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A Renewal Theorem and Its Applications to Some Sequential Procedures

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Summary. A renewal theorem of the elementary type for some stopping times which arise from some statistical estimation problems has been established. It is applied to prove the asymptotic risk efficiency for the problem of estimating the mean when the loss function is a weighted sum of the squared error and the sample size, and the variance is unknown. It is also applied to verify a conjecture of Robbins and Siegmund (1974) on the evaluation of the variance of the estimator of the logarithm of the odds ratio for a sequential procedure.

1. Introduction and Summary

Let (X, Y), (X_1, Y_1), ... be independent and identically distributed (i.i.d.) random vectors with $EX=\mu\in(0,\infty)$ and $EY=0$. Let $\{\xi_n, n\geqq 1\}$ be a sequence of uniformly bounded random variables. Assume that a_n and b_n are constants with $b_n=o(1)$ and $n^{-\alpha}a_n=1+o(1)$ as n tends to infinity for some $\alpha>0$. For some real constant c assume that $h(t)$ is a function continuous in some neighborhood of c. For each $n\geqq 1$, put

$$U_n=\bar{X}_n+h(c+\bar{Y}_n\xi_n)\,\bar{Y}_n^2+b_n, \tag{1}$$

where $\bar{X}_n=n^{-1}(X_1+\ldots+X_n)$ and similarly for $\bar{Y}_n$. For each $\lambda>0$, define

$$N=N_\lambda=\inf\{n\geqq n_\lambda\colon U_n\geqq(a_n\lambda)^{-1}\}, \tag{2}$$

where $n_\lambda=O(\lambda^{-1/\alpha})$ as $\lambda\to 0$.

The formulation of (1) and (2) has been motivated by a sequential procedure for estimating the mean when the variance is unknown, and the work of Lai and Siegmund (1977, 1979) on the nonlinear renewal theory. The main results, Theorems 1, 2 and 3, will be summarized in this section and their proofs will be given in the next two sections.

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Theorem 1. *Assume that* $E(X^-)^p < \infty$ *and* $E|Y|^p < \infty$ *for some* $p \geqq 1$. *Then* $\{(\lambda^{1/\alpha} N)^p,\ 1 \geqq \lambda > 0\}$ *is uniformly integrable. If moreover* $\limsup_{\lambda \to 0} \lambda^{1/\alpha} n_\lambda < \mu^{-1/\alpha}$, *then as* $\lambda \to 0$

$$E|\lambda^{1/\alpha} N - \mu^{-1/\alpha}|^p \to 0.$$

Corollary 1. *Under the same conditions as Theorem* 1 *and assume that* $Y=0$, $b_n = 0$ *and* $n_\lambda = 1$. *Then* $\{(\lambda^{1/\alpha} N)^p,\ 1 \geqq \lambda > 0\}$ *is uniformly integrable and as* $\lambda \to 0$

$$E|\lambda^{1/\alpha} N - \mu^{-1/\alpha}|^p \to 0.$$

For $0 < \alpha \leqq 1$ Corollary 1 has been established by Siegmund (1967) and Gut (1974). Their method does not seem to be able to handle the case when $\alpha > 1$.

Corollary 2. *Let* $Z, Z_1, \ldots$ *be independent and normally distributed with a common mean* $\theta \neq 0$ *and variance* 1. *For* $a > 0$, $r > 0$, *define*

$$T = \inf\{n \geqq 1 : (Z_1 + \ldots + Z_n)^2 \geqq 2a(r+n)\}.$$

Then for every $p \geqq 1$ $\{(a^{-1} T)^p,\ a \geqq 1\}$ *is uniformly integrable, and as* $a \to \infty$

$$E(T/2a)^p \to |\theta|^{-2p}. \tag{3}$$

Proof. Notice that $\bar{Z}_n^2 = (\bar{Z}_n - \theta)^2 + 2\theta(\bar{Z}_n - \theta) + \theta^2$. Put $X_n = \theta^2 + 2\theta\ (Z_n - \theta)$, $Y_n = Z_n - \theta$, $a_n = n^2/(r+n)$, $b_n = 0$, $\lambda = (2a)^{-1}$ and $h(t) = 1$ for all t. Then $\bar{Z}_n^2 = \bar{X}_n + \bar{Y}_n^2$ and T is simply N in (2) with $n_\lambda = 1$. $\{(a^{-1} T)^p,\ a \geqq 1\}$ is uniformly integrable by Theorem 1, or alternatively by Corollary 1, noting that $T \leqq \inf\{n \geqq 1 : \bar{X}_n \geqq (\lambda a_n)^{-1}\}$. Since $T/2a \to \theta^{-2}$ a.s. as $a \to \infty$ (Lai and Siegmund (1977)), (3) follows.

When $p = 1$, Corollary 2 is established by Lai and Siegmund (1977), using a different method.

Corollary 3. *Let* $Z, Z_1, \ldots$ *be i.i.d. random variables with mean* $\theta \in (0,1)$ *and* $E|Z|^p < \infty$ *for some* $p \geqq 1$. *Put* $S_n = Z_1 + \ldots + Z_n$. *For* $\alpha > 0$, $\lambda > 0$ *define* $N = \inf\{n \geqq 1 : S_n(n - S_n) \geqq n^{(2-\alpha)}/\lambda\}$. *Then* $\{(\lambda^{1/\alpha} N)^p,\ 1 \geqq \lambda > 0\}$ *is uniformly integrable, and as* $\lambda \to 0$.

$$E(\lambda^{1/\alpha} N)^p \to (\theta(1-\theta))^{-p/\alpha}. \tag{4}$$

Proof. Put $X_n = \theta^2 + (1 - 2\theta) Z_n$ and $Y_n = Z_n - \theta$. Then N can be written as $\inf\{n \geqq 1:\ \bar{X}_n - \bar{Y}_n^2 \geqq (n^\alpha \lambda)^{-1}\}$. By Theorem 1, $\{(\lambda^{1/\alpha} N)^p,\ 1 \geqq \lambda > 0\}$ is uniformly integrable. By a standard argument using the strong law of large numbers and the definition of N, as $\lambda \to 0$

$$\lambda^{1/\alpha} N \to (\theta(1-\theta))^{-1/\alpha} \text{ a.s.}$$

Hence (4) follows.

When Z has a Bernoulli distribution and $\alpha = 1$ or 2 N is the stopping time studied by Robbins and Siegmund (1974).

Corollary 4. *Let* $Z, Z_1, \ldots$ *be i.i.d. random variables. For each* $\lambda > 0$, *put*

$$N = \inf\left\{n \geqq n_\lambda :\ n^{-1} \sum_1^n (Z_i - \bar{Z}_n)^2 + b_n \geqq (a_n \lambda)^{-1}\right\},$$

where a_n and b_n are constants such that $b_n = o(1)$ and $n^{-\alpha} a_n = 1 + o(1)$ as $n \to \infty$ for some $\alpha > 0$, and $n_\lambda = O(\lambda^{-1/\alpha})$ as $\lambda \to 0$. Then for all $p > 0$, $\{(\lambda^{1/\alpha} N)^p, 1 \geqq \lambda > 0\}$ is uniformly integrable. If moreover $EZ = \theta$, $\operatorname{Var} Z = \sigma^2 \in (0, \infty)$ and $\limsup_\lambda \lambda^{1/\alpha} n_\lambda < \sigma^{-2/\alpha}$ then as $\lambda \to 0$

$$E(\lambda^{1/\alpha} N)^p \to \sigma^{-2p/\alpha}. \tag{5}$$

Proof. Put $X_n = (Z_n - \theta)^2$, $Y_n = Z_n - \theta$ and $h(t) = -1$ for all t. For $G > 0$ put $Z'_n = \max(-G, \min(G, Z_n))$ and choose G so that $E(Z'_n - EZ'_n)^2 > 0$. By a result of Chow and Studden (1969)

$$\begin{aligned} \bar{X}_n - \bar{Y}_n^2 &= n^{-1} \sum_1^n (Z_i - \bar{Z}_n)^2 \geqq n^{-1} \sum_1^n (Z'_i - \bar{Z}'_n)^2 \\ &= n^{-1} \sum_1^n (Z'_i - EZ'_i)^2 - \left(n^{-1} \sum_1^n (Z'_i - EZ'_i)\right)^2. \end{aligned}$$

Put $X'_n = (Z'_n - EZ'_n)^2$, $Y'_n = Z'_n - EZ'_n$ and

$$N' = \inf\{n \geqq_\lambda : \bar{X}'_n - (\bar{Y}'_n)^2 + b_n \geqq (a_n \lambda)^{-1}\}.$$

Then $N' \leqq N$ and since X' and Y' are bounded, by Theorem 1 $\{(\lambda^{1/\alpha} N')^p, 1 \geqq \lambda > 0\}$ is uniformly integrable for any $p > 0$ and hence so is $\{(\lambda^{1/\alpha} N)^p, 1 \geqq \lambda > 0\}$. From the definition $N < \infty$ and $N \to \infty$ a.s. as $\lambda \to 0$. And also

$$N^{-1} \sum_1^N (Z_i - \bar{Z}_N)^2 \geqq (a_N \lambda)^{-1}. \tag{6}$$

On $A \equiv \{N > n_\lambda\}$

$$(N-1)^{-1} \sum_1^{N-1} (Z_i - \bar{Z}_{N-1})^2 \leqq (a_{N-1} \lambda)^{-1}. \tag{7}$$

Since $n^{-\alpha} a_n = 1 + o(1)$ (6) and (7) give

$$\lim_\lambda ((N^\alpha \lambda)^{-1} - \sigma^2) I_A = 0, \text{ a.s.}$$

Since $\limsup_\lambda \lambda^{1/\alpha} n_\lambda < \sigma^{-2/\alpha}$, it follows that

$$\lim \lambda^{1/\alpha} N = \sigma^{-2/\alpha}. \tag{8}$$

Hence (5) follows.

Corollary 4 is rather surprising since the uniform integrability of $\{(\lambda^{\frac{1}{\alpha}} N)^p, 1 \geqq \lambda > 0\}$ does not require any moment conditions.

Theorem 2. *Let $Z, Z_1, \ldots$ be i.i.d. random variables with mean θ and variance $\sigma^2 \in (0, \infty)$. For each $\lambda > 0$, put*

$$N = N_\lambda = \inf\{n \geqq n_\lambda : n^{-1} \sum_1^n (Z_i - \bar{Z}_n)^2 \geqq (n^{2/\delta} \lambda)^{-1}\}, \tag{9}$$

where $\delta > 0$ *and* $\limsup_\lambda \lambda^{\delta/2} n_\lambda < \sigma^{-\delta}$. *Then as* $\lambda \to 0$, *for any* $r > 0$

$$\lambda^{\delta/2} N \to \sigma^{-\delta} \quad \text{a.s. and} \quad E(\lambda^{\delta/2} N)^r \to \sigma^{-r\delta}. \tag{10}$$

If moreover $E|Z|^{2p} < \infty$ *for* $p = 1 + \delta$, *then as* $\lambda \to 0$

$$E\lambda^{-\delta/2}(\bar{Z}_N - \theta)^2 \to \sigma^{2+\delta}. \tag{11}$$

Consequently as $\lambda \to 0$

$$E\left(\frac{\sigma^{-2-2\delta}(\bar{Z}_N - \theta)^2 + \lambda^\delta N}{2\lambda^{\frac{\delta}{2}}\sigma^{-\delta}}\right) \to 1. \tag{12}$$

Theorem 2 and Corollary 4 give the performance, in terms of the asymptotic risk efficiency, of the following sequential procedure for the estimation of the mean θ with a loss function $L_n = \sigma^{2\beta-2}(\bar{Z}_N - \theta)^2 + cn$ for some known constant $\beta \neq 0$, and c which is often allowed to go to 0. The optimal sample size when σ is known is $n_0 = \inf\{n \geqq 1: n \geqq c^{-1/2}\sigma^\beta\}$ or $n_0 + 1$, and the minimum risk is approximately $2c^{1/2}\sigma^\beta$ which is the denominator of (12) with $\lambda = c^{1/\delta}$ and $\delta = -\beta$ for the negative β case. The sequential procedure derived from the replacement of an unknown σ by its estimator, as suggested by Anscombe (1952) and Robbins (1959), has a sample size (9) and the numerator of (12) as its risk. Thus (12) gives the asymptotic risk efficiency. The case $\beta > 0$ has been settled in Chow and Yu (1981). See their article for further references about this problem.

Theorem 3. *Let* $Z, Z_1, \ldots$ *be i.i.d. random variables with* $\Pr[Z=1] = p = 1 - \Pr[Z=0]$, $0 < p < 1$ *and* $q = 1 - p$. *For each* $n \geqq 1$, *put* $S_n = Z_1 + \ldots + Z_n$. *For* $\alpha > 0, \lambda > 0$, *put*

$$N = N_\lambda = \inf\{n \geqq 1: S_n(n - S_n) \geqq \lambda^{-1} n^{2-\alpha}\}, \tag{13}$$

and

$$U = U_\lambda = (pq)^{(\alpha-1)/2\alpha} \lambda^{-1/2\alpha}(\log(S_N + a)/(N - S_N + a) - \log p/q) \tag{14}$$

where a *is a nonnegative constant. Then for any* $r > 0$, $\{|U|^r, 1 \geqq \lambda > 0\}$ *is uniformly integrable. Furthermore if* m_r *is the r-th absolute moment of a standard normal distribution, then*

(i) *as* $\lambda \to 0$, $E|U|^r \to m_r$, *and* $E(\log(S_N + a)/(N - S_N + a) - \log p/q)^k = o(\lambda^{k/2\alpha})$, *for any positive odd integer* k, *and*

(ii) $E|(pq)^{(\alpha-1)/2\alpha}\lambda^{-1/2\alpha}(S_N/N - p)|^r \to m_r$, *and* $E(S_N/N - p)^k = o(\lambda^{k/2\alpha})$, *for any positive odd integer* k.

Consequently as $\lambda \to 0$,

$$EU \to 0 \quad \text{and} \quad \operatorname{Var} U \to 1, \tag{15}$$

$$\lambda^{-1/2\alpha} E(S_N/N - p) \to 0 \quad \text{and} \quad \lambda^{-1/\alpha}\operatorname{Var}(S_N/N) \to (pq)^{(1-\alpha)/\alpha}. \tag{16}$$

When $\alpha = 1$, (13) and (14) have been studied by Robbins and Siegmund (1974), who have conjectured (15). They have also remarked that the asymptotic behavior of (13) for the case when $\alpha = 2$ can be treated with analogous results.

2. Proof of the Main Result

We shall need the following lemmas for the proof of Theorem 1.

Lemma 1. *Let $\{Z_n, n \geqq 1\}$ be a sequence of i.i.d. finite-dimensional random vectors and F_n be the σ-algebra generated by $\{Z_i, i \leqq n\}$ for each $n \geqq 1$. Let t be a finite F_n-stopping time and $t^{(1)}, t^{(2)}, \ldots$ be the copies of t. Put $t_0 = 0$ and $t_n = t^{(1)} + \ldots + t^{(n)}$, and $V_n = (t^{(n)}, Z_{t_{n-1}+1}, \ldots, Z_{t_n})$. Then $\{V_n, n \geqq 1\}$ are i.i.d. random vectors.*

The proof of this lemma follows standard arguments. For details see Chow and Teicher (1978), p. 136.

Lemma 2. *Let $Y, Y_1, \ldots$ be i.i.d. random variables, $S_n = Y_1 + \ldots + Y_n$ and $p \geqq 1$.*

(i) *Assume that $EY > 0$, $c \geqq 0$ and $T = \inf\{n \geqq 1: S_n \geqq c\}$. Then $ET^p < \infty$ if $E(Y^-)^p < \infty$; and $\{(c^{-1} S_T)^p, c \geqq 1\}$ is uniformly integrable if $E(Y^+)^p < \infty$.*

(ii) *If $EY = 0$, $E|Y|^p < \infty$ and t is a stopping time with $Et^{p/2} < \infty$ and $Et < \infty$, then $E|S_t|^p < \infty$.*

(iii) *If $E|Y|^p < \infty$ and t is a stopping time with $Et^p < \infty$, then $E|S_t|^p < \infty$.*

(iv) *If $EY = 0$, $E|Y|^{2p} < \infty$ and $\{T(\lambda), 1 \geqq \lambda > 0\}$ is a family of stopping times such that $\{(\lambda^{1/\alpha} T(\lambda))^p, 1 \geqq \lambda > 0\}$ is uniformly integrable where $\alpha > 0$, then $\{\lambda^{p/\alpha} |S_{T(\lambda)}|^{2p}, 1 \geqq \lambda > 0\}$ is uniformly integrable.*

(i) and (ii) are due to Gut (1974). (iii) follows from (i) and (ii). (iv) is due to Chow and Yu (1981).

Lemma 3. *Let (Y, Z), $(Y_1, Z_1), \ldots$ be i.i.d. random vectors with $EY = 0$, $Z \geqq 0$ and $EZ > 0$. Let $W_n = Y_1 + \ldots + Y_n$ and $V_n = Z_1 + \ldots + Z_n$. Define $M = \inf\{n \geqq 1: |W_n| \leqq V_n\}$. Assume that $E|Y|^p < \infty$ for some $p \geqq 1$. Then $EM^p < \infty$.*

Proof. Let $\tau = \inf\{n \geqq 1: W_n \leqq V_n\}$. Since $E(Z - Y) > 0$ and $E(Y^+)^p < \infty$, $E\tau^p < \infty$ by Lemma 2(i). Let $\tau^{(1)}, \tau^{(2)}, \ldots$ be the copies of τ, $\tau_0 = 0$, $\tau_n = \tau^{(1)} + \ldots + \tau^{(n)}$, $W_n' = Y_{\tau_{n-1}+1} + \ldots + Y_{\tau_n}$ and $V_n' = Z_{\tau_{n-1}+1} + \ldots + Z_{\tau_n}$. Then by Lemma 1, $\{(W_n', V_n'), n \geqq 1\}$ are i.i.d. with $EW_n' = EYE\tau = 0$ and $EV_n' = EZE\tau > 0$. Also $E|W_n'|^p < \infty$ by Lemma 2(ii) or (iii). Put

$$t = \inf\left\{n \geqq 1: \sum_1^n W_i' \geqq -\sum_1^n V_i'\right\}.$$

Since $E(W_n' + V_n') > 0$ and $E((W_n')^-)^p < \infty$, again by Lemma 2(i) $Et^p < \infty$. And

$$-V_{\tau_t} = -\sum_1^t V_i' \leqq \sum_1^t W_i' = W_{\tau_t} \leqq V_{\tau_t}.$$

Therefore $M \leqq \tau_t$. By Lemma 2(iii), $E(\tau_t)^p < \infty$ and it follows that $EM^p < \infty$.

Lemma 3 is the key to our proof of Theorem 1 and is interesting itself.

Proof of Theorem 1. We break the proof into the following six steps.

Step 1. Since $b_n = o(1)$ and $n^{-\alpha} a_n = 1 + o(1)$, and $\mu > 0$, there is a positive integer K such that $|b_n| < \mu/4$ and $n^\alpha < 2a_n$ for all $n \geqq K$. Put $n_\lambda' = \max(K, n_\lambda)$ and $X_n' = X_n - \mu/4$. Then $n_\lambda' = O(\lambda^{-1/\alpha})$ as $\lambda \to 0$, $EX' = 3\mu/4 > 0$ and $N \leqq \inf\{n \geqq n_\lambda': \bar{X}_n'$

$+h(c+\bar{Y}_n\xi_n)\,\bar{Y}_n^2 \geqq 2/n^\alpha\lambda\}$, where $\bar{X}'_n = n^{-1}(X'_1+\ldots+X'_n)$. Hence we can assume that $b_n=0$ and $a_n=n^\alpha$, i.e. $N=\inf\{n\geqq n_\lambda\colon \bar{X}_n+h(c+\bar{Y}_n\xi_n)\,\bar{Y}_n^2\geqq 1/n^\alpha\lambda\}$. By normalizing we can assume that $\mu=1$.

Step 2. Put $t=\inf\{n\geqq 1\colon |Y_1+\ldots+Y_n|\leqq n\eta\}$ for some $\eta>0$. By Lemma 3, $E|Y|^p<\infty$ implies that $Et^p<\infty$. Let $t^{(1)}, t^{(2)},\ldots$ be the copies of t, $t_0=0$, $t_n=t^{(1)}+\ldots+t^{(n)}$, $W_n=Y_{t_{n-1}+1}+\ldots+Y_{t_n}$ and $V_n=X_{t_{n-1}+1}+\ldots+X_{t_n}$. By Lemma 1, $\{(t^{(n)}, V_n, W_n), n\geqq 1\}$ are i.i.d. random vectors with $EW_1=EYEt=0$ and $E(V_1^-)^p\leqq E(X_1^-+\ldots+X_t^-)^p<\infty$ by Lemma 2(iii), and $EV_1=EXEt=Et$.

Step 3. Put $\tau=\inf\{n\geqq 1\colon V_1+\ldots+V_n\geqq(t^{(1)}+\ldots+t^{(n)})/2\}$. Since $E(V_1-t/2)=Et/2>0$ and $E((V_1-t/2)^-)^p<\infty$, by Lemma 2(i) $E\tau^p<\infty$. It can easily be seen that τ is a $(t^{(n)}, V_n, W_n)$-stopping time and t_τ an (X_n, Y_n)-stopping time. As before by Lemma 2(iii) $Et_\tau^p<\infty$.

Step 4. Let $\beta^{(1)}, \beta^{(2)},\ldots$ be the copies of t_τ, $\beta_0=0$ and $\beta_n=\beta^{(1)}+\ldots+\beta^{(n)}$. Choose $M>1$, $1/2>\varepsilon>0$ such that $h(x)$ is continuous and $|h(x)|\leqq M$ for $|x-c|\leqq\varepsilon$, and $|\xi_n|\leqq M$ for all $n\geqq 1$. Choose $\eta\leqq\varepsilon/M$. Then $|\bar{Y}_{\beta_n}|\leqq\varepsilon/M$, $\bar{X}_{\beta_n}\geqq 1/2$ and

$$\bar{X}_{\beta_n}+h(c+\bar{Y}_{\beta_n}\xi_n)\,\bar{Y}_{\beta_n}^2\geqq\bar{X}_{\beta_n}-M\eta^2\geqq 1/4. \tag{17}$$

Step 5. Let $T=\inf\{n\geqq 1\colon \lambda\beta_n^\alpha\geqq(4+B)\}$, where $B^{1/\alpha}=\sup_\lambda(\lambda^{1/\alpha}n_\lambda)$. Then $\lambda^{1/\alpha}\beta_T\geqq B^{1/\alpha}\geqq\lambda^{1/\alpha}n_\lambda$. Therefore $\beta_T\geqq n_\lambda$ and it is clear from (17) that $\beta_T\geqq N$. By Lemma 2(i), $\{(\lambda^{1/\alpha}\beta_T)^p, 1\geqq\lambda>0\}$ is uniformly integrable and hence so is $\{(\lambda^{1/\alpha}N)^p, 1\geqq\lambda>0\}$.

Step 6. As in the proof of (8) by the strong law of large numbers, $\lambda^{1/\alpha}N\to 1$ a.s. By uniform integrability $E|\lambda^{1/\alpha}N-1|^p\to 0$.

Remark 1. The $\bar{Y}_n$ in $h(c+\bar{Y}_n\xi_n)$ can be replaced by a more general $\bar{Z}_n$ with $EZ_n=0$ and $E|Z_n|^p<\infty$, where $\{(X_n, Y_n, Z_n), n\geqq 1\}$ are i.i.d. 3-dimensional random vectors. The whole argument in Theorem 1 modified to treat this case will go through by introducing an additional stopping time $\inf\{n\geqq 1\colon |\bar{Z}_n|\leqq\eta\}$ and its copies. The technique here can also be used to handle the case where $\bar{Y}_n^2$ in (1) is replaced by Y_n/n.

Remark 2. The formulation of the class of stopping times is partly motivated by the following problem of the nonlinear renewal theory studied by Lai and Siegmund (1977, 1979). Let $\{Z_n, n\geqq 1\}$ be i.i.d. random variables with mean θ. Let g be a function positive at θ and twice continuously differentiable in a neighborhood U of θ. Lai and Siegmund (1977, 1979) have investigated the asymptotic behavior of the stopping times

$$T=T_b=\inf\{n\geqq 1\colon ng(\bar{Z}_n)\geqq b\}$$

as $b\to\infty$, by linearizing $g(\bar{Z}_n)$ through the Taylor expansion as follows:

$$g(\bar{Z}_n)=g(\theta)+(\bar{Z}_n-\theta)\,g'(\theta)+(\bar{Z}_n-\theta)^2\,g''(W_n)/2$$

for $\bar{Z}_n\in U$, where W_n is between $\bar{Z}_n$ and θ. If we put $X_n=g(\theta)+(Z_n-\theta)\,g'(\theta)$, $Y_n=Z_n-\theta$, $c=\theta$, $h(t)=g''(t)/2$ and $\xi_n=(W_n-\theta)/(\bar{Z}_n-\theta)$ if $\bar{Z}_n\neq 0$, and 0 otherwise, then $g(\bar{Z}_n)=\bar{X}_n+h(c+\bar{Y}_n\xi_n)\bar{Y}_n^2$, with $EX_n=g(\theta)>0$, $EY_n=0$, and $|\xi_n|\leqq 1$, and T then is just a special case of N in (1) with $n_\lambda=1$, $b_n=0$, $\lambda=b^{-1}$ and $a_n=n$. Our method for treating the negligible term $h(c+\bar{Y}_n\xi_n)\bar{Y}_n^2$ is by introducing the first passage times rather than the last times as in Lai and Siegmund (1977, 1979). Hence the moment conditions required are weaker.

3. Applications to Sequential Analysis

We shall apply the main result to prove Theorems 2 and 3 which arise in two problems in statistical sequential analysis. Here Lemma 4 is a result about the uniform integrability of $S_T/T^{\frac{1}{2}}$.

Lemma 4. *Let $Y, Y_1, \ldots$ be i.i.d. random variable with $EY=0$, $E|Y|^{2p}<\infty$ for some $p\geqq 1$ and let $S_n=Y_1+\ldots+Y_n$. Let $\{T(\lambda), 1\geqq\lambda>0\}$ be a family of stopping times such that $\{(\lambda^{1/\alpha}T(\lambda))^p, 1\geqq\lambda>0\}$ and $\{(\lambda^{1/\alpha}T(\lambda))^{-(\beta+\frac{1}{2})2pr/(2p-r)}, 1\geqq\lambda>0\}$ are uniformly integrable, where $\alpha>0$, $\beta\geqq 0$ and $0<r<2p$. If $\lambda^{1/\alpha}T(\lambda)$ converges to some positive constant c in probability as $\lambda\to 0$, then $\{|(\lambda^{\frac{1}{\alpha}}T(\lambda))^{-\beta}S_{T(\lambda)}/T(\lambda)^{\frac{1}{2}}|^r, 1\geqq\lambda>0\}$ is uniformly integrable.*

Proof. We decompose $(\lambda^{\frac{1}{\alpha}}T(\lambda))^{-\beta}S_{T(\lambda)}/T(\lambda)^{\frac{1}{2}}$ into

$$\lambda^{\frac{1}{2}\alpha}S_{T(\lambda)}/c^{(\beta+\frac{1}{2})}+\lambda^{\frac{1}{2}\alpha}S_{T(\lambda)}((\lambda^{\frac{1}{\alpha}}T)^{-(\beta+\frac{1}{2})}-c^{-(\beta+\frac{1}{2})})\equiv V+W, \quad \text{say.}$$

Since $E|Y|^{2p}<\infty$ and $\{(\lambda^{1/\alpha}T(\lambda))^p, 1\geqq\lambda>0\}$ is uniformly integrable (u.i.), by Lemma 2(iv), $\{|V|^{2p}, 1\geqq\lambda>0\}$ is u.i. and a fortiori $\{|V|^r, 1\geqq\lambda>0\}$ is u.i. Hence

$$E|V|^{2p}=O(1).$$

Since $\lambda^{1/\alpha}T(\lambda)\to c$ and $\{(\lambda^{1/\alpha}T(\lambda))^{-(\beta+\frac{1}{2})2pr/(2p-r)}, 1\geqq\lambda>0\}$ is u.i.

$$E|(\lambda^{\frac{1}{\alpha}}T(\lambda))^{-(\beta+\frac{1}{2})}-c^{-(\beta+\frac{1}{2})}|^{2pr/(2p-r)}=o(1).$$

Therefore by Holder's inequality with $t+s=ts$, and choose $s=2p/r$, then

$$\begin{aligned}E|W|^r&\leqq E^{1/s}|\lambda^{1/2\alpha}S_{T(\lambda)}|^{rs}E^{1/t}|(\lambda^{\frac{1}{\alpha}}T(\lambda))^{-(\beta+\frac{1}{2})}-c^{-(\beta+\frac{1}{2})}|^{rt}\\&=O(1)\,o(1).\end{aligned}$$

Hence $\{|W|^r, 1\geqq\lambda>0\}$ is u.i., and the desired result follows.

Proof of Theorem 2. We can assume that $\sigma=1$. Put $\alpha=2/\delta$. As in the proof of (8) by the strong law of large numbers, $\lambda^{1/\alpha}N\to 1$ a.s. By Corollary 4, $\{(\lambda^{1/\alpha}N)^s, 1\geqq\lambda>0\}$ is uniformly integrable (u.i.) for every $s>0$, and $E(\lambda^{1/\alpha}N)^s\to 1$ as $\lambda\to 0$.

From the definition and $p=1+\delta$, (put $q=p/(p-1)$),

$$(\lambda^{1/\alpha}N)^{-2q}\leqq\left(N^{-1}\sum_1^N(Z_i-\theta)^2\right)^p\leqq\sup_n\left(n^{-1}\sum_1^n(Z_i-\theta)^2\right)^p. \tag{18}$$

Since $E|Z|^{2p}<\infty$, by Doob's martingale maximal inequality,

$$E\sup_n\left(n^{-1}\sum_1^n(Z_i-\theta)^2\right)^p<\infty. \tag{19}$$

Hence $\{(\lambda^{\frac{1}{\alpha}}N)^{-2q},\ 1\geqq\lambda>0\}$ is dominated.

If $\beta=\frac{1}{2}$, $r=2$, then $2q=(\beta+\frac{1}{2})2pr/(2p-r)$ and therefore by Lemma 4 with $N=T(\lambda)$,

$$\{\lambda^{-\frac{\delta}{2}}(\bar{Z}_N-\theta)^2,\ 1\geqq\lambda>0\}\quad\text{is uniformly integrable.}$$

By Anscombe's central limit theorem,

$$\{\lambda^{-\frac{\delta}{4}}(\bar{Z}_N-\theta)\}^2\rightarrow^L\{N(0,\sigma^{2+\delta})\}^2.$$

Hence we obtain (11). (12) follows immediately.

Lemma 5. *Let* $Z, Z_1,\ldots$ *be i.i.d. random variables with* $\Pr[Z=1]=p=1-\Pr[Z=0]$, $0<p<1$ *and* $q=1-p$. *For each* $n\geqq1$, *put* $S_n=Z_1+\ldots+Z_n$. *For* $\alpha>0$, $\lambda>0$, *put*

$$N=N_\lambda=\inf\{n\geqq1: S_n(n-S_n)\geqq\lambda^{-1}n^{2-\alpha}\}$$

and

$$U=U_\lambda=(pq)^{(\alpha-1)/2\alpha}\lambda^{-1/2\alpha}\{\log(S_N+a)/(N-S_N+a)-\log p/q\},$$

where a *is a nonnegative constant. Then as* $\lambda\rightarrow0$, U *converges in distribution to a standard normal distribution, (denoted by* $U\rightarrow^L N(0,1)$).

Proof. Put $W_N=\log(S_N+a)/pN$ and $V_N=\log(N-S_N+a)/qN$ as in Corollary 3, $\lambda^{1/\alpha}N\rightarrow(pq)^{-1/\alpha}$ a.s. By the strong law of large numbers, $S_N/N\rightarrow p$ a.s. Therefore for small λ, by series expansion,

$$W_N=\log(1+(S_N-Np)/pN+a/pN)=(S_N-Np+a)(pN)^{-1}(1+o(1)).$$

Similarly

$$V_N=(Np-S_N+a)(qN)^{-1}(1+o(1)).$$

Therefore

$$W_N-V_N=(1/p+1/q)(S_N-Np)/N+o(|S_N-Np+a|/N).$$

Hence by Anscombe's central limit theorem, as $\lambda\rightarrow0$

$$(pqN)^{1/2}(W_N-V_N)\rightarrow^L N(0,1).$$

Now

$$\begin{aligned}U&=(pq)^{(\alpha-1)/2\alpha}\lambda^{-1/2\alpha}(W_N-V_N)\\&=((pq)^{(\alpha-1)/2\alpha}\lambda^{-1/2\alpha}/(pqN)^{1/2})(pqN)^{1/2}(W_N-V_N).\end{aligned}$$

Clearly $((pq)^{(\alpha-1)/2\alpha}\lambda^{-1/2\alpha})/(pqN)^{1/2}\rightarrow1$ a.s. since $\lambda^{1/\alpha}N\rightarrow(pq)^{-1/\alpha}$. Hence the result follows.

Remark 3. When $\alpha=1$, $a=1/2$, the asymptotic normality of U has been established in Robbins and Siegmund (1974). The proof here is slightly different from theirs.

Proof of Theorem 3. We first note that

$$\begin{aligned}(pq)^{(1-\alpha)/2\alpha}U &= \lambda^{-1/2\alpha}\log(S_N+a)/Np-\lambda^{-1/2\alpha}\log(N-S_N+a)/Nq\\ &\equiv V+V', \quad \text{say.}\end{aligned}$$

Put $W=((S_N+a)/N-p)p^{-1}$. Then $V=\lambda^{-1/2\alpha}\log(1+W)$, and since V' is of the same form as V, it suffices to show that $\{|V|^r, 1\geqq\lambda>0\}$ is uniformly integrable (u.i.). Next we observe that for any $s>0$, by Corollary 3, $\{(\lambda^{1/\alpha}N)^s, 1\geqq\lambda>0\}$ is u.i. and $\lambda^{1/\alpha}N\to(pq)^{-1/\alpha}$ a.s., and by the same reasoning as in (18) and (19), $\{(\lambda^{1/\alpha}N)^{-s}, 1\geqq\lambda>0\}$ is u.i. Therefore by Lemma 4

$$\{|\lambda^{-1/2\alpha}(S_N-Np)/N|^s, 1\geqq\lambda>0\} \text{ is u.i.} \tag{20}$$

Now we consider the case when $W<0$. Then $S_N/N<p$ and since $S_N/N>1/N^\alpha\lambda$, it follows that $pN^\alpha\lambda>1$. Hence if $W<0$, $\lambda^{1/2\alpha}|V|\leqq\log pN^\alpha\lambda\leqq pN^\alpha\lambda$. Therefore for any $r>0$, for $0<\varepsilon<1$, by Corollary 3 and then (20) and for $A\equiv[W\leqq-\varepsilon]$

$$\begin{aligned}E|V|^r I_A &\leqq \lambda^{-r/2\alpha}E((N^\alpha\lambda)p)^r I_A\\ &\leqq O(\lambda^{-r/2\alpha})E^{1/2}(N^\alpha\lambda)^{2r}P^{1/2}[Np-S_N\geqq Np\varepsilon]\\ &\leqq O(\lambda^{-r/2\alpha})P^{1/2}[\lambda^{-1/2\alpha}|Np-S_N|/N\geqq p\varepsilon\lambda^{-1/2\alpha}]\\ &\leqq O(\lambda^{-r/2\alpha}\lambda^{s/4\alpha}(p\varepsilon)^{-s/2}E^{1/2}|\lambda^{-1/2\alpha}(Np-S_N)/N|^s\\ &= O(\lambda^{(s-2r)/4\alpha})=o(1)\end{aligned} \tag{21}$$

if $s>2r$. And

$$\begin{aligned}|V|^r I_{[-\varepsilon<W<0]} &= \lambda^{-r/2\alpha}(-\log(1+W))^r I_{[-\varepsilon<W<0]}\\ &\leqq O(\lambda^{-r/2\alpha})(|S_N-Np|/N)^r.\end{aligned} \tag{22}$$

By (20), (21) and (22), $\{|V|^r I_{[W<0]}, 1\geqq\lambda>0\}$ is u.i. For the case $W\geqq0$, since $\log(1+y)\leqq y$ for all $y\geqq0$, for some constant $C=C_{r,a,p}$,

$$\begin{aligned}|V|^r I_{[W\geqq0]} &\leqq |\lambda^{-1/2\alpha}W|^r I_{[W\geqq0]}\\ &\leqq |p^{-1}\lambda^{-1/2\alpha}(S_N-Np+a)/N|^r\\ &\leqq C(|\lambda^{-1/2\alpha}(S_N-Np)/N|^r+\lambda^{r/2\alpha}(\lambda^{1/\alpha}N)^{-r}).\end{aligned}$$

By (20), $\{|\lambda^{-1/2\alpha}(S_N-Np)/N|^r, 1\geqq\lambda>0\}$ is u.i. Since $\{(\lambda^{1/\alpha}N)^{-r}, 1\geqq\lambda>0\}$ is u.i., therefore $\{|V|^r I_{[W\geqq0]}, 1\geqq\lambda>0\}$ is u.i. This completes the proof that $\{|U|^r, 1\geqq\lambda>0\}$ is u.i. By Lemma 5, U obeys the central limit theorem, hence

$$E|U|^r\to m_r, \quad \text{and} \quad E(\log(S_N+a)/(N-S_N+a)-\log p/q)=o(\lambda^{k/2\alpha}),$$

for any positive odd integer. In particular $EU\to0$ and $EU^2\to1$ and $\operatorname{Var}U=EU^2-(EU)^2\to1$. Also by Lemma 4 and Anscombe's central limit theorem,

$$E|\lambda^{-1/2\alpha}(S_N - Np)/N|^r \to (pq)^{(1-\alpha)r/2\alpha} m_r$$

and

$$E(S_N/N - p)^k = o(\lambda^{k/2\alpha}),$$

for any positive odd integer k. In particular $\lambda^{-1/2\alpha} E((S_N/N) - p) \to 0$ and $\lambda^{-1/\alpha} E(S_N/N - p)^2 \to (pq)^{(1-\alpha)/\alpha}$, consequently $\lambda^{-1/\alpha} \operatorname{Var}(S_N/N) \to (pq)^{(1-\alpha)/\alpha}$.

Remark 4. When $\alpha = 1$, $a = 1/2$, (15) has been conjectured by Robbins and Siegmund (1974) in sequentially estimating the probability p and the odds for a Bernoulli distribution. The case $\alpha = 2$ has also been mentioned in their work. For details about motivation and application, see their article.

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