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A Stochastic Version of Center Manifold Theory

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Summary. Random dynamical systems arise naturally if the influence of white or real noise on the parameters of a nonlinear deterministic dynamical system is studied. In this situation Lyapunov exponents attached to the linearized flow replace the real parts of the eigenvalues and describe the stability behavior of the linear system. If at least one of them vanishes then it is possible to prove the existence of a stochastic analogue of the deterministic center manifold. The asymptotic behavior of the entire system can then be derived from the lower dimensional system restricted to this stochastic center manifold. A dynamical characterization of the stochastic center manifold is given and approximation results are proved.

1. Introduction

Consider the following ordinary differential equation

$$\dot{x} = F(x), \qquad x \in \mathbb{R}^d, \qquad F(x_0) = 0, \tag{1.1}$$

where $F\colon \mathbb{R}^d \to \mathbb{R}^d$ is assumed to be sufficiently smooth and to have a steady state at x_0. For convenience we take $x_0 = 0$.

We ask whether the asymptotic behavior of (1.1) and in particular the stability of the zero solution may be derived from a lower dimensional system. In the case of a *linear* system the real parts of the eigenvalues describe the stability behavior of the system: If for example all the real parts are negative then the zero solution will be asymptotically stable for $t \to +\infty$, i.e. initial values will finally tend to 0 forwards in time.

In the case of a *nonlinear* system we first investigate the eigenvalues of the Jacobian matrix $DF|_{x_0=0}$. If all the real parts of eigenvalues of this linearized system are negative (resp. positive) then already Lyapunov [26] proved that the stability properties of this system carry over to the original nonlinear system. However, this does no longer hold true if one of the real parts vanishes. This situation is the usual one encountered in bifurcation theory which deals with nonlinear systems that depend on a parameter. For a certain parameter value the zero solution may lose its stability and new nontrivial stable solutions may branch off.

Center manifold theory allows to decouple the system and to separate that part which corresponds to the vanishing real parts of the linearization. The heuristic reason for doing so is that the stability behavior of the part of the system corresponding to negative resp. positive real parts is well-known. In fact it can be shown that in order to determine the stability behavior of the original system it is sufficient to investigate the asymptotic behavior of the lower dimensional system obtained as the restriction of the original equation to the center manifold. In bifurcation theory this system will usually be of dimension 1 or 2, independently of the dimension of the original system. Details may be found, e.g., in Carr [9], Iooss [21], Marsden and McCracken [28] or in Vanderbauwhede [33].

In this paper we would like to examine what happens to the questions sketched above if a system like (1.1) or a difference equation (i.e., a discrete time system) is influenced by noise which might either be white noise or any stationary ergodic process. We will thus have to deal with random dynamical systems, a notion which will be made precise in Sect. 2. The aim will be to establish a *stochastic center manifold theory* for these random dynamical systems. Due to the fact that the elements of the probability space Ω are related by a so-called shift operator, the deterministic proofs can, however, not be carried over trajectory-wise. Nevertheless we will be able to show that the basic objects of such a theory, the stochastic center manifolds, share all the nice properties of their deterministic counterparts. In particular they enable us to restrict the investigations concerning the asymptotic behavior of the system to a reduced system of lower dimension. As in the deterministic context a stochastic center manifold theory will be a cornerstone in the construction of a stochastic bifurcation theory because it will justify focussing mainly onto low dimensional problems and it tells us how to reduce higher dimensional systems.

The paper is organized as follows:

After having recalled the notion of a random dynamical system and its linearization in the beginning of Sect. 2 we will introduce the notion of Lyapunov exponents and Oseledec spaces associated with the linearized system in order to replace the real parts of the eigenvalues and the eigenspaces in the deterministic situation. We will explain why they describe the dynamical behavior of this linear system. In various different contexts these objects have already been successfully used by many authors, e.g., by Baxendale [3], Bougerol and Lacroix [6], Carverhill [10], Crauel [14], Ledrappier [25], Mañé [27], Oseledec [29] and Ruelle [31]. There also exist several proofs of the multiplicative ergodic theorem originally due to Oseledec which ensures the existence of Lyapunov exponents and Oseledec spaces $E_i(\omega)$ and describes their main properties. In contrast to most of these authors we will, however, consider random dynamical systems on the entire time axis, i.e. $T=\mathbb{R}$ or $T=\boldsymbol{Z}$, because both directions of time are necessary to construct stochastic center manifolds.

Usually the influence of noise on an ordinary differential equation can be modeled either by considering a stochastic differential equation (white noise) or an ordinary differential equation with random coefficients (real noise). In the case of a difference equation products of random diffeomorphisms will be a suitable stochastic model. It is the purpose of Sect. 3 to explain under which conditions these models are covered by the general set up of Sect. 2 and how the extension to the entire time axis may be obtained.

If all the Lyapunov exponents are negative then the stochastic stable manifold theorem (see Ruelle [31] and Carverhill [10]) tells us that the stability behavior of the linearized system will carry over to the original nonlinear system. As in the deterministic case this will, however, no longer hold true if one of the Lyapunov exponents vanishes. For this reason the notion of a (global) stochastic center manifold is introduced in Sect. 4 to cope with this situation. Its definition reflects the fact that stochastic center manifolds may be viewed as the nonlinear version of the space $E_c(\omega)$ corresponding to the zero Lyapunov exponent. We will construct them in Sect. 5 as maps between measurable bundles. The structure of these spaces takes into account that it is not possible to repeat the deterministic construction pathwise. In fact, in order to obtain the stochastic center manifold for one fixed $\omega \in \Omega$ the entire probability space has to be used. For this reason the main task of Sect. 4 consists in finding suitable spaces to work in. The crucial step is to introduce random norms on the fibres of the bundle, i.e. on the spaces $E_i(\omega)$, and then to make the bundle a metric space.

Section 5 is the heart of the paper and contains the proof of the existence of global stochastic center manifolds, provided that the nonlinear part of the random dynamical system is sufficiently small. The influences determining the required smallness will be figured out explicitly. The main step in the proof is an application of the contraction mapping theorem to an appropriate operator acting on a subset of the bundle defined in Sect. 4. Hence as in the deterministic case the proof that stochastic center manifolds exist follows lines which are entirely different from those which lead to an existence proof for stochastic stable manifolds.

The smallness of the nonlinear part required in the existence result is rather restrictive. We can, however, get rid of this assumption by restricting ourselves to a suitable (random) neighborhood of the origin. Thus we obtain *local* stochastic center manifolds which are just the right tool to handle applications especially in bifurcation theory. The existence of local stochastic center manifolds is ensured under very weak conditions.

In Sect. 7 properties of the (global) stochastic center manifold are collected. It is shown that it is attracting, i.e. that in case of a nonpositive (resp. nonnegative) Lyapunov spectrum solutions starting away from the stochastic center manifold will approach it exponentially fast for $t \to \infty$ ($t \to -\infty$, resp.). This is particularly important for numerical purposes. In contrast to the geometrical characterization of a stochastic center manifold given in Sect. 4 we may also deduce a dynamical characterization as the set of all those initial values in $\mathbb{R}^d$ whose exponential growth rate is smaller than a certain $\delta > 0$ in both directions of time. This dynamically characterized stochastic center manifold is unique which is not the case for the geometrically characterized one. Finally we prove that it is in fact sufficient to examine the asymptotic behavior of the system restricted to the stochastic center manifold to derive the stability properties of the entire system. The error committed by considering this restricted system is shown to decay exponentially fast.

The last section deals with approximation results. First we show by means of a nonconstructive existence proof that, in principle, the stochastic center manifold may be approximated by polynomials up to any degree of accuracy. Since this result is not very helpful for practical purposes we also explain how an explicit approximation may be derived. This procedure is illustrated by means of two examples which also show that the stochastic center manifold is not just the deterministic center manifold plus noise but that completely new stochastic effects occur.

2. Stochastic Framework

Let $(\Omega, \mathscr{F}, P)$ be a probability space. For $T=\mathbb{R}$ or $T=\mathbb{Z}$ consider a group of bimeasurable measure preserving bijections (i.e. a flow) $\{\vartheta_t | t\in T\}$ on Ω. In the sequel $\{\vartheta_t\}$ is supposed to be ergodic.

Definition 2.1. A map $\varphi: \mathbb{R}\times\Omega\times\mathbb{R}^d \to \mathbb{R}^d$, $(t, \omega, x) \to \varphi(t, \omega, x)$ is called a *random dynamical system* (or a *cocycle*) of $\mathscr{C}^k$ (resp. $\mathscr{C}^{k,1}$)-diffeomorphisms ($k\geqq 1$) on $\mathbb{R}^d$ over the flow $\{\vartheta_t: t\in T\}$ if the following properties are satisfied:

(i) $\varphi(t, \cdot, x)$ is $\mathscr{F}, \mathscr{B}(\mathbb{R}^d)$-measurable for any $t\in T$, $x\in\mathbb{R}^d$.

There is a ϑ_t-invariant set $\Omega_0\subset\Omega$, $P(\Omega_0)=1$, such that for all $\omega\in\Omega_0$:

(ii) $\varphi(t, \omega, x)$ is continuous in $(t, x)\in T\times\mathbb{R}^d$.

(iii) $\varphi(t, \omega, \cdot)$ is a $\mathscr{C}^k$ (resp. $\mathscr{C}^{k,1}$)-diffeomorphism for any $t\in T$. Here $\mathscr{C}^{k,1}$ means that $D^k\varphi(t, \omega, \cdot)$ satisfies a global Lipschitz condition.

(iv) $\varphi(t+s, \omega, x) = \varphi(s, \vartheta_t\omega, \cdot)\circ\varphi(t, \omega, x)$ for any $t, s\in T$, $x\in\mathbb{R}^d$ (cocycle property).

Remarks.

(i) For notational convenience we will also write $\varphi(t, \omega)x$ instead of $\varphi(t, \omega, x)$.

(ii) The cocycle property implies that for all $\omega\in\Omega_0$ and for all $t\in T$ $\varphi(0, \omega) = \mathrm{Id}$ and $\varphi^{-1}(t, \omega) = \varphi(-t, \vartheta_t\omega)$.

The random dynamical system φ gives rise to a skew-product flow

$$\Theta_t: \Omega\times\mathbb{R}^d \to \Omega\times\mathbb{R}^d,$$
$$(\omega, x) \to (\vartheta_t\omega, \varphi(t, \omega, x)).$$

Let μ be a Θ_t-invariant probability measure on $\Omega\times\mathbb{R}^d$ (i.e. $\Theta_t\mu=\mu$ for all $t\in T$) satisfying $\pi_\Omega\mu = P$, where π_Ω: $\Omega\times\mathbb{R}^d\to\Omega$ denotes projection, and suppose that μ is ergodic.

Denote by $T_x\mathbb{R}^d$ the tangent space to $\mathbb{R}^d$ at $x\in\mathbb{R}^d$ and by $T\mathbb{R}^d$ the tangent bundle. Consider the map $T\varphi(t, \omega)$: $T\mathbb{R}^d\to T\mathbb{R}^d$ where $T\varphi(t, \omega, x)$: $T_x\mathbb{R}^d \to T_{\varphi(t,\omega,x)}\mathbb{R}^d$ is the linearization (derivative) of $\varphi(t, \omega)$ at x. For (t, ω, x) fixed this is a well-defined linear isomorphism which constitutes, by the chain rule, a linear random dynamical system on $\Omega\times\mathbb{R}^d$ over the base flow Θ_t, i.e.

$$T\varphi(t+s, \omega, x) = T\varphi(t, \Theta_s(\omega, x))\circ T\varphi(s, \omega, x) \quad \text{for all } t, s\in T,\ x\in\mathbb{R}^d,\ \omega\in\Omega_0.$$

Definition 2.2. The *Lyapunov exponent associated with the linearized cocycle* $T\varphi(t, \omega, x)$ *at* $x \in \mathbb{R}^d$ *under* $\omega \in \Omega$ *in the direction* $v \in \mathbb{R}^d$, $v \neq 0$, is defined by

$$\lambda^{\pm}(\omega, x, v) := \limsup_{t \to \pm\infty} \frac{1}{t} \log \| T\varphi(t, \omega, x) v \|.$$

The next theorem will ensure the existence of the limits defined above. It will also elucidate that the Lyapunov exponents are the stochastic counterparts of the real parts of the eigenvalues and it will provide us with stochastic analogues to the generalized eigenspaces. These objects completely describe the asymptotic behavior of the linearized system. Before stating the theorem we will, however, need some preparations:

Let $G_n(T\mathbb{R}^d)$, $n \in \{1, \ldots, d\}$, be the Grassmann bundle of n-dimensional subspaces of tangent spaces to $\mathbb{R}^d$ (see, e.g., Boothby [5], p. 63) endowed with the following topology: a set $U \subset G_n(T\mathbb{R}^d)$ is open if the set of all vectors contained in some subspace belonging to U is an open set in the tangent bundle $T\mathbb{R}^d$.

A mapping $E: \Omega \times \mathbb{R}^d \to G_n(T\mathbb{R}^d)$ is measurable if the preimages of Borel sets of $G_n(T\mathbb{R}^d)$ are measurable subsets of $\Omega \times \mathbb{R}^d$.

Theorem 2.1 (Oseledec's multiplicative ergodic theorem). *Consider a random dynamical system* φ *on* $\mathbb{R}^d$ *and an invariant ergodic probability measure* μ *on* $\Omega \times \mathbb{R}^d$. *Assume the following integrability condition:*

$$\int_{\Omega \times \mathbb{R}^d} [\log^+ \sup_{0 \leqq t \leqq 1} \| T\varphi(t, \omega, x) \| + \log^+ \sup_{0 \leqq t \leqq 1} \| [T\varphi(t, \omega, x)]^{-1} \|] \, d\mu(\omega, x) < \infty.$$

Then there are r real numbers $\lambda_r < \ldots < \lambda_1$ $(1 \leqq r \leqq d)$ *with multiplicities* d_i, $\sum_{i=1}^{r} d_i = d$, *such that for a* Θ_t*-invariant subset* $\Gamma \subset \Omega \times \mathbb{R}^d$, $\mu(\Gamma) = 1$, *there is a family of measurable mappings* $E_i: \Gamma \to G_{d_i}(T\mathbb{R}^d)$, $1 \leqq i \leqq r$, *such that for each* $(\omega, x) \in \Gamma$:

$$T_x\mathbb{R}^d = E_1(\omega, x) \oplus \ldots \oplus E_r(\omega, x), \qquad \dim E_i(\omega, x) = d_i,$$

$$E_i(\Theta_t(\omega, x)) = T\varphi(t, \omega, x)\, E_i(\omega, x) \qquad \text{for all } t \in T,$$

$$\lambda^{\pm}(\omega, x, v) = \lim_{t \to \pm\infty} \frac{1}{t} \log \| T\varphi(t, \omega, x) v \| = \lambda_i \qquad \text{iff } v \in E_i(\omega, x).$$

In order to prove this theorem one starts by slightly modifying Carverhill's [10] proof for $t \geqq 0$. Carverhill's assumption of a compact manifold instead of $\mathbb{R}^d$ is replaced by the integrability condition. Furthermore he supposes μ to be a product measure but an examination of his proof shows that this is not crucial. The statement of the theorem follows now as in Ruelle [31], Theorem 3.1, p. 35.

Remark. The collection $(\lambda_i, d_i)_{i=1,\ldots,r}$ is called the *Lyapunov spectrum of* φ *with respect to* μ and the spaces $E_i(\omega, x)$ are called *Oseledec spaces.*

3. Particular Cases

In this section we will explain under which conditions the usual models describing the influence of noise onto a deterministic dynamical system are covered by the stochastic framework developed in the last section.

3.1. *White Noise*

Since stochastic differential equations are in general only defined for $t \geqq 0$ we will first have to extend the probability space to obtain a random dynamical system in the sense of Definition 2.1.

For this we consider two independent copies $(\Omega^+, \mathscr{F}^+, P^+)$ and $(\Omega^-, \mathscr{F}^-, P^-)$ of canonical Wiener spaces with time $\mathbb{R}^+$, $\mathbb{R}^-$, resp. and put $(\Omega, \mathscr{F}, P) := (\Omega^- \times \Omega^+, \mathscr{F}^- \otimes \mathscr{F}^+, P^- \otimes P^+)$. Then $W(t, \omega) := \omega(t)$ will be a Brownian motion with time parameter $t \in \mathbb{R}$.

There is a canonical flow $\{\vartheta_t \mid t \in \mathbb{R}\}$ of shifts on Ω defined by

$$\vartheta_t\, \omega(s) := \omega(t+s) - \omega(t) \qquad \text{for all } \omega \in \Omega,\, t, s \in \mathbb{R},$$

with respect to which the Wiener measure P is invariant and ergodic. Invariance follows almost immediately for (suitably generalized) cylinder sets whereas the proof of ergodicity may be carried out along the same lines as the proof of ergodicity of the Bernoulli shift in Mañé [27], p. 101. For details see Boxler [8], Lemma 3.3.

Let X_i, $i = 0, \ldots, m$, be vector fields on $\mathbb{R}^d$. For $k \geqq 1$ and $\alpha > 0$, X_0 is supposed to be a $\mathscr{C}^{k,\alpha}$-function (i.e. a k-times continuously differentiable function whose k-th derivative is globally Hölder continuous of order α) and X_i, $i = 1, \ldots, m$, to be $\mathscr{C}^{k+1,\alpha}$-functions. Moreover the derivatives of X_i, $i = 0, \ldots, m$, up to k-th ($(k+1)$-th, resp.) order are assumed to be bounded.

Consider the following stochastic differential equation on $\mathbb{R}^d$ which is defined for all $t \in \mathbb{R}$ and is interpreted in the sense of Stratonovich:

$$d x_t = X_0(x_t)\, d t + \sum_{i=1}^{m} X_i(x_t) \circ d W_i(t), \qquad x_0 = x \in \mathbb{R}^d, \; t \in \mathbb{R}. \tag{3.1}$$

For each fixed $x \in \mathbb{R}^d$ the solution $\varphi(t, \omega, x)$ will be $\mathscr{F}^+$-measurable for $t \geqq 0$ and $\mathscr{F}^-$-measurable for $t \leqq 0$. The solution $\varphi(t, \omega, x)$ for $t \leqq 0$ has to be understood as the solution of a backward equation (see, e.g., Kunita [24]). The solution of (3.1) defines a cocycle of $\mathscr{C}^k$-diffeomorphisms for which the k-th derivative is locally Hölder continuous of order $\beta < \alpha$ (Kunita [24], p. 241). If the vector fields only satisfy local Lipschitz conditions then the solution of (3.1) will also induce a random dynamical system, provided there are appropriate conditions to prevent explosion. For further details see Kunita [24], II.5 and II.7.

If $X_i(x)=\sum_{j=1}^{d}\alpha_{ji}(x)\frac{\partial}{\partial x_j}$ put $A_i(x):=\left(\frac{\partial \alpha_{ji}}{\partial x_k}\right)_{j,k=1,\ldots,d}$. Then the random dynamical system $v_t:=T\varphi(t,\omega,x)v$ obtained by linearizing along the solution $\varphi(t,\omega,x)$ also satisfies:

$$d v_t = A_0(\varphi(t,\omega,x))\,v_t\,dt + \sum_{i=1}^{m} A_i(\varphi(t,\omega,x))\,v_t \circ dW_i(t), \qquad v_0 = v.$$

In order to apply Theorem 2.1 we need a Θ_t-invariant measure on $\Omega\times\mathbb{R}^d$. Assume that there is an invariant solution ρ^+ of the Fokker-Planck equation $L_+^*\rho^+=0$, L_+^* being the formal adjoint of the operator $L_+:=X_0+\frac{1}{2}\sum_{i=1}^{m}X_i^2$. From this the desired invariant measure on $\Omega\times\mathbb{R}^d$ may be constructed following Crauel [14], 6.3.2, by defining $\mu(d\omega,dx):=\mu_\omega(dx)\,P(d\omega)$, where $\mu_\omega:=\lim_{t\to\infty}\varphi^{-1}(-t,\omega)(\rho^+)$ P-a.s.

3.2. *Real Noise*

Let $\{\xi_t \mid t\in\mathbb{R}\}$ be a measurable stationary ergodic process on a probability space $(\Omega,\mathcal{F},P)$ taking values in a measurable space $(Y,\mathcal{Y})$. Ω will be identified with the space of trajectories of ξ. Furthermore we let $\xi\to f(\xi,\cdot)$ be a family of functions on $\mathbb{R}^d$ depending measurably on the parameter $\xi\in Y$ and satisfying global Lipschitz conditions with a Lipschitz constant which is almost surely integrable over every finite time interval. Consider the equation

$$\dot{x}_0 = f(\xi_t(\omega),x_t), \qquad x_0 = x\in\mathbb{R}^d,\ t\in\mathbb{R}, \tag{3.2}$$

which may be understood pathwise as an ordinary differential equation with time dependent coefficients. Assume $P\left[\int_0^T \|f(\xi_t(\omega),0)\|\,dt<\infty\right]=1$ for every $T>0$.

Then (3.2) has a unique solution which is absolutely continuous for any fixed t (see, e.g., Has'minskii [20], Theorem 3.1, p. 10). The uniqueness of the solution implies that it defines a cocycle of homeomorphisms with respect to the shift $\vartheta_t\,\omega(s):=\omega(t+s)$ (see, e.g., Coddington and Levinson [12], remark after Theorem 7.1, p. 23).

If the functions $f(\xi,\cdot)$ are $\mathscr{C}^k$, $k\geqq 1$, then the random dynamical system consists of $\mathscr{C}^k$-diffeomorphisms. This follows from Theorem 7.2, p. 25, in Coddington and Levinson [12] which is stated for $k=1$ but which can immediately be generalized to the case $k>1$. An examination of the proof shows that the assumption of continuity of f with respect to the first variable required there is not necessary in our case since we do neither claim nor need differentiability of the cocycle with respect to t.

Put $A(\xi, x_0):=\frac{\partial f}{\partial x}(\xi, \cdot)\Big|_{x_0}$. Then the linearized cocycle $v \to v_t := T\varphi(t, \omega, x)v$ satisfies the equation $\dot{v}_t = A(\xi_t(\omega), \varphi(t, \omega, x))v_t$, $v_0(\omega, v) = v$.

The integrability condition required in the multiplicative ergodic theorem is satisfied if $\|A(\xi_0(\omega), x)\| \in L^1(\mu)$, μ being as in Sect. 2 (see Oseledec [29], p. 214, for a proof).

3.3. Products of Random Diffeomorphisms

Consider a stationary ergodic sequence $\{\xi_n | n \in \boldsymbol{Z}\}$ of random variables on a probability space $(\Omega, \mathscr{F}, P)$ taking values in a measurable space $(Y, \mathscr{Y})$. If we identify once again the probability space with $Y^{\boldsymbol{Z}}$ then we will obtain a measure preserving transformation ϑ_n if we put $\vartheta_n\, \omega(\cdot) := \omega(\cdot + n)$ for any $n \in \boldsymbol{Z}$.

For $k \geqq 1$, denote by $\mathrm{Diff}^k(\mathbb{R}^d)$ the group of $\mathscr{C}^k$-diffeomorphisms on $\mathbb{R}^d$. Let $G\colon Y \to \mathrm{Diff}^k(\mathbb{R}^d)$ be measurable, write $G_n(\omega) := G(\xi_n(\omega)) = G(\xi_0 \circ \vartheta_n(\omega))$ and define

$$\varphi(n, \omega) := \begin{cases} G_n(\omega) \circ G_{n-1}(\omega) \circ \ldots \circ G_1(\omega) & \text{if } n > 0 \\ \mathrm{Id} & \text{if } n = 0\,. \\ G_{n+1}^{-1}(\omega) \circ G_{n+2}^{-1}(\omega) \circ \ldots \circ G_{-1}^{-1}(\omega) \circ G_0^{-1}(\omega) & \text{if } n < 0 \end{cases}$$

Obviously φ defines a random dynamical system of $\mathscr{C}^k$-diffeomorphisms over $\{\vartheta_n\}$. Assume that there is a Θ_n-invariant measure μ on $\Omega \times \mathbb{R}^d$. Combining Kifer [22], V, Theorem 1.2, p. 159, with some elementary calculations yields that the integrability condition of Oseledec's theorem may be written:

$$\int [\log^+ \|TG(\xi_1(\omega), x)\| + \log^+ \|TG^{-1}(\xi_0(\omega), x)\|]\, d\mu(\omega, x) < \infty.$$

4. Definition of Stochastic Center Manifolds

We return to the general situation described in Sect. 2. In a first step we reduce this situation to the case of a cocycle having an equilibrium point at 0. For this we identify $T_x\mathbb{R}^d$ with $\mathbb{R}^d$ itself and introduce a moving coordinate system attached to the orbit $\varphi(t, \omega, x)$. Let $\tilde{\varphi}(t, \omega, x, v) := \varphi(t, \omega, x+v) - \varphi(t, \omega, x)$ for any $x \in \mathbb{R}^d$, $v \in T_x\mathbb{R}^d \cong \mathbb{R}^d$.

We assume that Oseledec's theorem applies to the random dynamical system φ. Our aim will be an application of the multiplicative ergodic theorem to $\tilde{\varphi}$. For this we first take $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}) := (\Omega \times \mathbb{R}^d, \mathscr{F} \otimes \mathscr{B}^d, \mu)$ as a new probability space. Then a straight-forward calculation shows that $\tilde{\varphi}$ is a random dynamical system on $\mathbb{R}^d$ over $\{\Theta_t\}$ which means in particular that for $\tilde{\omega} = (\omega, x) \in \Omega \times \mathbb{R}^d = \tilde{\Omega}$, $\tilde{\varphi}(t, \tilde{\omega}) = \tilde{\varphi}(t, \omega, x)$ satisfies the cocycle property with respect to $\{\Theta_t\}$.

Furthermore, $\tilde{\varphi}(t, \omega, x, 0) = 0$ for all $t \in T$, $x \in \mathbb{R}^d$ P-a.s. and we may thus consider the linearization of $\tilde{\varphi}$ at 0. This linearization $T\tilde{\varphi}$ is a linear cocycle over Ξ_t, where $\Xi_t(\tilde{\omega}, 0) = (\Theta_t\tilde{\omega}, \tilde{\varphi}(t, \tilde{\omega}, 0)) = (\Theta_t(\omega, x), 0)$.

Since μ is Θ_t-invariant and ergodic $\tilde{\mu} := \tilde{P} \otimes \delta_0 = \mu \otimes \delta_0$ will be a Ξ_t-invariant and ergodic measure on $\tilde{\Omega} \times \mathbb{R}^d$.

The integrability condition is satisfied for $T\tilde{\varphi}$ as well because of $T\tilde{\varphi}(t, \tilde{\omega}, 0) = T\tilde{\varphi}(t, \omega, x, 0) = T\varphi(t, \omega, x)$ and thus Oseledec's theorem may be applied to the new system $\tilde{\varphi}$. Since the linearizations agree it provides us with Lyapunov exponents identical with those of φ and with Oseledec spaces $\tilde{E}_i(\tilde{\omega}, 0) = \tilde{E}_i((\omega, x), 0)$ for $\tilde{\mu}$-a.a. $(\tilde{\omega}, 0) \in \tilde{\Omega} \times \mathbb{R}^d$.

For notational convenience we will drop the tilde in the sequel and write ϑ_t and Θ_t instead of Θ_t and Ξ_t, resp. Finally we put $E_i(\omega) := E_i(\omega, 0)$. Thus we have shown:

Proposition 4.1. *Without loss of generality the random dynamical system φ may be assumed to have an equilibrium point at $x = 0$.*

In a second step the Oseledec spaces will be used to introduce a random coordinate system. For this we define

$$E_c(\omega) := \bigoplus_{\lambda_i = 0} E_i(\omega), \qquad E_s(\omega) := \bigoplus_{\lambda_i < 0} E_i(\omega), \qquad E_u(\omega) := \bigoplus_{\lambda_i > 0} E_i(\omega).$$

Throughout this paper we will always assume that $E_c(\omega) \neq \{0\}$. For any $t \in T$ Oseledec's theorem yields a decomposition of $T_0 \mathbb{R}^d \cong \mathbb{R}^d$:

$$\mathbb{R}^d = E_c(\vartheta_t \omega) \oplus E_s(\vartheta_t \omega) \oplus E_u(\vartheta_t \omega).$$

Projecting the random dynamical system onto E_s along $E_c \oplus E_u$ and so on and denoting the projections by subscripts we obtain for any $t \in T$:

$$\varphi(t, \omega) = \underbrace{[\varphi(t, \omega)]_c}_{\in \{E_c(\vartheta_t \omega)} + \underbrace{[\varphi(t, \omega)]_s}_{\in \{E_s(\vartheta_t \omega)} + \underbrace{[\varphi(t, \omega)]_u}_{\in \{E_u(\vartheta_t \omega)} =: \varphi_c(t, \omega) + \varphi_s(t, \omega) + \varphi_u(t, \omega).$$

Linearize the random dynamical system at the equilibrium point $x = 0$, put $\Psi(t, \omega) := T\varphi(t, \omega, 0)$ and $\Phi(t, \omega, x) := \varphi(t, \omega, x) - \Psi(t, \omega)x$. Hence $\varphi(t, \omega, x) = \Psi(t, \omega)x + \Phi(t, \omega, x)$ and by construction we know that $\Phi(t, \omega, 0) = D\Phi(t, \omega, 0) = 0$ where $D\Phi$ denotes the derivative with respect to x. Since $\Psi(t, \omega) E_i(\omega) = E_i(\vartheta_t \omega)$, Ψ has block diagonal structure in the coordinate system described by the Oseledec spaces we obtain the following decomposition for an initial value $x = x_c(\omega) \oplus x_s(\omega) \oplus x_u(\omega)$ (the notational conventions being evident):

$$\varphi_c(t, \omega, x_c(\omega), x_s(\omega), x_u(\omega)) = \Psi_c(t, \omega)\, x_c(\omega) + \Phi_c(t, \omega, x_c(\omega), x_s(\omega), x_u(\omega)), \tag{4.1 a}$$

$$\varphi_s(t, \omega, x_c(\omega), x_s(\omega), x_u(\omega)) = \Psi_s(t, \omega)\, x_s(\omega) + \Phi_s(t, \omega, x_c(\omega), x_s(\omega), x_u(\omega)), \tag{4.1 b}$$

$$\varphi_u(t, \omega, x_c(\omega), x_s(\omega), x_u(\omega)) = \Psi_u(t, \omega)\, x_u(\omega) + \Phi_u(t, \omega, x_c(\omega), x_s(\omega), x_u(\omega)). \tag{4.1 c}$$

To keep notation simple we will write x_c instead of $x_c(\omega)$ although our initial values will still be random.

Let $\mathscr{P}_f(\mathbb{R}^d)$ be the set of all closed subsets of $\mathbb{R}^d$ endowed with the Hausdorff distance. Then $\mathscr{P}_f(\mathbb{R}^d)$ is a metric space (see e.g. Castaing and Valadier [11], II, § 1, p. 38). A mapping $\Gamma\colon \Omega \to \mathscr{P}_f(\mathbb{R}^d)$ will be measurable if it is $\mathscr{F}$, $\mathscr{B}(\mathscr{P}_f(\mathbb{R}^d))$-measurable (Castaing and Valadier [11], III, § 0, p. 61).

Definition 4.1. A *global stochastic $\mathscr{C}^k$-center manifold* ($k \geqq 1$) for the cocycle φ is a measurable mapping $M: \Omega \to \mathscr{P}_f(\mathbb{R}^d)$ for which there is a ϑ_t-invariant set $\Omega_0 \subset \Omega$, $P(\Omega_0) = 1$, such that for all $\omega \in \Omega_0$ we have:

(i) $M(\omega)$ is a $\mathscr{C}^k$-submanifold of $\mathbb{R}^d$ containing the origin.

(ii) $\varphi(t, \omega) M(\omega) \subset M(\vartheta_t \omega)$ for all $t \in T$ (invariance property).

(iii) $T_0 M(\omega) = E_c(\omega)$, where $T_0 M(\omega)$ is the tangent space of $M(\omega)$ at 0.

Remark. (ii) actually implies that $\varphi(t, \omega) M(\omega) = M(\vartheta_t \omega)$.

In fact: (ii) also holds for the arguments $-t$ and $\vartheta_t \omega$ instead of t and ω. Then the application of $\varphi(t, \omega)$ to both sides together with the cocycle property yields the result.

A key step in the construction of stochastic center manifolds consists in introducing new random norms. For this we start with some estimates:

Lemma 4.1. *For each $\beta > 0$ there exist measurable functions C_c, C_s, C_u: $\Omega \to [1, \infty[$, which depend on β, such that the following estimates hold:*

(i) $C_s^{-1}(\omega)\, e^{(\tilde{\lambda}_s - \beta)t} \|x\| \leqq \|\Psi_s(t, \omega) x\| \leqq C_s(\omega)\, e^{(\lambda_s + \beta)t} \|x\|$ *for all* $x \in E_s(\omega)$ *and* $t \geqq 0$ *a.s.*

(ii) $C_c^{-1}(\omega)\, e^{-\beta|t|} \|x\| \leqq \|\Psi_c(t, \omega) x\| \leqq C_c(\omega)\, e^{\beta|t|} \|x\|$ *for all* $x \in E_c(\omega)$ *and* $t \in T$ *a.s.*

(iii) $C_u^{-1}(\omega)\, e^{(\tilde{\lambda}_u + \beta)t} \|x\| \leqq \|\Psi_u(t, \omega) x\| \leqq C_u(\omega)\, e^{(\lambda_u - \beta)t} \|x\|$ *for all* $x \in E_u(\omega)$ *and* $t \leqq 0$ *a.s.*

Here we have put:

$$\lambda_s := \max_{\lambda_i < 0} \lambda_i, \qquad \tilde{\lambda}_s := \min_{\lambda_i < 0} \lambda_i, \qquad \lambda_u := \min_{\lambda_i > 0} \lambda_i, \qquad \tilde{\lambda}_u := \max_{\lambda_i > 0} \lambda_i.$$

Furthermore, $\|\cdot\|$ denotes the norm induced on the subspaces by the usual Euclidean norm in $\mathbb{R}^d$.

If $T = \boldsymbol{Z}$ a proof may be found in Pesin [30], Theorems 1.1.1 and 1.2.1. If $T = \mathbb{R}$ the estimates can easily be derived from Oseledec's theorem. The fact that the random dynamical system Ψ is continuous and measurable by definition ensures the existence and measurability of the functions C_c, C_s and C_u.

For a given $\beta > 0$ consider the condition

$$\text{(LE)} \qquad \lambda_s + 4\beta < 0, \qquad \lambda_u - 4\beta > 0.$$

Lemma 4.2. *For each $\beta > 0$ which satisfies condition* (LE) *we put in case $T = \mathbb{R}$:*

$$|x|_\omega^s := \int_0^\infty e^{-(\lambda_s + 2\beta)\tau} \|\Psi_s(\tau, \omega) x\| \, d\tau \qquad \textit{for any } x \in E_s(\omega),$$

$$|x|_\omega^c := \int_{-\infty}^\infty e^{-2\beta|\tau|} \|\Psi_c(\tau, \omega) x\| \, d\tau \qquad \textit{for any } x \in E_c(\omega),$$

$$|x|_\omega^u := \int_0^\infty e^{(\lambda_s - 2\beta)\tau} \|\Psi_u(-\tau, \omega) x\| \, d\tau \qquad \textit{for any } x \in E_u(\omega).$$

In case $T=\mathbf{Z}$ the integrals are replaced by sums. Then $|\cdot|^s_\omega$, $|\cdot|^c_\omega$ and $|\cdot|^u_\omega$ define measurable norms on $E_s(\omega)$, $E_c(\omega)$ and $E_u(\omega)$, resp. which are almost surely related to the norms induced by the usual Euclidean norm on $\mathbb{R}^d$ by the following estimate:

$$\frac{C_s^{-1}(\omega)}{3\beta+\lambda_s-\tilde{\lambda}_s}\|x\| \leqq |x|^s_\omega \leqq \frac{C_s(\omega)}{\beta}\|x\|,$$

and similarly for $|\cdot|^c_\omega$ and $|\cdot|^u_\omega$. Furthermore we obtain:

(i) $|\Psi_s(t,\omega)x|^s_{\vartheta_t\omega} \leqq e^{(\lambda_s+2\beta)t}|x|^s_\omega$ *for all $x\in E_s(\omega)$ and $t\geqq 0$ a.s.,*
(ii) $|\Psi_c(t,\omega)x|^c_{\vartheta_t\omega} \leqq e^{2\beta|t|}|x|^c_\omega$ *for all $x\in E_c(\omega)$ and $t\in T$ a.s.,*
(iii) $|\Psi_u(t,\omega)x|^u_{\vartheta_t\omega} \leqq e^{(\lambda_u-2\beta)t}|x|^u_\omega$ *for all $x\in E_u(\omega)$ and $t\leqq 0$ a.s.*

Remark. The usefulness of these random norms is due to the fact that the estimates above do no longer contain constants which depend on chance, in contrast to Lemma 4.1. This is possible because all the knowledge about the long term behavior of the random dynamical system is incorporated into the definition of the norm.

Proof. Since the discrete time case is completely analogous we restrict ourselves to the case $T=\mathbb{R}$. Lemma 4.1 together with the fact that $\|\cdot\|$ is a norm and that $\Psi_s(t,\omega)$ is an isomorphism imply that $|\cdot|^s_\omega$ defines a norm on $E_s(\omega)$. Since Ψ_s is measurable in ω and continuous in t the $\mathscr{B}\otimes\mathscr{F}$-measurability of the integrand and thus the $\mathscr{F}$-measurability of the norm will follow.

The estimate relating both norms follows immediately from Lemma 4.1. Estimate (i) holds because for $t\geqq 0$ we obtain:

$$\begin{aligned}|\Psi_s(t,\omega)x|^s_{\vartheta_t\omega} &= \int_0^\infty e^{-(\lambda_s+2\beta)\tau}\|\Psi_s(\tau,\vartheta_t\omega)\Psi_s(t,\omega)x\|\,d\tau\\ &= e^{(\lambda_s+2\beta)t}\int_0^\infty e^{-(\lambda_s+2\beta)(t+\tau)}\|\Psi_s(t+\tau,\omega)x\|\,d\tau\\ &\leqq e^{(\lambda_s+2\beta)t}\int_0^\infty e^{-(\lambda_s+2\beta)r}\|\Psi_s(r,\omega)x\|\,dr = e^{(\lambda_s+2\beta)t}|x|^s_\omega.\end{aligned}$$

The proofs of (ii) and (iii) are completely analogous and are thus omitted. □

We wish to obtain a stochastic center manifold as a graph and thus we will need suitable spaces to work in. In order to define them we recall that the Oseledec spaces are considered as random variables $E_i\colon \Omega\to G_{d_i}(\mathbb{R}^d)$. Hence we may define:

$$E := \{(\omega,x)\in\Omega\times\mathbb{R}^d \mid x\in E_c(\omega)\},$$

$$\mathscr{X} := \{h\colon E\to\mathbb{R}^d \text{ meas.} \mid h(\omega,\cdot)\colon$$

$$E_c(\omega)\to E_s(\omega)\oplus E_u(\omega) \text{ a.s. continuous and bounded}\}.$$

Boundedness of $h(\omega, \cdot)$ is understood with respect to the norm

$$|h(\omega,\cdot)|_{\infty,\omega} := \sup_{x\in E_c(\omega)} |h(\omega,x)|_\omega = \sup_{x\in E_c(\omega)} [|h_s(\omega,x)|_\omega^s + |h_u(\omega,x)|_\omega^u]$$

where $h_{s,u}(\omega,\cdot)$ denotes the projection of $h(\omega,\cdot)$ onto $E_{s,u}(\omega)$.

Furthermore let $|h_{s,u}(\omega,\cdot)|_{\infty,\omega}^{s,u} := \sup_{x\in E_c(\omega)} |h_{s,u}(\omega,x)|_\omega^{s,u}$ and put for any $h, \tilde{h}\in\mathscr{X}$: $d(h,\tilde{h}) := d(h_s,\tilde{h}_s) + d(h_u,\tilde{h}_u)$ where

$$d(h_{s,u},\tilde{h}_{s,u}) := \int \frac{|h_{s,u}(\omega,\cdot)-\tilde{h}_{s,u}(\omega,\cdot)|_{\infty,\omega}^{s,u}}{1+|h_{s,u}(\omega,\cdot)-\tilde{h}_{s,u}(\omega,\cdot)|_{\infty,\omega}^{s,u}} P(d\omega).$$

The fact that $x \to \frac{x}{1+x}$ is an increasing function for $x \geqq 0$ implies the inequality $\frac{|a+b|}{1+|a+b|} \leqq \frac{|a|+|b|}{1+|a|+|b|} \leqq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$ which enables us to check that d defines a pseudometric. Let X be the space of equivalence classes of $\mathscr{X}$ with respect to almost sure equality. It is therefore endowed with the metric of convergence in probability.

Lemma 4.3. *(X, d) is a complete metric space.*

Proof. In a first step a straightforward adaptation, the details of which will be omitted, of a proof of Federer [18], p. 79, shows the convergence of a Cauchy sequence with respect to the metric

$$d_\kappa(h,\tilde{h}) := \inf\{\varepsilon \mid P[K_\kappa(\omega)\, \|h(\omega,\cdot)-\tilde{h}(\omega,\cdot)\|_\infty > \varepsilon] \leqq \varepsilon\}, \qquad \kappa = 1, 2,$$

where $\|h(\omega,\cdot)\|_\infty := \sup_{x\in E_c(\omega)} \|h(\omega,x)\|$ and $K_\kappa\colon \Omega \to]0,\infty[$ is any measurable function. Let (h_n) be a Cauchy sequence with respect to d. An application of the first step to $K_1(\omega) := \frac{C_s^{-1}(\omega)}{3\beta+\lambda_s-\bar{\lambda}_s}$, $K_2(\omega) := \frac{C_s(\omega)}{\beta}$ yields because of Lemma 4.2 that $d_1(h_m,h_k) \leqq d(h_m,h_k) \leqq d_2(h_m,h_k)$. For this reason it remains to be shown that (h_n) is a Cauchy sequence with respect to d_2 as well. Putting $K(\omega) := \frac{K_2(\omega)}{K_1(\omega)}$ this follows if for $c>0$ we combine the estimate

$$\begin{aligned} d_2(h_m,h_k) &\leqq \int_{[K(\omega)\leqq c]} \frac{c\,|h_m(\omega,\cdot)-h_k(\omega,\cdot)|_\omega}{1+c\,|h_m(\omega,\cdot)-h_k(\omega,\cdot)|_\omega} P(d\omega) + P[K(\omega)>c] \\ &\leqq d(c\,h_m, c\,h_k) + P[K(\omega)>c] \end{aligned}$$

with the fact that $d(c\,h_m, c\,h_k) \leqq \max\{d(h_m,h_k), c\,d(h_m,h_k)\}$ and with the Cauchy property of (h_n). □

Let $k \geqq 1$ and $L > 0$ be given constants. Consider the following properties:

(1) $D^j h(\omega,\cdot)$ exists for all $j = 0, \ldots, k$;

(2) $h(\omega, 0) = 0$;

(3) $D^1 h(\omega, 0) = 0$;

(4) $|D^j h_{s,u}(\omega, \cdot)|^{s,u}_{\infty,\omega} \leqq \frac{L}{2}$ for all $j=0, \ldots, k$;

(5) $|D^k h_{s,u}(\omega, x) - D^k h_{s,u}(\omega, \tilde{x})|^{s,u}_{\omega} \leqq \frac{L}{2} |x-\tilde{x}|^c_{\omega}$ for all $x, \tilde{x} \in E_c(\omega)$;

where $D^j h(\omega, \cdot)$ denotes the j-th order derivative with respect to the second argument. Let

$$A_k(L) := \{h \in X \mid \exists\, \Omega_0 \subset \Omega,\ P(\Omega_0)=1,\ \forall\, \omega \in \Omega_0\colon (1)\text{–}(5) \text{ hold}\}.$$

Lemma 4.4. *Let (h_n) be a sequence in $A_k(L)$ which converges to $h \in X$. Then there is a set $\tilde{\Omega} \subset \Omega$, $P(\tilde{\Omega})=1$, such that for any $\omega \in \tilde{\Omega}$, h has the properties* (1)–(3) *and* (5).

Proof. Let (h_n) be a sequence in $A_k(L)$ such that $d(h_n, h) \to 0$ for $n \to \infty$. Since the metric d describes convergence in probability there is an almost surely convergent subsequence (h_{n_k}), i.e. $h_{n_k}(\omega, \cdot) \to h(\omega, \cdot)$ P-a.s. For notational simplicity we will, however, write h_n instead of h_{n_k}.

(2) is obvious, and the proof of the other assertions is now a purely analytic task:

$k=1$: The Lipschitz condition ensures equicontinuity and thus Ascoli's theorem (see, e.g., Dunford and Schwartz [16], Vol. I, IV.6.14, p. 269) implies the existence of a uniformly convergent subsequence of $((D^1 h_{n_k})_{s,u}(\omega, \cdot))$. If we combine this with a lemma concerning the derivative of the limit function (see Dieudonné [15], VIII, §6, Theorem 8.6.3, p. 163) then the desired properties will follow.

$k=2$: We apply the same reasoning as above to $(D^1 h_{n_k})$ instead of (h_n).

For $k>2$ the result follows by induction. □

5. An Existence Theorem for Global Stochastic Center Manifolds

Let Ψ be a linear cocycle, e.g., the linearization of a random dynamical system φ, and assume the situation described in Sect. 2. Choose $\beta > 0$ such that condition (LE) is satisfied, i.e. such that $\lambda_s + 4\beta < 0$, $\lambda_u - 4\beta > 0$. Since the long term behavior of Ψ will not necessarily carry over to $\varphi := \Psi + \Phi$ if Φ is of arbitrary size it will be important to determine the class of nonlinearities which do not destroy the asymptotic behavior of the linear system. For this we need some notations:

$$C(\omega) := \max\{C_c(\omega), C_s(\omega), C_u(\omega)\},$$

$C_{c,s,u}$ as in Lemma 4.1,

$$\|p(\omega)\| := \max\{\|p_c(\omega)\|, \|p_s(\omega)\|, \|p_u(\omega)\|\},$$

$p_{c,s,u}(\omega)\colon \mathbb{R}^d \to E_{c,s,u}(\omega)$ the projection map.

Let ε_j, $j=0, \ldots, k$, be positive constants and define the following random variables:

$$\varepsilon_j(\omega) := \frac{\beta}{C(\omega)\, \|p(\omega)\|}\, \varepsilon_j, \qquad j=0, \ldots, k.$$

Consider the following property:

For any $x, y \in \mathbb{R}^d$ and any $t \in T$ such that $-1 \leqq t \leqq 1$ let

$$(NL1)(\varepsilon_0, \ldots, \varepsilon_k) \quad \|D^j \Phi(t, \omega, x)\| \leqq \varepsilon_j(\vartheta_t \omega) \qquad \text{for all } j=0, \ldots, k;$$

$$(NL2)(\varepsilon_0, \ldots, \varepsilon_k) \quad \|D^k \Phi(t, \omega, x) - D^k \Phi(t, \omega, y)\| \leqq C^{-1}(\omega)\, \varepsilon_k(\vartheta_t \omega)\, \|x-y\|.$$

Denote by $NL_{\varepsilon_0, \ldots, \varepsilon_k}$ the class of those random dynamical systems $\varphi := \Psi + \Phi$ for which Φ satisfies $(NL1)(\varepsilon_0, \ldots, \varepsilon_k)$ and $(NL2)(\varepsilon_0, \ldots, \varepsilon_k)$ for any $x, y \in \mathbb{R}^d$ and any $t \in \mathbb{R}$ such that $-1 \leqq t \leqq 1$.

After these preparations we are now able to formulate the key result of this paper. Its proof will occupy the rest of this section.

Theorem 5.1. *There is a constant L_0 such that for any L, $0 < L \leqq L_0$, and any β, $0 < \beta < \frac{1}{4}\min(-\lambda_s, \lambda_u)$, there are constants $\varepsilon_0(L) \geqq \ldots \geqq \varepsilon_k(L) > 0$, which depend on β, and for any random dynamical system of class $NL_{\varepsilon_0(L), \ldots, \varepsilon_k(L)}$ which consists of $\mathscr{C}^{k,1}$-diffeomorphisms there exists a global stochastic $\mathscr{C}^{k,1}$-center manifold for P-a.a. $\omega \in \Omega$. It may be written in the form*

$$M(\omega) = \{(x, h_s(\omega, x), h_u(\omega, x)) \mid x \in E_c(\omega)\}$$

with a function $h \in A_k(L)$.

Before entering into the proof we are going to express conditions $(NL1, 2)(\varepsilon_0, \ldots, \varepsilon_k)$ in terms of the random norms introduced in Lemma 4.2. For this we combine the continuity of the projection map with the inequality relating the new norms and the original Euclidean norm. Hence we obtain as an equivalent formulation that for any $x, y \in \mathbb{R}^d$ and any $t \in T$ such that $-1 \leqq t \leqq 1$ we have:

$$(CM1)(\varepsilon_0, \ldots, \varepsilon_k) \quad |D^j \Phi_{c,s,u}(t, \omega, x)|^{c,s,u}_{\vartheta_t \omega} \leqq \varepsilon_j, \qquad j=0, \ldots, k;$$

$$(CM2)(\varepsilon_0, \ldots, \varepsilon_k) \quad |D^k \Phi_{c,s,u}(t, \omega, x) - D^k \Phi_{c,s,u}(t, \omega, y)|^{c,s,u}_{\vartheta_t \omega} \leqq \gamma\, \varepsilon_k\, |x-y|_\omega,$$

where we have put $\gamma := 3(3\beta + \lambda_s - \lambda_s)$.

Remark. If we compare these two formulations we see that two stochastic effects have to be taken into account:

The random variable C is due to the fact that Oseledec's theorem only describes the long term behavior of the system whereas the system may behave very irregularly until a random time $t(\omega)$. The bigger the maximal value which may be reached before time $t(\omega)$ the smaller $\varepsilon_j(\omega)$ has to be.

The random coordinate system described by the spaces $E_{c,s,u}(\omega)$ is not fixed but moves according to the multiplicative ergodic theorem. If the coordinate axes come close together in the course of time the random variable p will take big values. Once again $\varepsilon_j(\omega)$ has to be shrunk to cope with this phenomenon.

Proof of Theorem 5.1. We will construct the stochastic center manifold as the fixed point of an appropriate operator acting on $A_k(L)$. The following lemma will enable us to define this operator.

Lemma 5.1. *Given $L>0$ and $\beta>0$ satisfying* (LE). *Then there is an $\varepsilon_0(L)=\varepsilon_0(L,\beta)$ such that for any φ of class $NL_{\varepsilon_0(L)}$ and any $h\in A_k(L)$ there is a set $\tilde{\Omega}\subset\Omega$, $P(\tilde{\Omega})=1$, such that for any $\omega\in\tilde{\Omega}$ the map*

$$S_h(\omega)\colon\ E_c(\omega)\to E_c(\vartheta\,\omega),$$
$$x\to\varphi_c(1,\omega,x,h(\omega,x))$$

is a $\mathscr{C}^{k,1}$-diffeomorphism. Here we have put $\vartheta\,\omega:=\vartheta_1\,\omega$.

Remark. An analogous statement holds for the map $x\to\varphi_c(-1,\omega,x,h(\omega,x))$.

Proof. Since φ and hence φ_c is a $\mathscr{C}^{k,1}$-map by assumption it remains to be shown that $S_h(\omega)$ is a bijection. But $S_h(\omega)(x)=\Psi_c(1,\omega)\,x+\Phi_c(1,\omega,x,h(\omega,x))$ where Ψ_c is a linear isomorphism. By construction Φ_c has an equilibrium point at 0. Thus, if we choose $\varepsilon_0(L)$ sufficiently small, the smallness being measured in a norm which depends on β, the inverse mapping theorem applies and yields the result. See, e.g., Abraham et al. [1], Theorem 2.5.2, p. 102. □

As a consequence of this lemma we derive some estimates which will be extensively used in the sequel. For a fixed $L>0$ we choose $\varepsilon_0:=\varepsilon_0(L)$ sufficiently small such that Lemma 5.1 holds and such that $\gamma(1+L)\,\varepsilon_0\,e^{2\beta}<1$.

Lemma 5.2. *Assume the situation of Lemma* 5.1. *Let $h,\tilde{h}\in A_k(L)$ and $y,\tilde{y}\in E_c(\vartheta\,\omega)$.*

(i) *Choose $x,\tilde{x}\in E_c(\omega)$ such that $S_h(\omega)(x)=y$, $S_h(\omega)(\tilde{x})=\tilde{y}$. Then we have:*

$$|x-\tilde{x}|_\omega^c\leqq\frac{e^{2\beta}}{1-\gamma(1+L)\,\varepsilon_0\,e^{2\beta}}\,|y-\tilde{y}|_{\vartheta\omega}^c,\tag{5.1}$$

(ii) *Choose $x,\tilde{x}\in E_c(\omega)$ such that $S_h(\omega)(x)=y=S_{\tilde{h}}(\omega)(\tilde{x})$. Then we have:*

$$|x-\tilde{x}|_\omega^c\leqq\frac{\gamma\,\varepsilon_0\,e^{2\beta}}{1-\gamma(1+L)\,\varepsilon_0\,e^{2\beta}}\,|h(\omega,\cdot)-\tilde{h}(\omega,\cdot)|_{\infty,\omega}.\tag{5.2}$$

Proof. Lemma 5.1 enables us to write

$$x=\Psi_c^{-1}(1,\omega)\,y-\Psi_c^{-1}(1,\omega)\,\Phi_c(1,\omega,x,h(\omega,x))$$

and similarly for $\tilde{x}$. We take into account that $\Psi_c^{-1}(1,\omega)=\Psi_c(-1,\vartheta\,\omega)$ and make use of Lemma 4.2 and assumption $(CM2)(\varepsilon_0)$ to obtain:

$$|x-\tilde{x}|_\omega^c\leqq e^{2\beta}[|y-\tilde{y}|_{\vartheta\omega}^c+\gamma\,\varepsilon_0(|x-\tilde{x}|_\omega^c+|h(\omega,x)-h(\omega,\tilde{x})|_\omega)].$$

h satisfies a Lipschitz condition since $h\in A_k(L)$. Together with our choice of ε_0 this yields the result.

The second case follows by similar arguments from the fact that

$$|x-\tilde{x}|_\omega^c=|\Psi_c^{-1}(1,\omega)(\Phi_c(1,\omega,x,h(\omega,x))-\Phi_c(1,\omega,\tilde{x},\tilde{h}(\omega,\tilde{x})))|_\omega^c$$

if we make use of

$$|h(\omega, x)-\tilde{h}(\omega, \tilde{x})|_\omega \leqq |h(\omega, x)-\tilde{h}(\omega, x)|_\omega + |\tilde{h}(\omega, x)-\tilde{h}(\omega, \tilde{x})|_\omega$$
$$\leqq |h(\omega, \cdot)-\tilde{h}(\omega, \cdot)|_{\infty,\omega} + L|x-\tilde{x}|^c_\omega. \quad \square$$

Next we define an operator $T: A_k(L) \to X, h \to Th$, where we put for any $y \in E_c(\omega)$:

$$(Th)(\omega, y) := \varphi_s(1, \vartheta_{-1}\omega, x_{-1}, h(\vartheta_{-1}\omega, x_{-1})) + \varphi_u(-1, \vartheta\omega, x_1, h(\vartheta\omega, x_1)).$$

Here $x_{-1} \in E_c(\vartheta_{-1}\omega)$ and $x_1 \in E_c(\vartheta\omega)$ are chosen such that

$$\varphi_c(1, \vartheta_{-1}\omega, x_{-1}, h(\vartheta_{-1}\omega, x_{-1})) = y = \varphi_c(-1, \vartheta\omega, x_1, h(\vartheta\omega, x_1)) \in E_c(\omega).$$

Various proofs of the deterministic center manifold theorem (see, e.g., Marsden and McCracken [28], p. 30, or Iooss [21], p. 146) will serve as a guideline for the proof we are going to carry out now. It is structured as follows:

(1) For fixed β, $L>0$ and $h \in A_k(L)$ choose $\tilde{\varepsilon}_0(L)$ according to Lemma 5.1. We show that there are constants $\tilde{\varepsilon}_0(L) \geqq \varepsilon_0(L) \geqq \ldots \geqq \varepsilon_k(L) > 0$, which depend on β, such that $T(A_k(L)) \subset A_k(L)$ for any random dynamical system φ of class $NL_{\varepsilon_0(L), \ldots, \varepsilon_k(L)}$.

(2) We prove that there is a constant $L_0 > 0$ such that for any L, $0 < L \leqq L_0$, and any cocycle φ of class $NL_{\varepsilon_0(L), \ldots, \varepsilon_k(L)}$, T is a contraction on $A_k(L)$. For these values of L (1) provides us with the required constants $\varepsilon_j(L)$, $j=0, \ldots, k$. Hence the contraction mapping theorem ensures the existence of a unique fixed point h which is an element of the completion of $A_k(L)$ with respect to d. Thus h has the required properties because of Lemma 4.4.

(3) We show that the graph of h is invariant, i.e. that $\varphi(t, \omega) M(\omega) = M(\vartheta_t \omega)$ for any $t \in T$. In terms of h this means that we have to prove that for any $t \in T$

$$\varphi_{s,u}(t, \omega, x_c, h_s(\omega, x_c), h_u(\omega, x_c)) = h_{s,u}(\vartheta_t \omega, \varphi_c(t, \omega, x_c, h_s(\omega, x_c), h_u(\omega, x_c))).$$

Step (1)

(i) Th is $\mathscr{B}_{A_k(L)}, \mathscr{B}_X$-measurable:

Here $\mathscr{B}_{A_k(L)} := \mathscr{B}_X \cap A_k(L)$, where $\mathscr{B}_X$ denotes the Borel sets of the metric space X. The assertion follows since h and the random dynamical system φ are measurable by assumption and φ depends continuously on the initial values.

(ii) $(Th)(\omega, \cdot): E_c(\omega) \to E_s(\omega) \oplus E_u(\omega)$ is P-a.s. continuous and bounded:

By definition $(Th)(\omega, y) \in E_s(\omega) \oplus E_u(\omega)$ for all $y \in E_c(\omega)$. The continuity follows from the continuity of h and φ together with Lemma 5.1. An application of Lemma 4.2 and assumption $(CM1)(\tilde{\varepsilon}_0(L))$ yields:

$$\begin{aligned}|(Th)(\omega, y)|_\omega \leqq\, & |\Psi_s(1, \vartheta_{-1}\omega)\, h_s(\vartheta_{-1}\omega, x_{-1})|^s_\omega \\ & + |\Phi_s(1, \vartheta_{-1}\omega, x_{-1}, h(\vartheta_{-1}\omega, x_{-1}))|^s_\omega \\ & + |\Psi_u(-1, \vartheta\omega)\, h_u(\vartheta\omega, x_1)|^u_\omega + |\Phi_u(-1, \vartheta\omega, x_1, h(\vartheta\omega, x_1))|^u_\omega \\ \leqq\, & e^{(\lambda_s+2\beta)} |h_s(\vartheta_{-1}\omega, x_{-1})|^s_{\vartheta_{-1}\omega} + e^{-(\lambda_u-2\beta)} |h_u(\vartheta\omega, x_1)|^u_{\vartheta\omega} + 2\tilde{\varepsilon}_0(L).\end{aligned}$$

Thus the almost sure boundedness of $h(\omega, \cdot)$ implies the result.

(iii) $(Th)(\omega, 0)=0$:

This is a consequence of the facts that φ has a fixed point at 0, φ_c is a bijection by Lemma 5.1 and $h(\omega, 0)=0$.

(iv) $D^j(Th)(\omega, \cdot)$ exists for all $j=0, \ldots, k$ P-a.s.:

The assertion follows if we take into account Lemma 5.1 because the random dynamical system consists of $\mathscr{C}^k$-diffeomorphisms and $h \in A_k(L)$.

(v) $D^1(Th)(\omega, 0)=0$ P-a.s.:

For a given $y \in E_c(\omega)$, Lemma 5.1 enables us to consider x_{-1} and x_1 as functions of y. Hence we may write:

$$D_y x_{-1}(y)=[\Psi_c(1, \vartheta_{-1}\omega)+D_{x_{-1}}\Phi_c(1, \vartheta_{-1}\omega, x_{-1}, h(\vartheta_{-1}\omega, x_{-1}))]^{-1}$$

and similarly for $D_y x_1(y)$. On the other hand we have

$$\begin{aligned} D_y(Th)(\omega, y)=[&\Psi_s(1, \vartheta_{-1}\omega) D_{x_{-1}} h_s(\vartheta_{-1}\omega, x_{-1}) \\ &+D_{x_{-1}}\Phi_s(1, \vartheta_{-1}\omega, x_{-1}, h(\vartheta_{-1}\omega, x_{-1}))] D_y x_{-1}(y) \\ &+[\Psi_u(-1, \vartheta\omega) D_{x_1} h_u(\vartheta\omega, x_1) \\ &+D_{x_1}\Phi_u(-1, \vartheta\omega, x_1, h(\vartheta\omega, x_1))] D_y x_1(y). \end{aligned}$$

The same reasoning as in (iv) will thus enable us to conclude.

(vi) $|D^j(Th)_{s,u}(\omega, \cdot)|^{s,u}_{\infty,\omega} \leqq \frac{L}{2}$ for all $j=0, \ldots, k$ P-a.s.:

$j=0$: Since $h \in A_k(L)$ the estimate established in (ii) yields

$$|(Th)_s(\omega, y)|^s_\omega \leqq \frac{L}{2} e^{(\lambda_s+2\beta)}+\tilde{\varepsilon}_0(L).$$

Assumption (LE) ensures that $e^{(\lambda_s+2\beta)}<1$ and thus the right-hand side becomes $\leqq \frac{L}{2}$ if we replace $\tilde{\varepsilon}_0(L)$ by $\varepsilon_0^{(1)} := \varepsilon_0^{(1)}(L) \leqq \tilde{\varepsilon}_0(L)$ sufficiently small if necessary. An analogous argument holds true for $(Th)_u(\omega, \cdot)$ and yields an $\varepsilon_0^{(2)}$. Hence we may choose $\varepsilon_0=\varepsilon_0(L) := \varepsilon_0^{(1)} \wedge \varepsilon_0^{(2)}$.

$j=1$: For any $\varepsilon_1 \leqq \varepsilon_0$ Lemma 4.2 and assumption $(CM1)(\varepsilon_0, \varepsilon_1)$ imply that

$$|\Psi_c^{-1}(\pm 1, \vartheta_{\pm 1}\omega) D_{x_{\pm 1}}\Phi_c(\pm 1, \vartheta_{\pm 1}\omega, x_{\pm 1}, h(\vartheta_{\pm 1}\omega, x_{\pm 1}))|^c_{\vartheta_{\pm 1}\omega} \leqq e^{2\beta}\varepsilon_1.$$

For any $c<1$ choose $\varepsilon_1(L, c)$ such that $e^{2\beta}\varepsilon_1(L, c) \leqq c<1$. Using an argument from the theory of matrix algebras (see, e.g., Dieudonné [15], VIII, §3, p. 154) we obtain:

$$|[Id+\Psi_c^{-1}(\cdot) D_{x_{\pm 1}}\Phi_c(\cdot)]^{-1}|=\left|\sum_{n=1}^{\infty}(-\Psi_c^{-1}(\cdot) D_{x_{\pm 1}}\Phi_c(\cdot))^{n-1}\right| \leqq \frac{1}{1-e^{2\beta}\varepsilon_1(L, c)}.$$

If we combine this with the estimate derived in (v) and Lemma 4.2 then we will find that under the assumption $(CM1)(\varepsilon_0, \varepsilon_1(L, c))$

$$|D_y(Th)_s(\omega, y)|_\omega^s \leqq \tilde{c}\left[\frac{L}{2} e^{(\lambda_s+4\beta)} + \varepsilon_1(L, c)\, e^{2\beta}\right], \qquad \tilde{c} := \frac{1}{1-c}.$$

An analogous estimate holds for the unstable part and assumption (LE) enables us to choose c and thus $\varepsilon_1 := \varepsilon_1(L, c)$ sufficiently small such that both expressions become $\leqq \frac{L}{2}$.

$j>1$: The calculations are completely analogous and are thus omitted. Once the estimations have been carried out for $j=0, \ldots, k$ the desired constants $\varepsilon_0 \geqq \ldots \geqq \varepsilon_k$ are determined in dependence on the given L.

(vii) $|D^k(Th)_{s,u}(\omega, y) - D^k(Th)_{s,u}(\omega, \tilde{y})|_\omega^{s,u} \leqq \frac{L}{2} |y-\tilde{y}|_\omega^c$ for all $y, \tilde{y} \in E_c(\omega)$ P-a.s.:

The proof follows the same lines as the proof of (vi) and will thus be omitted. Instead of making use of assumption $(CM1)$ we apply $(CM2)(\tilde{\varepsilon}_k)$, $\tilde{\varepsilon}_k \leqq \varepsilon_k$ sufficiently small. For a given $L>0$ we will thus work in the class $NL_{\varepsilon_0, \ldots, \tilde{\varepsilon}_k}$ but for notational simplicity we will drop the tilde in the sequel.

Step (2)

We have to prove the existence of a constant $L_0>0$ such that for any L, $0<L\leqq L_0$, T is a contraction on $A_k(L)$. For this we start with an arbitrary $L>0$ and consider $h, \tilde{h} \in A_k(L)$. We are going to show the existence of a constant c, $0<c<1$, such that $d(Th, T\tilde{h}) \leqq c\, d(h, \tilde{h})$. In a preliminary step we will thus estimate

$$|Th(\omega, \cdot) - T\tilde{h}(\omega, \cdot)|_{\infty,\omega} = \sup_{x \in E_c(\omega)} |Th(\omega, y) - T\tilde{h}(\omega, y)|_\omega,$$

where

$$S_h(\vartheta_{-1}\omega)(x_{-1}) = y = S_h(\vartheta\omega)(x_1), \qquad S_{\tilde{h}}(\vartheta_{-1}\omega)(\tilde{x}_{-1}) = y = S_{\tilde{h}}(\vartheta\omega)(\tilde{x}_1).$$

The estimates derived in Lemma 4.2 combined with assumption $(CM2)(\varepsilon_0(L))$ yield:

$$\begin{aligned} |(Th)(\omega, \cdot) - (Th)_s(\omega, \cdot)|_{\infty,\omega}^s \leqq \sup_{x \in E_c(\omega)} [&e^{(\lambda_s+2\beta)} |h_s(\vartheta_{-1}\omega, x_{-1}) \\ &- \tilde{h}_s(\vartheta_{-1}\omega, \tilde{x}_{-1})|_{\vartheta_{-1}\omega}^s \\ &+ \gamma\,\varepsilon_0(|x_{-1} - \tilde{x}_{-1}|_{\vartheta_{-1}\omega}^c + |h(\vartheta_{-1}\omega, x_{-1}) \\ &- \tilde{h}(\vartheta_{-1}\omega, \tilde{x}_{-1})|_{\vartheta_{-1}\omega})] \end{aligned}$$

and similarly for the unstable part. For simplicity we have written ε_0 instead of $\varepsilon_0(L)$. Because of assumption (LE) there is a $\delta>0$ such that $\lambda_s+4\beta \leqq \delta < 0$

and $-(\lambda_u - 4\beta) \leqq \delta < 0$. Using this together with the same reasoning as in the proof of Lemma 5.2 we obtain after a straightforward calculation:

$$|(Th)_s(\omega,\cdot)-(T\tilde{h})_s(\omega,\cdot)|^s_{\infty,\omega} \leqq e^{(\delta-2\beta)}\,|h_s(\vartheta_{-1}\omega,\cdot)-\tilde{h}_s(\vartheta_{-1}\omega,\cdot)|^s_{\infty,\vartheta_{-1}\omega} + K\,|h(\vartheta_{-1}\omega,\cdot)-\tilde{h}(\vartheta_{-1}\omega,\cdot)|_{\infty,\vartheta_{-1}\omega},$$

where $K := \dfrac{\gamma\,\varepsilon_0\left(1+\frac{L}{2}\,e^{\delta}\right)}{1-\gamma\,\varepsilon_0(1+L)\,e^{2\beta}}$. A similar estimate holds for the unstable part.

Combining the fact that $x \to \dfrac{x}{1+x}$ is an increasing function with the invariance of P with respect to ϑ we may write:

$$d(Th, T\tilde{h}) \leqq E\,\frac{e^{(\delta-2\beta)}\,|h_s(\omega,\cdot)-\tilde{h}_s(\omega,\cdot)|^s_{\infty,\omega}}{1+e^{(\delta-2\beta)}\,|h_s(\omega,\cdot)-\tilde{h}_s(\omega,\cdot)|^s_{\infty,\omega}} + 2E\,\frac{K\,|h(\omega,\cdot)-\tilde{h}(\omega,\cdot)|_{\infty,\omega}}{1+K\,|h(\omega,\cdot)-\tilde{h}(\omega,\cdot)|_{\infty,\omega}} + E\,\frac{e^{(\delta-2\beta)}\,|h_u(\omega,\cdot)-\tilde{h}_u(\omega,\cdot)|^u_{\infty,\omega}}{1+e^{(\delta-2\beta)}\,|h_u(\omega,\cdot)-\tilde{h}_u(\omega,\cdot)|^u_{\infty,\omega}}.$$

In Step (1) we have seen that the dependence of ε_0 on L implies that if we shrink L then ε_0 will automatically be shrunk as well. Since $e^{(\delta-2\beta)} < 1$ we may thus choose $K > 0$ such that $e^{(\delta-2\beta)} + 2K < 1$.

Furthermore we have:

$$H := |h_{s,u}(\omega,\cdot)-\tilde{h}_{s,u}(\omega,\cdot)|^{s,u}_{\infty,\omega}\;|h_{s,u}(\omega,\cdot)|^{s,u}_{\infty,\omega} + |\tilde{h}_{s,u}(\omega,\cdot)|^{s,u}_{\infty,\omega} \leqq \frac{L}{2}+\frac{L}{2} = L.$$

Since an application of Hölder's inequality yields

$$E\,\frac{\eta H}{1+\eta H} = E\left[\frac{\eta+\eta H}{1+\eta H}\cdot\frac{H}{1+H}\right] \leqq \frac{\eta+\eta L}{1+\eta L}\,E\,\frac{H}{1+H}$$

for any constant η, $0<\eta\leqq 1$, we obtain:

$$d(Th, T\tilde{h}) \leqq \left[\frac{e^{(\delta-2\beta)}(1+L)}{1+e^{(\delta-2\beta)}L} + 2\,\frac{K(1+L)}{1+KL}\right]\cdot d(h,\tilde{h}) =: c(L)\cdot d(h,\tilde{h}).$$

We have already chosen K such that $c(0) = e^{(\delta-2\beta)} + 2K < 1$. For this K we determine $L_0 > 0$ and the continuity of $c(\cdot)$ ensures that $c(L) < 1$ for any L which satisfies $0 < L \leqq L_0$.

As explained earlier, the contraction mapping theorem, which is now applicable, yields the assertion of Step (2).

Step (3)

By construction we know that $Th = h$, and this means that

$$\varphi_s(1, \vartheta_{-1}\omega, x_{-1}, h(\vartheta_{-1}\omega, x_{-1})) = h_s(\omega, \varphi_c(1, \vartheta_{-1}\omega, x_{-1}, h(\vartheta_{-1}\omega, x_{-1}))),$$

$$\varphi_u(-1, \vartheta\omega, x_1, h(\vartheta\omega, x_1)) = h_u(\omega, \varphi_c(-1, \vartheta\omega, x_1, h(\vartheta\omega, x_1))), \quad (5.3\,\text{a, b})$$

In order to prove the desired invariance property we will thus proceed in several steps:

(i) We are going to prove that for any $y \in E_c(\omega)$ we have:

$$\varphi_s(-1, \omega, y, h(\omega, y)) = h_s(\vartheta_{-1}\omega, \varphi_c(-1, \omega, y, h(\omega, y)))$$

$$\varphi_u(1, \omega, y, h(\omega, y)) = h_u(\vartheta\omega, \varphi_c(1, \omega, y, h(\omega, y))).$$

We restrict ourselves to the proof of the second assertion. From Eq. (5.3b) we know that

$$\begin{aligned}\varphi_u(-1, \vartheta\omega, x, h(\vartheta\omega, x)) &= \Psi_u(-1, \vartheta\omega)\, h_u(\vartheta\omega, x)\\ &\quad + \Phi_u(-1, \vartheta\omega, x, h(\vartheta\omega, x)) = h_u(\omega, y),\end{aligned}$$

where $y = \varphi_c(-1, \vartheta\omega, x, h(\vartheta\omega, x))$.

Now let $y \in E_c(\omega)$ be given. After having determined $x \in E_c(\vartheta\omega)$ such that

$$y = \varphi_c(-1, \vartheta\omega, x, h(\vartheta\omega, x)) = \Psi_c(-1, \vartheta\omega)\, x + \Phi_c(-1, \vartheta\omega, x, h(\vartheta\omega, x))$$

we may write:

$$\begin{aligned}\varphi_u(1, \omega, y, h(\omega, y)) &= \Psi_u(1, \omega)\, h_u(\omega, y) + \Phi_u(1, \omega, y, h(\omega, y))\\ &= \Psi_u(1, \omega)\, \Psi_u(-1, \vartheta\omega)\, h_u(\vartheta\omega, x)\\ &\quad + \Psi_u(1, \omega)\, \Phi_u(-1, \vartheta\omega, x, h(\vartheta\omega, x)) + \Phi_u(1, \omega, y, h(\omega, y))\\ &= \Psi_u(1-1, \vartheta\omega)\, h_u(\vartheta\omega, x) + \Phi_u(1-1, \vartheta\omega, x, h(\vartheta\omega, x))\\ &= \varphi_u(0, \vartheta\omega, x, h(\vartheta\omega, x)) = h_u(\vartheta\omega, x).\end{aligned}$$

Here we have made use of the fact that the cocycle property implies that

$$\begin{aligned}\Psi(t+s, \omega)\, z + \Phi(t+s, \omega, z) &= \varphi(t, \vartheta_s\omega, \varphi(s, \omega, z))\\ &= \Psi(t, \vartheta_s\omega)\, \Psi(s, \omega)\, z + \Psi(t, \vartheta_s\omega)\, \Phi(s, \omega, z) + \Phi(t, \vartheta_s\omega, \varphi(s, \omega, z)).\end{aligned}$$

It remains to be shown that $x = \varphi_c(1, \omega, y, h(\omega, y))$. But this follows by the same arguments if we write $\varphi_c(1, \omega, y, h(\omega, y)) = \Psi_c(1, \omega)\, y + \Phi_c(1, \omega, y, h(\omega, y))$ and insert the expression for y above.

(ii) Applying the cocycle property several times a straightforward induction proof, the details of which are omitted, yields:

$$\varphi_{s,u}(n, \omega, x, h(\omega, x)) = h_{s,u}(\vartheta_n\omega, \varphi_c(n, \omega, x, h(\omega, x))) \qquad \text{for all } n \in \boldsymbol{Z}. \tag{5.4}$$

(iii) In case $T = \boldsymbol{Z}$ we are already done; thus assume $T = \mathbb{R}$ in the sequel. We are going to prove that the invariance property (5.4) will also hold true for all $t \in \mathbb{R}$. Because of (i) and (ii) it is sufficient to consider t such that $0 < t < 1$ since an arbitrary $t > 0$ can be written as $t = n + (t - n)$ where $n \leqq t < n + 1$, i.e. $0 \leqq t - n < 1$.

Since $(CM1, 2)(\varepsilon_0, \ldots, \varepsilon_k)$ are assumed to be satisfied not only at time $t = \pm 1$ but for any $t = \pm\tau$, $0 < \tau < 1$, we may repeat steps (1) and (2) of the proof for such a τ and obtain a fixed point h_τ which satisfies

$$\varphi_s(\tau, \omega, x, h_\tau(\omega, x)) = h_{\tau,s}(\vartheta_\tau \omega, \varphi_c(\tau, \omega, x, h_\tau(\omega, x))), \tag{5.5}$$

and similarly for the unstable part. Thus we have to show that $h_\tau = h_1 = h$ for any τ such that $0 < \tau < 1$. Since the arguments remain the same for the general case we will assume for notational simplicity that $E_u(\omega) = \{0\}$.

Case 1: $\tau = \frac{1}{2}$

The cocycle property together with property (5.5) applied several times yields:

$$\begin{aligned}
&\varphi_s(\tau + \tau, \omega, x, h_\tau(\omega, x)) = \varphi_s(\tau, \vartheta_\tau \omega, \varphi_c(\tau, \omega, x, h_\tau(\omega, x)),\\
&h_\tau(\vartheta_\tau \omega, \varphi_c(\tau, \omega, x, h_\tau(\omega, x))))\\
&\quad = h_\tau(\vartheta_\tau(\vartheta_\tau \omega),\\
&\varphi_c(\tau, \vartheta_\tau \omega, \varphi_c(\tau, \omega, x, h_\tau(\omega, x)), \underbrace{h_\tau(\vartheta_\tau \omega, \varphi_c(\tau, \omega, x, h_\tau(\omega, x))))}_{=\varphi_s(\tau, \omega, x, h_\tau(\omega, x))})\\
&\quad = h_\tau(\vartheta \omega, \varphi_c(1, \omega, x, h_\tau(\omega, x))).
\end{aligned}$$

Thus, h_τ and h are both invariant with respect to $\varphi_s(1, \omega)$ and the uniqueness of the fixed point of the operator T implies that $h_\tau = h$.

Case 2: $\tau = \dfrac{p}{q}$, $p, q \in \mathbb{N}$

Working with $\varphi_s(p, \omega)$ instead of $\varphi_s(1, \omega)$ we may repeat the same reasoning as above if we take into account (ii).

Case 3: $\tau \in \mathbb{R}$, $0 < \tau < 1$

We approximate τ by a sequence (t_n) of rational numbers.

For any t such that $-1 \leqq t \leqq 1$ we define $T_t^{s,u}$: $A_k(L) \to X$, $h \to T_t^{s,u} h$, where for any $y \in E_c(\vartheta_t \omega)$ we put:

$$(T_t^{s,u} h)(\vartheta_t \omega, y) := \varphi_{s,u}(t, \omega, x, h(\omega, x)),$$

$x \in E_c(\omega)$ being chosen such that $y = \varphi_c(t, \omega, x, h(\omega, x))$.

$T_t^{s,u}$ is understood as an operator on X which we endow with the following metric:

$$d(h, \tilde{h}) := E \frac{\|h(\omega, \cdot) - \tilde{h}(\omega, \cdot)\|_\infty}{1 + \|h(\omega, \cdot) - \tilde{h}(\omega, \cdot)\|_\infty},$$

where $\|\cdot\|_\infty$ denotes the supremum norm derived from the Euclidean norm.

In order to show invariance we have to make sure that $T_\tau^{s,u} h = h_{s,u}$ for any $h \in A_k(L)$. Up to now we know that $T_{t_n}^{s,u} h = h_{s,u}$ for any $n \in \mathbb{N}$. This implies:

$$d(T_\tau^{s,u} h, h_{s,u}) \leqq d(T_\tau^{s,u} h, T_{t_n}^{s,u} h) + d(T_{t_n}^{s,u} h, h_{s,u}) = d(T_\tau^{s,u} h, T_{t_n}^{s,u} h).$$

For $n \to \infty$, t_n tends to τ and since the random dynamical system is continuous in (t, x) we obtain that $\|\varphi_{s,u}(t_n, \omega, x_{t_n}, h(\omega, x_{t_n})) - \varphi_{s,u}(\tau, \omega, x_\tau, h(\omega, x_\tau))\| \to 0$ P-

a.s. and thus also with respect to the metric d. This yields $d(T_\tau^{s,u} h, h_{s,u})=0$, and this was to be shown.

Since the invariance property is completely established now the theorem is proven. □

Remark. We see from the proof that the class of possible nonlinearities may become smaller if β grows. However, if a nonlinear part is admissible for different values of β then the corresponding stochastic center manifolds will remain the same.

6. Local Stochastic Center Manifolds

As we have already mentioned in the last section the conditions that we have to impose to obtain *global* stochastic center manifolds are quite restrictive. It is, however, possible to deduce the existence of *local* stochastic center manifolds from the global results. As we learn from the deterministic case they are just the right objects for applications, for example in bifurcation theory. We start with some definitions which will enable us to localize the entire construction:

Recall that we have denoted the set of closed subsets of $\mathbb{R}^d$ by $\mathscr{P}_f(\mathbb{R}^d)$. Then we consider set-valued maps $U\colon \Omega\to\mathscr{P}_f(\mathbb{R}^d)$ and $W\colon \Omega\to\mathscr{P}_f(T\times\mathbb{R}^d)$ with solid values, i.e. $\mathring{U}(\omega):=\operatorname{int} U(\omega)\neq\emptyset$ for almost all $\omega\in\Omega$, and similarly for W.

The *graph* of these maps is defined as $\operatorname{Gr} U:=\{(\omega, x)\in\Omega\times\mathbb{R}^d \mid x\in\mathring{U}(\omega)\}$, and analogously for W. If we are only interested in $\omega\in\Omega_0\subset\Omega$ then we denote this by $\operatorname{Gr} U_0$.

As in Engl [17], Definition 2, *measurability* of a map $\varphi_{\text{loc}}\colon \operatorname{Gr} W\to\mathbb{R}^d$ with stochastic domain is understood in the sense that for any $(t, x)\in T\times\mathbb{R}^d$ and any $B\in\mathscr{B}(\mathbb{R}^d)$, $\{\omega\in\Omega \mid (t, x)\in\mathring{W}(\omega)$ and $\varphi_{\text{loc}}(t, \omega, x)\in B\}\in\mathscr{F}$.

Definition 6.1. Let U and W be given measurable maps as above. Then the map $\varphi_{\text{loc}}\colon \operatorname{Gr} W\to\mathbb{R}^d$ is called a *local random dynamical system* (or a *local cocycle*) *of $\mathscr{C}^k$-maps* ($k\geq 1$) *on the random neighborhood U over* $\{\vartheta_t \mid t\in T\}$ if there are a ϑ_t-invariant set $\Omega_0\subset\Omega$, $P(\Omega_0)=1$, and maps $t^\pm\colon \operatorname{Gr} U_0\to T\cap\overline{\mathbb{R}}_*^\pm$ where $\overline{\mathbb{R}}_*^\pm=(\mathbb{R}^\pm\setminus\{0\})\cup\{\pm\infty\}$, such that φ_{loc} is measurable and such that the following properties hold for any $\omega\in\Omega_0$:

(i) $\mathring{U}(\omega)$ is a neighborhood of the origin in $\mathbb{R}^d$ which satisfies $\varphi_{\text{loc}}(t, \omega, x)\in\mathring{U}(\vartheta_t\omega)$ for any $(t, x)\in\mathring{W}(\omega)$.

(ii) $W(\omega)=\{(t, x)\in T\times\mathbb{R}^d \mid t^-(\omega, x)\leq t\leq t^+(\omega, x), x\in U(\omega)\}$.

(iii) $\varphi_{\text{loc}}(\cdot, \omega, \cdot)\colon \mathring{W}(\omega)\to\mathbb{R}^d$ is continuous.

(iv) φ_{loc} is a $\mathscr{C}^k$-map with respect to x.

(v) If $(s, x)\in\mathring{W}(\omega)$ then $t^\pm(\vartheta_s\omega, \varphi_{\text{loc}}(s, \omega, x))=t^\pm(\omega, x)-s$, and for any t such that $t^-(\omega, x)<t+s<t^+(\omega, x)$, $\varphi_{\text{loc}}(t+s, \omega, x)$ is defined and satisfies:

$$\varphi_{\text{loc}}(t+s, \omega, x)=\varphi_{\text{loc}}(t, \vartheta_s\omega, \varphi_{\text{loc}}(s, \omega, x)).$$

Let $D_t(\omega)$ be the domain of definition of the map $\varphi_{\text{loc}}(t, \omega)\colon D_t(\omega)\subset\mathbb{R}^d\to\mathbb{R}^d$, i.e. $D_t(\omega):=\{x\in\mathbb{R}^d \mid (t, x)\in\mathring{W}(\omega)\}$. Then we can show:

Lemma 6.1. *For any $t\in T$ and $\omega\in\Omega_0$, $D_t(\omega)$ is an open set and $\varphi_{\text{loc}}(t, \omega)\colon D_t(\omega)\to D_{-t}(\vartheta_t\omega)\subset\mathring{U}(\vartheta_t\omega)$ is a diffeomorphism.*

Proof. If we make use of (i) of Definition 6.1 and take into account that (v) implies $\varphi_{\text{loc}}^{-1}(t, \omega) = \varphi_{\text{loc}}(-t, \vartheta_t \omega)$ then we may proceed along the same lines as in the proof of Theorem (3.12), p. 127, in Boothby [5]. Since this is straightforward we omit the details here. □

Assume that we are given a random dynamical system φ on $\mathbb{R}^d$ as in Definition 2.1. Then we define:

Definition 6.2. A *local stochastic* $\mathscr{C}^k$*-center manifold* ($k \geq 1$) for the random dynamical system φ is a measurable mapping $M_{\text{loc}}: \Omega \to \mathscr{P}_f(\mathbb{R}^d)$ for which the following property holds:

There are measurable maps U, W as above and measurable maps $\tilde{\varphi}$: $T \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, $\tilde{M}$: $\Omega \to \mathscr{P}_f(\mathbb{R}^d)$ such that on a ϑ_t-invariant set $\Omega_0 \subset \Omega$, $P(\Omega_0) = 1$, we have:

(i) $\tilde{\varphi}|_{\text{Gr}\,W}$ is a local random dynamical system on U which coincides with φ on $\mathring{W}$.

(ii) $\tilde{M}(\omega)$ is a $\mathscr{C}^k$-submanifold of $\mathbb{R}^d$ containing the origin.

(iii) $T_0 \tilde{M}(\omega) = E_c(\omega)$, where $T_0 \tilde{M}(\omega)$ is the tangent space at 0.

(iv) $M_{\text{loc}}(\omega) = \tilde{M}(\omega) \cap U(\omega)$.

(v) $\varphi(t, \omega, x) \in M_{\text{loc}}(\vartheta_t \omega)$ for all $(t, x) \in \mathring{W}(\omega) \cap (T \times M_{\text{loc}}(\omega))$.

Theorem 6.1. *Let φ be a random dynamical system on $\mathbb{R}^d$ which consists of $\mathscr{C}^k$-diffeomorphisms ($k \geq 2$) and satisfies the assumptions of Sect. 2. If one of the Lyapunov exponents of the linearization vanishes then there will be a local stochastic $\mathscr{C}^{k-1}$-center manifold for φ.*

Remark. If the local stochastic center manifold is described by a map h then $D^{k-1} h(\omega, \cdot)$ will P-a.s. satisfy a local Lipschitz condition by construction. This means that there is not really a loss of smoothness. Our construction would also yield the existence of a local stochastic $\mathscr{C}^{0,1}_{\text{loc}}$-center manifold (i.e. described by a continuous map h which satisfies a local Lipschitz condition) for a $\mathscr{C}^1$-diffeomorphism but since we have only defined $\mathscr{C}^k$-center manifolds for $k \geq 1$ we will not go into detail here.

Proof. We proceed in several steps:

(1) Fix $\beta > 0$ and determine L_0 and constants $\varepsilon_j(L_0)$, $j = 0, \ldots, k-1$, according to Theorem 5.1. Then we show that there are constants τ, $\delta_0 > 0$ such that the transformed random dynamical system $\varphi^\tau(t, \omega, x) := \dfrac{1}{\tau} \varphi(t, \omega, \tau x)$ has the same Lyapunov exponents as φ and satisfies $(CM1, 2)(\varepsilon_0(L_0), \ldots, \varepsilon_{k-1}(L_0))$ for any $x, y \in \mathbb{R}^d = E_c(\omega) \oplus E_s(\omega) \oplus E_u(\omega)$ such that $|x|_\omega < \delta_0$, $|y|_\omega < \delta_0$.

It is easily checked that the cocycle property is satisfied for φ^τ and that the Lyapunov exponents remain unchanged because the linearization is invariant under this transformation.

In order to prove the assertion we make use of the following lemma which is interesting in its own right:

Lemma 6.2. *For almost all $\omega \in \Omega$ the map $(t, x) \to |\Phi_{c,s,u}(t, \omega, x)|^{c,s,u}_{\vartheta_t \omega}$ is jointly continuous.*

Proof. The cocycle property enables us to write:

$$|\Phi_s(t,\omega,x)|^s_{\vartheta_t\omega} = \int_0^\infty e^{-(\lambda_s+2\beta)\tau} \|\Psi_s(\tau,\vartheta_t\omega)\,\Phi_s(t,\omega,x)\|\,d\tau$$

$$= \int_0^\infty e^{-(\lambda_s+2\beta)\tau} \|\Psi_s(\tau+t,\omega)[\Psi_s^{-1}(t,\omega)\,\Phi_s(t,\omega,x)]\|\,d\tau.$$

Hence the statement follows from the joint continuity of φ and hence of Ψ and Φ in (t, x). □

Now we are ready to establish what we have claimed in step (1) above. Let us consider condition $(CM1)$ first:

For $j=0, 1$ Lemma 6.2 (which, by analogous arguments, also holds true for the derivatives of Φ) combined with the fact that $\Phi(t,\omega,0)=D^1\,\Phi(t,\omega,0)=0$ ensures that condition $(CM1)$ is satisfied for any x in a suitable neighborhood of the origin and for any t such that $-t_0 \leqq t \leqq t_0$ for some $t_0>0$. After a change of time scale, if necessary, we may assume without loss of generality that $t_0=1$.

For $j\geqq 2$, $D^j\,\Phi(t,\omega,0)$ will usually be $\neq 0$. But since the chain rule implies $D^j\,\Phi^\tau(t,\omega,0)=\tau^{(j-1)}\,D^j\,\Phi(t,\omega,0)$ we may argue as in the first case if we choose τ sufficiently small.

If, besides the arguments above, we make use of the mean value theorem then the assertion concerning $(CM2)$ will follow as well.

(2) In order to construct the map $\tilde{\varphi}^\tau$ we generalize the cut-off procedure described in Sell [32], p. 381, to higher order derivatives by means of a Taylor series expansion. Taking into account Lemma 6.2 this procedure provides us with constants δ and $\tilde{\delta}$ such that $0<\delta<\tilde{\delta}\leqq\delta_0$ and with a $\mathscr{C}^{k-1,1}$-map $\tilde{\Phi}^\tau$ which has the following properties for any t, $|t|\leqq 1$:

(i) $\tilde{\Phi}^\tau(t,\omega,x)=\Phi^\tau(t,\omega,x)$ for any $x\in\bar{B}_\delta(\omega)$,

(ii) $\tilde{\Phi}^\tau(t,\omega,x)=0$ for any $x\notin\bar{B}_{\tilde{\delta}}(\omega)$,

(iii) $\tilde{\Phi}^\tau := \Psi^\tau+\tilde{\Phi}^\tau$ is of class $NL_{\varepsilon_0(L),\ldots,\varepsilon_{k-1}(L)}$.

Here we have put $B_\delta(\omega):=\{x\,|\,|x|_\omega<\delta\ P\text{-a.s.}\}$.

(3) In this step we determine the maps U and W required in the theorem. For any $s\in T$ we consider the set

$$U(\vartheta_s\omega) := \{x\in\bar{B}_\delta(\vartheta_s\omega)\,|\,|\tilde{\varphi}^\tau(t,\vartheta_s\omega,x)|_{\vartheta_{t+s}\omega}\leqq\delta \text{ for all } t \text{ such that } |t|\leqq 1-|s|\}$$

which contains 0 and is closed as an intersection of closed sets. Since φ depends measurably on ω we have thus defined a measurable map U. The same reasoning as in Lemma 6.2 shows that U has solid values. Note that for $|s|\geqq 1$, $\mathring{U}(\vartheta_s\omega) = B_\delta(\vartheta_s\omega)$.

For given $\omega\in\Omega$, $x\in U(\omega)$ let $t^-(\omega,x):=\sup\{t<0\,|\,\tilde{\varphi}^\tau(t,\omega,x)\notin\bar{B}_\delta(\vartheta_t\omega)\}$ and $t^+(\omega,x):=\inf\{t>0\,|\,\tilde{\varphi}^\tau(t,\omega,x)\notin\bar{B}_\delta(\vartheta_t\omega)\}$. If $\tilde{\varphi}^\tau(t,\omega,x)\in\bar{B}_\delta(\vartheta_t\omega)$ for any $t<0$ ($t>0$, resp.) then we put $t^-(\omega,x):=-\infty$ ($t^+(\omega,x):=+\infty$, resp.). By construction we know that $|t^\pm(\omega,x)|>1$.

Define the map W as in Definition 6.1, i.e.

$$W(\omega)=\{(t,x)\in T\times\mathbb{R}^d\,|\,t^-(\omega,x)\leqq t\leqq t^+(\omega,x),\ x\in U(\omega)\}.$$

Let $\varphi_{\text{loc}} := \tilde{\varphi}|_{\text{Gr} W}$ and let $\Omega_0 \subset \Omega$ be the set of full measure for which φ^τ satisfies the cocycle property. If $(s, x) \in \mathring{W}(\omega)$ and $|s| < 1$ then $|\tilde{\varphi}^\tau(t, \vartheta_s \omega, \tilde{\varphi}^\tau(s, \omega, x))|_{\vartheta_{t+s}\omega} = |\varphi^\tau(t+s, \omega, x)|_{\vartheta_{t+s}\omega} < \delta$ for all t such that $|t+s| \leqq 1$ and this implies that $\tilde{\varphi}^\tau(s, \omega, x) \in \mathring{U}(\vartheta_s \omega)$. If $|s| \geqq 1$ then $\tilde{\varphi}^\tau(s, \omega, x) \in B_\delta(\vartheta_s \omega) = \mathring{U}(\vartheta_s \omega)$ and thus (i) of Definition 6.1 is satisfied.

The properties of φ^τ obviously yield (iii) and (iv) of Definition 6.1. Furthermore it is an immediate consequence of the cocycle property that for any $(s, x) \in \mathring{W}$, $t^{\pm}(\vartheta_s \omega, \varphi_{\text{loc}}(s, \omega, x)) = t^{\pm}(\omega, x) - s$, and that (v) of Definition 6.1 holds. Hence φ_{loc} is a local random dynamical system which coincides with φ^τ on $\mathring{W}$ because inside $U(\cdot) \subset B_\delta(\cdot)$ the system has not been altered.

(4) After these preparations we may repeat the first two steps of the proof of Theorem 5.1, for which the cocycle property is not needed, for the transformed system $\tilde{\varphi}^\tau$ which is defined on all of $\mathbb{R}^d$, satisfies $(CM1, 2)$ but does not have the cocycle property.

This procedure provides us with a map $\tilde{h} \in A_{k-1}(L)$ the graph of which describes a set $\tilde{M}(\cdot)$ satisfying properties (ii) and (iii) of Definition 6.2 by construction. Let

$$M^\tau_{\text{loc}}(\omega) := \tilde{M}(\omega) \cap U(\omega) \quad \text{and} \quad h^\tau(\omega, \cdot) := \tilde{h}^\tau(\omega, \cdot)|_{U(\omega)}.$$

It remains to be shown that $M^\tau_{\text{loc}}(\cdot)$ satisfies (v) of Definition 6.2, i.e. the local invariance property. At present we know by construction that it holds true for $|t| \leqq 1$. But since on $\mathring{U}(\cdot)$ $\tilde{\varphi}^\tau$ coincides with the cocycle φ^τ step (3) of the proof of Theorem 5.1 may be repeated for any t up to the first exit from $B(\cdot)$.

Hence we have shown the existence of a local stochastic center manifold $M^\tau_{\text{loc}}(\cdot)$ for φ^τ described by a map h^τ. For this reason a local stochastic center manifold for the original system φ is described by $h(\omega, x) := \tau\, h^\tau\left(\omega, \frac{1}{\tau} x\right)$. □

Remarks. (i) The proof has thus shown as well that the restriction of a *global* random dynamical system φ to a random neighborhood U which satisfies property (i) of Definition 6.1 up to the first exit from $U(\omega)$ is a *local* random dynamical system.

(ii) Theorem 6.1 also applies to the special cases described in Sect. 3, provided that the vector fields in 3.1, 3.2, resp. satisfy appropriate Lipschitz conditions such that the solution generates a cocycle of $\mathscr{C}^k$-diffeomorphisms. See the references in Sect. 3 for more information.

7. Dynamical Properties of Stochastic Center Manifolds

7.1. Global Attractivity

In this section we are going to investigate what happens if we pick an initial value which is not on the stochastic center manifold. For this we return to the situation described in Sects. 2 and 5, respectively.

Theorem 7.1. *Let $M(\omega)$ be the stochastic center manifold of Theorem 5.1. Then we obtain:*

(i) *If all Lyapunov exponents are $\leqq 0$ then there is a map $c_s\colon \mathbb{R}^+ \to]0, \infty[$ satisfying $\lim_{t\to\infty} \frac{1}{t} \log c_s(t) < 0$ such that for any initial values $x_c \in E_c(\omega)$ and $x_s \in E_s(\omega)$:*

$$|\varphi_s(t, \omega, x_c, x_s) - h_s(\vartheta_t \omega, \varphi_c(t, \omega, x_c, x_s))|^s_{\vartheta_t \omega} \leqq c_s(t)\, |x_s - h_s(\omega, x_c)|^s_\omega$$

for any $t \geqq 0$ a.s.

(ii) *If all Lyapunov exponents are $\geqq 0$ then there is a map $c_u\colon \mathbb{R}^- \to]0, \infty[$ satisfying $\lim_{t\to-\infty} \frac{1}{t} \log c_u(t) > 0$ such that for any initial values $x_c \in E_c(\omega)$ and $x_u \in E_u(\omega)$:*

$$|\varphi_u(t, \omega, x_c, x_u) - h_u(\vartheta_t \omega, \varphi_c(t, \omega, x_c, x_u))|^u_{\vartheta_t \omega} \leqq c_u(t)\, |x_u - h_u(\omega, x_c)|^u_\omega$$

for any $t \leqq 0$ a.s.

Proof. Since the proof of (ii) is completely analogous we will only show (i).

Let x_c, x_s be given. Then Lemma 5.1 ensures the existence of an initial value $x \in E_c(\omega)$ such that $\varphi_c(1, \omega, x_c, x_s) = y = \varphi_c(1, \omega, x, h_s(\omega, x))$. Estimating $|x_c - x|^c_\omega$ as in Lemma 5.2 and making use of the invariance of the stochastic center manifold we obtain for δ chosen such that $\lambda_s + 4\beta \leqq \delta < 0$:

$$\begin{aligned}
&|\varphi_s(1, \omega, x_c, x_s) - h_s(\vartheta \omega, \varphi_c(1, \omega, x_c, x_s))|^s_{\vartheta\omega} \\
&\quad \leqq |\varphi_s(1, \omega, x_c, x_s) - \varphi_s(1, \omega, x, h_s(\omega, x))|^s_{\vartheta\omega} \\
&\qquad + |h_s(\vartheta\omega, \varphi_c(1, \omega, x, h_s(\omega, x))) - h_s(\vartheta\omega, \varphi_c(1, \omega, x_c, x_s))|^s_{\vartheta\omega} \\
&\quad \leqq [e^{(\delta - 2\beta)} + K_1]\, |x_s - h_s(\omega, x_c)|^s_\omega,
\end{aligned}$$

where arguments as in the proof of the contraction property (Theorem 5.1) imply the last inequality and where we have put $K_\tau := \dfrac{\gamma \varepsilon_0 \left(1 + \frac{L}{2} e^{\delta\tau}\right)}{1 - \gamma\varepsilon_0 \left(1 + \frac{L}{2}\right) e^{2\beta\tau}}$.

The cocycle property enables us to iterate this procedure and yields estimates for any $n \in \mathbb{N}$. An arbitrary $t > 0$ is decomposed as $t = n + (t - n)$, $0 \leqq t - n < 1$. Thus the cocycle property together with an estimate at time $t - n$, which is derived like the one for time $t = 1$, will imply the result if we take into account that $e^{(\delta - 2\beta)} + K_1 < 1$ (see the proof of Theorem 5.1). □

Remark. Making use of the inequality relating the norms $|\cdot|^{c,s,u}_\omega$ and the Euclidean norm we may also express the estimates of Theorem 7.1 in terms of the Euclidean norm. In this case the functions $c_{s,u}$ are replaced by maps $c_{s,u}(\cdot, \cdot)$ such that for any $t \geqq 0$ (resp. $t \leqq 0$) $c_{s,u}(t, \cdot)$ is a random variable. For the Euclidean norm it is easily seen that the stochastic center manifold is attracting exponentially fast with respect to the metric of convergence in probability. Moreover, if $C_{s,u}(\cdot)$ (see Lemma 4.1) is integrable then the exponential attractivity will even hold true with respect to the L^1-norm.

7.2. Dynamical Characterization of Stochastic Center Manifolds

Instead of describing a stochastic center manifold by its geometric properties, i.e. invariance and tangency to a certain subspace, one may also understand it as the collection of those initial values which show a certain dynamical behavior in both directions of time. In the deterministic case such a characterization may be found, e.g., in Sell's paper [32].

Given a random dynamical system φ as in Sect. 2 and assume condition (LE) of Sect. 4, i.e. choose $\beta>0$ such that $\lambda_s+4\beta<0$ and $\lambda_u-4\beta>0$. For a $\delta>0$ we define the following random set:

$$W_{\text{dyn}}^{\delta}(\omega):=\left\{x\in\mathbb{R}^d \;\middle|\; \limsup_{t\to+\infty}\frac{1}{t}\log|\varphi_{c,s,u}(t,\omega,x)|_{\vartheta_t\omega}^{c,s,u}\leqq\delta \text{ and } \liminf_{t\to-\infty}\frac{1}{t}\log|\varphi_{c,s,u}(t,\omega,x)|_{\vartheta_t\omega}^{c,s,u}\geqq-\delta\right\}.$$

Let the assumptions of Theorem 5.1 be satisfied, which means in particular that φ is a random dynamical system of class $NL_{\varepsilon_0(L),\dots,\varepsilon_k(L)}$. Before we are going to prove a theorem relating W_{dyn}^{δ} and the stochastic center manifold of Theorem 5.1 we will investigate in which way exponential growth rates carry over from the linear to the nonlinear random dynamical system:

Lemma 7.1. *Let Ψ be a linear cocycle for which there is a constant $\eta>0$ such that*

$$|\Psi(t,\omega)x|_{\vartheta_t\omega}\leqq e^{\eta t}\,|x|_{\omega} \quad \textit{for any } x\in\mathbb{R}^d \textit{ and any } t\geqq 0 \textit{ a.s.} \tag{7.1}$$

Then we obtain for any random dynamical system $\varphi=\Psi+\Phi$ of class $NL_{\varepsilon_0(L),\dots,\varepsilon_k(L)}$:

$$\limsup_{t\to\infty}\frac{1}{t}\log|\varphi(t,\omega,x)|_{\vartheta_t\omega}\leqq\eta \quad \textit{for any } x\in\mathbb{R}^d \textit{ a.s.}$$

Remark. Obviously an analogous statement holds true for $\liminf_{t\to-\infty}$ if there is an estimate of the linear cocycle for $t\leqq 0$.

Proof. In order to derive an estimate for $|\varphi(n,\omega,x)|_{\vartheta_n\omega}$ we make use of (7.1) and assumption $(CM2)$ together with the cocycle property and write:

$$\begin{aligned}|\varphi(n,\omega,x)|_{\vartheta_n\omega}&\leqq e^{\eta}\,|\varphi(n-1,\omega,x)|_{\vartheta_{n-1}\omega}+\tilde{\varepsilon}\,|\varphi(n-1,\omega,x)|_{\vartheta_{n-1}\omega}\\&\leqq e^{\eta}\,|\varphi(n-1,\omega,x)|_{\vartheta_{n-1}\omega}+\tilde{\varepsilon}\,e^{\eta}\,|\varphi(n-2,\omega,x)|_{\vartheta_{n-2}\omega}\\&\quad+\tilde{\varepsilon}^2\,|\varphi(n-2,\omega,x)|_{\vartheta_{n-2}\omega}\end{aligned}$$

where we have denoted $\gamma\varepsilon_0$ by $\tilde{\varepsilon}$. We go on applying the cocycle property and the estimates (7.1) and $(CM2)$ to the last term in this sum and finally obtain:

$$|\varphi(n,\omega,x)|_{\vartheta_n\omega} \leqq \tilde{\varepsilon}^n\,|x|_\omega + e^\eta \sum_{k=0}^{n-1} \tilde{\varepsilon}^k\,|\varphi(n-k-1,\omega,x)|_{\vartheta_{n-k-1}\omega}.$$

This implies that

$$\begin{aligned} &e^{-\eta n}\,|\varphi(n,\omega,x)|_{\vartheta_n\omega} \\ &\leqq e^{-\eta n}\,\tilde{\varepsilon}^n\,|x|_\omega + \sum_{k=0}^{n-1} e^{-\eta k}\,\tilde{\varepsilon}^k (e^{-\eta(n-k-1)}\,|\varphi(n-k-1,\omega,x)|_{\vartheta_{n-k-1}\omega}) \end{aligned}$$

and if we estimate $e^{-\eta n}$ by 1 and put $y_n := e^{-\eta n}\,|\varphi(n,\omega,x)|_{\vartheta_n\omega}$ we arrive at

$$y_n \leqq \tilde{\varepsilon}^n\,y_0 + \sum_{k=0}^{n-1} \tilde{\varepsilon}^k\,y_{n-k-1} \leqq \tilde{\varepsilon}\,y_0 \exp\left(\sum_{k=0}^{n-1} \tilde{\varepsilon}^k\right),$$

where the last estimate is a consequence of a discrete version of Gronwall's inequality (see Beesack [4], Corollary 10.1, p. 96). This leads to

$$\log|\varphi(n,\omega,x)|_{\vartheta_n\omega} \leqq \eta\,n + \log\tilde{\varepsilon}\,|x|_\omega + \sum_{k=0}^{n-1} \tilde{\varepsilon}^k.$$

For a given $t \geqq 0$ choose n such that $n \leqq t < n+1$, i.e. $0 \leqq t-n < 1$. Hence

$$|\varphi(t,\omega,x)|_{\vartheta_t\omega} = |\varphi(t-n,\vartheta_{-n}\omega,\varphi(n,\omega,x))|_{\vartheta_t\omega} \leqq (e^{\eta(t-n)} + \tilde{\varepsilon})\,|\varphi(n,\omega,x)|_{\vartheta_n\omega}$$

and thus the result follows if we take into account that $\tilde{\varepsilon} = \gamma\varepsilon_0 < 1$. □

Theorem 7.2. *Let* $\Lambda_\beta := \min(-(\lambda_s + 2\beta), \lambda_u - 2\beta)$. Then we obtain for any δ such that $2\beta < \delta < \Lambda_\beta$:

(i) W^δ_{dyn} *is invariant with respect to* φ, *i.e.*

$$\varphi(t,\omega)\,W^\delta_{\mathrm{dyn}}(\omega) = W^\delta_{\mathrm{dyn}}(\vartheta_t\omega) \qquad \textit{for any } t \in T \textit{ a.s.}$$

(ii) *Let* $M(\omega)$ *be the stochastic center manifold constructed in Theorem* 5.1 *which depends on* β *via the random norms. Then we obtain:*

$$W^\delta_{\mathrm{dyn}}(\omega) = M(\omega) \quad P\text{-a.s.}$$

Remark. The theorem means that W^δ_{dyn} may be interpreted as the *dynamical characterization* of the stochastic center manifold.

Proof. (i) is an immediate consequence of the cocycle property.

(ii) a) We are going to show that $M(\omega) \subset W^{\delta}_{\rm dyn}(\omega)$ P-a.s. For this we write $x \in E_c(\omega)$ in the form $x=(x_c, h(\omega, x_c))$. Since the invariance of the stochastic center manifolds yields

$$|\varphi_{s,u}(\pm t, \omega, x_c, h(\omega, x_c))|^{s,u}_{\vartheta_{\pm t}\omega}$$
$$= |h_{s,u}(\vartheta_{\pm t}\omega, \varphi_c(\pm t, \omega, x_c, h(\omega, x_c)))|^{s,u}_{\vartheta_{\pm t}\omega} \leqq \frac{L}{2} |\varphi_c(\pm t, \omega, x_c, h(\omega, x_c))|^{c}_{\vartheta_{\pm t}\omega}$$

it is sufficient to estimate the center component.

Lemma 4.2 yields $|\Psi_c(t, \omega) x_c|^c_{\vartheta_t\omega} \leqq e^{2\beta|t|} |x_c|_\omega$ and thus the assertion follows from Lemma 7.1.

b) In order to show that $W^{\delta}_{\rm dyn}(\omega) \subset M(\omega)$ P-a.s. we need the notions of *stochastic center-stable* resp. *center-unstable manifolds* $M^{cs}(\omega)$, $M^{cu}(\omega)$, resp. Since these objects may be constructed along the same lines as the stochastic center manifold we omit the details. We will make use of the fact that they may be written as graphs $M^{cs}(\omega)=\{(x_{cs}, h^{cs}(\omega, x_{cs})) | x_{cs} \in E_{cs}(\omega)\}$ (and analogously for M^{cu}) where $E_{cs}(\omega) := \bigoplus_{\lambda_i \leqq 0} E_i(\omega)$ and $h^{cs}(\omega, \cdot)\colon E_{cs}(\omega) \to E_u(\omega)$.

We will prove that $\tilde{x} \notin M(\omega)$ implies that there cannot be a δ, $2\beta < \delta < \Lambda_\beta$, such that x is an element of $W^{\delta}_{\rm dyn}(\omega)$. Since $M(\omega)=M^{cs}(\omega) \cap M^{cu}(\omega)$, $\tilde{x} \notin M(\omega)$ implies that $\tilde{x}=(\tilde{x}_{cu}, \tilde{x}_s) \notin M^{cu}(\omega)$, say (otherwise one has to reverse time in the arguments below), i.e. $\tilde{x}_s \neq h^{cu}(\omega, \tilde{x}_{cu})$.

The same considerations as in the proof of Theorem 7.1 provide us with an estimate for $|\varphi_s(t, \vartheta_{-t}\omega, x_{cu}, x_s) - h^{cu}(\omega, \varphi_{cu}(t, \vartheta_{-t}\omega, x_{cu}, x_s))|^s_\omega$ for any $t \geqq 0$ and any $x_{cu} \in E_{cu}(\vartheta_{-t}\omega)$, $x_s \in E_s(\vartheta_{-t}\omega)$. Since the arguments are by now familiar to us we will not go into details.

We apply this estimate to $x_{cu} := \varphi_{cu}(-t, \omega, \tilde{x}_{cu}, \tilde{x}_s)$, $x_s := \varphi_s(-t, \omega, \tilde{x}_{cu}, \tilde{x}_s)$ and obtain because of the cocycle property:

$$|\tilde{x}_s - h^{cu}(\omega, \tilde{x}_{cu})|^s_\omega \leqq \tilde{c}_s(t) |\varphi_s(-t, \omega, \tilde{x}_{cu}, \tilde{x}_s) - h^{cu}(\vartheta_{-t}\omega, \varphi_{cu}(-t, \omega, \tilde{x}_{cu}, \tilde{x}_s))|^s_{\vartheta_{-t}\omega},$$

where $\tilde{c}_s(t)$ converges to 0 exponentially fast for $t \to \infty$ with an exponential growth rate of $\lambda_s + 2\beta$. After having multiplied both sides of this inequality with $e^{-(\lambda_s+2\beta)t}$ we see that the left-hand side grows exponentially fast for $t \to \infty$ because by assumption $|\tilde{x}_s - h^{cu}(\omega, \tilde{x}_{cu})|^s_\omega \neq 0$. Since $h^{cu}(\vartheta_{-t}\omega, \cdot)$ is bounded by construction this implies that $\varphi_s(-{\rm t}, \omega, \tilde{\rm x}_{cu}, \tilde{\rm x}_s)$ has to grow exponentially fast with growth rate $\geqq -(\lambda_s + 2\beta) \geqq \Lambda_\beta$. Hence $\tilde{x} \notin W^{\delta}_{\rm dyn}(\omega)$, which was to be proved. □

Remarks. (i) Once again it is possible to express $W^{\delta}_{\rm dyn}(\omega)$ in terms of the Euclidean norm (see Boxler [8], p. 117, for details). However, we will have to require an extra integrability condition if we wish to write

$$W^{\delta}_{\rm dyn}(\omega) = \left\{ x \in \mathbb{R}^d \,\middle|\, \limsup_{t \to +\infty} \frac{1}{t} \log \|\varphi_{s,u}(t, \omega, x)\| \leqq \delta, \right.$$
$$\left. \liminf_{t \to -\infty} \frac{1}{t} \log \|\varphi_{s,u}(t, \omega, x)\| \geqq -\delta \right\}.$$

(ii) If β changes this will only imply that the upper and lower bound for δ is altered. Theorem 7.2 will still hold true, as long as the random dynamical system belongs to the class $NL_{\varepsilon_0(L),\dots,\varepsilon_k(L)}$ which may have become smaller (compare the remark after the proof of Theorem 5.1).

(iii) As in the deterministic case, geometrically characterized stochastic center manifolds $M(\omega)$ are in general not unique (see, e.g., Vanderbauwhede [33], p. 27, for a deterministic example). But among them there is precisely one, $W^{\delta}_{\text{dyn}}(\omega)$, which is singled out by its dynamical behavior. On the other hand, Theorem 7.2 also tells us that the set $W^{\delta}_{\text{dyn}}(\omega)$ is in fact a manifold.

7.3. *Reduction Principle*

In this subsection we will see that it is sufficient to examine the asymptotic behavior of the system restricted to the stochastic center manifold to deduce the stability properties of the entire system. Consider

$$\varphi_c(t, \omega, z_c, h(\omega, z_c)) = \Psi_c(t, \omega)\, z_c + \Phi_c(t, \omega, z_c, h(\omega, z_c)), \tag{7.2}$$

where $z_c \in E_c(\omega)$. Then the following theorem holds:

Theorem 7.3. a) *Let all Lyapunov exponents be* $\leqq 0$. *Then the zero solution of the original system described by* (4.1) *is asymptotically stable* (*resp. unstable*) *for* $t \geqq 0$ *if and only if the zero solution of* (7.2) *has this property.*

Let the zero solution of (7.2) *be asymptotically stable. Then there are maps* $k_s, k_c\colon \mathbb{R}^+ \to]0, \infty[$, $\lim\limits_{t\to\infty} \frac{1}{t} \log k_{s,c}(t) < 0$, *such that for given initial values* x_c, x_s *there is a* $z_c \in E_c(\omega)$ *for which the following estimates hold:*

(i) $|\varphi_c(t, \omega, x_c, x_s) - \varphi_c(t, \omega, z_c, h(\omega, z_c))|^c_{\vartheta_t\omega} \leqq k_c(t)$ *for all* $t \geqq 0$ *a.s.*

(ii) $|\varphi_s(t, \omega, x_c, x_s) - h_s(\vartheta_t\omega, \varphi_c(t, \omega, z_c, h(\omega, z_c)))|^s_{\vartheta_t\omega} \leqq k_s(t)$ *for all* $t \geqq 0$ *a.s.*

b) *Let all Lyapunov exponents be* $\geqq 0$. *Then results analogous to those in* a) *hold true for* $t \leqq 0$ *and for* φ_u *instead of* φ_s.

Proof. The proof of b) being analogous, we restrict ourselves to a).

Obviously the stability behavior of the entire system also fixes the stability behavior of (7.2). The other direction of the proof follows from the invariance of the stochastic center manifold combined with its attractivity property.

For the proof of the estimates (i) and (ii) we consider the following error functions:

$$c(t, \omega, c_0) := \varphi_c(t, \omega, c_0 + z_c, s_0 + h_s(\omega, c_0 + z_c)) - \varphi_c(t, \omega, z_c, h_s(\omega, z_c)), \tag{7.3a}$$

$$\begin{aligned} s(t, \omega, s_0) := {} & \varphi_s(t, \omega, c_0 + z_c, s_0 + h_s(\omega, c_0 + z_c)) \\ & - h_s(\vartheta_t\omega, \varphi_c(t, \omega, c_0 + z_c, s_0 + h_s(\omega, c_0 + z_c))) \end{aligned} \tag{7.3b}$$

where we have put: $x_c = c_0 + z_c$ and $x_s = s_0 + h_s(\omega, x_c)$.

We already know from Theorem 7.1 that for any fixed c_0 in (7.3b), s has the desired growth. For given z_c and s_0 we will thus have to determine a c_0

for which c in (7.3a) has the desired growth. This c_0 will do for both equations. This yields initial values x_c and x_s, and we will see at the end of the proof why this reasoning enables us to show the assertion.

Let z_c and s_0 be fixed. Then (7.3a) is solved by applying Schauder's fixed point theorem to the operator $V(\omega,\cdot)$: $Y(\omega)\to Y(\omega)$,

$$V(\omega,f)(t):=\varphi_c(t,\omega,f(\omega,0)+z_c,s_0+h_s(\omega,f(\omega,0)+z_c))-\varphi_c(t,\omega,z_c,h_s(\omega,z_c)),$$

where $Y(\omega):=\{f(\omega,\cdot)\in\mathscr{C}(\mathbb{R}^+,\mathbb{R}^d)\,|\,f(\omega,t)\in E_c(\vartheta_t\omega),\ \lim_{t\to\infty}|f(\omega,t)\,a(t)|^c_{\vartheta_t\omega}=0\}$, and where we have put $a(t):=c_s^{-1}(t)$ (see Theorem 7.1). For any $\omega\in\Omega$ we endow $Y(\omega)$ with the norm $\|f(\omega,\cdot)\|:=\sup_{t\geqq 0}|f(\omega,t)\,a(t)|^c_{\vartheta_t\omega}$ and consider the convex bounded subset $B(\omega):=\{f(\omega,\cdot)\in Y(\omega)\,|\,\|f(\omega,\cdot)\|\leqq K_\omega$ for a suitable constant $K_\omega\}$. We are going to show that $V(\omega,\cdot)$ is a compact operator which satisfies $V(\omega,B(\omega))\subset B(\omega)$ for almost all $\omega\in\Omega$. Thus, let $f(\omega,\cdot)\in B(\omega)$ be given.

1) $\|V(\omega,f)(\cdot)\|\leqq K_\omega$ a.s.:

In order to obtain the desired growth rate we proceed as follows. We put:

$$x_n^{c,s}:=\varphi_{c,s}(t,\omega,f(\omega,0)+z_c,s_0+h_s(\omega,f(\omega,0)+z_c))-\varphi_{c,s}(t,\omega,z_c,h_s(\omega,z_c)),$$

$$\Psi^n_{c,s}:=\Psi_{c,s}(t,\vartheta_{nt}\omega),\qquad \varepsilon:=\varepsilon_0.$$

The cocycle property together with assumption $(CM2)$ implies that

$$|x_n^{c,s}|\leqq(|\Psi^{n-1}_{c,s}|+\varepsilon)\,|x^c_{n-1}|+\varepsilon\,|x^s_{n-1}|,$$

where we have omitted the indices at the norms for notational simplicity.

We iterate this estimate for $|x^s_{n-1}|$ and obtain an estimate for $|x^c_n|$ containing only center terms and the term $|x^s_1|$. The invariance of the stochastic center manifold together with Theorem 7.1 yields $|x^s_1|\leqq c_s(t)\,|s_0|+\frac{L}{2}\,|x^c_1|$ and thus we are able to proceed as in the proof of Lemma 7.1 using a discrete version of Gronwall's inequality (see, e.g., Beesack [4], Corollary 10.1, p. 96) in order to derive an estimate for $|V(\omega,f)(n)|^c_{\vartheta_n\omega}$. For an arbitrary t the desired inequality follows once again as in Lemma 7.1. For more details see Boxler [8], Theorem 8.1.

2) $V(\omega,\cdot)$ is a continuous operator:

This is shown by means of estimates completely analogous to those used in 1). For this, we omit the details.

3) $V(\omega,B(\omega))$ is compact:

We apply a theorem that may be found e.g. in Bourbaki [7], X, § 2.5, Corollary 3, p. 292, and which tells us that the desired compactness will be established as soon as we will have proved the equicontinuity of $B(\omega)$.

Assume that this does not hold true for some $t_0\in\mathbb{R}^+$. Then this implies by the definition of equicontinuity that there is a $\eta>0$ such that for any $\delta>0$ there is a $f(\omega,\cdot)\in B(\omega)$ such that

$$|f(\omega,t_0+\delta)\,a(t_0+\delta)-f(\omega,t_0)\,a(t_0+\delta)|^c_{\vartheta_t\omega}\geqq\eta\,a(t_0+\delta).$$

But since $f(\omega,\cdot)\in B(\omega)$ the left hand side is $\leqq K_\omega$, and this cannot hold true for all $\delta>0$ because a grows exponentially fast. Hence we arrive at a contradiction and the desired equicontinuity is erstablished.

The assumptions of Schauder's theorem are thus satisfied and hence the existence of a fixed point $f(\omega,\cdot)\in B(\omega)$ is ensured for almost all $\omega\in\Omega$. By construction, it has the required properties and depends measurably on ω.

It remains to be shown that the map G: $(z_c, s_0)\to(z_c+f(\omega,0), s_0)=(x_c, s_0)$ is one-to-one because then we will be able to determine z_c for given initial values x_c and x_s. For this we assume that $z_c+f(\omega,0)=\tilde z_c+\tilde f(\omega,0)$ which implies that for any $t\geqq 0$

$$\varphi_c(t,\omega,\tilde z_c, h_s(\omega,\tilde z_c))-\varphi_c(t,\omega,z_c,h_s(\omega,z_c))=f(\omega,t)-\tilde f(\omega,t). \tag{7.4}$$

On the other hand we may proceed as in the proof of Theorem 7.2 to derive the following estimate in which $K(t)$ grows more slowly than $a(t)$:

$$|z_c-\tilde z_c|^c_\omega\leqq K(t)\,|\varphi_c(t,\omega,z_c,h_s(\omega,z_c))-\varphi_c(t,\omega,\tilde z_c,h_s(\omega,\tilde z_c))|^c_{\vartheta_t\omega}.$$

This implies that $\lim_{t\to\infty} a(t)\,|\varphi_c(t,\omega,z_c,h_s(\omega,z_c))-\varphi_c(t,\omega,\tilde z_c,h_s(\omega,\tilde z_c))|^c_{\vartheta_t\omega}=\infty$ if $z_c\neq\tilde z_c$, whereas the definition of f yields: $a(t)\,|f(\omega,t)|^c_{\vartheta_t\omega}\leqq K_\omega$ for any $t\geqq 0$. Hence Eq. (7.4) can only be satisfied for $z_c=\tilde z_c$. □

8. Approximation of the Stochastic Center Manifold

8.1. Existence of an Approximation

A stochastic center manifold, which is described by a map h, has to satisfy the following equations which, in general, cannot be solved explicitly:

$$h_s(\omega,\varphi_c(1,\vartheta_{-1}\omega,x_{-1},h(\vartheta_{-1}\omega,x_{-1})))=\varphi_s(1,\vartheta_{-1}\omega,x_{-1},h(\vartheta_{-1}\omega,x_{-1})),$$

$$h_u(\omega,\varphi_c(-1,\vartheta\omega,x_1,h(\vartheta\omega,x_1)))=\varphi_u(-1,\vartheta\omega,x_1,h(\vartheta\omega,x_1)).$$

For a map $\zeta\in A_k(L)$ we consider an operator V defined by

$$(V\zeta)(\omega,y):=\varphi_s(1,\vartheta_{-1}\omega,x_{-1},\zeta(\vartheta_{-1}\omega,x_{-1}))-\varphi_u(-1,\vartheta\omega,x_1,\zeta(\vartheta\omega,x_1))-\zeta(\omega,y),$$

where x_{-1} and x_1 satisfy

$$\varphi_c(1,\vartheta_{-1}\omega,x_{-1},\zeta(\vartheta_{-1}\omega,x_{-1}))=y=\varphi_c(-1,\vartheta\omega,x_1,\zeta(\vartheta\omega,x_1)).$$

Obviously we have $(Vh)(\omega,y)=0$.

Denote by $f(\omega,y)=O(\|y\|)$ the fact that $f(\omega,y)\leqq C(\omega)\,\|y\|$ a.s. for a suitable random variable C: $\Omega\to]0,\infty[$ and y sufficiently small. Then we can show:

Theorem 8.1. *For a given $\zeta\in A_k(L)$ assume $(V\zeta)(\omega,y)=O(\|y\|^q)$ for some $q>1$, almost all $\omega\in\Omega$ and all $y\in E_c(\omega)$ sufficiently small. Then we obtain:*

$$\|h(\omega,y)-\zeta(\omega,y)\|=O(\|y\|^q).$$

Proof. The proof is a straightforward adaptation of a proof for differential equations in Carr [9] (Theorem 7, p. 35). A brief description of the arguments will thus be sufficient.

For a constant $K_0 > 0$ (to be specified later) consider the set

$$Y := \{\Gamma \in A_k(L) \mid \exists K_\Gamma, 0 < K_\Gamma \leqq K_0 : |\Gamma(\omega, y)|_\omega \leqq K_\Gamma(|y|_\omega^c)^q \text{ a.s. for all } y \in E_c(\omega)\}$$

and define an operator U acting on Y by $U(\Gamma) := T(\Gamma + \zeta) - \zeta$, T as in the proof of Theorem 5.1.

Since convergence in the metric d implies that there is an almost surely convergent subsequence we see immediately that Y is a closed subset of $A_k(L)$.

Furthermore we can show that there is a constant $K_0 > 0$ such that the set $Y := Y(K_0)$ satisfies: $U(Y) \subset Y$. For this we write

$$\begin{aligned}
|(U\Gamma)(\omega, y)|_\omega &= |T(\Gamma + \zeta)(\omega, y) - \zeta(\omega, y)|_\omega \\
&= |\varphi_s(1, \vartheta_{-1}\omega, \tilde{x}_{-1}, \Gamma(\vartheta_{-1}\omega, \tilde{x}_{-1}) + \zeta(\vartheta_{-1}\omega, \tilde{x}_{-1})) \\
&\quad + \varphi_u(-1, \vartheta\omega, \tilde{x}_1, \Gamma(\vartheta\omega, \tilde{x}_1) + \zeta(\vartheta\omega, \tilde{x}_1)) - \zeta(\omega, y)|_\omega \\
&\leqq |\varphi_s(1, \vartheta_{-1}\omega, \tilde{x}_{-1}, \Gamma(\vartheta_{-1}\omega, \tilde{x}_{-1}) + \zeta(\vartheta_{-1}\omega, \tilde{x}_{-1})) \\
&\quad - \varphi_s(1, \vartheta_{-1}\omega, x_{-1}, \zeta(\vartheta_{-1}\omega, x_{-1}))|_\omega^s \\
&\quad + |\varphi_u(-1, \vartheta\omega, \tilde{x}_1, \Gamma(\vartheta\omega, \tilde{x}_1) + \zeta(\vartheta\omega, \tilde{x}_1)) \\
&\quad - \varphi_u(-1, \vartheta\omega, x_1, \zeta(\vartheta\omega, x_1))|_\omega^u + |(V\zeta)(\omega, y)|_\omega
\end{aligned}$$

by definition of $V\zeta$. Here $x_{\pm 1}$ and $\tilde{x}_{\pm 1}$ are determined such that

$$\varphi_c(1, \vartheta_{-1}\omega, x_{-1}, \zeta(\vartheta_{-1}\omega, x_{-1})) = y = \varphi_c(-1, \vartheta\omega, x_1, \zeta(\vartheta\omega, x_1)),$$

$$\begin{aligned}
&\varphi_c(1, \vartheta_{-1}\omega, \tilde{x}_{-1}, \Gamma(\vartheta_{-1}\omega, \tilde{x}_{-1}) + \zeta(\vartheta_{-1}\omega, \tilde{x}_{-1})) = y \\
&\quad = \varphi_c(-1, \vartheta\omega, \tilde{x}_1, \Gamma(\vartheta\omega, \tilde{x}_1) + \zeta(\vartheta\omega, \tilde{x}_1)).
\end{aligned}$$

A combination of the estimates obtained in Lemma 4.2 and Lemma 5.2 with condition ($CM2$) and the Lipschitz conditions satisfied by Γ and ζ will then yield the result if we take into account that $\Gamma \in Y$. Since this type of arguments is now familiar to us we do not have to go into detail (see Boxler [8], Theorem 9.1).

Since T is a contraction on $A_k(L)$ this implies that U is a contraction on Y which has a unique fixed point Γ_0. Hence the definition of U yields: $T(\Gamma_0 + \zeta) = \Gamma_0 + \zeta$. Therefore the assertion follows because of the uniqueness of the fixed point of T. □

8.2. Explicit Calculation of an Approximation

Since the existence proof given in the last subsection is not constructive we will show now how to derive an explicit polynomial approximation up to second order. The same procedure works for higher orders as well but since the generalization does not contain any new ideas we prefer omitting the lengthy calculations.

By construction $h(\omega, y)=\frac{1}{2}D^2 h(\omega, 0)(y, y)+O(\|y\|^3)$ for any $y\in E_c(\omega)$. In order to calculate the second order derivative we make use of the following consequence of the chain rule (see e.g. Abraham, Marsden and Ratiu [1], p. 92):

$$D^2(g\circ f)(0)(v, w)=D^1 g(f(0))\, D^2 f(0)(v, w)+D^2 g(f(0))(D^1 f(0)\, v, D^1 f(0)\, w).$$

Furthermore we take into account that $\Phi_{c,s,u}(t, \omega, 0)=D^1 \Phi_{c,s,u}(t, \omega, 0)=0$. The expression we obtain is used to calculate $D^2 h_s(\vartheta_{-1}\omega, 0)(\Psi_c(-1, \omega)v, \Psi_c(-1, \omega)v)$. The cocycle property enables us to iterate this procedure such that we may write:

$$\begin{aligned} D^2 h_s(\omega, 0)(v, v) &= \Psi_s(n, \vartheta_{-n}\omega)\, D^2 h_s(\vartheta_{-n}\omega, 0)(\Psi_c(-n, \omega)v, \Psi_c(-n, \omega)v) \\ &\quad + \sum_{k=0}^{n-1} \Psi_s(k, \vartheta_{-k}\omega)\, D^2_{x_c}\Phi_s(1, \vartheta_{-k-1}\omega, 0, 0)(\Psi_c(-k-1, \omega)v, \Psi_c(-k-1, \omega)v). \end{aligned}$$

Lemma 4.2 together with assumption $(CM1)(\varepsilon_0, \varepsilon_1, \varepsilon_2)$ implies that

$$|D^2 h_s(\omega, 0)(v, v)|^s_\omega \leqq \frac{L}{2}\, e^{(\lambda_s+2\beta)n}+\sum_{k=0}^{n-1} \varepsilon_2\, e^{(\lambda_s+2\beta)k}.$$

Hence we have shown, after a repetition of the same procedure for the unstable part:

Theorem 8.2. *Let $h\in A_k(L)$ define a stochastic center manifold $(k\geqq 2)$ for the random dynamical system φ. Then for almost all $\omega\in\Omega$ and any $y\in E_c(\omega)$ $h(\omega, y)$ may be written:*

$$\begin{aligned} h(\omega, y) &= \tfrac{1}{2}\sum_{n=0}^{\infty} [\Psi_s(n, \vartheta_{-n}\omega)\, D^2_{x_c}\Phi_s(1, \vartheta_{-n-1}\omega, 0, 0)\,(\Psi_c(-n-1, \omega)y, \Psi_c(-n-1, \omega)y) \\ &\quad + \Psi_u(-n, \vartheta_n\omega)\, D^2_{x_c}\Phi_u(-1, \vartheta_{n+1}\omega, 0, 0)(\Psi_c(n+1, \omega)y, \Psi_c(n+1, \omega)y)]+O(\|y\|^3). \end{aligned}$$

Remark. An expression which is more closely related to the usual Taylor series expansion in the deterministic case may be obtained from an interpretation of $D^2\Phi_{s,u}(\pm 1, \vartheta_k\omega, 0, 0)$ as a real quadratic form described by a symmetric linear operator, which can be transformed into a diagonal one. See Gantmacher [19], X, § 5, p. 308.

8.3. Examples

a) Real noise:

We consider an ordinary differential equation disturbed by real noise:

$$\dot{x}_t = A(\xi_t(\omega))\, x_t + f(\xi_t(\omega), x_t), \qquad x(0)=x_0\in\mathbb{R}^2,$$

where $A(\cdot):=\begin{pmatrix} a(\cdot) & 0 \\ 0 & b(\cdot) \end{pmatrix}$, $E\,a=0$, $E\,b<0$, $f\left(\cdot,\begin{pmatrix} u \\ v \end{pmatrix}\right):=\begin{pmatrix} 0 \\ c(\cdot)\,u^2 \end{pmatrix}$, with integrable functions a, b, c and a measurable stationary ergodic process $\xi_t(\cdot)$.

Put $\alpha(t,\omega):=\int_0^t a(\xi_\tau(\omega))\,d\tau$, $\beta(t,\omega):=\int_0^t b(\xi_\tau(\omega))\,d\tau$. Then the Lyapunov exponents of the system above are $\lambda_c=E\,a=0$, $\lambda_s=E\,b<0$ (see, e.g., Crauel [13], p. 244). Since

$$\varphi(t,\omega,x)=\Psi(t,\omega)\,x+\int_0^t \Psi(t-s,\omega)\,f(\varphi(s,\omega,x))\,ds, \qquad \Psi(t,\omega)=\begin{pmatrix} e^{\alpha(t,\omega)} & 0 \\ 0 & e^{\beta(t,\omega)} \end{pmatrix}$$

we obtain the following approximation:

$$h(\omega,y)=\left[\sum_{n=0}^{\infty} e^{\beta(n,\vartheta_{-n}\omega)}\left(\int_0^1 e^{\beta(1-s,\vartheta_{-n-1}\omega)}\,e^{2\alpha(s,\vartheta_{-n-1}\omega)}\,c(\xi_s(\vartheta_{-n-1}\omega))\,ds\right)\cdot e^{2\alpha(-n-1,\omega)}\right]y^2+O(\|y\|^3), \qquad y\in E_c(\omega).$$

b) White noise:
We consider the following two-dimensional system:

$$d\,u_t=a\,u_t\,dt+\sigma_1\,u_t\circ dW_1(t),$$

$$d\,v_t=(b\,v_t+c\,u_t^2)\,dt+\sigma_2\,v_t\circ dW_2(t),$$

where $\sigma_1>0$ and $\sigma_2>0$ describe the noise intensities and where W_1 and W_2 are supposed to be independent Wiener processes. We require the Lyapunov exponents to be $=0$ and <0, resp. This yields $a=0$, $b<0$.

It is easily checked that

$$\Phi_s(t,\omega,x_c,x_s)=c\,x_c^2\int_0^t e^{b(t-s)+\sigma_2 W_2(t-s)(\omega)}\,e^{2\sigma_1 W_1(s)(\omega)}\,ds \quad \text{and} \quad \Phi_c(t,\omega,x_c,x_s)=0.$$

In order to apply Theorem 5.1 we have to make sure that Φ_s satisfies the boundedness, resp. Lipschitz, conditions required there (alternatively we have to work in a suitable neighborhood of the origin and to restrict ourselves to local results). This will impose conditions on the size of b, σ_1 and σ_2, which we assume in the sequel without stating them in detail. Hence Theorem 8.2 yields:

$$h(\omega,y)=c\left[\sum_{n=0}^{\infty} e^{bn}\,e^{-\sigma_2 W_2(-n)(\omega)}\int_0^1 e^{b(1-s)}\,e^{\sigma_2 W_2(1-s)(\vartheta_{-n-1}\omega)}\,e^{2\sigma_1 W_1(s-n-1)(\omega)}\,ds\right]y^2+O(\|y\|^3).$$

This means that in the case of multiplicative white noise the stochastic center manifold is also influenced by the noise in the stable equation. If we switch off the noise (i.e. put $\sigma_1=\sigma_2=0$) then we will get back the expression for the approximation of the deterministic center manifold.

Finally we make use of the independence of $\vartheta_{-n-1} W_2(1-s)=W_2(-s-n)-W_2(-s-n)$, $W_2(-n)$ and $W_1(-s-n-1)$ for $0\leqq s\leqq 1$ and apply $E\,e^{\sigma(W(t+s)-W(s))}=e^{(\sigma^2/2)|t|}$ in order to calculate the expectation $E\,h(\cdot, y)$. We obtain:

$$E\,h(\cdot,y)=\frac{-c}{b+2\sigma_1^2+\frac{\sigma_2^2}{2}}\,y^2+O(\|y\|^3).$$

The last expression makes sense if $b+2\sigma_1^2+\frac{\sigma_2^2}{2}\neq 0$. But it is immediately checked that otherwise $\Phi_s(t,\cdot,x)$ is not integrable and in this case the expectation will evidently not exist either.

If the expectation exists we realize that it does not coincide with the deterministic center manifold. This means that the influence of the noise cannot be interpreted as a perturbation of an otherwise deterministic situation. The noise produces a new effect which can only be explained within the stochastic framework. Hence the approach suggested by Knobloch and Wiesenfeld [23], who are the first to try to include noise into center manifold theory, is not quite satisfactory because they replace the deterministic center manifold by a normal distribution with mean equal to the deterministic center manifold in order to describe the stochastic situation.

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