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Norming Constants for the Finite Mean Supercritical Bellman-Harris Process

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Summary. Let $\{Z(t)\}$ be a supercritical Bellman-Harris process with offspring distribution $\{p_k\}$ and lifetime distribution G. It is shown that the finiteness of the offspring mean guarantees the existence of norming constants $\{C(t)\}$ such that $\lim_{t\to\infty} Z(t)/C(t)=W$ a.s. for some nondegenerate random variable W. $C(t)$ is the μ-quantile of the distribution function of $Z(t)$, where $q<\mu<1$, q being the extinction probability of the process. As a by-product of the proof, $\{Z(t)/C(t)\}$ is shown to be "asymptotic" Markov. The theory of weakly stable sums of i.i.d. is used to get characterizations of W and $\{C(t)\}$.

1. Introduction

Let $\{Z(t)\}$ be a supercritical age-dependent branching process assuming lifetime distribution G and offspring distribution $\{p_k\}$. We shall assume that G is non-lattice, $G(0)=0$, $Z(0)\equiv 1$, $p_k \neq 1$ for all k and $m=\sum_{k=0}^{\infty} k p_k<\infty$. According to the Bellman-Harris model each particle lives a random length of time having distribution function G and upon death gives rise to a random number of offspring according to distribution $\{p_k\}$. The lifetimes of the particles and their number of offspring are supposed to be mutually independent random variables. We assume that we deal with a right continuous separable version of $\{Z(t)\}$.

It has been noticed by Harris [9] that the vector process $\{\hat{Z}(t)\}$ of the ages of particles alive at time t, i.e. $\hat{Z}(t)=(x_1(t), \ldots, x_{Z(t)}(t))$ is a Markov process. $Z(t)$ is a function of $\hat{Z}(t)$, being the number of components of $\hat{Z}(t)$. This function is measurable, but obviously not one-to-one and $\{Z(t)\}$, unlike $\{\hat{Z}(t)\}$, is not, in general, Markovian. If the lifetime distribution G is exponential, $\{Z(t)\}$ turns out to be Markovian and such a property makes its investigations easier to carry out. In the general case the underlying Markov process can be used to

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derive properties of $\{Z(t)\}$. The study of $\{\hat{Z}(t)\}$ involves the ages variables $\{x_i(t)\}$ which satisfy a kind of law of large numbers: if $Z(x,t)$ denotes the number of particles alive at time t whose ages do not exceed x and $A(x)$ is defined by

$$A(x)=\frac{\int_0^x e^{-\alpha u}[1-G(u)]\,du}{\int_0^\infty e^{-\alpha u}[1-G(u)]\,du}, \tag{1}$$

then $\left\{\frac{Z(x,t)}{Z(t)}\right\}$ converges almost surely to $A(x)$ for any $x>0$. $A(x)$ is called limiting age distributions and α is the so-called Malthusian parameter defined as the unique solution of the equation

$$m\int_0^\infty e^{-\alpha u}\,dG(u)=1. \tag{2}$$

This property of the ages has been derived by Harris [9] under some restrictions on G and assuming $\sum_{k=1}^{\infty} k^2 p_k<\infty$, by Jagers [11] assuming only $\sum_{k=0}^{\infty} k^2 p_k<\infty$, by Athreya and Kaplan [3] in the case $\sum_{k=1} k \log k p_k<\infty$, by Athreya and Kaplan [4] under some restriction on G and assuming $\sum_{k=0}^{\infty} k p_k<\infty$, and finally by Nerman [14] and Kuczek [12] assuming only the finiteness of the offspring mean.

The object of this paper is to investigate the asymptotic behaviour of $\{Z(t)\}$. It was proved by Harris [9] that if $\sum_{k=0}^{\infty} k^2 p_k<\infty$, then $\{Z(t)/e^{\alpha t}\}$ converges almost surely to a random variable W. Athreya and Kaplan [3] have relaxed Harris' condition to $\sum_{k=1}^{\infty} k \log k p_k<\infty$. Harris' proof is based on renewal arguments, while Athreya and Kaplan's proof uses the convergence in distribution of $\{Z(t)/e^{\alpha t}\}$ and $\{V_t/e^{\alpha t}\}$ as well as the martingale property of $\{V_t/e^{\alpha t}\}$ where

$$V_t=\sum_{i=1}^{Z(t)} V(x_i(t)) \tag{3}$$

$$V(x)=m\int_0^\infty e^{-\alpha u}\,dG_x(u) \tag{4}$$

and

$$G_x(y)=\frac{G(x+y)-G(x)}{1-G(x)}. \tag{5}$$

If $\sum_{k=1}^{\infty} k \log k p_k=\infty$, Athreya [1] has proved that $\{Z(t)/e^{\alpha t}\}$ converges in probability to 0. Thus $\{e^{\alpha t}\}$ is not the right normalization in this case and the problem of finding constants $\{C(t)\}$ such that $\{Z(t)/C(t)\}$ converges to a non-degenerate limit has been known as "the Seneta problem". We shall prove here that such constants exist and that $\{Z(t)/C(t)\}$ converges almost surely to a non-degenerate random variable W. This result parallels the Seneta-Heyde theorem for the simple Galton-Watson process (see [16] and [10]) where the constants $\{C_n\}$ were obtained analytically. Such an analytical approach, however, does not seem to carry over to the age-dependent branching processes.

Our result will also provide a unified approach of the cases

$$\sum_{k=1}^{\infty} k \log k p_k < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} k \log k p_k = \infty,$$

since in the former case it will follow that $C(t) \sim e^{\alpha t}$. The idea of the proof consists in defining the norming constants as the quantiles of $\{Z(t)\}$, proving convergence to a non-degenerate limit for subsequences and then extending this convergence to the whole process. The main ingredient of the approach is the convergence of the empirical age distribution. The method used will also reveal the fact that while $\{Z(t)\}$ is not in general Markovian, it is always "asymptotical Markovian", in view of the behaviour of the age process.

The Laplace transform ϕ of the limiting distribution function of $\{Z(t)/C(t)\}$ is shown to satisfy the functional equation

$$\phi(t) = \int_0^{\infty} f(\phi(t e^{-\alpha x}))\, dG(x), \tag{6}$$

where f is the generating function of the offspring distribution $\{p_k\}$. Characterizations of the tail of the distribution of W and of $\{C(t)\}$ are also obtained.

2. Results

Theorem 1. *Let $\{Z(t)\}$ be a Bellman-Harris process with lifetime distribution G, $G(0)=0$, offspring distribution $\{p_k\}$ and $1 < \sum_{k=0}^{\infty} k p_k < \infty$. Then there exist some norming constants $\{C(t)\}$ such that $\lim_{t\to\infty} Z(t)/C(t) = W$ a.s., where W is a random variable assuming distribution function F. F is continuous on $(0, \infty)$ and $F(0)=q$. ϕ, the Laplace transform of F, satisfies the functional equation* (6).

Before giving the next result we need introduce the notion of slow variation: a function L will be said to be slowly varying if for any $\lambda > 0$, $\lim_{x\to\infty} L(\lambda x)/L(x) = 1$.

Theorem 2. *Suppose that the assumptions of Theorem* 1 *are satisfied.*

(i) *If $\sum_{k=1}^{\infty} k \log k p_k < \infty$, then $E(W) < \infty$ and $C(t) \sim e^{\alpha t}$, α being the Malthusian parameter defined by* (2).

(ii) *If $\sum_{k=1}^{\infty} k \log k p_k = \infty$, then $L(x) = \int_0^x P(W > u)\, du$ is a slowly varying function, $E(W) = \infty$, $E(W^\beta) < \infty$ for any $\beta < 1$, and $C(t) \sim e^{\alpha t} L(e^{\alpha t})$.*

3. Proofs

The proofs of the above given results are based on eight lemmas.

We define the norming constants $\{C(t)\}$ to be the μ-quantiles of the variables $\{Z(t)\}$, i.e. $C(t)$ is a number with the property

$$P(Z(t) \leqq C(t)) \leqq \mu < P(Z(t) \leqq C(t)+1) \tag{7}$$

μ will be assumed to lie in the interval $(q, 1)$. Because $\{Z(t)=0\}$ is an increasing sequence of events and

$$\lim_{t\to\infty} P(Z(t)=0) = q = 1 - P(\lim_{t\to\infty} Z(t)=\infty) < 1$$

we conclude that $\lim_{t\to\infty} C(t)=\infty$.

Lemma 1. *From any subsequence of* $\{Z(t)/C(t)\}$, *say* $\{Z(t_n)/C(t_n)\}$, *with* $\lim_{n\to\infty} t_n = \infty$ *one can extract a further subsequence,* $\{Z(t'_n)/C(t'_n)\}$, *converging in distribution to a proper, non-degenerate distribution function* F, *with* $F(0)=P(\lim_{t\to\infty} Z(t)=0)$. ϕ, *the Laplace transform of* F, *satisfies the functional equation*

$$\phi(u) = \int_0^\infty f(\phi(u e^{-\alpha x}))\, dG(x). \tag{8}$$

Proof. By the well-known weak compactness theorem, from any subsequence $\{Z(t'_n)/C(t'_n)\}$ one can extract another subsequence $\{Z(t'_n)/C(t'_k)\}$ converging vaguely to a limit F.

Making use of the decomposition of the Bellman-Harris process according to the first split time l we get

$$Z(t)/C(t) = \sum_{i=1}^{v} Z_i(t-l)/Z_i(t)\, Z_i(t)/C(t) \tag{9}$$

where $\{Z_i(t-l)\}$ and $\{Z_i(t)\}$ are, conditional on v, independent, identically distributed random variables, distributed like $Z(t-l)$ and $Z(t)$ respectively, v is an integer-valued random variable with distribution function $\{p_k\}$ and l is a random variable assuming distribution G, and independent of Z_i's and v.

According to a result by Kuczek (Theorem 3.5.2 of [13]) for any $s>0$, $\lim_{t\to\infty} Z(t-s)/Z(t)=e^{-\alpha s}$ a.s. on the set of non-extinction $\{\lim_{t\to\infty} Z(t)=\infty\}$. This implies that $\lim_{t\to\infty} Z_i(t-l)/Z_i(t)=e^{-\alpha l}$ for almost all $\omega\in\{\lim_{t\to\infty} Z_i(t)=\infty\}$. Taking now the Laplace transform of (9), employing the total probability formula and then taking the limit over t'_n as $n\to\infty$ we get (8).

Letting $u\downarrow 0$ and $u\uparrow\infty$ in (8) we get

$$F(0)=f(F(0)) \quad \text{and} \quad F(\infty)=f(F(\infty)). \tag{10}$$

It is easy to see that

$$F(0) = \lim_{n\to\infty} P(Z(t'_n)/C(t'_n) \leqq x) \leqq \lim_{n\to\infty} P(Z(t'_n)/C(t'_n) \leqq 1) \leqq \mu < 1 \tag{11}$$

where x is a continuity point of F with $0<x<1$. Also

$$\begin{aligned} F(\infty) &\geqq \lim_{n\to\infty} P(Z(t_n')/C(t_n')\leqq x) \\ &\geqq \liminf_{n\to\infty} P(Z(t_n')\leqq C(t_n')+1) \\ &\geqq \mu>0 \end{aligned} \tag{12}$$

where x is a continuity point of F with $x>1$. Since by a well-known result (see [9]) the equation $s=f(s)$ assumes only the solutions $s=q$ and $s=1$ and because $\mu>q$ we get from (10), (11) and (12) that $F(0)=q$ and $F(\infty)=1$, which proves that F is a proper distribution function.

To prove that F is non-degenerate notice that if $\sum_{k=1}^{\infty} k\log k p_k<\infty$ F was proved to be absolutely continuous on $(0,\infty)$ and therefore non-degenerate (see [1]). If $\sum_{k=1}^{\infty} k\log k p_k=\infty$, by a result of Athreya [2] asserting that $\int_0^{\infty} x F(dx)<\infty$ if and only if $\sum_{k=1}^{\infty} k\log k p_k<\infty$, F being a solution to (8), we deduce that F corresponds to a random variable with infinite mean which is therefore non-degenerate and the proof is complete.

Lemma 2. *F is continuous on* $(0,\infty)$.

Proof. Choose $t_n'>t$ for a given t. By the Markov and branching property of $\{\hat{Z}(t)\}$ we get

$$P(Z(t_n')/C(t_n')\leqq x\,|\,\mathfrak{F}_t)=P\left(\sum_{i=1}^{Z(t)} Z(t_n', x_i(t))\leqq C(t_n')\,x\,|\,\mathfrak{F}_t\right) \tag{13}$$

where $Z(t_n', x_i(t))$ is the number of individuals at time t_n' in the line of descent of the individual aged $x_i(t)$ at time t and $\mathfrak{F}_t=\sigma(\hat{Z}(t))$. $Z(t_n', x_1(t)), \dots, Z(t_n', x_{Z(t)}(t_n'))$ are, conditional on $Z(t)$, independent random variables.

For $\omega\in\{Z(t_n')>0\}$ we can write

$$P\left(\sum_{i=1}^{Z(t)} Z(t_n', x_i(t))\leqq C(t_n')\,x\,|\,\mathfrak{F}_t\right)=P\left(\sum_{i=1}^{Z(t)} Z(t_n', x_i(t))/Z(t_n')\,Z(t_n')/C(t_n')\leqq x\,|\,\mathfrak{F}_t\right). \tag{14}$$

Notice that the i-th individual alive at time t was born at time $t-x_i(t)$ and by time t_n' his line of descent has evolved for time $t_n'+(t-x_i(t))$. Using this and again Theorem 3.5.2 of [13] when passing to limit in (14) we get that for any continuity point x of F

$$\lim_{n\to\infty} P(Z(t_n')/C(t_n')\leqq x\,|\,\mathfrak{F}_t)=P\left(\sum_{i=1}^{Z(t)} e^{\alpha(x_i(t)-t)}\,W_i\leqq x\,|\,\mathfrak{F}_t\right) \tag{15}$$

for almost all $\omega\in\{Z(t)>0\}$, where $\{W_i\}$ are, conditional on $\mathfrak{F}_t$, independent, identically distributed random variables, distributed according to F and independent of $\{x_i(t)\}$.

Assume now that F has a jump point x_0. Then there exist positive numbers $\{\varepsilon_n\}$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that

$$\lim_{n\to\infty} P(Z(t_n')/C(t_n') \in (x_0 - \varepsilon_n, x_0 + \varepsilon_n)) = F(x_0) - F(x_0 -) = \delta > 0. \tag{16}$$

By (15) we get

$$\lim_{n\to\infty} P(Z(t_n')/C(t_n') \in (x_0 - \varepsilon_n, x_0 + \varepsilon_n) \mid \mathfrak{F}_t) = P\left(\sum_{i=1}^{Z(t)} e^{\alpha(x_i(t)-t)} W_i = x_0 \mid \mathfrak{F}_t\right) \tag{17}$$

for almost all $\omega \in \{Z(t) > 0\}$.

Notice that we can always find some constants $\{B(t)\}$ with $\lim_{t\to\infty} B(t) = \infty$ such that $\lim_{t\to\infty} P(Z(t) \in (1, B(t))) = 0$. Then (16) and (17) yield

$$\delta = \lim_{t\to\infty} \int_{\{Z(t) > B(t)\}} P\left(\sum_{i=1}^{Z(t)} e^{\alpha(x_i(t)-t)} W_i = x_0 \mid \mathfrak{F}_t\right) dP. \tag{18}$$

We shall prove that (18) with $\delta > 0$ is impossible by using some properties of the concentration functions of sums of independent random variables. Write $Q(X, x) = \sup_y P(y < X \leqq y + x)$ for the concentration function of X. It is well known that

$$Q(S_n, L) \leqq \frac{CL}{\lambda \left[\sum_{i=1}^{n} (1 - Q(X_i, 2\lambda)\right]^{1/2}}$$

where $S_n = X_1 + \ldots + X_n$, $\{X_i\}$ being independent random variables, C is a universal constant and λ an arbitrary number in the interval $(0, L)$ (see e.g. [15]). It is easy to see that for $Q(S_n, L)$ to be bounded away from 0 it is necessary that $Q(X_i, 2\lambda)$ goes to 1 as $i \to \infty$. However this is not the case for the sums that appear on the right hand side of (18). Indeed, if we assume that $q > 0$, then $Q(e^{\alpha(x(t)-t)} W_i, L) \leqq \max(q, 1-q)$ in view of the fact that $e^{\alpha(x(t)-t)} > 0$. If $q = 0$ we can take the logarithm of $e^{\alpha(x(t)-t)} W_i$ which turns out to be a sum of two independent random variables $\alpha(x(t)-t) + \log W_i$. It is known that the concentration function of a sum of independent variables is smaller than the concentration function of one of its components, say $\log W_i$. Since by Lemma 1 W_i is a non-degenerate random variable, this property implies that the concentration function of $\log\{e^{\alpha(x(t)-t)} W_i\}$ stays away from 1 and so does the concentration function of $e^{(x(t)-t)\alpha} W_i$. Thus

$$\lim_{t\to\infty} P\left(\sum_{i=1}^{Z(t)} e^{\alpha(x_i(t)-t)} W_i = x_0 \mid \mathfrak{F}_t\right) = 0 \quad \text{a.s.}$$

and (18) is invalidated. We have got a contradiction that finishes the proof.

Corollary. *Suppose that* $\{p_k\} \in \mathfrak{C}$, *where* $\mathfrak{C}$ *is the class of finite mean offspring distributions for which the functional equation* (8) *assumes a unique solution up to a scale factor. Then* $\{Z(t)/C(t)\}$ *converges in distribution to* F.

Proof. By Lemma 1 any sequence $\{t_n\}$ contains a subsequence $\{t'_n\}$ such that $\{Z(t'_n)/C(t'_n)\}$ converges in distribution to a distribution function F satisfying (8). By Lemma 2, F is continuous and this property in conjunction with (7) implies $\lim_{t\to\infty} P(Z(t)\leqq 1)=\mu$. Thus the limits in distribution of subsequences $\{Z(t'_n)/C(t'_n)\}$ cannot differ by a scale factor and the proof is complete.

Remark. It has been proved by Harris [9] that $\mathfrak{C}$ contains the $\{p_k\}$'s for which $\sum_{k=1}^{\infty} k^2 p_k<\infty$. Athreya [1] has shown that this restriction can be relaxed to $\sum_{k=1}^{\infty} k\log k p_k<\infty$, but it has remained an open problem whether such a result holds in the case $\sum_{k=1}^{\infty} k\log k p_k=\infty$ and $\sum_{k=1}^{\infty} k p_k<\infty$. We shall circumvent the uniqueness problem by proving in a different way the convergence of $\{Z(t)/C(t)\}$. Our result seems to lend additional credence to the conjecture that uniqueness extends here as well.

Lemma 3. *For any* $s>0$, $\lim_{t\to\infty} C(t+s)/C(t)=e^{\alpha s}$.

Proof. Notice first that by Lemma 2, 1 is a continuity point for F and if we take into account that the reasoning for F is valid for any subsequence $\{t_n\}$ we get by (7) that $\lim_{t\to\infty} P(Z(t)\leqq C(t))=\mu$. We next show that it can always be assumed that $(F(1+\varepsilon)-F(1))(F(1)-F(1-\varepsilon))>0$ for all $\varepsilon>0$. Indeed, if this is not the case, by Lemmas 1 and 2 there must exist a point w such that $(F(w+\varepsilon)-F(w))(F(w)-F(w-\varepsilon))>0$ for any $\varepsilon>0$. Let $\{C'(t)\}$ be the constants defined by (7) for $\mu=F(w)$. By Lemma 1, $F(o)=q$ and $F(\infty)=1$ and therefore one must have

$$0<\liminf_{t\to\infty} C(t)/C'(t)\leqq\limsup_{t\to\infty} C(t)/C'(t)<\infty.$$

Thus we can choose a subsequence of $\{t'_n\}$, say $\{t''_n\}$, such that $\lim_{n\to\infty} C(t''_n)/C'(t''_n)=b$ for some positive b. Then

$$\lim_{n\to\infty} P(Z(t''_n)/C'(t''_n)\leqq x)=\lim_{n\to\infty} P(Z(t''_n)/C(t''_n)\, C(t''_n)/C'(t''_n)\leqq x)=F(x/b).$$

Putting $x=1$ in this equality we get $F(1/b)=F(w)$, where from $1/b=w$. Thus, if necessary, by a change from $F(x)$ to $F(xw)$ we can always arrange to have a limit distribution F for a subsequence of $\{t_n\}$ such that $(F(1)-F(1-\varepsilon))(F(1+\varepsilon)-F(1))>0$ for any $\varepsilon>0$.

Fix now $s>0$ and denote $\psi(t)=C(t+s)/C(t)$. We shall complete the proof by showing that for any subsequence $\{t_n\}$ one can extract another subsequence $\{t''_n\}$ such that $\psi(t''_n)\to e^{\alpha s}$ as $n\to\infty$.

By Lemma 1 and Theorem 3.5.2 of [13] we get

$$\lim_{t\to\infty} P(e^{-\alpha s} Z(t'_n+s)/C(t'_n)\leqq x)=\lim_{t\to\infty} P(e^{-\alpha s} Z(t'_n+s)/Z(t'_n)\, Z(t'_n)/C(t'_n)\leqq x)=F(x) \quad (19)$$

for any $x>0$. However by Lemma 1 one can extract from $\{t'_n+s\}$ another subsequence, say $\{t''_n+s\}$, such that $\{Z(t''_n+s)/C(t''_n+s)\}$ converges in distribution to a proper non-degenerate law H. By (19) and a well known property by Khintchine concerning the uniqueness, up to an equivalence, of norming constants of this kind we get that $C(t''_n+s)\sim K e^{\alpha s} C(t''_n)$ where the symbol $\sim$ links numbers whose ratio tends to 1 as $n\to\infty$.

This implies by (19) that $F(K^{-1})=H(1)$. However by (7), $F(1)=H(1)$ and in view of the properties of F at 1 we conclude that $K=1$. Thus $\lim_{n\to\infty} \psi(t''_n)=e^{\alpha s}$ and the proof is complete.

Lemma 4. *Suppose that $\mathfrak{F}'_t$ is the σ-algebra generated by $\{\hat{Z}(s)\colon s\leqq t\}$. Then for any continuity point x of F*

$$\lim_{n\to\infty} P(Z(t'_n)/C(t'_n)\leqq x\,|\,\mathfrak{F}'_t)=P(\xi_1(t)\,\tilde{W}_1+\ldots+\xi_{Z(t)}(t)\,\tilde{W}_{Z(t)}\leqq x\,e^{\alpha t}\,|\,\mathfrak{F}_t) \tag{20}$$

almost surely, where $\xi_i(t)=e^{-\alpha y_i(t)}$, y_i being the excess life of the i-th particle alive at time t, i.e. the time elapsed from t to the death of the particle aged $x_i(t)$ at time t; $\tilde{W}_1,\ldots,\tilde{W}_{Z(t)}$ are identically distributed random variables assuming the distribution F_1, where $F_1(x)=\sum_{k=0}^{\infty} F^{*(k)}(x)\,p_k$, $F^{*(0)}(x)=1$ and $F^{*(k)}(x)$, for $k\geqq 1$, is the k-th convolution of F. $\tilde{W}_1,\ldots,\tilde{W}_{Z(t)}$ are, conditional on $\mathfrak{F}_t$, independent random variables.

Proof. Choose a number n such that

$$t'_n>\max_{1\leqq i\leqq Z(t)} (x_i(t)+y_i(t))$$

$\max_{1\leqq i\leqq Z(t)} (x_i(t)+y_i(t))$ is random, but since G is a proper distribution function it must be finite with probability one. By the Markov and branching properties of $\{\hat{Z}(t)\}$ we get

$$\begin{aligned} P(Z(t'_n)/C(t'_n)\leqq x\,|\,\mathfrak{F}'_t)&=P(Z(t'_n,x_1(t))+\ldots+Z(t'_n,x_{Z(t)}(t))\\ &\qquad\leqq C(t'_n)\,x\,|\,\mathfrak{F}_t)\quad \text{a.s.} \end{aligned} \tag{21}$$

where $Z(t'_n,x_i(t))$ is the number of objects at time t'_n in the line of descent of the particle aged $x_i(t)$ at time t. $Z(t'_n,x_1(t)),\ldots,Z(t'_n,x_{Z(t)}(t))$ are, conditional on $\mathfrak{F}_t$, independent random variables.

Denote now by $l_1,\ldots,l_{Z(t)}$ the death times of the $Z(t)$ particles alive at time t. Then we can write

$$\begin{aligned} &P(Z(t'_n,x_1(t))+\ldots+Z(t'_n,x_{Z(t)}(t))\leqq C(t'_n)\,x\,|\,\mathfrak{F}_t)\\ &\quad=P\left(\sum_{i=1}^{Z(t)}\frac{Z(t'_n,x_i(t))}{C(t'_n-(l_i+t))}\,\frac{C(t'_n-(l_i+t))}{C(t'_n-l_i)}\,\frac{C(t'_n-l_i)}{C(t'_n)}\leqq x\,|\,\mathfrak{F}_t\right). \end{aligned} \tag{22}$$

Taking in (22) the limit as $n\to\infty$, using Lemma 2(i) and (ii), and (21) we get (20). The distribution function F_1 of the limiting random variable of

$$\left\{\frac{Z(t'_n,x_i(t))}{C(t'_n-(l_i+t))}\right\}$$

is accounted by the fact that upon death, at time l_i+t, the i-th particle gives rise to a number of offspring according to the distribution $\{p_k\}$ and each line of descent of these particles accounts, in the limit, for a distribution equal to F. This completes the proof.

Lemma 5. *Let $\eta(t)$ be a random variable defined by the equation*

$$\xi_1(t)\tilde{W}_1+\ldots+\xi_{Z(t)}(t)\tilde{W}_{Z(t)}=\eta(t)(\tilde{W}_1+\ldots+\tilde{W}_{Z(t)}) \tag{23}$$

Then

$$0\leqq\eta(t)\leqq 1 \quad \text{and} \quad \lim_{t\to\infty} E(\eta(t)|\mathfrak{F}_t)=\frac{n_1}{m} \quad \text{a.s.,}$$

where

$$n_1=\int_0^\infty e^{-\alpha u}(1-G(u))\,du/m\int_0^\infty u\,e^{-\alpha u}\,dG(u).$$

Proof. Since $\xi_1(t),\ldots,\xi_{Z(t)}(t)$ are positive and smaller than 1, the Eq. (23) must be satisfied for a certain variable $\eta(t)$ with $0\leqq\eta(t)\leqq 1$.

Consider now the ratio

$$\frac{\xi_1(t)\tilde{W}_1+\ldots+\xi_{Z(t)}(t)\tilde{W}_{Z(t)}}{\tilde{W}_1+\ldots+\tilde{W}_{Z(t)}} \tag{24}$$

which will be defined as equal to 1 in the case when the denominator equals 0. Such a situation occurs if and only if $W_1=\ldots=W_{Z(t)}=0$, in which case the numerator is also 0. Thus (24) equals $\eta(t)$ and its expectation turns out to be

$$\begin{aligned}E(\eta(t)|\mathfrak{F}_t)=&E\left(\frac{\tilde{W}_1}{\tilde{W}_1+\ldots+\tilde{W}_{Z(t)}}\middle|\mathfrak{F}_t\right)E(\xi_1(t)|\mathfrak{F}_t)\\&+E\left(\frac{\tilde{W}_{Z(t)}}{\tilde{W}_1+\ldots+\tilde{W}_{Z(t)}}\middle|\mathfrak{F}_t\right)E(\xi_{Z(t)}(t)|\mathfrak{F}_t).\end{aligned} \tag{25}$$

If we take into account the symmetry of the sequence of, conditional on $\mathfrak{F}_t$, i.i.d. random variables $\{\tilde{W}_n\}$ we get

$$E\left(\frac{\tilde{W}_1}{\tilde{W}_1+\ldots+\tilde{W}_{Z(t)}}\middle|\mathfrak{F}_t\right)=\ldots=E\left(\frac{\tilde{W}_{Z(t)}}{\tilde{W}_1+\ldots+\tilde{W}_{Z(t)}}\middle|\mathfrak{F}_t\right)=\frac{1}{Z(t)}.$$

Thus

$$E(\eta(t)|\mathfrak{F}_t)=\frac{V_t}{m\,Z(t)}.$$

Corollary 2 p. 47 of [3], in view of the a.s. convergence of $\{Z(x,t)/Z(t)\}$, applies and yields $\lim_{t\to\infty}\frac{V_t}{Z(t)}=c_1$ a.s. and this completes the proof.

Lemma 6. *Suppose that $\{X_{ni},\ 1\leqq i\leqq n,\ 1\leqq n<\infty;\ W_j,\ 1\leqq j<\infty\}$ are nonnegative independent random variables, $X_{ni}\leqq 1$ for any n and i, and $\{W_j\}$ are i.i.d.. Write*

$$S_n=\sum_{i=1}^n W_i,\ T_n=\sum_{i=1}^n X_{ni}W_i,\quad V_n=\sum_{i=1}^n X_{ni} \quad \text{and} \quad \Lambda(n)=\{V_n>n\lambda\}$$

for a certain $\lambda>0$. *Then there exists a constant a with* $0<a<\lambda$ *such that for any* $x>0$, $P(\{T_n>x\}\,|\,A(n))\geqq P(a\,S_{[n\nu]}>x)$, *where* $\nu=(\lambda-a)/(1-a)>0$.

Proof. Write $\Gamma_n=\{i: X_{ni}>a\}$ and let β_n be the number of elements in Γ_n. Notice that Γ_n is a random set and β_n is a random variable. The obvious equality $\sum_{i\in\Gamma_n} X_{ni}+\sum_{i\notin\Gamma_n} X_{ni}=V_n$ yields that for $\omega\in A(n)$, $(n-\beta_n)\,a+\beta_n>n\,\lambda$ which entails $\beta_n>n(\lambda-a)/(1-a)\geqq[n\,\nu]$. Thus, if $\mathfrak{G}_n$ is the σ-field generated by the random variables $\{X_{ni},\ 1\leqq i\leqq n\}$, for $\omega\in A(n)$ and $x>0$ we get

$$P(T_n>x\,|\,\mathfrak{G}_n)\geqq P(a\sum_{i\in\Gamma_n} W_i>x\,|\,\mathfrak{G}_n)\geqq P(a\,S_{[n\nu]}>x). \tag{26}$$

The proof can now be finished on integrating (26) on $A(n)$.

The following result, under the slightly more restrictive assumption that $\{X_{ni}\}$ are constants, was given in [6]. The proof, essentially extracted from [6], will be included here for the sake of self-containedness. We use the same notations as in Lemma 6.

Lemma 7. *Suppose that* $\{X_{ni},\ 1\leqq i\leqq n,\ 1\leqq n<\infty;\ W_j,\ 1\leqq j<\infty\}$ *are nonnegative independent random variables,* $X_{ni}\leqq K$ *for any n and i, K being a positive constant, and* $\{W_j\}$ *are i.i.d. Assume further that there exist a sequence of integers* $\{n_k\}$ *and a positive constant c such that* $V_{n_k}/n_k\xrightarrow{P}c$ *as* $k\to\infty$ *and that for a sequence of norming constants* $\{b_k\}$, $S_{n_k}/b_k\xrightarrow{P}\mu$ *as* $k\to\infty$, μ *being a positive constant. Then* $T_{n_k}/b_k\xrightarrow{P}c\,\mu$ *as* $k\to\infty$ *uniformly with respect to the class of random variables* $\{X_{ni}\}$ *defined above (i.e.* $P(|T_{n_k}/b_k-c\,\mu|>\varepsilon)$ *tends to* 0 *uniformly for any* $\varepsilon>0$.)

Proof. Consider the ratios W_i/S_{n_k}, $1\leqq i\leqq n_k$ which for $S_{n_k}=0$ will be defined to equal 1. This happens if and only if $W_1=\ldots=W_{n_k}=0$, so that the symmetry of the joint distribution of $\{W_i/S_{n_k},\ i=1,\ldots,n_k\}$ known for strictly positive variables $\{W_j\}$ is preserved by this definition. Thus

$$E(T_{n_k}/c\,S_{n_k})=1/c\sum_{i=1}^{n_k}E(X_{n_k i})\,E(W_i/S_{n_k})=1/c\,n_k\sum_{i=1}^{n_k}E(X_{n_k i}).$$

Since $\{X_{ni}\}$ are uniformly bounded and assumed to have the property $V_{n_k}/n_k\xrightarrow{P}c$ as $k\to\infty$ we get that

$$\lim_{k\to\infty}E(T_{n_k}/c\,S_{n_k})=1.$$

Notice now that

$$T_{n_k}/S_{n_k}=\sum_{i=1}^{n_k}X_{n_k i}(W_i/S_{n_k})\leqq\sum_{i=1}^{n_k}W_i/S_{n_k}=K.$$

Thus $\{T_{n_k}/c\,S_{n_k}\}$ is a uniformly bounded sequence of random variables and therefore $T_{n_k}/c\,S_{n_k}\xrightarrow{P}1$ if and only if $E(T_{n_k}/c\,S_{n_k}-1)^2\to 0$ as $k\to\infty$.

Write now $\delta_n=E(W_1/S_n)^2$ and $\gamma_n=E(W_1\,W_2/S_n^2)$. Then

$$E(T_{n_k}/c\,S_{n_k}-1)^2=\frac{1}{c^2}\left[\delta_{n_k}\sum_{i=1}^{n_k}E(X_{n_k i}-c)^2+\gamma_{n_k}\sum_{i\neq j}\sum E(X_{n_k i}-c)\,E(X_{n_k j}-c)\right]. \tag{27}$$

Since $1=E(S_{n_k}/S_{n_k})^2=n_k\,\delta_{n_k}+n_k(n_k-1)\,\gamma_{n_k}$ and all the quantities in this equality are nonnegative we get $\gamma_{n_k}<1/n_k(n_k-1)$. Further

$$|\gamma_{n_k}\sum_{i\neq j}\sum E(X_{n_k i}-c)\,E(X_{n_k j}-c)|$$
$$\leqq E^2\left(\sum_{i=1}^{n_k}(X_{n_k i}-c)\right)\Big/ n_k(n_k-1)+E\left(\sum_{i=1}^{n_k}(X_{n_k i}-c)^2\right)\Big/ n_k(n_k-1)$$

which because $V_{n_k}/n_k \xrightarrow{P} c$ as $k\to\infty$ and the bounded convergence theorem tends to 0 as $k\to\infty$.

Thus, by (27), to complete the proof it will suffice to show that $\delta_{n_k} n_k\to 0$ as $k\to\infty$. To that aim let us choose a particular sequence $\{X_{ni}\}$, say $\{X'_{ni}\}$, defined as $X'_{ni}=2$ or 0 according as i is even or odd. Write T'_{n_k} for T_{n_k} with $X'_{n_k i}$ in place of $X_{n_k i}$. It is easy to see that $c'=1$ and that $T'_{n_k}/S'_{n_k}\xrightarrow{P} 1$ as $k\to\infty$. Therefore $E(T'_{n_k}/S'_{n_k}-1)^2\to 0$ as $k\to\infty$. Finally if we write (27) for T'_{n_k} we necessarily get that $\delta_{n_k} n_k\to 0$ as $k\to\infty$ and the proof is complete.

Lemma 8. (i) $(n_1/m)(\tilde W_1+\ldots+\tilde W_{[C(t)]})/e^{\alpha t}\xrightarrow{\tilde P} 1$ *as* $t\to\infty$, *where* $\tilde P$ *is the probability measure corresponding to a sequence of i.i.d. random variables* $\{\tilde W_i\}$ *assuming distribution function* F_1.
(ii) $\{Z(t)/C(t)\}$ *converges in distribution to* F *as* $t\to\infty$.

Proof. We choose two arbitrary sequences of positive numbers $\{t_n\}$ and $\{s_n\}$ with $\lim_{n\to\infty} t_n=\lim_{n\to\infty} s_n=\infty$ and then extract a subsequence of $\{t_n\}$, say $\{t'_n\}$, such that $\{Z(t'_n)/C(t'_n)\}$ converges in distribution to F, and a subsequence of $\{s_n\}$, say $\{s'_n\}$, such that $\{Z(s'_n)/C(s'_n)\}$ converges in distribution to H. We shall prove that $F\equiv H$ and this will clearly imply (ii). Since the proof of (ii) involves expressions of the type considered in (i), (i) and (ii) will result simultaneously.

Let us recall that F and H are proper non-degenerate distributions (by Lemma 1) and continuous on $(0,\infty)$ (by Lemma 2). Choose now a number $y>0$ with the property $H(y+\delta_1)-H(y-\delta_2)>0$ for any $\delta_1,\delta_2>0$ and write

$$A_u=\{Z(u)/C(u)\in(y-\delta_2,y+\delta_1)\}\quad\text{and}\quad T_{Z(u)}=\xi_1(u)\,\tilde W_1+\ldots+\xi_{Z(u)}\,\tilde W_{Z(u)}.$$

By Lemma 4 we get for any $x>0$.

$$\lim_{n\to\infty} P(\{Z(t'_n)/C(t'_n)>x\}\cap A_{s'_k})=\int_{A_{s'_k}} P(T_{Z(s'_k)}>x\,e^{\alpha s'_k}\,|\,\mathfrak{F}_{s'_k})\,dP \tag{28}$$

and

$$\lim_{n\to\infty} P(\{Z(t'_n)/C(t'_n)>x\}\cap A_{s'_k+s})=\int_{A_{s'_k+s}} P(T_{Z(s'_k+s)}>x\,e^{\alpha(s'_k+s)}\,|\,\mathfrak{F}_{s'_k+s})\,dP. \tag{29}$$

Lemma 2 and the continuity of H imply

$$\lim_{k\to\infty} P(A_{s'_k}\,\Delta\,A_{s'_k+s})=0 \tag{30}$$

for any $s>0$, Δ being the symbol of symmetric difference of two sets.

(28), (29), (30) and Lemma 5 yield

$$\lim_{k\to\infty}\Big[\int_{A_{s'_k}} P(T_{Z(s'_k)}>x\,e^{\alpha s'_k}\,|\,\mathfrak{F}_{s'_k})\,dP - \int_{A_{s'_k+s}} P(\eta(s'_k+s)\,S_{Z(s'_k+s)}>x\,e^{\alpha(s'_k+s)}\,|\,\mathfrak{F}_{s'_k+s})\,dP\Big]=0 \tag{31}$$

where $S_n=\tilde{W}_1+\ldots+\tilde{W}_n$.

Extract now from $\{s'_k\}$ a further subsequence, say $\{s''_k\}$, such that $\{S_{[C(s''_k)]}/e^{\alpha s''_k}\}$ converges vaguely, under $\tilde{P}$, to a limit distribution D. From (28) through (31) one gets

$$\lim_{n\to\infty} P(\{Z(t'_n)/C(t'_n)>x\}\cap A_{s''_k}) \geqq \int_{A_{s''_k+s}} P(\eta(s''_k+s)\,S_{[C(s''_k+s)(y-\delta_1)]}/e^{\alpha s''_k}\,e^{\alpha s}>x\,|\,\mathfrak{F}_{s''_k+s})\,dP \tag{32}$$

(32) will help us prove that D is a proper distribution and assumes finite mean. Indeed, by Lemma 3, $C(s''_k+s)(y-\delta_1)\sim e^{\alpha s}\,C(s''_k)(y-\delta_1)$. If we choose s such that $e^{\alpha s}$ is an integer the limit in distribution of $S_{[C(s''_k+s)(y-\delta_1)]}/e^{\alpha s''_k}\,e^{\alpha s}$ as $k\to\infty$, under $\tilde{P}$, turns out to be $S^*_{[e^{\alpha s}(y-\delta_1)]}/e^{\alpha s}$, where S^*_n is the sum of n i.i.d. copies of a random variable assuming distribution D. Since s is at our disposal we can take $s\to\infty$ and get by the law of large numbers that $\{S^*_{e^{\alpha s}/e^{\alpha s}}\}$ converges almost surely to a (possibly infinite) constant v. This implies that $\{S^*_{[e^{\alpha s}(y-\delta_1)]/e^{\alpha s}}\}$ converges almost surely to $(y-\delta_1)\,v$. We show next that v is finite. Since $\lim_{k\to\infty} E(\eta(s''_k+s)\,|\,\mathfrak{F}_{s''_k+s})=n_1$ a.s. and the variables $\{\eta(s''_k+s)\}$ are positive and bounded by 1 we deduce that there must exist β with $0<\beta<n_1$ and $\gamma>0$ such that

$$\liminf_{k\to\infty} P(\eta(s''_k+s)>\beta\,|\,\mathfrak{F}_{s''_k+s})>\gamma \quad \text{a.s.}$$

Indeed, this is a simple consequence of the elementary inequality $E(X)\leqq z+P(X\geqq z)$ for any random variable X with $0\leqq X\leqq 1$ and $0<z\leqq 1$.

It follows that

$$\liminf_{k\to\infty} P(\eta(s''_k+s)\,S_{[C(s''_k+s)(y-\delta_1)]}/e^{\alpha s''_k}\,e^{\alpha s}>x\,|\,\mathfrak{F}_{s''_k+s})>\gamma \quad \text{a.s.} \tag{33}$$

If $v=\infty$, (32) and (33) imply that for any $x>0$

$$\liminf_{k\to\infty}\lim_{n\to\infty} P(\{Z(t'_n)/C(t'_n)>x\}\cap A_{s''_k})>\gamma(H(y+\delta_2)-H(y-\delta_1)) \tag{34}$$

and this inequality contradicts the fact that F is a proper distribution function.

We shall next prove that D is the distribution of a positive constant v. From (28) and (31) we get

$$\limsup_{k\to\infty}\lim_{n\to\infty} P(\{Z(t'_n)/C(t'_n)>x\}\cap A_{s'_k})$$

$$\leq \limsup_{k\to\infty} \int_{A_{s_k''+s}} P(\eta(s_k''+s)\, S_{[C(s_k''+s)(y+\delta_2)]}/e^{\alpha s_k''}\, e^{\alpha s} > x \mid \mathfrak{F}_{s_k''+s})\, dP$$
$$\leq \limsup_{k\to\infty} \int_{A_{s_k''+s}} P(S_{[C(s_k''+s)(y+\delta_2)]}/e^{\alpha s_k''}\, e^{\alpha s} > x \mid \mathfrak{F}_{s_k''+s})\, dP$$
$$= \tilde{P}(S^*_{[e^{\alpha s}(y+\delta_2)]}/e^{\alpha s} > x)\,(H(y+\delta_2)-H(y-\delta_1)). \tag{35}$$

Since $S^*_{e^{\alpha s}}/e^{\alpha s} \xrightarrow{\text{a.s.}} v$ as $s\to\infty$ with v finite we deduce that $S^*_{[e^{\alpha s}(y+\delta_2)]}/e^{\alpha s} \xrightarrow{\text{a.s.}} v(y+\delta_2)$. Thus for $x > v(y+\delta_2)$ the last term in (35) is null and

$$\lim_{k\to\infty} \lim_{n\to\infty} P(\{Z(t_n')/C(t_n') > x\} \cap A_{s_k''}) = 0 \tag{36}$$

for $x > v(y+\delta_2)$. (28) and (36) together imply

$$\lim_{k\to\infty} \int_{A_{s_k''}} P(T_{Z(s_k'')}/e^{\alpha s_k''} > x \mid \mathfrak{F}_{s_k''})\, dP = 0. \tag{37}$$

Assume, by way of contradiction, that $D(x) > 0$ for any $x > 0$. If we apply Lemma 6 with $\lambda = n_1/m - \varepsilon$, ε being a positive number with $n_1/m - \varepsilon - a > 0$ we get

$$\int_{A_{s_k''}} P(T_{Z(s_k'')}/e^{\alpha s_k''} > x \mid \mathfrak{F}_{s_k''})\, dP$$
$$\geq \int_{A_{s_k''}} P(\{T_{Z(s_k'')}/e^{\alpha s_k''} > x\} \cap \Lambda(Z(s_k'')) \mid \mathfrak{F}_{s_k''})\, dP$$
$$\geq \int_{A_{s_k''}} \tilde{P}(a\, S_{[C(s_k'')(y-\delta_1)v]}/e^{\alpha s_k''} > x)\, P(\Lambda(Z(s_k'') \mid \mathfrak{F}_{s_k''})\, dP. \tag{38}$$

Since $E(\xi_i(t) \mid \mathfrak{F}_t) = V(x_i(t))/m$,

$$E\left(\sum_{i=1}^{Z(t)} \xi_i(t) \mid \mathfrak{F}_t\right)/Z(t)) = V_t/m Z(t) \xrightarrow{\text{a.s.}} n_1/m$$

as $t\to\infty$. By the law of large numbers

$$P\left(\left|\sum_{i=1}^{Z(t)} \xi_i(t)/Z(t) - n_1/m\right| > \varepsilon \mid \mathfrak{F}_t\right) \to 0 \quad \text{a.s.}$$

for any $\varepsilon > 0$. This implies that $P(\Lambda(Z(s_k'') \mid \mathfrak{F}_{s_k''}) \xrightarrow{\text{a.s.}} 1$ as $k\to\infty$ and by (38)

$$\liminf_{k\to\infty} \int_{A_{s_k''}} P(T_{Z(s_k'')}/e^{\alpha s_k''} > x \mid \mathfrak{F}_{s_k''})\, dP$$
$$\geq \lim_{k\to\infty} \tilde{P}(a S_{[C(s_k'')(y-\delta_1)v]}/e^{\alpha s_k''} > x)\,(H(y+\delta_2)-H(y-\delta_1)). \tag{39}$$

Recalling that $\{S_{[C(s_k'')]}/e^{\alpha s_k''}\}$ converges in distribution to D under $\tilde{P}$ and since the sum $S_{[C(s_k'')]}$ consists of i.i.d. random variables we get that $D(x) > 0$ for any $x > 0$ implies that the last term in (39) is positive for any $x > 0$ which contradicts (37). Thus D has finite interval support and since it is infinitely divisible it has to be constant (see [8] p. 177). We have therefore proved that D is the distribution of a positive constant v.

We shall next prove that v does not depend on the choice of the subsequence $\{s_n\}$. To that aim notice first that we can always assume that $(F(1+\varepsilon)-F(1))(F(1)-F(1-\varepsilon))>0$ for all $\varepsilon>0$, by a reasoning already used in the proof of Lemma 2.

Lemma 4 and the total probability formula yield

$$\begin{aligned} F(1+\delta) &= \lim_{k\to\infty} \int P(T_{Z(s_k'')}/e^{\alpha s_k''}<1+\delta \mid \mathfrak{F}_{s_k''})\, dP \\ &= \lim_{k\to\infty} \Big(\int_{B_k}+\int_{C_k}+\int_{D_k}\Big) \end{aligned} \tag{40}$$

where

$$B_k=\{Z(s_k'')/C(s_k'')\leqq (v n_1/m)(1+\delta)-\varepsilon\},$$
$$C_k=\{(v n_1/m)(1+\delta)-\varepsilon<Z(s_k'')/C(s_k'')\leqq (v n_1/m)(1+\delta)+\varepsilon\}$$

and

$$D_k=\{Z(s_k'')/C(s_k'')>(v n_1/m)(1+\delta)+\varepsilon\}.$$

By Lemma 7 we get

$$\lim_{k\to\infty} [P(T_{Z(s_k'')}/e^{\alpha s_k''}<1+\delta \mid \mathfrak{F}_{s_k''})\, 1_{B_k}-1_{B_k}]=0 \quad \text{a.s.} \tag{41}$$

where 1_{B_k} is the indicator of the set B_k, and

$$\lim_{k\to\infty} [P(T_{Z(s_k'')}/e^{\alpha s_k''}<1+\delta \mid \mathfrak{F}_{s_k''})\, 1_{D_k}=0 \quad \text{a.s.} \tag{42}$$

Notice that by the continuity of H

$$\lim_{\varepsilon\to 0}\limsup_{k\to\infty} \int_{C_k} P(T_{Z(s_k'')}/e^{\alpha s_k''}<1+\delta \mid \mathfrak{F}_{s_k''})\, dP=0 \tag{43}$$

and that (41), (42) and (43) used in (40) yield $F(1+\delta)=H((v n_1/m)(1+\delta))$. Since we can then take $\delta=\pm\eta$ with $\eta>0$, $F(1-\eta)<F(1)<F(1+\eta)$ and because by Lemma 2, F and H are continuous we get $F(1)=H(v n_1/m)$. But $F(1)=H(1)$ by the definition of $\{C(t)\}$. It follows that $v n_1/m=1$. Thus $v=m/n_1$ and since v is independent of the subsequence $\{s_n\}$, Lemma 8 (i) follows. On the other hand, $v=m/n_1$ implies $F(x)=H(x)$ for any $x>0$ and this finishes the proof.

Proof of Theorem 1. We shall first prove that for an arbitrary sequence $\{t_n\}$, $\{Z(t_n)/C(t_n)\}$ converges almost surely. By Lemma 8(ii), $\{Z(t)/C(t)\}$ converges in distribution to F. This fact allows one to replace t_n by t in (28), s_k'' by s in (40), (41), (42) and (43) and to deduce that

$$F(x)=\lim_{t\to\infty}\lim_{s\to\infty} P(\{Z(t)/C(t)\leqq x\}\cap\{Z(s)/C(s)\leqq x\}).$$

If we use the notation $A_n(x)=\{Z(t_n)/C(t_n)\leqq x\}$, we realize that we have a situation identical to that described in the proof of Theorem 1 of [5], p. 75. Thus, we can conclude that there exists an event $A(x)$ such that

$$A(x)=\lim_{n\to\infty}\{Z(t_n')/C(t_n')\leqq x\} \quad \text{a.s.}$$

for a subsequence $\{t'_n\}$ of $\{t_n\}$, where a.s. convergence for a sequence of events $\{C_n\}$ to C is defined by the property $P(C \Delta \limsup_{n\to\infty} C_n) = P(C \Delta \liminf_{n\to\infty} C_n) = 0$. By Lemma 4 and Lemma 7 we get

$$\lim_{n\to\infty} P(A(x)|\mathfrak{F}'_{t_n}) = \lim_{n\to\infty} P(m/n_1(\tilde{W}_1 + \ldots + \tilde{W}_{Z(t_n)})/e^{\alpha t_n} \leqq x | \mathfrak{F}_{t_n}). \tag{44}$$

It is easy to see that by the Markov property of $\{\hat{Z}(t)\}$,

$$P(m/n_1(\tilde{W}_1 + \ldots + \tilde{W}_{Z(t_n)})/e^{\alpha t_n} \leqq x | \mathfrak{H}_{t_n}) \\ = P(m/n_1(\tilde{W}_1 + \ldots + \tilde{W}_{Z(t_n)})/e^{\alpha t_n} \leqq x | \mathfrak{F}_{t_n}) \quad \text{a.s.}$$

where $\mathfrak{H}_{t_n}$ is the σ-field generated by the random variable $Z(t_n)$. Since by the martingale convergence theorem $\lim_{n\to\infty} P(A(x)|\mathfrak{F}'_{t_n}) = 1_{A(x)}$ a.s. we conclude that

$$\lim_{n\to\infty} P(m/n_1(\tilde{W}_1 + \ldots + \tilde{W}_{Z(t_n)})/e^{\alpha t_n} \leqq x | \mathfrak{H}_{t_n}) = 1_{A(x)} \quad \text{a.s.}. \tag{45}$$

We have reached a situation of the kind considered in the proof of Theorem 1 of [5], (45) being a sort of asymptotic Markov property that suffices to prove a.s. convergence. In view of Lemma 8(i) and Remark 1 p. 80 of [5] the conditions of Theorem 1 of [5] are satisfied and we conclude that $\{Z(t_n)/C(t_n)\}$ converges almost surely. We can take $\{n\delta, n \geqq 1\}$ for the sequence $\{t_n\}$ for any $\delta > 0$. The almost sure convergence of $Z(t+s)/Z(t)$ to $e^{\alpha s}$ as $t \to \infty$ implies $\lim_{\delta\to 0} Z((n+1)\delta)/Z(n\delta) \to 1$ a.s. $n \to \infty$ which in conjunction with Lemma 3 yields that $\{Z(t)/C(t)\}$ converges almost surely as $t \to \infty$.

Proof of Theorem 2. According to a result by Athreya [2], $\sum k \log k p_k < \infty$ if and only if $E(W) < \infty$, W being a random variable whose distribution satisfies the functional equation (6). But $E(W) < \infty$ implies $E(\tilde{W}) = mE(W) < \infty$ and combining Lemma 8(i) with the classical law of large numbers we get $C(t) \sim e^{\alpha t}$ and (i) is proved.

On the other hand $E(W) = \infty$ in the case $\sum k \log k p_k = \infty$ is an immediate consequence of (6) and the above mentioned result by Athreya [2].

Choose now a sequence of positive numbers $\{t_n\}$ with $C(t_n) = n$, $n \geqq 1$. Lemma 8(i) implies

$$n_1/m(\tilde{W}_1 + \ldots + \tilde{W}_n)/e^{\alpha t_n} \xrightarrow{P} 1 \qquad \text{as } n \to \infty \tag{46}$$

property which is usually referred to as relative stability of the sequence $\{\tilde{W}_i\}$. By a result of Feller ([8], Theorem 2, p. 236, see also the footnote) the relative stability of $\{\tilde{W}_i\}$ is equivalent to the slow variation of $L_1(x) = \int_0^x u\,dF_1(u)$. But

$$L_1(x) = \sum_{k=0}^{\infty} \int_0^x u\,dF^{*(k)}_{(u)} p_k = \sum_{k=0}^{\infty} F^k_{(x)} k p_k \int_0^x u\,dF(u) \sim mL(x)$$

on using the Laplace transform. Thus $L(x)$ is also slowly varying and by the same result of Feller mentioned above $\{W_i\}$ turns out to be relatively stable.

The characterization of the norming constants given by the formula (7.12) p. 236 of [8] yields in this case that $C(t_n)=e^{\alpha t_n}/L(m/n_1 e^{\alpha t_n})$. Since

$$C(t+s)/C(t)\to e^{\alpha s} \quad \text{as } t\to\infty$$

we conclude that the gaps between t_{n-1} and t_n in the definition of $C(t)$ can be filled by taking $C(t)=e^{\alpha t}/L(m/n_1 e^{\alpha t})\sim e^{\alpha t}/L(e^{\alpha t})$ for $t>0$.

Finally the proof is completed by noticing that $L(x)\leqq x(1-F(x))$ implies $E(W^\beta)<\infty$ for any $\beta<1$.

The property $E(W^\beta)<\infty$ for $\beta<1$ parallels a result given for the simple branching processes by Dubuc [7]. An alternative proof can be found in [17]. The methods of [7] and [17], however, do not seem to apply in the setting of a Bellman-Harris process.

Remark. Most of the proofs given in this paper were required by the case $\sum k\log kp_k=\infty$. If we assume $\sum k\log kp_k<\infty$, then a much simpler proof can be devised as an alternative to the one given by Athreya [1] and Athreya and Kaplan [3]. Indeed, by the Corollary after Lemma 2, Athreya's result [2] concerning the unicity of the solution to the functional equation (6) combined with Lemma 3 yield convergence in distribution for $\{Z(t)/C(t)\}$. One can further use Lemma 4 to deduce that $C(t)\sim e^{\alpha t}$, since if $E(W_1)<\infty$ the classical law of large numbers implies that $\{\xi_1\tilde{W}_1+\ldots+\xi_{[e^{\alpha t}]}\tilde{W}_{[e^{\alpha t}]}/e^{\alpha t}\}$ converges in probability to a finite constant. Further, Lemmas 2, 5, 6, 7 and 8(i) are no longer necessary in this case and (40) can be proved easily. The almost sure convergence can be proved as in the proof of Theorem 1, or, alternatively, one can use the martingale $\{V_t/e^{\alpha t}\}$ as in [3] to complete the proof.

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