon_a] > 0,$$

because of the irreducibility of $\{m_i\}$. Using the translation invariance of ν, we see

$$\begin{aligned}\langle \nu, f_{\varepsilon_0} \rangle &= \mathbf{E}_{\varepsilon_0}[\langle \nu, f_{\alpha(t)} \rangle] \\ &\leqq \langle \nu, f_{2\varepsilon_0} \rangle \mathbf{P}_{\varepsilon_0}[\alpha(t) = 2\varepsilon_0] + \langle \nu, f_{\varepsilon_0 + \varepsilon_a} \rangle \mathbf{P}_{\varepsilon_0}[\alpha(t) = \varepsilon_0 + \varepsilon_a] \\ &\quad + \langle \nu, f_{\varepsilon_0} \rangle \mathbf{P}_{\varepsilon_0}[\alpha(t) \notin \{2\varepsilon_0, \varepsilon_0 + \varepsilon_a\}],\end{aligned}$$

which implies that $\langle \nu, f_{\varepsilon_0} \rangle = \langle \nu, f_{2\varepsilon_0} \rangle = \langle \nu, f_{\varepsilon_0 + \varepsilon_a} \rangle$; hence $\langle \nu, f_\alpha \rangle = \langle \nu, f_{\varepsilon_0} \rangle$ for all $\alpha \neq 0$. Thus we have

$$\nu = \lambda \delta_{\mathbf{1}} + (1 - \lambda) \delta_{\mathbf{0}}, \text{ where } \lambda = \langle \nu, f_{\varepsilon_0} \rangle.$$

Remark. If $d = 1$ or $d = 2$, and if $\sum_{i \in Z^d} i m_i = 0$ together with the certain moment condition, we can show that $\mathscr{S}_{\text{ext}} = \{\delta_{\mathbf{0}}, \delta_{\mathbf{1}}\}$, by making use of the fact that any excessive function of a recurrent random walk is constant.

Theorem 4.2. *Assume, in addition, that* $\sum_{i \in Z^d} |i| m_i < \infty$. *Then*

$$\mathbf{P}_\alpha[\lim_{t \to \infty} |\operatorname{supp} \alpha(t)| = \infty] = 1 \qquad \textit{for all } \alpha \neq 0.$$

Once we have shown this theorem the proof of Theorem 1.4 follows immediately. In fact by the duality together with Lemma 2.5 and Theorem 4.2 for all $\mu \in \mathscr{I}^{Z^d}$ with $\mu[\{\mathbf{1}\}] = 0$

$$\lim_{t \to \infty} \langle T_t^* \mu, f_\alpha \rangle = \lim_{t \to \infty} \langle T_{t+s}^* \mu, f_\alpha \rangle = \lim_{t \to \infty} \mathbf{E}_\alpha[\langle T_s^* \mu, f_{\alpha(t)} \rangle] = 0.$$

The idea of our proof of Theorem 4.2 is to define arbitrarily many one-particle processes which are embedded in $\alpha(t)$ and move with mutually distinct asymptotic velocities so that they eventually occupy different sites of Z^d. Fix $a \neq 0$ of Z^d satisfying $m_a > 0$. For $0 \leqq c \leqq 1$ arbitrarily given let us introduce a Markov chain $(\beta(t), \bar{\mathbf{P}}_\beta^c)$ taking values in $I_0 = \{\alpha \in I \colon \alpha = \varepsilon_i,$ *or* $\alpha = 2\varepsilon_i$ *for some* $i \in Z^d\}$, with the infinitesimal matrix $\{\bar{Q}_{\alpha, \beta}\}$:

$$\bar{Q}_{\varepsilon_i, 2\varepsilon_i} = \gamma, \; \bar{Q}_{\varepsilon_i, \varepsilon_j} = m_{i-j} \quad (i \neq j), \quad \text{and}$$

$$\bar{Q}_{\varepsilon_i, \beta} = 0 \quad \text{for any other } \beta (\neq \varepsilon_i);$$

$$\bar{Q}_{2\varepsilon_i, \varepsilon_i} = 1 + 2 \sum_{j \neq a} m_j - 2c m_a, \; \bar{Q}_{2\varepsilon_i, \varepsilon_{i+a}} = 2c m_a, \quad \text{and}$$

$$\bar{Q}_{2\varepsilon_i, \beta} = 0 \quad \text{for any other } \beta (\neq 2\varepsilon_i).$$

It is easily seen that $\beta(t)$ can be embedded in $\alpha(t)$. (See Appendix for the proof.)

Let ζ be the first jump time of $\{\beta(t), \bar{\mathbf{P}}^c_\beta\}$, namely $\zeta = \inf\{t > 0 : \beta(t) \neq \beta(0)\}$. And let $\sigma_1 = \inf\{t \geqq \zeta : |\beta(t)| = 1\}$ and $\sigma_n = \sigma_{n-1} + \sigma_1 \cdot \theta_{\sigma_{n-1}}$ for $n \geqq 2$.

Note that for $j \neq i$

$$\text{(4.1)} \qquad \bar{\mathbf{P}}^c_{\varepsilon_i}[\beta(\sigma_1) = \varepsilon_j] = \frac{m_{j-i}}{\gamma + m} + \frac{\gamma}{\gamma + m}\frac{2cm_a}{1+2m}\delta_{j,i+a},$$

$$\bar{\mathbf{P}}^c_{\varepsilon_i}[\beta(\sigma_1) = \varepsilon_i] = \frac{\gamma}{\gamma + m}\frac{1+2m-2cm_a}{1+2m},$$

where $m = \sum_{j \neq 0} m_j$.

Denoting by $X^c(t)$ the (unique) site which is occupied by a particle of $\beta(t)$, we see that $X^c(\sigma_n)$ is a spatially homogeneous Markov chain (a random walk) with values in Z^d and with the transition law (4.1) so that by the law of large numbers we get

$$\text{(4.2)} \qquad \lim_{n \to \infty} \frac{X^c(\sigma_n)}{n} = \xi(c) \in R^d \quad \text{a.s.} \quad (\bar{\mathbf{P}}^c_\beta),$$

where

$$\xi(c) = \bar{\mathbf{E}}^c_{\varepsilon_0}[X^c(\sigma_1)] = \frac{1}{\gamma + m}\left(\sum_{j \in Z^d} m_j j + \frac{2\gamma c m_a}{1+2m} a\right).$$

Also, the distribution of $(\sigma_1, \bar{\mathbf{P}}^c_{\varepsilon_i})$ obviously is independent of $i \in Z^d$. Hence $\{\sigma_n\}_{n=1}^\infty$ also is a random walk with the mean increment

$$\bar{\mathbf{E}}^c_{\varepsilon_i}[\sigma_1] = \frac{1}{\gamma + m}\left(1 + \frac{\gamma}{1+2m}\right).$$

Accordingly, using the law of large numbers again, we have

$$\text{(4.3)} \qquad \lim_{n \to \infty} \frac{\sigma_n}{n} = \bar{\mathbf{E}}^c_{\varepsilon_i}[\sigma_1] \quad \text{a.s.} \quad (\bar{\mathbf{P}}^c_\beta).$$

From (4.2) and (4.3) it follows that

$$\lim_{t \to \infty} \frac{X^c(t)}{t} = \kappa(c) \quad \text{a.s.} \quad (\bar{\mathbf{P}}^c_\beta),$$

where

$$\kappa(c) = \{(1+2m)\sum_j m_j j + 2\gamma c m_a a\}/(1+2m+\gamma).$$

This implies that if $0 \leqq c \leqq 1$, then with probability one we can trace among particles of the configuration process $\alpha(t)$ a specified particle that moves with an asymptotic velocity $\kappa(c)$. Such tracing can be made simultaneously for countably many values of c, completing the proof of Theorem 4.2.

In below the state 0 is discarded from the state space of the dual process $\alpha(t)$, since it does not communicate with the other states at all.

Theorem 4.3. *Suppose that* $\sum_{j\in Z^d} |j|^2 m_j < \infty$ *and*

$$\sup_{|\theta|=1} \xi_\theta \cdot \theta < 0, \tag{4.4}$$

where

$$\xi_\theta = \sum_{j\in Z^d} m_j j + \frac{2\gamma}{1+2m}\Big(\sum_{j\cdot\theta<0} m_j j\Big) \qquad \Big(m = \sum_{j\neq 0} m_j\Big),$$

the dot · indicates the inner product in R^d *and the supremum is taken over all unit vectors in* R^d. *Let* ζ *be the first jump time of* $\alpha(t)$ *and* τ^* *the first passage time through the set* $\{\alpha\in I: \alpha_0>0\}$ *after the first jump:* $\zeta = \inf\{t\geqq 0: \alpha(t)\neq\alpha(t-)\}$ *and* $\tau^* = \inf\{t\geqq\zeta: \alpha_0(t)>0\}$. *Then* $\mathbf{E}_{\varepsilon_0}[\tau^*]<\infty$; *in particular* $\{\alpha: \alpha_0>0\}$ *is a positive recurrent set for* $\{\alpha(t), \mathbf{P}_\alpha; \alpha\neq 0\}$, *i.e.*

$$\varliminf_{t\to\infty} \mathbf{P}_\alpha[\alpha_0(t)>0]>0 \qquad \textit{for } \alpha\neq 0. \tag{4.5}$$

If $\{\alpha: \alpha_0>0\}$ is a recurrent set, then every non-increasing bounded harmonic function for the process $\alpha(t)$ is constant. In fact for such a harmonic function $f(\alpha)$ we have $h(\alpha)=\mathbf{E}_\alpha[h(\alpha(\tau_j))]$ where $\tau_j=\inf\{t\geqq 0: \alpha_j(t)>0\}$. Taking $\alpha = \varepsilon_i$ and using monotonicity of h we see that $h(\varepsilon_i)\leqq h(\varepsilon_j)$. Since i and j are arbitrary, $h(\varepsilon_i)=h(\varepsilon_0)$ for all $i\in Z^d$. This shows that the maximum value of h is attained at ε_0. Therefore h is constant, for every point of I other than 0 can be reached from ε_0 with positive probability.

Since $v\in\mathcal{S}$ implies that $h(\alpha)=\langle v, f_\alpha\rangle$ is a harmonic function for the dual process, we have the following Corollary of Theorem 4.3.

Corollary 4.4. *Under the assumption of Theorem* 4.3, $\mathcal{S}_{\text{ext}}=\{\delta_{\mathbf{0}}, \delta_{\mathbf{1}}\}$.

Proof of Theorem 4.3. Let N be a (large) positive integer. Define a Markov process $\bar\alpha(t)$ on I_0 as in the proof of Theorem 4.2, but this time the infinitesimal matrix $\{\bar Q_{\alpha,\beta}\}$ is not spatially homogeneous; it is given by

$$\begin{aligned}
\bar Q_{\varepsilon_i,\varepsilon_j} &= m_{i-j} \quad \text{for all } i \text{ and } j \text{ with } i\neq j,\\
\bar Q_{\varepsilon_i, 2\varepsilon_i} &= \gamma, \quad \bar Q_{2\varepsilon_i,\varepsilon_i} = 1+2\sum_{j\cdot i\geqq 0} m_j \quad \text{if } |i|\geqq N,\\
\bar Q_{2\varepsilon_i,\varepsilon_{i+j}} &= 2m_j \quad \text{if } |i|\geqq N \text{ and } i\cdot j<0, \text{ and}\\
\bar Q_{\alpha,\beta} &= 0 \quad \text{otherwise} \quad (\alpha\neq\beta).
\end{aligned}$$

By tracing the (unique) site that is occupied by at least one particle along the successive times σ_n at which the occupancy changes so that $|\bar\alpha(\sigma_n)|=1$ we have a Markov chain X_n on Z^d (see the proof of Theorem 4.2 for the formal description). We can take N so large that X_n is irreducible. By the same computation as before we have

$$E[X_{n+1}-X_n \mid X_n=i] = \frac{1}{\gamma+m}\,\xi_{i/|i|} \qquad \text{if } |i|\geqq N,$$

and

$$\sup_{i\in Z^d} E[|X_{n+1}-X_n|^2 \mid X_n=i] \leqq \frac{1+2m+2\gamma}{(\gamma+m)(1+2m)} \sum_{j\in Z^d} m_j|j|^2 \equiv M,$$

so that if $|i|\geqq N$

$$\begin{aligned} E[|X_{n+1}|^2 \mid X_n=i] &= |i|^2 + E[|X_{n+1}-X_n|^2 \mid X_n=i] \\ &\quad + \frac{2}{\gamma+m}\, i\cdot\xi_{i/|i|} \\ &\leqq |i|^2 + \frac{2|i|}{\gamma+m} \sup_{|\theta|=1} \xi_\theta\cdot\theta + M. \end{aligned}$$

If N is large enough, the last member above is less than $|i|^2-1$. Clearly we also have $E[|X_1|^2 \mid X_0=i]<\infty$ for $|i|<N$. These together imply that X_n is positive recurrent (see Tweedie [17], Theorem 6.1). Set $T=\inf\{n\geqq 1\colon X_n=0\}$ and let $\bar\tau^*$ be the first time for $\bar\alpha(t)$ to arrive at the set $\{\alpha\in I_0\colon \alpha_0>0\}$ after its first jump. Then

$$\begin{aligned} E[\bar\tau^* \mid \bar\alpha(0)=\varepsilon_0] &\leqq \sum_{k=1}^{\infty} E\left[\sum_{n=1}^{k} (\sigma_n-\sigma_{n-1});\ T=k \mid \bar\alpha(0)=\varepsilon_0\right] \\ &\leqq E[T \mid X_0=0] \sup_i E[\sigma_1 \mid X_0=i] \\ &< \infty \qquad (\sigma_0=0). \end{aligned}$$

Here the second inequality is verified by using either of the strong Markov property of $\bar\alpha(t)$ (after changing the order of the summation) or the fact that the distribution of the minimum of independent exponential holding times is not affected by knowing which one attains the minimum. Since $\bar\alpha(t)$ can be embedded in $\alpha(t)$ so that the first jump of $\alpha(t)$ precedes that of $\bar\alpha(t)$, we have $\mathbf{E}_{\varepsilon_0}[\tau^*]<\infty$. Now (4.5) follows from an application of the renewal theorem. In fact by Lemma 2.2 (i) $u(t)=\mathbf{P}_{\varepsilon_0}[\alpha_0(t)>0]$ satisfies

$$u(t) \geqq \mathbf{P}_{\varepsilon_0}[\zeta>t] + \int_0^t u(t-s)\,\mathbf{P}_{\varepsilon_0}[\tau^*\in ds]$$

so it is bounded by the solution of the corresponding renewal equation, which by virtue of the celebrated renewal theorem converges to a positive constant $\mathbf{E}_{\varepsilon_0}[\zeta]/\mathbf{E}_{\varepsilon_0}[\tau^*]$ as t approaches infinity. The proof of Theorem 4.3 is complete.

Remarks. 1) The condition (4.4) is far from being necessary for (4.5). One can improve it by allowing the multiple occupancy in each site for the embedded process defined in the proof of Theorem 4.3, but still stays off critical criteria. (4.4) is trivially fulfilled if $\sum_{j\in Z^d} m_j j=0$ and $\gamma>0$ (under the irreducibility of $\{m_j\}$). If $\sum_{j\in Z^d} m_j j\neq 0$ and the cumulant generating function of $\{m_j\}$ is finite in a neighbourhood of the origin of the parameter space, then there is a positive γ such that for any $\alpha\in I$

$$\mathbf{P}_\alpha[\alpha_0(t)=0 \quad \textit{for all sufficiently large } t]=1,$$

as is easily seen by comparing $\alpha(t)$ with a branching random walk (see e.g. [18], ii) of Sect. 7).

2) If the state space of the process $\alpha(t)$ is $I_\infty=\{\alpha: |\operatorname{supp}\alpha|=\infty, \alpha_i<\infty$ *for all* $i\in Z^d\}$ rather than I, it has an stationary distribution, say π, which is the direct product measure of the Poisson distributions on $\{0,1,2,\ldots\}$ with common parameter 2γ (providing $\sum_{j\in Z^d} m_j|j|<\infty$ as well as $u=v=0$). This fact implies that for any $\alpha\in I$

$$\lim_{N\to\infty}\sup_{t>0}\mathbf{P}_\alpha[\alpha_0(t)>N]=0. \tag{4.6}$$

3) The relation (4.5) together with (4.6) has an important implication to the stepping stone model. Let x be an element of X such that $0\leqq x_i<1$ if $i\in B$ and $x_i=1$ if $i\notin B$ for some nonempty finite subset B of Z^d. Then from (4.5) and (4.6) it follows that for $\alpha\in I$, $\alpha\neq 0$

$$0<\varliminf_{t\to\infty}\langle T_t^*\delta_x, f_\alpha\rangle\leqq\varlimsup_{t\to\infty}\langle T_t^*\delta_x, f_\alpha\rangle<1.$$

It is quite plausible that $\langle T_t^*\delta_x, f_\alpha\rangle$ would converge to $\int_{I_\infty} f_\alpha(x)\,\pi(d\alpha)$, but we do not known how to prove it.

5. Regularity of Stationary States

In this section we will discuss regularity of finite dimensional marginal distributions of stationary states of a certain class of infinite dimensional diffusion processes which includes the stepping stone model. The subject of this sort is studied by Holley and Stroock [4], (their results do not (at least formally) apply to our processes), but our method is much simpler and more elementary than theirs.

Let us consider the following stochastic differential equation (5.1) restricted on the space $X=[0,1]^{Z^d}$ under the conditions (5.2) to (5.4).

$$\begin{aligned} &dx_i(t)=\sqrt{a}(x_i(t))\,dB_i(t)+b_i(x(t))\,dt \qquad (i\in Z^d),\\ &\text{with } x(0)=\{x_i\}\in X, \end{aligned} \tag{5.1}$$

where $\{B_i(t)\}_{i\in Z^d}$ is an independent system of standard Brownian motions on a complete probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$,

(5.2) $a(0)=a(1)=0$, $a(u)>0$ for $0<u<1$, and $\sqrt{a}(u)$ is 1/2 Hölder continuous on $[0,1]$,

(5.3) $\{b_i(x)\}_{i\in Z^d}$ are functions defined on X, and there exists a matrix $Q=\{Q_{ij}\}_{i,j\in Z^d}$ satisfying that $Q_{ij}\geqq 0$ for all i and j, $\sup_{i\in Z^d}\sum_{j\in Z^d} Q_{ij}<\infty$, and

$$|b_i(x)-b_i(y)|\leqq\sum_{j\in Z^d} Q_{ij}|x_j-y_j| \qquad \text{for } x \text{ and } y\in X,$$

(5.4) $b_i(x)\geqq 0$ if $x_i=0$, and $b_i(x)\leqq 0$ if $x_i=1$.

Then it is known [14] that (5.1) has a pathwise unique solution, which defines a diffusion process $(x(t), P_x)$ taking values in X, governed by the generator L,

$$L=\frac{1}{2}\sum_{i\in Z^d} a(x_i)\frac{\partial^2}{\partial x_i^2}+\sum_{i\in Z^d} b_i(x)\frac{\partial}{\partial x_i}.$$

We furthermore assume that

$$(5.5)\qquad a(u)\in C^\infty(0,1) \quad \text{and} \quad \int_0^1 \frac{du}{\sqrt{a(u)}}<\infty,$$

$$(5.6)\qquad b_i(x)\in C_f^\infty(X) \quad \text{for all } i\in Z^d.$$

(The definition of $C_f^m(X)$ is given in the introduction.) Under these conditions (5.2) to (5.6) we obtain

Theorem 5.1. *Let ν be a stationary state of the diffusion process $(x(t), P_x)$ that satisfies*

$$(5.7)\qquad \nu[0<x_i<1]=1 \quad \textit{for all } i\in Z^d.$$

Then, for $A \Subset Z^d$, denoting by ν_A the marginal distribution on $X_A=[0,1]^A$ of ν, ν_A is absolutely continuous with respect to the Lebesgue measure on X_A. Furthermore, its probability density admits the continuous version $p_A(x_A)$ that is strictly positive and of C^∞-class in $\mathring{X}_A=(0,1)^A$.

Proof. We first show that ν_A has a C^∞-probability density in $\mathring{X}_A$. By (5.7) it suffices to show that the restriction of ν_A on each compact subset of $\mathring{X}_A$ has a C^∞-density. Let

$$\xi_i(x)=\int_0^{x_i}\frac{du}{\sqrt{a(u)}}, \qquad \xi_A(x)=\{\xi_i(x)\}_{i\in A}, \quad \text{and set}$$

$$e_{\mathbf{n}}(x)=\exp\frac{2\pi\sqrt{-1}}{M}\langle\xi_A(x),\mathbf{n}\rangle \qquad \text{for each } \mathbf{n}=\{n_i\}_{i\in A}\in Z^A$$

($Z^A\equiv\{0,\pm1,\pm2,\dots\}^A$), where $M=\int_0^1\frac{du}{\sqrt{a(u)}}$.

Let $D_A=\{\phi\in C_f^\infty(X);$ *for some compact set K of* $(0,1)^A$, $\phi(x)=0$ *if* $x_A=\{x_i\}_{i\in A}\notin K\}$. For $\phi\in D_A$

$$(5.8)\qquad \begin{aligned} L(\phi e_{\mathbf{n}}) &=(L\phi)e_{\mathbf{n}}+\sum_{i\in Z^d} a(x_i)\frac{\partial}{\partial x_i}\phi\frac{\partial}{\partial x_i}e_{\mathbf{n}} \\ &\quad +\frac{1}{2}\sum_{i\in Z^d} a(x_i)\phi(x)\frac{\partial^2}{\partial x_i^2}e_{\mathbf{n}}+\sum_{i\in Z^d} b_i(x)\phi(x)\frac{\partial}{\partial x_i}e_{\mathbf{n}} \\ &=(L\phi)e_{\mathbf{n}}-\frac{2\pi^2}{M^2}|\mathbf{n}|^2\phi e_{\mathbf{n}}+\sum_{i\in A} n_i(K_i\phi)e_{\mathbf{n}}, \end{aligned}$$

where

$$K_i\phi(x)=\frac{2\pi\sqrt{-1}}{M}\left(\sqrt{a}(x_i)\frac{\partial}{\partial x_i}\phi+\frac{b_i(x)\phi(x)}{\sqrt{a}(x_i)}-\frac{1}{4}\frac{a'(x_i)\phi(x)}{\sqrt{a}(x_i)}\right).$$

Thus $L\phi\in D_A$ and $K_i\phi\in D_A$ for all $i\in A$ if $\phi\in D_A$. Since ν is a stationary state of $(x(t), P_x)$, it holds that

(5.9) $$\langle\nu, L\phi\rangle=0 \quad \text{for all } \phi\in C_f^2(X).$$

By (5.8) and (5.9) we have

(5.10) $$\langle\nu, \phi e_{\mathbf{n}}\rangle=\frac{M^2}{2\pi^2}\left(\frac{1}{|\mathbf{n}|^2}\langle\nu, (L\phi)e_{\mathbf{n}}\rangle+\sum_{i\in A}\frac{n_i}{|\mathbf{n}|^2}\langle\nu, (K_i\phi)e_{\mathbf{n}}\rangle\right).$$

Apply this relation with $L\phi$ and $K_i\phi$, $i\in A$, in place of ϕ and substitute those obtained thereby into the right-hand side of (5.10). By repeating such procedure, it follows that for every $p>0$ there exists a constant $C_p=C(p, A, \phi)$ satisfying that

(5.11) $$|\langle\nu, \phi e_{\mathbf{n}}\rangle|\leqq\frac{C_p}{|\mathbf{n}|^p} \quad \textit{for all } \mathbf{n}\in Z^A.$$

Let $\tilde{\nu}_A$ be a signed measure on $[0,1]^A$ defined by

$$\langle\tilde{\nu}_A, f\rangle=\int f\left(\frac{\xi_A(x)}{M}\right)\phi(x)\,\nu(dx) \quad \text{for any } f\in C([0,1]^A).$$

Then by (5.11) we obtain

$$\left|\int_{X_A}\exp(2\pi\sqrt{-1}\langle y_A, \mathbf{n}\rangle)\,\tilde{\nu}_A(dy_A)\right|\leqq\frac{C_p}{|\mathbf{n}|^p}.$$

Hence, $\tilde{\nu}_A$ has a density $\tilde{p}_A^\phi(x_A)$ of $C^\infty((0,1)^A)$-class with respect to the Lebesgue measure on X_A, which is given by

$$\tilde{p}_A^\phi(x_A)=\sum_{\mathbf{n}}\exp(-2\pi\sqrt{-1}\langle x_A, \mathbf{n}\rangle)\int_{X_A}\exp(2\pi\sqrt{-1}\langle y_A, \mathbf{n}\rangle)\,\tilde{\nu}_A(dy_A).$$

Therefore, ν_A also has a probability density of C^∞-class in $\mathring{X}_A$, because $x_A\to\xi_X(x)/M$ is a C^∞-diffeomorphism from $(0,1)^A$ onto itself.

We next claim that $p_A(x_A)$ is strictly positive in $\mathring{X}_A$. Otherwise, $p_A(\bar{x}_A)=0$ for some $\bar{x}_A=\{\bar{x}_i\}_{i\in A}\in\mathring{X}_A$. Since p_A is smooth in $\mathring{X}_A$ there are positive constants ε_0 and C_1 such that

$$p_A(x_A)\leqq C_1|x_A-\bar{x}_A| \quad \text{if} \quad |x_i-\bar{x}_i|<\varepsilon_0 \quad \text{for all } i\in A.$$

This implies that for some $C_2>0$

(5.12) $$\nu_A[U_\varepsilon(\bar{x}_A)]\leqq C_2\,\varepsilon^{|A|+1} \quad \text{if } \varepsilon<\varepsilon_0,$$

where $U_\varepsilon(\bar{x}_A)=\prod_{i\in A}(\bar{x}_i-\varepsilon, \bar{x}_i+\varepsilon)$.

We choose $\delta>0$ so that $U_{\varepsilon_0}(\bar{x}_A)\subset(\delta,1-\delta)^A$ and $\nu_A[(\delta,1-\delta)^A]>0$. Let $\tau=\inf\{t>0\colon x_i(t)\in\left(\frac{\delta}{2},1-\frac{\delta}{2}\right)$ *for some* $i\in A\}$, and set

$$M_t=\exp\left\{-\sum_{i\in A}\int_0^t\frac{b_i(x(s\wedge\tau))}{\sqrt{a(x_i(s\wedge\tau))}}dB_i(s)-\tfrac{1}{2}\sum_{i\in A}\int_0^t\frac{b_i(x(s\wedge\tau))^2}{a(x_i(s\wedge\tau))}ds\right\}.$$

Then it is easy to check that if $\delta<x_i<1-\delta$ for all $i\in A$, M_t is a P_x-martingale with $E_x[M_t]=1$, and for some constant $C_3>0$ depending on δ, $t>0$ and $p>1$

$$E_x[M_t^p]\leqq C_3. \tag{5.13}$$

For $x\in X$ satisfying that $\delta\leqq x_i\leqq 1-\delta$ for all $i\in A$, we define a new probability measure $\tilde{P}_x$ on $(\Omega,\mathscr{F},\mathscr{F}_t)$ and new processes $\{\tilde{B}_i(t)\}_{i\in A}$ by

$$\tilde{P}_x(d\omega)=M_t(\omega)P_x(d\omega)\quad\text{on}\quad\mathscr{F}_t\quad\text{for each } t>0,\quad\text{and}$$

$$\tilde{B}_i(t)=B_i(t)+\int_0^t\frac{b_i(x(s\wedge\tau))}{\sqrt{a(x_i(s\wedge\tau))}}ds.$$

Then $\tilde{P}_x$ is well-defined as a probability measure on $(\Omega,\mathscr{F},\mathscr{F}_t)$ and $\{\tilde{B}_i(t)\}_{i\in A}$ turns into an independent system of standard Brownian motions with respect to $\tilde{P}_x$. Let us consider the following system of stochastic differential equations:

$$y_i(t)-x_i=\int_0^t\sqrt{a(y_i(s))}\,d\tilde{B}_i(s)\qquad i\in A. \tag{5.14}$$

By the condition (5.2), (5.14) has a pathwise unique solution, so that $\{y_i(t)\}_{i\in A}$ is mutually independent. Moreover, setting $\tau_i=\inf\left\{t>0\colon\ y_i(t)\in\left(\frac{\delta}{2},1-\frac{\delta}{2}\right)\right\}$ for $i\in A$, we see that $\tau=\min\{\tau_i\colon i\in A\}$ and $y_i(t)=x_i(t)$ for $0\leqq t\leqq\tau$. Accordingly, if $\delta\leqq x_i\leqq 1-\delta$ for all $i\in A$

$$\begin{aligned}\tilde{P}_x&[x_A(t)\in U_\varepsilon(\bar{x}_A),\,t<\tau]\\&=\tilde{P}_x[|y_i(t)-\bar{x}_i|<\varepsilon,\,t<\tau_i\quad\textit{for all }i\in A]\\&=\prod_{i\in A}\tilde{P}_x[|y_i(t)-\bar{x}_i|<\varepsilon,\,t<\tau_i]\\&\geqq C_4\,\varepsilon^{|A|},\end{aligned}\tag{5.15}$$

where $C_4>0$ is a constant depending on t and δ. Here the last inequality is deduced from the fact that the stopped diffusion $y_i(t\wedge\tau_i)$ has a transition density $p(t,x,y)$ which is positive and continuous in $(x,y)\in\left(\frac{\delta}{2},1-\frac{\delta}{2}\right)^2$.

On the other hand, if $\delta\leqq x_i\leqq 1-\delta$ for all $i\in A$, we have by Hölder's inequality

(5.16)
$$\begin{aligned}\tilde{P}_x[x_A(t)\in U_\varepsilon(\bar{x}_A), t<\tau]\\ &= E_x[M_t; x_A(t)\in U_\varepsilon(\bar{x}_A), t<\tau]\\ &\leqq E_x[M_t^p]^{1/p} P_x[x_A(t)\in U_\varepsilon(\bar{x}_A)]^{1/q},\end{aligned}$$

where $p>1$, $q>1$ and $1/p+1/q=1$.

Combining (5.15) and (5.16) with (5.13) we have some constant $C_5>0$ satisfying that if $\delta\leqq x_i\leqq 1-\delta$ for all $i\in A$,

(5.17)
$$P_x[x_A(t)\in U_\varepsilon(\bar{x}_A)]\geqq C_5\,\varepsilon^{q|A|}.$$

Using the stationarity of ν and (5.17)

(5.18)
$$\begin{aligned}\nu_A[U_\varepsilon(\bar{x}_A)] &\geqq \int_{\delta\leqq x_i\leqq 1-\delta,\, i\in A} \nu(dx)\, P_x[x_A(t)\in U_\varepsilon(\bar{x}_A)]\\ &\geqq C_5\,\varepsilon^{q|A|}\,\nu_A[(\delta, 1-\delta)^A].\end{aligned}$$

Choosing q such as $1<q<(|A|+1)/|A|$, we see that (5.18) and (5.12) contradict each other. Therefore it holds that $p_A(x_A)>0$ for any $x_A\in \mathring{X}_A$, and the proof of Theorem 5.1 is complete.

Remark. Although $b_i(x)$ of the stepping stone model does not satisfy (5.6) if $m_i \neq 0$ for infinitely many i, the above argument is valid with a slight modification so that Theorem 5.1 holds for it.

Theorem 5.2. *Let ν be a stationary state of the diffusion process $(x(t), P_x)$ corresponding to* (5.1) *under the conditions* (5.2) *to* (5.4). *For $i\in Z^d$ we set*

$$\underline{b}_i(0)=\min\{b_i(x)\colon x\in X, x_i=0\}, \quad \textit{and}$$
$$\bar{b}_i(1)=\max\{b_i(x)\colon x\in X, x_i=1\}.$$

If $\underline{b}_i(0)>0$, then $\nu[x_i>0]=1$, while if $\bar{b}_i(1)<0$, then $\nu[x_i<1]=1$.

Proof. We will show it only in case that $\underline{b}_i(0)>0$. Since $\int_0^{1/2}\frac{du}{a(u)}=\infty$ by (5.2), there exists a sequence $\{\varepsilon_n\}$ satisfying that $\varepsilon_n>\varepsilon_{n+1}>\dots\to 0$ as $n\to\infty$, and

$$\int_{\varepsilon_{n+1}}^{\varepsilon_n}\frac{du}{a(u)}=2.$$

Let $g_n(u)$ be a continuous function on $[0,1]$ such that the support of g_n is contained in $[\varepsilon_{n+1}, \varepsilon_n]$, $\int g_n(u)\,du=1$, and

$$g_n(u)\leqq\frac{1}{a(u)} \quad \text{for } \varepsilon_{n+1}\leqq u\leqq\varepsilon_n.$$

Set $\phi_n(u)=\int_0^u\int_t^1 g_n(s)\,ds\,dt$. Then $a(u)\,\phi_n''(u)$ converges to zero boundedly as $n\to\infty$, and

$$\lim_{n\to\infty} b_i(x)\,\phi_n'(x_i)=0 \quad \text{boundedly in } x_i>0, \quad \text{and}$$
$$b_i(x)\,\phi_n'(x_i)\geqq\underline{b}_i(0)>0 \quad \text{if } x_i=0.$$

Hence, applying (5.9) for $\psi_n(x) \equiv \phi_n(x_i)$, we get

$$0 = \lim_{n\to\infty} \langle \nu, L\psi_n \rangle \geqq \underline{b}_i(0)\, \nu[x_i = 0],$$

which completes the proof of Theorem 5.2.

Now, we are in position to prove Theorem 1.5. Let ν be an extremal stationary state distinct from $\delta_{\mathbf{1}}$ and $\delta_{\mathbf{0}}$, appearing in Theorems 1.1 to 1.3. By Remark of Theorem 5.1 it suffices to check the condition (5.7). We first notice that if $\delta_{\mathbf{1}} \in \mathscr{S}$, then $\nu[\{\mathbf{1}\}] = 0$, since ν has a non-trivial decomposition unless $\nu[\{\mathbf{1}\}] = 0$. Indeed, the conditional distribution $\nu[\cdot \mid \{\mathbf{1}\}^c]$ is stationary and $\nu = \lambda\delta_{\mathbf{1}} + (1-\lambda)\nu[\cdot \mid \{\mathbf{1}\}^c]$ with $\lambda = \nu[\{\mathbf{1}\}]$. In particular, if the selection parameter s is non-positive, by virtue of Theorem 2.6 it holds that $\nu[x_i < 1 \textit{ for all } i] = 1$.

Hence (5.7) is verified for $\nu = \nu_c$ $(0 < c < 1)$ in Case 1, because $\mathscr{S}_{\mathrm{ext}} = \{\nu_c : 0 \leqq c \leqq 1\}$, $\nu_0 = \delta_{\mathbf{0}}$ and $\nu_1 = \delta_{\mathbf{1}}$ (by Theorem 1.1), and the process is invariant under the transformation $\{x_i\} \to \{1 - x_i\}$. In Case 3, if $u = 0$, $v > 0$, and $s < s_c (< 0)$, $(\mathscr{S} \cap \mathscr{I}^{Z^d})_{\mathrm{ext}} = \{\nu, \delta_{\mathbf{1}}\}$ (by Theorem 1.3), so by the above observation we have $\nu[x_i < 1 \textit{ for all } i] = 1$. Moreover, we note that Theorem 5.2 is applicable in this case, because $\underline{b}_i(0) > 0$ for all i, and we get $\nu[x_i > 0 \textit{ for all } i] = 1$. Thus the condition (5.7) is verified in the subcase of Case 3: $u = 0$, $v > 0$ and $s < s_c$. In another subcase of Case 3 is also fulfilled the condition (5.7) since it is reduced to the former subcase by the transformation $\{x_i\} \to \{1 - x_i\}$. In Case 2 there exists a unique stationary state ν by Theorem 1.2, and both $\underline{b}_i(0) > 0$ and $\bar{b}_i(1) < 0$ hold because $u > 0$ and $v > 0$. Hence by Theorem 5.2, the condition (5.7) is verified. We complete the proof of Theorem 1.5.

Remark. In the situation of Theorem 1.5 let $A = \{0\}$. Then, the probability density of the one-dimensional marginal distribution $p_0(x_0)$ of an extremal stationary state ν other than $\delta_{\mathbf{0}}$ and $\delta_{\mathbf{1}}$, appearing in Theorems 1.1 to 1.3, satisfies the following equation:

$$p_0(x_0) = C \cdot x_0^{2v-1}(1 - x_0)^{2u-1} \times \exp\left(2sx_0 + \int_{1/2}^{x_0} \frac{2m(r(y) - y)}{y(1-y)}\, dy\right), \tag{5.19}$$

where $C > 0$ is a normalizing constant, $m = \sum_{j \neq 0} m_j$, and

$$r(x_0) = \sum_{j \neq 0} m_j \int_0^1 x_j\, p_{\{0,j\}}(x_0, x_j)\, dx_j / m p_0(x_0).$$

Here we note that $0 < r(y) < 1$ for any $0 < y < 1$.

(5.19) is deduced from (5.9) or specifically from the equation

$$\int_0^1 (a\phi'' + b\phi')(x)\, p_0(x)\, dx = 0 \tag{5.20}$$

where $a(x) = x(1-x)/2$ *and*

$$b(x) = v - (u + v)x + sx(1-x) + m(r(x) - x).$$

In fact the Eq. (5.20) which is valid for all ϕ from $C^{\infty}[0,1]$ is uniquely solved among Lebesgue integrable functions on $[0,1]$ up to a constant factor; a solution is given by $C\cdot a(x)^{-1}\exp\left\{\int_{1/2}^{x}(b/a)(y)\,dy\right\}$, which coincides with the expression in (5.19).

To obtain a more information concerning $p_0(x_0)$ we need to investigate the function $r(x_0)$ in detail, but this problem remains open.

Appendix

In Sects. 2–4 we frequently applied certain comparison arguments as a basic technique, whereas none of them so far are proved, leaving proofs depending on them incomplete. So in this appendix we will establish some criteria upon which to test the validity of comparisons which are made in this paper.

Let $(\alpha(t),\mathbf{P})$ and $(\alpha(t),\mathbf{P}^*)$ be two stochastic processes on the state space $I:=\{\alpha=(\alpha_i)_{i\in S}\colon \alpha_i=0,1,2,\dots\ (i\in S),\ \sum\alpha_i<\infty\}$ where S is a given countable set. For $\alpha,\beta\in I$ we write $\alpha\leqq\beta$ if $\alpha_i\leqq\beta_i$ for all $i\in S$. Let $\Lambda=\{(\alpha,\beta)\in I\times I\colon \alpha\leqq\beta\}$. We will say that the process $(\alpha(t),\mathbf{P})$ can be embedded in $(\alpha(t),\mathbf{P}^*)$ and write $(\alpha(t),\mathbf{P})\prec(\alpha(t),\mathbf{P}^*)$, if one can construct a process $((\alpha(t),\alpha^*(t)),\tilde{\mathbf{P}})$ on the state space Λ such that $(\alpha(t),\tilde{\mathbf{P}})$ and $(\alpha^*(t),\tilde{\mathbf{P}})$ are stochastically equivalent to $(\alpha(t),\mathbf{P})$ and $(\alpha(t),\mathbf{P}^*)$, respectively.

Let $(\alpha(t),\mathbf{P}_\alpha)_{\alpha\in I}$ be a Markov process on I and $Q_{\alpha,\beta}$ its infinitesimal generator. Here and below it is supposed that all Markov processes are minimal and conservative without instantaneous states, so that $-Q_{\alpha,\alpha}=\sum_{\beta\neq\alpha}Q_{\alpha,\beta}<\infty$ for all α, and $(\alpha(t),\mathbf{P}_\alpha)$ is the unique Markov process on I generated by $\{Q_{\alpha,\beta}\}$. Let $(\alpha(t),P^*_\alpha)_{\alpha\in I}$ be another Markov process on I with a generator $\{Q^*_{\alpha,\beta}\}$. Set

$$I(\alpha)=\{\beta\in I\colon \mathbf{P}_\alpha(\alpha(t)=\beta)>0 \text{ for some (all) } t>0\}$$

and similarly for $I^*(\alpha)$. Given $(\alpha_0,\alpha_0^*)\in\Lambda$ we also set

$$\Lambda(\alpha_0,\alpha_0^*)=\{(\alpha,\alpha^*)\in\Lambda\colon \alpha\in I(\alpha_0),\ \alpha^*\in I^*(\alpha_0^*)\}.$$

Then for the order relation

$$(\alpha(t),\mathbf{P}_{\alpha_0})\prec(\alpha(t),\mathbf{P}^*_{\alpha_0^*}) \tag{A.1}$$

to hold it is sufficient that there exists a family of non-negative numbers $R_{\xi,\eta}$: $\xi,\eta\in\Lambda(\alpha_0,\alpha_0^*)$, $\xi\neq\eta$ such that

$$\begin{aligned}\sum_{\gamma\in I^*(\alpha_0^*),\ \gamma\geqq\beta} R_{(\alpha,\alpha^*),(\beta,\gamma)}&=Q_{\alpha,\beta}\\ \sum_{\gamma\in I(\alpha_0),\ \gamma\leqq\beta^*} R_{(\alpha,\alpha^*),(\gamma,\beta^*)}&=Q^*_{\alpha^*,\beta^*}\end{aligned} \tag{A.2}$$

for all (α,α^*), $(\beta,\beta^*)\in\Lambda(\alpha_0,\alpha_0^*)$ with $\alpha\neq\beta$ and $\alpha^*\neq\beta^*$. In fact $\{R_{\xi,\eta}\}$ generates a Markov process on Λ which realizes the required embedding, provided that it satisfies (A.2).

We write $\alpha \nleqq \beta$ as a complementary statement of $\alpha \leqq \beta$; and allow β_i to take negative integers under the convention that $Q^*_{\alpha,\beta}=0$ if $\beta \ngeqq 0$. (The addition, inequality etc. are extended naturally.)

Theorem A.1. *Let* $(\alpha_0, \alpha_0^*) \in \Lambda$. i) *Assume that for all* $(\alpha, \alpha^*) \in \Lambda(\alpha_0, \alpha_0^*)$ *and* $\beta \neq \alpha$ (*even for* $\beta \ngeqq 0$) *the following condition is satisfied:*

$$\text{(A.3)} \qquad \begin{cases} Q_{\alpha,\beta} \geqq Q^*_{\alpha+\gamma,\beta+\gamma} & \text{if } -\gamma \nleqq \beta-\alpha \leqq \gamma \\ Q_{\alpha,\beta} \leqq Q^*_{\alpha+\gamma,\beta+\gamma} & \text{if } -\gamma \leqq \beta-\alpha \nleqq \gamma \\ Q_{\alpha,\beta} = Q^*_{\alpha+\gamma,\beta+\gamma} & \text{if } -\gamma \nleqq \beta-\alpha \nleqq \gamma \end{cases}$$

where $\gamma=\alpha^*-\alpha$. *Then* $(\alpha(t), \mathbf{P}_{\alpha_0}) \prec (\alpha(t), \mathbf{P}^*_{\alpha_0^*})$. ii) *Assume that* $Q^*_{\alpha,\beta}=0$ *if* $\alpha \in I^*(\alpha_0^*)$ *and* $\sum_{i \in S} \min\{\beta_i-\alpha_i, 0\} \leqq -2$. *Then* (A.3) *holds for all* $\beta \ngeqq 0$, *so that* (A.3) *restricted to* $\beta \in I$ ($\beta \neq \alpha$) *implies the same conclusion as in* i).

Proof. Given $\xi=(\alpha, \alpha^*) \in \Lambda(\alpha_0, \alpha_0^*)$ we must define non-negative numbers $R_{\xi,\eta}$ appropriately. In below positive numbers will be assigned to $R_{\xi,\eta}$ only when η is of the form

$$\eta=(\beta, \alpha^*),\ (\alpha, \alpha^*+\beta-\alpha) \quad \text{or} \quad (\beta, \alpha^*+\beta-\alpha);$$

and zero otherwise. Writing $\gamma=\alpha^*-\alpha$, we set for $\beta \neq \alpha$

$$\left.\begin{aligned} R_{\xi,(\beta,\alpha^*)} &= Q_{\alpha,\beta} \\ R_{\xi,(\alpha,\beta+\gamma)} &= Q^*_{\alpha^*,\beta+\gamma} \end{aligned}\right\} \quad \text{if } \alpha-\gamma \leqq \beta \leqq \alpha+\gamma$$

$$\left.\begin{aligned} R_{\xi,(\beta,\alpha^*)} &= Q_{\alpha,\beta} - Q^*_{\alpha^*,\beta+\gamma} \\ R_{\xi,(\beta,\beta+\gamma)} &= Q^*_{\alpha^*,\beta+\gamma} \end{aligned}\right\} \quad \text{if } \alpha-\gamma \nleqq \beta \leqq \alpha+\gamma$$

$$\left.\begin{aligned} R_{\xi,(\alpha,\beta+\gamma)} &= Q^*_{\alpha^*,\beta+\gamma} - Q_{\alpha,\beta} \\ R_{\xi,(\beta,\beta+\gamma)} &= Q_{\alpha,\beta} \end{aligned}\right\} \quad \text{if } \alpha-\gamma \leqq \beta \nleqq \alpha+\gamma$$

$$R_{\xi,(\beta,\beta+\gamma)} = Q_{\alpha,\beta} \qquad \text{if } \alpha-\gamma \nleqq \beta \nleqq \alpha+\gamma$$

and $R_{\xi,\eta}=0$ for all the other transitions. Clearly the condition (A.2) is satisfied, so i) is proved. For the assertion ii) it suffices to check, under its assumption, the claim that if $\beta \ngeqq 0$, $\beta+\gamma \nleqq \alpha$ and $\beta+\gamma \geqq 0$, then $Q^*_{\alpha^*,\beta+\gamma}=0$. But, if one sets $\delta=\beta+\gamma-\alpha^*$, the former three relations in this claim are transformed into $\delta+\alpha \ngeqq 0$, $\delta+\gamma \nleqq 0$ and $\delta+\alpha+\gamma \geqq 0$, respectively, which together imply that $\sum_{i \in S} \min\{\beta_i-\alpha_i, 0\} \leqq -2$; hence, by the assumption of the part ii), $Q^*_{\alpha^*,\alpha^*+\delta}=0$, completing the proof of the part ii).

Corollary A.1. *Let* $(\alpha_0, \alpha_0^*) \in \Lambda$. *Assume that* $Q^*=Q$ *and for all* $(\alpha, \alpha^*) \in \Lambda(\alpha_0, \alpha_0^*)$ *and* $\beta \neq \alpha$, $\beta \in I$ *the following conditions hold:*

(A.4) $Q_{\alpha,\beta}>0$ *only if* β *is of the form:*

$$\beta=\alpha \pm \varepsilon_i \quad \text{or} \quad \alpha \pm \varepsilon_i + \varepsilon_j \quad (i \neq j),$$

(A.5) $Q_{\alpha,\beta} \geqq Q_{\alpha+\gamma,\beta+\gamma}$ *if* $\beta \leqq \alpha+\gamma$ *and* $\beta_i<\alpha_i$ *for some* $i \notin \operatorname{supp}\gamma$, (A.6) $\qquad Q_{\alpha,\beta} \leqq Q_{\alpha+\gamma,\beta+\gamma}$ *if* $\beta+\gamma \geqq \alpha$ *and* $\beta_i > \alpha_i$ *for some* $i \notin \operatorname{supp}\gamma$,

(A.7) $\qquad Q_{\alpha,\beta} = Q_{\alpha+\gamma,\beta+\gamma}$ *if* $\alpha_i = \beta_i$ *for all* $i \in \operatorname{supp}\gamma$,

where $\gamma = \alpha^* - \alpha$. *Then* $(\alpha(t), \mathbf{P}_{\alpha_0}) \prec (\alpha(t), \mathbf{P}_{\alpha_0^*})$.

Proof. If $-\gamma \nleqq \beta - \alpha$, then either $\beta_i < \alpha_i$ for some $i \notin \operatorname{supp}\gamma$ or $\beta_i \leqq \alpha_i - 2$ for some $i \in \operatorname{supp}\gamma$. Thus by (A.4) and (A.5) we have the first inequality of (A.3); and similarly for the second. If $-\gamma \nleqq \beta - \alpha \nleqq \gamma$, then either there are two distinct sites i and j outside of $\operatorname{supp}\gamma$ such that $\beta_i > \alpha_i$ and $\beta_j < \alpha_j$ or $|\alpha_i - \beta_i| \geqq 2$ for some $i \in \operatorname{supp}\gamma$, proving the last relation of (A.3) in view of (A.4) and (A.7).

In the next theorem will be compared two Markov processes $(\alpha(t), \mathbf{P}_\alpha)_{\alpha\in I}$ and $(\alpha(t), \mathbf{P}_\alpha^*)_{\alpha\in I}$ governed by infinitesimal generators $Q_{\alpha,\beta}$ and $Q_{\alpha,\beta}^*$, respectively. We will suppose the stochastic monotonicity for one of them:

(A.8) $$(\alpha(t), \mathbf{P}_\alpha^*) \leqq (\alpha(t), P_\beta^*) \qquad \textit{whenever } \alpha \leqq \beta.$$

(See (2.4) for the meaning of the inequality above.)

Theorem A.2. *Let* $\alpha_0 \in I$ *be given. Assume that* (A.8) *holds and that for every* $\alpha \in I(\alpha_0)$ *and for every finite sequence* $\beta^1, \ldots, \beta^n$ *in* I

(A.9) $$\begin{aligned} &\sum_{\beta:\, \beta \leqq \beta^k \text{ for } \exists k} (Q_{\alpha,\beta}^* - Q_{\alpha,\beta}) \leqq 0 \quad \textit{if } \alpha \nleqq \beta^k \quad \textit{for all } k \\ &\sum_{\beta:\, \beta \nleqq \beta^k \text{ for } \forall k} (Q_{\alpha,\beta}^* - Q_{\alpha,\beta}) \geqq 0 \quad \textit{if } \alpha \leqq \beta^k \quad \textit{for some } k. \end{aligned}$$

Then $(\alpha(t), \mathbf{P}_\alpha) \prec (\alpha(t), \mathbf{P}_\beta^*)$ *whenever* $\alpha \in I(\alpha_0)$ *and* $\alpha \leqq \beta$.

Remark. i) For the conclusion of Theorem A.2 the condition (A.9) is necessary regardless of whether the other assumption is valid. (Observe that if (A.9) is violated, then dw/dt in the proof of Theorem A.2 actually attains a positive value at $t=0$ for a non-increasing f.) ii) Instead of the condition (A.8) one can assume the corresponding monotonicity condition for $(\alpha(t), \mathbf{P}_\alpha)$ without changing the rest of the theorem.

For the proof of Theorem A.2 we prepare two lemmas, which are variations of well known facts.

Lemma A.1. *Let* $\{a_\alpha\}$ *be a family of real numbers indexed by* $\alpha \in I$ *such that* $\sum_{\alpha\in I} |a_\alpha| < \infty$. *In order that the inequality* $\sum_{\alpha\in I} a_\alpha f(\alpha) \leqq 0$ *holds for every non-negative, bounded and monotone non-increasing function* f *defined on* I *it is necessary and sufficient that every finite sequence* $\alpha^k \in I$, $k = 1, \ldots, n$

(A.10) $$\sum_{\alpha:\, \alpha \leqq \alpha^k \text{ for } \exists k} a_\alpha \leqq 0.$$

Proof. Set $\Gamma(t) = \{\alpha \in I : f(\alpha) > t\}$ and $h(t) = \sum_{\alpha \in \Gamma(t)} a_\alpha$. Then $\sum_\alpha a_\alpha f(\alpha) = \int_0^\infty h(t)\, dt$. $\Gamma(t)$ is a monotone non-increasing subset of I; hence it is a monotone limit of a set of the form $\{\alpha : \alpha \leqq \alpha^k \textit{ for some } k, 1 \leqq k \leqq n\}$. (To see this make a truncation of $\Gamma(t)$ and take as $\{\alpha^k\}$ all the elements of the truncated.) Therefore (A.10) implies $h(t) \leqq 0$, so $\sum_\alpha a_\alpha f(\alpha) \leqq 0$.

Lemma A.2. *Assume* $(\alpha(t), \mathbf{P}_\alpha) \leqq (\alpha(t), \mathbf{P}^*_\beta)$ *whenever* $\alpha \leqq \beta$ *and* $\alpha \in I(\alpha_0)$. *Then* $(\alpha(t), \mathbf{P}_\alpha) \prec (\alpha(t), \mathbf{P}^*_\beta)$ *whenever* $\alpha \leqq \beta$ *and* $\alpha \in I(\alpha_0)$.

Proof. We will apply a compactness argument to a sequence of probability measures of stochastic processes taking values in I. Let Ω be the space of right-continuous step functions from $[0, \infty)$ into I which is equipped with the Skorohod topology. For two elements ω_1 and ω_2 from Ω write $\omega_1 \leqq \omega_2$ if $\omega_1(t) \leqq \omega_2(t)$ for all $t \geqq 0$. This is a partial order relation, which is closed (i.e. the order relation "$\leqq$" is preserved in the procedure of taking a limit), for the convergence in the Skorohod topology implies the convergence at every point of continuity of the limit function. Now let the assumption of the lemma hold. For simplicity we assume $I(\alpha_0) = I$. For each n there exists a stochastic kernel $p^{(n)}(\xi, \eta)$, $\xi, \eta \in I \times I$ such that if $\xi = (\alpha, \beta) \in \Lambda$, then $\sum_{\eta \in \Lambda} p^{(n)}(\xi, \eta) = 1$ and the first and the second marginals of the probability $p^{(n)}(\xi, \cdot)$ on the product space $I \times I$ coincides with the laws of $\alpha(1/n)$ induced from $\mathbf{P}_\alpha$ and $\mathbf{P}^*_\beta$, respectively (set $p^{(n)}(\xi, \cdot) = \delta_\xi(\cdot)$ e.g. if $\xi \notin \Lambda$). The kernel $p^{(n)}$ is constructed by applying an allocation lemma as found in Pollard [10] together with a truncation argument; also its existence is a special case of Theorem 11 of Strassen [16] (c.f. also Theorem 2.4 of Liggett [9]). This kernel determines a discrete time Markov chain on the state space $I \times I$. For each $(\alpha, \beta) \in \Lambda$ it naturally induces a probability measure $P^{(n)}$ on the product space $\Omega \times \Omega$ (equipped with the product topology) in such a way that the first and second marginals of $P^{(n)}$ agree with the laws on Ω induced by $\{\alpha([nt]/n), t \geqq 0\}$ from $\mathbf{P}_\alpha$ and $\mathbf{P}^*_\beta$, respectively, and $P^{(n)}$ is concentrated on $\Delta = \{(\omega_1, \omega_2) \in \Omega \times \Omega \colon \omega_1 \leqq \omega_2\}$. $\{P^{(n)}\}$ is relatively compact in the weak topology of probability measures, for its marginals are relatively compact. Let $\tilde{P}$ be a limit point of $P^{(n)}$. By the well-known lower semi-continuity property of probability measures evaluated for a closed set the probability $\tilde{P}$ is concentrated on the closed set Δ. Clearly the marginals of $\tilde{P}$ are identical to $(\alpha(t), \mathbf{P}_\alpha)$ and $(\alpha(t), \mathbf{P}^*_\beta)$. Thus $\tilde{P}$ realize the required embedding, completing the proof of Lemma A.2.

Proof of Theorem A.2. Let f be a bounded non-increasing function defined on I and set $u(t, \alpha) = \mathbf{E}_\alpha[f(\alpha(t))]$ and $u^*(t, \alpha) = \mathbf{E}^*_\alpha[f(\alpha(t))]$. Let us prove that $w(t, \alpha) := u^*(t, \alpha) - u(t, \alpha) \leqq 0$ for $\alpha \in I(\alpha_0)$. To this end we can assume $f \geqq 0$ so that $u^* \geqq 0$. In view of the differential-difference equation $dw/dt = Qw + (Q^* - Q) u^*$ with $w(0) = 0$ it suffices to show that $(Q^* - Q) u^* \leqq 0$ (on $I(\alpha_0)$). But this inequality is immediate by an application of Lemma A.1 with the help of the assumptions (A.8) and (A.9), if it is noted that (A.9) is a paraphrase of (A.10) with $a_\beta = Q^*_{\alpha,\beta} - Q_{\alpha,\beta}$. Now Theorem A.2 follows from Lemma A.2 and the monotonicity (A.8).

The next corollary is intuitive.

Corollary A.2. *Let* $\alpha_0 \in I$ *be given. Assume that* (A.8) *holds and that for all* $\alpha \in I(\alpha_0)$ *the following conditions are satisfied:*

(A.11) $$Q^*_{\alpha,\beta} > 0 \quad \text{only if } \beta \geqq \alpha - \varepsilon_i \quad \text{for some } i$$

(A.12) $$Q^*_{\alpha,\beta} \geqq Q_{\alpha,\beta} \quad \text{if } \beta \nleqq \alpha$$

$$\text{(A.13)} \qquad \sum_{\substack{\beta:\alpha-\varepsilon_i\leqq\beta\not\leqq\alpha \\ \beta_i=\alpha_i-\varepsilon_i}} (Q^*_{\alpha,\beta}-Q_{\alpha,\beta})\leqq Q_{\alpha,\alpha-\varepsilon_i}-Q^*_{\alpha,\alpha-\varepsilon_i} \qquad \text{for all } i.$$

Then $(\alpha(t),\mathbf{P}_\alpha)\prec(\alpha(t),\mathbf{P}^*_\beta)$ *whenever* $\alpha\in I(\alpha_0)$ *and* $\alpha\leqq\beta$.

Proof. We must check the condition (A.9) of Theorem A.2. The second inequality of (A.9) trivially follows from (A.12). To check the first we note that the sets of β under the summation sign in (A.13) are mutually disjoint for different i's. Let c_i denote the left-hand side of (A.13) minus the right-hand one. Then by (A.11) and (A.12) the left-hand side of the first inequality of (A.9) is dominated above by the sum of c_i over those i for which $\alpha_i-1\geqq 0$ and $\beta^k\geqq\alpha-\varepsilon_i$ for some k. Thus (A.9) is proved. The proof of Corollary A.2 is complete.

Let $(\alpha(t),\mathbf{P}_\alpha)_{\alpha\in I}$, $(\alpha^1(t),\mathbf{P}^1_\alpha)_{\alpha\in I}$ and $(\alpha^2(t),P^2_\alpha)_{\alpha\in I}$ be three Markov processes with their generators $Q_{\alpha,\beta}$, $Q^1_{\alpha,\beta}$ and $Q^2_{\alpha,\beta}$, respectively. Let $I^1(\alpha)$ and $I^2(\alpha)$ be defined in the same way as $I(\alpha)$. Recall the convention that $Q^i_{\alpha,\beta}=0$ if $\beta\not\geqq 0$.

Theorem A.3. *Let* α_0 *and* $\beta_0\in I$ *be given. Assume that* $(\alpha(t),\mathbf{P}_\alpha)$ *satisfies the monotonicity condition* (A.8) *with* $\mathbf{P}_\cdot$ *in place of* $\mathbf{P}^*_\cdot$ *and that for every* $\alpha\in I^1(\alpha_0)$ *and* $\beta\in I^2(\beta_0)$ *and for every finite sequence* $\gamma^1,\ldots,\gamma^n$ *in* I

$$\sum_{\gamma:\,\gamma\leqq\gamma^k \text{ for } \exists k} (Q^1_{\alpha,\gamma-\beta}+Q^2_{\beta,\gamma-\alpha}-Q_{\alpha+\beta,\gamma})\leqq 0 \qquad \text{if } \alpha+\beta\not\leqq\gamma^k \quad \text{for all } k,$$

$$\sum_{\gamma:\,\gamma\not\leqq\gamma^k \text{ for } \forall k} (Q^1_{\alpha,\gamma-\beta}+Q^2_{\beta,\gamma-\alpha}-Q_{\alpha+\beta,\gamma})\geqq 0 \qquad \text{if } \alpha+\beta\leqq\gamma^k \quad \text{for some } k.$$

Then $(\alpha(t),\mathbf{P}_{\alpha+\beta})\leqq(\alpha^1(t)+\alpha^2(t),\ \mathbf{P}^1_\alpha\otimes\mathbf{P}^2_\beta)$ *for all* $\alpha\in I^1(\alpha_0)$ *and all* $\beta\in I^2(\beta_0)$.

Remark. The first one of Remarks to Theorem A.2 applies to Theorem A.3, while the second one does not (at least as far as our proof is concerned). The latter is because the sum of the two independent processes is no longer Markovian.

Proof. We will proceed as in the proof of Theorem A.2. For a bounded, non-negative and monotone non-increasing function f on I set

$$u(t,\alpha)=\mathbf{E}_\alpha[f(\alpha(t))], \qquad \tilde{u}(t,\alpha,\beta)=\mathbf{E}^1_\alpha\otimes\mathbf{E}^2_\beta[f(\alpha^1(t)+\alpha^2(t))]$$

where the last expression means the expectation by $\mathbf{P}^1_\alpha\otimes\mathbf{P}^2_\beta$. Let $\tilde{Q}_{\xi,\eta}$ be the infinitesimal generator of $((\alpha^1(t),\alpha^2(t)),\ \mathbf{P}^1_\alpha\otimes\mathbf{P}^2_\beta)$:

$$\tilde{Q}_{(\alpha,\beta),(\alpha',\beta')}=\begin{cases} Q^1_{\alpha,\alpha'} & \text{if } \alpha\neq\alpha' \text{ and } \beta=\beta' \\ Q^2_{\beta,\beta'} & \text{if } \beta\neq\beta' \text{ and } \alpha=\alpha' \\ 0 & \text{otherwise (but } (\alpha,\beta)\neq(\alpha',\beta')). \end{cases}$$

Then $w(t,\alpha,\beta):=\tilde{u}(t,\alpha,\beta)-u(t,\alpha+\beta)$ satisfies the equation $dw/dt=\tilde{Q}w+R$, where

$$R(t,\alpha,\beta)=\sum_{\alpha',\beta'}\tilde{Q}_{(\alpha,\beta),(\alpha',\beta')}u(t,\alpha'+\beta')-\sum_\gamma Q_{\alpha+\beta,\gamma}u(t,\gamma).$$

But R is rewritten $\sum_\gamma h_{\alpha,\beta,\gamma} u(t,\gamma)$, where

$$h_{\alpha,\beta,\gamma} = Q^1_{\alpha,\gamma-\beta} + Q^2_{\beta,\gamma-\alpha} - Q_{\alpha+\beta,\gamma}; \tag{A.14}$$

so the assumptions of the theorem verify that $R \leqq 0$ as in the proof of Theorem A.2. Accordingly $w \leqq 0$.

We can state a corollary parallel to Corollary A.2, but here give another version.

Corollary A.3. *Let* $\alpha_0, \beta_0 \in I$. *Assume that the condition* (A.8) *with* $\mathbf{P}$. *in place of* $\mathbf{P}^*_.$ *is satisfied and that for all* $\alpha \in I^1(\alpha_0)$ *and all* $\beta \in I^2(\beta_0)$ *the following relations hold*

$$Q_{\alpha+\beta,\gamma} > 0 \quad \text{only if } \gamma \leqq \alpha+\beta+\varepsilon_i \quad \text{for some } i,$$

$$h_{\alpha,\beta,\gamma} \leqq 0 \quad \text{if } \gamma \not\leqq \alpha+\beta,$$

$$\sum_{\substack{\gamma:\alpha+\beta \not\leqq \gamma \leqq \alpha+\beta+\varepsilon_i \\ \gamma_i = \alpha_i+\beta_i+1}} h_{\alpha,\beta,\gamma} \geqq -h_{\alpha,\beta,\alpha+\beta+\varepsilon_i} \quad \text{for all } i,$$

where h *is defined in* (A.14). *Then the conclusion of Theorem* A.3 *holds.*

The proof of Corollary A.3 is similar to that of Corollary A.2.

Remark. We so far restricted the state spaces of processes to elements of I, i.e., configurations of non-negative integers on S, because only comparison between such processes were concerned in the main body of this paper. When one allows configurations to take negative integers too, do Theorem A.1 to Theorem A.3 remain valid? To make statements precise we set $\bar{I} = \{\alpha \in Z^S : \sum_{i \in S} |\alpha_i| < \infty\}$. Then this is correct for Theorems A.2 and A.3 as well as for their corollaries without any change except that $\bar{I}$ replaces I. To see this we must extend Lemmas A.1 and A.2. The extension of the latter together with its proof is valid with no alternation. Lemma A.1 also is extended, but for its proof we need to think out a suitable manner of truncation: given a monotone non-increasing subset Γ of $\bar{I}$, set $\Gamma^n = \{\alpha \in \Gamma : \alpha \leqq \beta$ *for some* $\beta \in \Gamma$ *such that* $\beta_i \leqq n$ *if* $i \in S_n$ *and* $\beta_i = 0$ *if* $i \notin S_n\}$, where S_n is an initial segment up to the n-th in any enumeration of S, to have that $\Gamma^n \uparrow \Gamma$, Γ^n is a non-increasing subset of $\bar{I}$ and the maximal elements of Γ^n is finite (the last of these three statements may be proved by induction on n with a little difficulty). As for Theorem A.1 its first part is even more naturally stated for $\bar{I}$ than for I; the second part is rather motivated by restricting it to I. Corollary A.1 as well as i) of Theorem A.1 remains valid after replacing I with $\bar{I}$. Finally we add one more remark: in Theorem A.1 and its corollary one can replace I by $I^{(n)} = \{\alpha \in \{0, 1, 2, \dots\}^S : \sum_{i \in S} |\alpha_i - n| < \infty\}$ $(n = 1, 2, \dots)$.

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