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On the Construction of Hunt Processes from Resolvents

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Let $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a standard process with state space E (locally compact with a countable base) and $(V_\lambda)_{\lambda>0}$ its resolvent. It is known and easy to check that the following relation holds:

(1) $$\lim_{\lambda\to\infty} \lambda V_\lambda f(x) = f(x), \quad \text{for each } f\in\mathscr{C}_c(E) \text{ and each } x\in E.$$

If (K_n) is an increasing sequence of compact sets such that $K_n \subset \mathring{K}_{n+1}$ and $\bigcup_n K_n = E$, it follows from the quasileft-continuity of the process that the sequence of hitting times T_{CK_n} converges to ζ. Therefore if we denote by Γ_λ the family of all λ-excessive functions s which satisfy the inequality $V_\lambda 1 \leqq s$ on the complement of some compact set $K = K(s)$, depending on s, then from Hunt's balayage theorem one easily deduces the following relation:

(2) $$\inf \Gamma_\lambda = 0, \quad \text{for each } \lambda > 0.$$

The aim of this paper is to prove a partial converse. Namely Theorem 4.3 in the text states that each sub-Markov resolvent $(V_\lambda)_{\lambda>0}$ satisfying the properties (1), (2) and the following one

(3) $$V_\lambda \mathscr{C}_c(E) \subset \mathscr{C}_b(E) \quad \text{for each } \lambda > 0,$$

produces a Hunt process. This result generalises the classical theorem which associates a Hunt process to each Feller semigroup (see [1] or [4]).

The paper is divided into five sections. The first section recalls a result about convex cones of functions. Then in Sect. 2 we study the excessive functions under the assumption that the resolvent has a finite potential kernel. This assumption is also kept through Sect. 3 and 4 because it produces much simplification in the exposition, although it is not necessary for the proof of the main result. Two essential properties are presented in Proposition 3.2 and Corollary 3.4. They state that each continuous function with compact support can be approximated in a suitable sense with continuous excessive functions. These properties are substitutes for the fact that each continuous function with compact support could be uniformly approximated with continuous excessive functions in the case of a Feller semigroup.

Section 4 contains the construction of the process. First we use a general theorem about Ray resolvents on locally compact spaces in order to produce a semigroup of kernels. Then the construction of the process follows the lines of Meyer's proof from [4]. Therefore the details are omitted. We emphasise only two points that are different from the classical situation. The first point concerns the existence of right and left limits near ζ. The second point is quasileft-continuity at ζ. Section 5 is somewhat independent of the preceding sections in the sense that it does not use any of the previous results. However the main result of this section is essentially used in Sect. 4. It retakes the construction of a semigroup from a Ray resolvent in the case of a locally compact space.

While the probabilists were satisfied with the construction theorem which associates a Hunt process to each Feller semigroup, other authors working in potential theory were interested in constructing semigroups and processes from resolvents which act on spaces of functions having a more complicated behaviour near infinity. So our paper follows the works of Boboc-Constantinescu-Cornea [8], Hansen [9] and Taylor [7].

We note that J.C. Taylor was the first to study Ray resolvents and Ray processes on locally compact spaces and two of his results (namely Theorems 1.7 and 3.4 from [7]) are generalised here.

Thanks are do to K. Janssen who pointed out the error from [6], which was the starting point of this paper.

1. Preliminaries on Cones of Functions

Let E be a locally compact space with a countable base and let $\mathscr{S}$ be a convex cone of lower semicontinuous nonnegative numerical functions on E such that the constant function 1 belongs to $\mathscr{S}$. Denote by $\mathscr{S}^*$ the family of all universally measurable non-negative numerical functions f such that $\mu(f) \leqq f(x)$ for each $x \in E$ and each measure μ which fulfils the inequalities $\mu(s) \leqq s(x)$ for all $s \in \mathscr{S}$. Obviously $\mathscr{S}^*$ is a convex cone stable under infimum. Further let $\mathscr{T}$ be another convex cone such that $\mathscr{S} \subset \mathscr{T} \subset \mathscr{S}^*$. If $f: E \to R$ is an arbitrary function we use the notation

$$^{\mathscr{T}}Rf = \inf\{s \in \mathscr{T} / f \leqq s\},$$

provided the set in brackets is not empty. The function $^{\mathscr{T}}Rf$ is called the reduite function. If $f \leqq 0$ we have $^{\mathscr{T}}Rf = 0$. We shall denote by $D(\mathscr{T})$ the family of all continuous functions f which have the following properties:

- there exists $s \in \mathscr{T}$ such that $|f| \leqq s$ and
- $\inf\{^{\mathscr{T}}R(\chi_{CK}|f|)/K \text{ compact set}\} = 0$.

It is easy to see that $D(\mathscr{T})$ is a vector lattice which contains the space $\mathscr{C}_c(E)$ and that $^{\mathscr{T}}Rf < \infty$ for each $f \in D(\mathscr{T})$.

The next result was proved in [6 p. 76] in the case $\mathcal{T}=\mathcal{S}^*$. The proof given there is still valid in the more general case stated here.

Theorem 1.1. *Let $f\colon E\to R$ be an upper semicontinuous function such that there exists a function $g\in D(\mathcal{T})$ with $f\leqq g$. Then for each point $x\in E$ there exists a nonnegative measure μ such that*

$$\mu(s)\leqq s(x) \quad \text{for each } s\in\mathcal{S} \quad \text{and} \quad {}^{\mathcal{T}}Rf(x)=\mu(f).$$

2. The Cone of Excessive Functions

Further let $(V_\lambda)_{\lambda>0}$ be a sub-Markov resolvent of kernels on E satisfying the following two conditions:

(2.1) $V_\lambda \mathscr{C}_c(E)\subset \mathscr{C}_b(E)$, for each $\lambda\geqq 0$, where $V_0=\sup\limits_\lambda V_\lambda$ and

(2.2) $\lim\limits_{\lambda\to\infty} \lambda V_\lambda f(x)=f(x)$, for each $f\in\mathscr{C}_c(E)$ and $x\in E$.

A function $s\colon E\to[0,\infty]$ will be called excessive provided:

1^0 s is universally measurable,
2^0 $\lambda V_\lambda s\leqq s$, $\lambda>0$ and
3^0 $s(x)=\lim\limits_{\lambda\to\infty}\lambda V_\lambda s(x)$, for each $x\in E$.

It follows from (2.2) that each lower semicontinuous function $s\colon E\to[0,\infty]$ satisfying conditions 1^0 and 2^0 is in fact excessive. The family of all excessive functions is a convex cone that will be denoted by $\mathscr{E}$. The subcone of all real valued continuous excessive functions will be denoted by $\mathscr{E}_c$. Following G. Mokobodzki [5 Chap. III] we next present some basic properties related to the cones $\mathscr{E}$ and $\mathscr{E}_c$. We have $V_0\mathscr{C}_c^+(E)\subset\mathscr{E}_c$. The monotone class theorem shows that $V_0 f\in\mathscr{E}_c^*$ for each Borel nonnegative function f, and hence the same is true for f universally measurable and nonnegative. By standard arguments on excessive functions it follows $\mathscr{E}\subset\mathscr{E}_c^*$.

Theorem 2.1. 1^0 *If f is a lower semicontinuous function, then the reduite ${}^{\mathscr{E}}Rf$ is also lower semicontinuous and excessive.*

2^0 *If f is an upper semicontinuous function and there exists a function $g\in D(\mathscr{E}_c)$ such that $f\leqq g$, then*

$${}^{\mathscr{E}_c}Rf={}^{\mathscr{E}}Rf.$$

Particularly the function ${}^{\mathscr{E}}Rf$ is upper semicontinuous.

3^0 *If f belongs to $D(\mathscr{E}_c)$, then ${}^{\mathscr{E}}Rf$ is continuous.*

The proof of the theorem uses Theorem 1.1 and the method of proof of the similar results from [5 Chap. III].

Further we shall use the following simple lemma.

Lemma 2.2. *Let μ, ν be two finite measures such that $\mu(V_0 f)=\nu(V_0 f)$ for each $f\in\mathscr{C}_c(E)$. Then $\mu=\nu$.*

Proof. Let $g \in \mathscr{C}_b(E)$, $g \geqq 0$ be such that $V_0 g$ is bounded. Then clearly $\mu(V_0 g) = \nu(V_0 g)$. Therefore if $f \in \mathscr{C}_c(E)$, $f \geqq 0$ on account of the resolvent equation we deduce that $V_0 V_\lambda f$ is bounded, and consequently $\mu(V_0 V_\lambda f) = \nu(V_0 V_\lambda f)$. Then we deduce $\mu(V_\lambda f) = \nu(V_\lambda f)$ for each $\lambda > 0$ and letting $\lambda \to \infty$ from (2.2) we deduce $\mu(f) = \nu(f)$. Since f was arbitrary the equality $\mu = \nu$ follows.

Consequence 2.3. The measures $V_\lambda(x, \cdot)$ are nonzero for each $\lambda \geqq 0$ and $x \in E$.

3. Approximation of Continuous Functions with Excessive Functions

In this section we assume that the resolvent satisfies the conditions (2.1), (2.2) and the following one:

$$V_0 f \in D(\mathscr{E}) \qquad \text{for each } f \in \mathscr{C}_c(E). \tag{3.1}$$

Lemma 3.1. *Let $f \in \mathscr{C}_b(E)$ be such that $f > 0$ and $V_0 f \in \mathscr{C}_b(E)$.*

If U is an open set and $x \in U$, then

$$^{\mathscr{E}}R(\chi_{CU} V_0 f)(x) < V_0 f(x).$$

Proof. First we note that $V_0 f \in D(\mathscr{E})$. Indeed let $y \in E$ and $\varepsilon > 0$. We can choose $g \in \mathscr{C}_c(E)$, $0 \leqq g \leqq 1$ such that $V_0 f(y) - V_0(gf)(y) < \varepsilon/2$. From (3.1) we get a compact set K and $s \in \mathscr{E}$ such that $s(y) < \varepsilon/2$ and $V(gf) \leqq s$ on CK. Putting $t = s + V_0((1-g)f)$ we have $t(y) < \varepsilon$ and $V_0 f \leqq t$ on CK. Therefore $V_0 f \in D(\mathscr{E})$. Then from Theorem 1.1 we get a measure μ on E such that $\mu(s) \leqq s(x)$ for each $s \in \mathscr{E}$ and $\mu(\chi_{CU} V_0 f) = h(x)$, where we denote by h the reduite function

$$h = {}^{\mathscr{E}}R(\chi_{CU} V_0 f).$$

Now let us suppose that $h(x) = V_0 f(x)$. Then we have

$$V_0 f(x) = \mu(\chi_{CU} V_0 f) \leqq \mu(V_0 f) \leqq V_0 f(x).$$

and hence

$$\mu(V_0 gf) \leqq V_0 gf(x), \qquad \mu(V_0(1-g)f) \leqq V_0(1-g)f(x)$$

and

$$\mu(V_0 gf) + \mu(V_0(1-g)f) = \mu(V_0 f) = V_0 f(x) = V_0 gf(x) + V_0(1-g)f(x).$$

We deduce $\mu(V_0 gf) = V_0 gf(x)$ and on account of Lemma 2.2, we get $\mu = \varepsilon_x$. Since $\chi_{CU}(x) = 0$ it follows $h(x) = 0$.

On the other hand Consequence 2.3 shows that $V_0 f(x) > 0$. The assumption $V_0 f(x) = h(x)$ fails and this finishes the proof.

Proposition 3.2. *Let U be an open set and K a compact set such that $U \subset K$ and denote by T_0 the subspace of $\mathscr{C}_b(E)$ consisting of all functions of the form*

$$r \wedge V_0 f_1 \wedge \ldots V_0 f_n - q \wedge V_0 g_1 \wedge \ldots V_0 g_p,$$

with $r, q \in R$, r, $q \geqq 0$, f_i, $g_j \in \mathscr{C}_c(E)$, $f_i \geqq 0$, $g_j \geqq 0$, $i \leqq n$, $j \leqq p$. *Then the subspace* T *of* $C_0(U)$ *defined by*

$$T = \{f_{|U} / f \in T_0, f = 0 \text{ on } K \setminus U\}$$

is dense in $\mathscr{C}_0(U)$.

Proof. The space T_0 is a sub-vector lattice of $\mathscr{C}_b(E)$ having the property that if $f \in T_0$, then $f \wedge 1 \in T_0$. Therefore T is a sub-vector lattice of $\mathscr{C}_0(U)$ with the same property and the Stone-Weierstrass theorem will imply the proposition provided we prove that T separates the points of U.

Let $x \in U$ and let W be an open neighbourhood of x such that $\overline{W} \subset U$. We shall construct a function $f \in T_0$ such that $f(x) > 0$ and $f = 0$ on $K \setminus W$. First from condition (2.1) we deduce the existence of a function $g \in \mathscr{C}_b(E)$, $g > 0$ such that $V_0 g \in \mathscr{C}_b(E)$. Put $u = V_0 g$ and choose a compact neighbourhood M of x such that $M \subset W$. The function

$$v = {}^{\mathscr{E}}R(\chi_{CM} u)$$

has the following properties: $v = u$ on CM, $u(x) > v(x)$ on account of Lemma 3.1 and v is lower semicontinuous and excessive on account of Theorem 2.1.1^0. Then we have the following relation: $v = \sup\{\lambda V_\lambda h / 0 \leqq h \leqq v,\ h \in \mathscr{C}_c(E),\ \lambda > 0\}$. If we fix $\varepsilon > 0$ such that $u(x) > v(x) + \varepsilon$, since v is continuous on CW, we can choose $h \in \mathscr{C}_c(E)$ and $\lambda > 0$ satisfying $0 \leqq h \leqq v$ on E and $v \leqq \lambda V_\lambda h + \varepsilon/2$ on $K \setminus W$. We also have $\lambda V_\lambda h \leqq \lambda V_\lambda v \leqq v$. Further the relation

$$\lambda^2 V_0 V_\lambda h = \sup\{\lambda^2 V_0 h' V_\lambda h / h' \in \mathscr{C}_c(E), 0 \leqq h' \leqq 1\}$$

shows that we can choose a function $h' \in \mathscr{C}_c(E)$ such that $V_0 h' \leqq \lambda^2 V_0 V_\lambda h$ and $\lambda^2 V_0 V_\lambda h \leqq V_0 h' + \varepsilon/2$ on K. Then we have

$$\lambda V_\lambda h = \lambda V_0 h - \lambda^2 V_0 V_\lambda h \leqq \lambda V_0 h - V_0 h'$$

and

$$\lambda V_0 h - V_0 h' \leqq \lambda V_\lambda h + \varepsilon/2 \quad \text{on } K.$$

Further we deduce

$$u = v \leqq \lambda V_\lambda h + \varepsilon/2 \leqq \lambda V_0 h - V_0 h' + \varepsilon/2 \quad \text{on } K \setminus W$$

and

$$\lambda V_0 h(x) - V_0 h'(x) + \varepsilon/2 \leqq \lambda V_\lambda h(x) + \varepsilon \leqq v(x) + \varepsilon < u(x).$$

The function $f = u - u \wedge (\lambda V_0 h - V_0 h' + \varepsilon/2)$ posseses the asserted properties. Thus T separates the points of U.

Corollary 3.4. *Let* U *be an open set,* K *a compact set and* $f \in \mathscr{C}_c(E)$ *a function such that* $\operatorname{supp} f \subset U \subset K$. *For each point* $x \in E$ *and each* $\varepsilon > 0$, *there exist* $s, t \in \mathscr{E}_c$ *such that*

$$|s - t - f| < \varepsilon \quad \text{on } U, \ s = t \quad \text{on } K \setminus U$$

and

$${}^{\mathscr{E}}R(|s - t - f|)(x) < \varepsilon.$$

Proof. First we choose a function $g \in \mathscr{C}_b(E)$ such that $g>0$ and $V_0 g \in \mathscr{C}_b(E)$. Since $V_0 g>0$ we may suppose that $|f| \leqq V_0 g$. Then we choose a compact set K' such that $K \subset K'$ and

$$^{\mathscr{E}}R(\chi_{CK'} V_0 g)(x) < \varepsilon/2.$$

Using the above proposition we get $s', t' \in \mathscr{E}_c$ such that $s'=t'$ on $K' \setminus U$ and $|s'-t'-f| < \varepsilon/2$ on U. Further putting $s=s' \wedge (V_0 g + t')$ and $t = t' \wedge (V_0 g + s' \wedge (V_0 g + t'))$ we get $s=s'=t'=t$ on $K' \setminus U$ and $s-t=((s'-t') \vee (-V_0 g)) \wedge V_0 g$, which implies $|s-t| \leqq V_0 g$. Since $|f| \leqq V_0 g$ we have $|s-t-f| < \varepsilon/2$ on U. On the other hand we have

$$^{\mathscr{E}}R(|s-t-f|) \leqq {}^{\mathscr{E}}R(\chi_{K'}|s-t-f|) + {}^{\mathscr{E}}R(\chi_{CK'}|s-t|) \leqq \varepsilon/2 + {}^{\mathscr{E}}R(\chi_{CK'} V_0 g).$$

The Corollary is proved.

4. Construction of the Process

In this section we are going to associate a Hunt process to the resolvent $(V_\lambda)_{\lambda>0}$ satisfying the conditions (2.1), (2.2) and (3.1). First we are going to associate a semigroup $(P_t)_{t \geqq 0}$ of sub-Markov kernels by using Theorem 5.3 proved in the next section. Condition (A_1) of Theorem 5.3 follows from Corollary 3.4 and condition (A_2) is a consequence of (3.1) (the assumption (3.1) shows that for each $f \in \mathscr{C}_c(E)$, $V_0 f$ belongs to the cone $\mathscr{P}_0$ defined above Theorem 5.3). From Theorem 5.3 we have the semigroup $(P_t)_{t \geqq 0}$ satisfying for each $f \in \mathscr{C}_c(E)$ and each $x \in E$,

- the map $t \to P_t(x)$ is right continuous
- $V_\lambda f(x) = \int_0^\infty \exp(-\lambda t) P_t f(x)\, dt, \ \lambda \geqq 0.$

From condition (2.2) follows $P_0 = I$.

Now we begin the construction of the process. We set $E_\Delta = E \cup \{\Delta\}$, where Δ is the Alexandrov point if E is noncompact or an isolated point adjoined to E if it is compact. As usual we extend the semigroup (P_t) to a Markov one on E_Δ, which will still be denoted by (P_t). Furthermore we make the convention that each numerical function f defined on E is tacitly extended on E_Δ taking $f(\Delta) = 0$. Now we prove the analogous of Theorem 3 from p. 27 in [4].

Lemma 4.1. *Let (X_t) be a Markov process on a complete probability space $(\Omega, \mathscr{F}, P)$ having (P_t) as transition function. Let S be a countable dense subset of $(0, \infty)$. Then almost all $\omega \in \Omega$ have the following properties:*

(1) *there exists* $X_{t+}(\omega) = \lim_{S \ni s > t, s \to t} X_s(\omega)$, *for each* $t \in R_+$,

(2) *there exists* $X_{t-}(\omega) = \lim_{S \ni s < t, s \to t} X_s(\omega)$, *for each* $t > 0$,

(3) $X_s(\omega) = \Delta$ *if* $s \in S$ *and* $s > \zeta(\omega)$,

where $\zeta(\omega) = \inf\{t \in S / X_t(\omega) = \Delta$ *or* $X_{t+}(\omega) = \Delta$ *or* $X_{t-}(\omega) = \Delta\}$ *and the limits in* (1) *and* (2) *are taken in* E_Δ.

Proof. For each excessive function $f(f\in\mathscr{E})$ the process $(f(X_t))_{t\geqq 0}$ is a supermartingale with respect to the family $(\mathscr{F}_t)_{t\geqq 0}$ where $\mathscr{F}_t$ is the P-completion of $\sigma(X_s, s\leqq t)$ in $\mathscr{F}$.

For a compact set $K\subset E$ we know from Proposition 3.2 that the space $\{f_{|K}-g_{|K}/f,g\in\mathscr{E}_c\}$ is dense in $\mathscr{C}(K)$. If we define $T_K=\inf\{t\in S/X_t\in E_\Delta\setminus K\}$, the classical arguments used in the construction of standard processes (see [1] or [4]) can be used to show that almost all $\omega\in\Omega$ have the following properties:

(4) there exists $\lim\limits_{S\ni s>t, s\to t} X_s(\omega)$ for each $t<T_K$,

(5) there exists $\lim\limits_{S\ni s<t, s\to t} X_s(\omega)$ for each $t\leqq T_K$.

Now let (K_n) be a sequence of compact sets such that $K_n\subset \mathring{K}_{n+1}$ and $E=\bigcup\limits_n K_n$ and put $T_n=\inf\{t\in S/X_t\in E\setminus K_n\}$ and $T=\sup\limits_n T_n$. We choose a continuous function $f\in\mathscr{C}_b(E)$ such that $f>0$ on E and $V_0 f\in\mathscr{C}_b(E)$. Then we have $V_0 f>0$ on E and if $Y_t=V_0 f(X_t)$, then $X_t(\omega)=\Delta$ if and only if $Y_t(\omega)=0$. Since $(Y_t)_{t\geqq 0}$ is a supermartingale and $t\to E[Y_t]$ is right continuous there exists a right continuous version of $(Y_t)_{t\geqq 0}$ which we denote by $(\hat{Y}_t)_{t\geqq 0}$.

Let us suppose that g is an excessive function satisfying the inequality $V_0 f\leqq g$ on CK_n, for some fixed $n\in N$. Then we are going to show that $E[\hat{Y}_{T_n}/T_n<\infty]\leqq E[g(X_0)]$. In order to prove this inequality we take an increasing sequence $(S_k)_k$ of finite subsets of S such that $\bigcup\limits_k S_k=S$ and define $R_k=\inf\{t\in S_k/X_t\in E_\Delta\setminus K_n\}$. Then $T_n=\inf\limits_k R_k$ and $X_{R_k}(\omega)\in E_\Delta\setminus K_n$ if $R_k(\omega)<\infty$. Therefore

$$E[\hat{Y}_{R_k}/R_k<\infty]=E[Y_{R_k}/R_k<\infty]\leqq E[g(X_{R_k})]\leqq E[g(X_0)].$$

Letting $k\to\infty$ we get the desired inequality. Further we deduce $\limsup\limits_{n\to\infty} E[\hat{Y}_{T_n}/T_n<\infty]\leqq E[g(X_0)]$.

Now taking $g_n={}^{\mathscr{E}}R(\chi_{CK_n} V_0 f)$, on account of relation (3.1) one can show that $\lim g_n=0$ (see the first part of the proof of Lemma 3.1), and hence $\lim E[g_n(X_0)]=0$, which implies that $\hat{Y}_{T_n}\to 0$ almost surely.

A well known result on supermartingales implies that almost all ω have the property that $Y_t(\omega)=0$ for each $t\in S$, $t>T(\omega)$, which is equivalent to

(7) $$X_t(\omega)=\Delta \quad \text{for each } t\in S,\ t>T(\omega).$$

On the other hand the properties (4) and (5) imply that almost all ω have the following properties:

(8) there exists $\lim\limits_{S\ni s>t, s\to t} X_s(\omega)$ for each $t<T$ and

(9) there exists $\lim\limits_{S\ni s>t, s\to t} X_s(\omega)$ for each $t<T$ and even for $t=T(\omega)$ provided $\omega\in\bigcup\limits_n \{T_n=T\}$.

It remains to study the left limit of the process $(X_s)_{s\in S}$ at T on the set $\bigcap\limits_n \{T_n<T\}$. Since almost surely $\hat{Y}_t=Y_t$ for all $t\in S$ from the relation $\hat{Y}_{T_n}\to 0$

follows $\lim_{S\ni t<T, t\to T} Y_t=0$ a.s. on $\bigcap_n \{T_n<T\}$. On the other hand for each compact set K there exists a constant $c>0$ such that $V_0 f \geqq c$ on K, because $V_0 f$ is continuous. Thus $Y_t(\omega)\geqq c$ provided $X_t(\omega)\in K$. This implies that

$$\lim_{S\ni t<T, t\to T} X_t=\Delta \text{ a.s. on } \bigcap_n \{T_n<T\}.$$

Now it is easy to see that the assertion of the lemma holds with $\zeta=T$.

Theorem 4.2. *There exists a Hunt process $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ associated to the resolvent $(V_\lambda)_{\lambda>0}$.*

Proof. On account of the previous lemma the method of P.A. Meyer [4, pages 30–51] can be applied in our situation in order to construct the process $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ having all the properties of a Hunt process, except possible the quasileftcontinuity. We shall check here that this process is quasileftcontinuous.

Let (R_k) be an increasing sequence of stopping times having a finite limit $R=\lim R_k<\infty$. For each function $f\in\mathscr{C}_b(E)$, $f\geqq 0$ such that $V_0 f<\infty$, we have

$$E^x[V_0 f(X_R)]=E^x\left[\int_R^\infty f(X_t)\,dt\right]=\lim_{k\to\infty} E^x\left[\int_{R_k}^\infty f(X_t)\,dt\right]$$
$$=\lim_{k\to\infty} E^x[V_0 f(X_{R_k})],$$

which implies that $V_0 f(X_{R_k})\to V_0 f(X_R)$ a.s.

As in the proof of the preceding lemma we consider a sequence of compact sets (K_n) such that $K_n\subset \mathring{K}_{n+1}$ and $E=\bigcup_n K_n$. Then $T_n=\inf\{t>0/X_t\in E_\Delta\setminus K_n\}$, $n\in N$ satisfy $\lim_{n\to\infty} T_n=\zeta$.

Further using Proposition 3.2 one easily deduces that $X_{R_k}\to X_R$ on the set $\{R<T_n\}$ for each $n\in N$ and therefore on $\{R<\zeta\}$.

Considering a function $f>0$ such that $V_0 f\in\mathscr{C}_b(E)$ we see that $V_0 f$ is bounded from below by a strictly positive constant on each compact subset of E. On the other hand $V_0 f(X_\zeta)=0$. The theorem is proved.

Now we state a more general version of the above theorem relaxing the assumption that V_0 is a finite kernel. Its proof results similarly by using Theorem 5.6 from the next section instead of Theorem 5.3 and by other standard modifications.

Theorem 4.3. *Let us assume that $(V_\lambda)_{\lambda>0}$ is a sub-Markov resolvent satisfying the following conditions:*

(2.1′) $\qquad V_\lambda \mathscr{C}_c(E) \subset \mathscr{C}_b(E)$ *for each* $\lambda>0$,

(2.2) $\qquad \lim_{\lambda\to\infty} \lambda V_\lambda f(x)=f(x)$ *for each* $f\in\mathscr{C}_c(E)$ *and each* $x\in E$,

(3.1′) $\qquad V_\lambda f\in D(\mathscr{E}_\lambda)$ *for each* $f\in\mathscr{C}_c(E)$ *and each* $\lambda>0$,

where $\mathscr{E}_\lambda$ is the cone of all λ-excessive functions.

Then there exists a Hunt process associated to the resolvent $(V_\lambda)_{\lambda>0}$.

Remark 4.4. A version of the above Theorem 4.2 was stated in [6 Chap. VI] as Theorem 2.4 completed by Corollary 2.8, but the reasoning given there do not cover the statements. However the arguments given in [6] work if the condition that $V_0 1 \in D(\mathscr{E})$ assumed there (with a different notation) is replaced by the stronger condition that $V_0 1 \in D(\mathscr{E}_c)$. An example making evident the difference between these two conditions was communicated to the author by K. Janssen.

5. Ray Resolvents on Locally Compact Spaces

Let E be a locally compact space with a countable base and $(V_\lambda)_{\lambda>0}$ a sub-Markov resolvent of kernels on $(E, \mathscr{B}(E))$. For each $\alpha \geqq 0$ we shall denote by $\mathscr{S}_\alpha$ the family of all Borel bounded α-supermean valued functions:

$$\mathscr{S}_\alpha = \{s: E \to [0, \infty), s \in B_b(E), \lambda V_{\lambda+\alpha} s \leqq s, (\forall)\, \lambda > 0\}.$$

It is wellknown that $V_\alpha \mathscr{B}_b^+(E) \subset S_\alpha$ if $\alpha > 0$, $\mathscr{S}_\alpha \supset \mathscr{S}_\beta$ if $\alpha > \beta$ and $1 \in \mathscr{S}_0$. We set $\mathscr{S}_\infty = \bigcup_{\alpha \geqq 0} \mathscr{S}_\alpha$.

It is also known that the space V_λ $(\mathscr{B}_b(E))$ does not depend on $\lambda > 0$. Its uniform closure will be denoted by $\mathscr{D}$, i.e. $\mathscr{D} = \overline{V_\lambda(B_b(E))}$. The Hille-Yosida theorem applied to the resolvent $(V_\lambda)_{\lambda>0}$ on the space $\mathscr{D}$ gives us a semigroup $(Q_t)_{t \geqq 0}$ of operators on $\mathscr{D}$ such that:

(1) $Q_0 = I$ and the map $t \to Q_t f$ is right continuous for each $f \in \mathscr{D}$.

(2) $V_\lambda f = \int_0^\infty \exp(-\lambda t) Q_t f\, dt$ for each $\lambda > 0$ and $f \in \mathscr{D}$.

(3) if $f \in \mathscr{D}$ is such that $0 \leqq f \leqq 1$, then $0 \leqq Q_t f \leqq 1$.

(4) if $f \in \mathscr{D} \cap S_\alpha$, $\alpha \geqq 0$, then $\exp(-\alpha t) Q_t f \leqq f$.

The properties (3) and (4) are proved for example in [2 p. 252].

If $f \in \mathscr{B}_b$, then $V_\lambda f \in \mathscr{D}$, and hence $Q_t \lambda V_\lambda f$ makes sense. We denote by $\hat{\mathscr{D}}$ the vector space of all functions $f \in \mathscr{B}_b$ which have the property that the limit

$$\hat{Q}_t f(x) = \lim_{\lambda \to \infty} Q_t \lambda V_\lambda f(x) \tag{5.1}$$

exists for each $t \geqq 0$ and each $x \in E$. The operator $\hat{Q}_t$, $t \geqq 0$ defined this way obviously map $\hat{\mathscr{D}}$ into $\mathscr{B}_b(E)$ and $\hat{Q}_{t|\mathscr{D}} = Q_t$. It should be noted that $\hat{Q}_0$ differs from I in general. If $f \in \mathscr{S}_\alpha$ we see that the map $\lambda \to Q_t \lambda V_{\lambda+\alpha} f$ is increasing and on account of the inequality $\|\lambda V_\lambda - \lambda V_{\lambda+\alpha}\| = \|V_\lambda V_{\lambda+\alpha}\| \leqq \alpha/(\lambda+\alpha)$ it follows $f \in \hat{D}$. Therefore we have $\mathscr{S}_\infty \subset \hat{D}$.

If $f \in \hat{D}$ and $x \in E$ we have

$$\begin{aligned}\int_0^\infty e^{-\alpha t} \hat{Q}_t f(x)\, dt &= \lim_{\lambda \to \infty} \int_0^\infty e^{-\alpha t} Q_t \lambda V_\lambda f(x)\\ &= \lim_{\lambda \to \infty} \lambda V_\alpha V_\lambda f(x) = V_\alpha f(x).\end{aligned}$$

Therefore relation (2) still holds for the operators $\hat{Q}_t$, $t \geqq 0$ and $f \in \hat{D}$. It is easy to see that relation (3) also holds for $\hat{Q}_t$. Further properties are presented in the next two lemmas.

Lemma 5.1. *If $f \in S_\alpha$ and $x \in E$, then the map $t \to e^{-\alpha t}\hat{Q}_t f(x)$ is right continuous, decreasing and bounded by $f(x)$.*

Proof. From the equality $V_{\lambda+\alpha}f = V_\alpha(f - \lambda V_{\lambda+\alpha}f)$ it follows $V_{\lambda+\alpha}f \in \mathcal{S}_\alpha \cap \mathcal{D}$. From the properties (1) and (4) of the semigroup we deduce that the map $t \to e^{-\alpha t} Q_t \lambda V_{\lambda+\alpha} f(x)$ is right continuous, decreasing and bounded by $f(x)$. The family of all these maps is increasing with respect to λ and its supremum is $t \to e^{-\alpha t}\hat{Q}_t f(x)$. It is an elementary lemma of real analysis which asserts that the limit of an increasing sequence of right continuous decreasing real functions on an interval is also a right continuous function. This implies the statement of the Lemma.

For $x \in E$ and $\alpha \geqq 0$ we define a semi-norm $p_{x\alpha}$ on $\mathcal{B}_b$ by putting for $f \in \mathcal{B}_b$,

$$p_{x\alpha}(f) = \inf\{s(x)/s \in \mathcal{S}_\alpha, |f| \leqq s\}.$$

Lemma 5.2. *Let $f \in \mathcal{B}_b$ and suppose that for each $x \in E$ there exist $\alpha = \alpha(x) \geqq 0$ and a sequence $(f_n^x)_{n \in N}$ in $\hat{D}$ such that*

$$\lim_{n \to \infty} p_{x\alpha}(f_n^x - f) = 0.$$

Then $f \in \hat{\mathcal{D}}$ and for each $x \in E$ and $t_0 \in R_+$ we have

$$\lim_{n \to \infty} \hat{Q}_t f_n^x(x) = \hat{Q}_t f(x) \qquad \textit{uniformly for } t \in [0, t_0].$$

Proof. First we are going to show that

$$\hat{Q}_t \lambda V_{\lambda+\alpha} g(x) \leqq e^{\alpha t} p_{x\alpha}(g),$$

for each $g \in \mathcal{B}_b(E)$. Indeed, if $s \in \mathcal{S}_\alpha$ is such that $|g| \leqq s$, then we have

$$\hat{Q}_t \lambda V_{\lambda+\alpha} g \leqq \hat{Q}_t \lambda V_{\lambda+\alpha} s \leqq \hat{Q}_t s \leqq e^{\alpha t} s,$$

which leads to the asserted inequality.

Further we deduce that for each $x \in E$ and $t \geqq 0$ we have

$$\lim_{n \to \infty} Q_t \lambda V_{\lambda+\alpha} f_n^x(x) = Q_t \lambda V_{\lambda+\alpha} f(x),$$

uniformly in λ. Then it is easy to deduce that $f \in \hat{\mathcal{D}}$. The first inequality also implies

$$\hat{Q}_t(f_n^x - f)(x) \leqq e^{\alpha t} p_{x\alpha}(f_n^x - f),$$

which leads to the uniform limit relation asserted by the lemma.

Before stating the main result of this section we introduce some notation. We denote by $\mathcal{P}_\alpha$ the family of all functions $s \in \mathcal{S}_\alpha$ having the property that for each $x \in E$ and each $\varepsilon > 0$, there exist $f \in \mathcal{C}_c(E)$ and $u \in \mathcal{S}_\alpha$ such that $f \leqq s \leqq f + u$ and $u(x) < \varepsilon$. It is easy to see that each function $s \in \mathcal{P}_\alpha$ is lower semicontinuous and if f, u and ε satisfy the above inequalities, then

$$p_{x\alpha}(s - f) \leqq u(x) < \varepsilon.$$

Let $\mathscr{A}$ be a family of functions in $\mathscr{B}_b(E)$. We shall denote by $p(\mathscr{A})$ the family of all functions f in $\mathscr{B}_b$ which can be approximated with functions from $\mathscr{A}$ in the following sense: for each $x \in E$ there exists $\alpha = \alpha(f, x)$ and a sequence (f_n^x) in $\mathscr{A}$ such that $\lim_{n \to \infty} p_{x\alpha}(f_n^x - f) = 0$. With this notation we have $\mathscr{P}_\alpha \subset p(\mathscr{C}_c(E))$.

Theorem 5.3. *Suppose that the potential kernel $V_0 = \sup_{\lambda > 0} V_\lambda$ satisfies the following finiteness property: $V_0 f < \infty$ for each $f \in \mathscr{C}_c$, $f \geqq 0$. Moreover assume that the following conditions are fulfilled:*

(A_1) $$\mathscr{C}_c(E) \subset p(\mathscr{S}_\infty - \mathscr{S}_\infty) \quad \text{and}$$

(A_2) for each $f \in \mathscr{C}_c(E)$, $f \geqq 0$, *there exists a sequence (s_n) in $\mathscr{P}_0$ such that* $\lim_{n \to \infty} s_n = V_0 f$ *and* $s_n \leqq V_0 f$ *for any* $n \in N$.

Then there exists a unique semigroup of kernels $(P_t)_{t \geqq 0}$ on E such that for each $f \in \mathscr{C}_c(E)$ and each $x \in E$ the following properties are satisfied:

(a) *the map $t \to P_t f(x)$ is right continuous,*

(b) $V_\lambda f(x) = \int_0^\infty \exp(-\lambda t) P_t f(x)\, dt$, *for each* $\lambda \geqq 0$.

Remark. In general we may have $P_0 \neq I$.

Proof. From Lemma 5.2 we have $p(\mathscr{S}_\infty - \mathscr{S}_\infty) \subset \hat{\mathscr{D}}$. Therefore the assumption (A_1) allows us to define a family of kernels $(P_t)_{t \geqq 0}$ on E such that $P_t f = \hat{Q}_t f$, for each $f \in \mathscr{C}_c(E)$. The property (b) is obviously satisfied for $f \in \mathscr{C}_c$ and a monotone class argument shows that it remains valid for any $f \in \mathscr{B}_b$. Property (a) follows from Lemma 5.1 and Lemma 5.2. It remains to prove the semigroup property. First we prove the following lemmas:

Lemma 5.4. *If $f \in \mathscr{B}_b$ is such that $f \geqq 0$ and $V_0 f < \infty$, then the map $t \to P_t V_0 f(x)$ is right continuous and bounded by $V_0 f(x)$ for each $x \in E$.*

Proof. If $s \in \mathscr{P}_0$ and $g \in \mathscr{C}_c(E)$ are such that $0 \leqq g \leqq s$, then $P_t g = \hat{Q}_t g \leqq \hat{Q}_t s$. Taking the supremum over g we get $P_t s \leqq \hat{Q}_t s$. Now for a fixed point $x \in E$ we choose a sequence $(f_n) \subset \mathscr{C}_c(E)$ such that $0 \leqq f_n \leqq s$ and $p_{x0}(s - f_n) \to 0$. Therefore $\hat{Q}_t s(x) = \lim_{n \to \infty} \hat{Q}_t f_n(x) \leqq P_t s(x)$ and we conclude that $\hat{Q}_t s = P_t s$.

Further let $g \in \mathscr{C}_c$, $g \geqq 0$. From assumption (A_2) we have a sequence $(s_n) \subset \mathscr{P}_0$ such that $\lim_{n \to \infty} s_n = V_0 g$ and $s_n \leqq V_0 g$, $n \in N$. Then $P_t V_0 g = \lim_{n \to \infty} P_t s_n$. Since the maps $t \to P_t s_n(x)$, $n \in N$ are decreasing, right continuous and $P_t s_n(x) \leqq s_n(x)$ their limit $t \to P_t V_0 g(x)$ is also decreasing, right continuous and $P_t V_0 g(x) \leqq V_0 g(x)$. Then a monotone class argument shows that the map $t \to P_t V_0 g(x)$ is decreasing and $P_t V_0 g(x) \leqq V_0 g(x)$ for each function $g \in \mathscr{B}_b$ such that $g \geqq 0$ and $\{g \neq 0\}$ is relatively compact. Furthermore for such a function g we can choose another function $g' \in \mathscr{B}_b$, $g' \geqq 0$ such that $g + g' \in \mathscr{C}_c(E)$. We have two nonnegative decreasing maps $t \to P_t V_0 f(x)$ and $t \to P_t V_0 g'(x)$ and their sum is right continuous. Thus both of them are right continuous. Now expressing f as the limit of an increasing sequence $(g_n)\subset\mathscr{B}_b^+$, with $\{g_n\neq 0\}$ relatively compact we deduce that $t\to P_t V_0 f(x)$ is right continuous and $P_t V_0 f(x)\leqq V_0 f(x)$.

Lemma 5.5. *$P_t f = Q_t f$ for each $f\in\mathscr{D}$.*

Proof. Let $g\in\mathscr{B}_b$ be such that $g\geqq 0$ and the set $\{g\neq 0\}$ is relatively compact. From the relation $V_0 g = V_\lambda g + V_0 V_\lambda g$ we deduce that $V_0 V_\lambda g<\infty$. Further from the preceding lemma we have $P_t V_0 g<\infty$ and $P_t V_0 V_\lambda<\infty$ and we deduce that the map $t\to P_t V_\lambda g(x)$ is right continuous for each $x\in E$. Since the Laplace transform of this map coincides with the Laplace transform of the map $t\to Q_t V_\lambda g(x)$, which is also right continuous we get $P_t V_\lambda g = Q_t V_\lambda g$. Moreover we deduce from Lemma 5.1 that the map $t\to e^{-\lambda t} P_t V_\lambda g(x)$ is decreasing.

Now for $f\in\mathscr{B}_b, f\geqq 0$ we can express this function as the supremum of an increasing sequence $(g_n)\subset\mathscr{B}_b$ satisfying the property that the sets $\{g_n\neq 0\}$, $n\in N$ are relatively compact. We deduce that the map $t\to e^{-\lambda t} P_t V_\lambda f(x)$ is right continuous. Again the unicity of the Laplace transform implies that $P_t V_\lambda f(x) = Q_t V_\lambda f(x)$. Then it is easy to deduce the equality $P_t f = Q_t f$ for each $f\in\mathscr{D}$.

Finally in order to prove the relation $P_t P_s = P_{t+s}$ let $f\in\mathscr{C}_c$. Then we have

$$P_t P_s f = P_t \hat{Q}_s f = \lim_{\lambda\to\infty} P_t Q_s \lambda V_\lambda f = \lim_{\lambda\to\infty} Q_{t+s}\lambda V_\lambda f = \hat{Q}_{t+s} f = P_{t+s} f.$$

Next we give a version of Theorem 5.3 for the case when V_0 is non-finite.

Theorem 5.6. *Suppose that $(V_\lambda)_{\lambda>0}$ is a sub-Markov resolvent satisfying property (A_1) and the following:*

(A_2') for each $\alpha>0$ there exists a sequence (s_n) in $\mathscr{P}_\alpha$ such that $\lim_{n\to\infty} s_n = V_\alpha 1$ and $s_n\leqq V_\alpha 1$, for any $n\in N$.

Then there exists a unique semigroup of kernels $(P_t)_{t\geqq 0}$ on E such that:

(a) *the* map $t\to P_t f(x)$ *is right continuous for each $f\in\mathscr{C}_c$, $x\in E$,*

(b) $V_\lambda f(x) = \int_0^\infty \exp(-\lambda t) P_t f(x)\,dt$, *for each $f\in\mathscr{C}_c$, $x\in E$, $\lambda>0$.*

Proof. Let $\alpha>0$. A slight variation of the arguments used in the proof of Theorem 5.3 imply the existence of a unique semigroup $(P_t^\alpha)_{t\geqq 0}$ such that $V_{\lambda+\alpha} = \int_0^\infty \exp(-\lambda t) P_t^\alpha\,dt$.

Now for $0<\beta<\alpha$ we deduce $P_t^\alpha = e^{-(\alpha-\beta)t} P_t^\beta$. Then the semigroup $(P_t)_{t\geqq 0}$ given by $P_t = e^{\alpha t} P_t^\alpha$ satisfies the properties (a) and (b).

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