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Probability Theory and Related Fields

Upper Classes for the Increments of Fractional Wiener Processes*

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Summary. Let $(X(t),\ t \geqq 0)$ be a centred Gaussian process with stationary increments and $EX^2(t) = C_0 t^{2\alpha}$ for some $C_0 > 0$, $0 < \alpha < 1$, and let $0 < a_t \leqq t$ be a nondecreasing function of t with a_t/t nonincreasing. The asymptotic behaviour of several increment processes constructed from X and a_t is studied in terms of their upper classes.

1. Introduction

Let $(X(t),\ t \geqq 0)$ be a centred Gaussian process with stationary increments, $X(0) = 0$ a.s. and define $\sigma^2(h) = EX^2(h) = E(X(t+h) - X(t))^2$. If $\sigma^2(h) = C_0 h^{2\alpha}$, $0 < \alpha < 1$ and $C_0 > 0$ then X is known as a fractional Wiener process of order α (FWP(α)). If $\alpha = 1/2$ and $C_0 = 1$ this is the standard Wiener process.

The purpose of this paper is the study of the asymptotic behaviour of the following increment processes: let a_t be a nondecreasing function of t with $0 < a_t \leqq t$, and a_t/t nonincreasing. We define the following processes in terms of X and a_t:

$$Y_1(t) = \frac{X(t+a_t) - X(t)}{\sigma(a_t)}$$

$$Y_2(t) = \sup_{0 \leqq s \leqq t} Y_1(s)$$

$$Y_3(t) = \sup_{0 \leqq u \leqq a_t} \frac{X(t+u) - X(t)}{\sigma(a_t)}$$

$$Y_4(t) = \sup_{0 \leqq s \leqq t - a_t} Y_3(s)$$

$$Y_5(t) = \sup_{0 \leqq s \leqq t} Y_3(s)$$

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and the processes $Y_i^*(t)$, $i=1, \ldots, 5$ defined using the absolute value of the increments. Thus, for example,

$$Y_1^*(t)=\frac{|X(t+a_t)-X(t)|}{\sigma(a_t)}$$

In [7] it was shown that if X is a FWP(α) then

$$\limsup_{T\to\infty} Y_1(T)\,\beta_T=\limsup_{T\to\infty} Y_5^*(T)\,\beta_T$$

$$=\limsup_{T\to\infty}\sup_{0\leqq t\leqq T}\sup_{0\leqq s\leqq a_T}\frac{|X(t+s)-X(t)|}{\sigma(a_T)}\beta_T=1 \quad \text{a.s.} \quad (1.1)$$

where $\beta_T=\left(2\log\left(\frac{T\log T}{a_T}\right)\right)^{-1/2}$. For the Wiener process these results were obtained in 1979 by Csörgö and Révész [4]. Also, (1.1) implies the LIL for FWP proved by Orey [5]. Note that if $a_t=t$, $Y_4(t)=\sup_{0\leqq s\leqq t} X(s)$.

We shall consider the asymptotic behaviour of these processes with respect to a nondecreasing function $\phi(t)$ in terms of their upper classes, which we define following Révész [13]:

Definition 1. The function ϕ belongs to the *upper-upper class* of the prozess $Z(\phi\in\mathrm{UUC}(Z))$ if, with probability one, there exists a $t_0(\omega)$ such that $Z(t)<\phi(t)$ for all $t>t_0$.

Definition 2. The function ϕ belongs to the *upper-lower class* of the process $Z(\phi\in\mathrm{ULC}(Z))$ if, with probability one, there exists a random sequence $0<t_1<t_2<\ldots$ with $t_i\to\infty$ as $i\to\infty$ such that $Z(t_i)\geqq\phi(t_i)$ for $i\geqq 1$.

In [6] the upper classes of the increments of the Wiener process were described by means of an integral test as follows.

Theorem A. *Let $\alpha=1/2$ and let Z be any of the processes Y_i, Y_i^*, $i=1, \ldots, 5$, then*

$$\phi\in\mathrm{UUC}(Z)\Leftrightarrow\int_1^\infty\frac{\phi(t)}{a(t)}\exp\left\{-\frac{\phi^2(t)}{2}\right\}dt<\infty.$$

The main result of this paper is the following:

Theorem 1. *Let X be a* FWP(α) *and Z any of the processes Y_i, Y_i^*, $i=1, \ldots, 5$, then*

$$\phi\in\mathrm{UUC}(Z)\Leftrightarrow I_\alpha(\phi)=\int_1^\infty\frac{(\phi(t))^{(1/\alpha)-1}}{a(t)}\exp\left\{-\frac{\phi^2(t)}{2}\right\}dt<\infty.$$

This result includes Theorem A and Theorem 5 in [15] as special cases, and implies the first two inequalities in (1.1). It also extends previous works of the author on the increments of FWP [7, 8].

In the next section we give some preliminary results and in Sect. 3 the proof of Theorem 1. We shall assume without loss of generality that $C_0=1$. In what follows Const denotes a positive constant which can take different values on each appearance and $\psi(x)=(1/\sqrt{2\pi}x)\exp\left(-\frac{x^2}{2}\right)$. We shall frequently use the well-known fact that if X is a standard Gaussian r.v. then $P(X>x)\leqq\psi(x)$ for $x>0$ and $\frac{P(X>x)}{\psi(x)}\to 1$ as $x\to\infty$.

2. Preliminary Results

In this section we give some results that will be used in the proof of Theorem 1. In particular, Lemma 5 is an asymptotic upper bound for the tail of the distribution of the oscillations of FWP which may be independent interest.

Lemma 1 (Qualls-Watanabe [11]). *Let $\{X(t),\ t\in\mathbb{R}^n\}$ be a continuous centred Gaussian random field with variance* 1 *satisfying*

$$E(X(p)-X(q))^2\leqq 2C_1\|p-q\|^{2\alpha}$$

for all p, q in $D\subset\mathbb{R}^n$ with $\|p-q\|<\delta_1$, where $0<\alpha\leqq 1$ and $\|x\|$ is the Euclidean norm of $x\in\mathbb{R}^n$. Then

$$\limsup_{u\to\infty}\frac{P(Z(D)>u)}{\mu(D)\,\psi(u)\,(c(u))^n}\leqq H_\alpha\, C_1^{n/2\alpha}$$

where $Z(D)=\sup\{X(p),\ p\in D\}$, D is an open bounded set with Lebesque measure $\mu(D)=\mu(\bar{D})$, $c(u)=u^{1/\alpha}$ and $0<H_\alpha<\infty$ is a constant which does not depend on u.

Let $(X(t),\ t\geqq 0)$ be a FWP(α), we define the biparametric processes γ and Δ by

$$\gamma(t,t')=X(t)-X(t');\qquad \Delta(t,t')=\frac{\gamma(t,t')}{\sigma(t'-t)}\quad\text{for } t<t'.$$

Then $E\gamma(t,t')=E\Delta(t,t')=0$ and $E\Delta^2(t,t')=1$.

Lemma 2. *If $\mathbf{t}=(t,t')$ and $\mathbf{s}=(s,s')$ with $t<t'$ and $s<s'$, the incremental variance of Δ satisfies*

$$\tilde{\sigma}^2(\mathbf{t},\mathbf{s})=E(\Delta(t,t')-\Delta(s,s'))^2\leqq\frac{4\|\mathbf{t}-\mathbf{s}\|^{2\alpha}}{|t'-t|^\alpha|s'-s|^\alpha}$$

Proof.

$$\begin{aligned}\tilde{\sigma}^2(\mathbf{t},\mathbf{s}) &= E\left(\frac{X(t')-X(t)}{\sigma(t'-t)}-\frac{X(s')-X(s)}{\sigma(s'-s)}\right)^2\\ &=\frac{E(X(t')-X(t)-X(s')+X(s))^2-(\sigma(t'-t)-\sigma(s'-s))^2}{\sigma(t'-t)\,\sigma(s'-s)}\\ &\leqq\frac{2(\sigma^2(|t'-s'|)+\sigma^2(|t-s|))}{\sigma(t'-t)\,\sigma(s'-s)}\\ &\leqq\frac{4(|t'-s'|^2+|t-s|^2)^\alpha}{|t'-t|^\alpha\,|s'-s|^\alpha}\\ &=\frac{4\,\|\mathbf{t}-\mathbf{s}\|^{2\alpha}}{|t'-t|^\alpha\,|s'-s|^\alpha}\quad\square\end{aligned}$$

We give now some results about the asymptotic distribution of the supremum of γ

Lemma 3. *Let $(X(t),\ t\geqq 0)$ be a $FWP(\alpha)$ with $\alpha>1/2$. Then for $h>0$,*

$$\lim_{u\to\infty}\frac{P\left(\sup\limits_{0\leqq t<t\leqq h}\gamma(t,t')\geqq\sigma(h)\,u\right)}{P\{X(1)>u\}}=1. \tag{2.1}$$

This lemma is a consequence of Theorem B below: let $\{X(t),\ t\in[0,\ 1]^n\}$ be a real separable centred Gaussian process with continuous covariance function, and put $\sigma^2(t)=EX^2(t)$. Suppose that there is a point τ in $[0,\ 1]^n$ such that $\sigma^2(t)$ has a unique maximum value at $t=\tau$, and put $\sigma^2=\sigma^2(\tau)$. Define the metric $\|s-t\|=\max_i|s_i-t_i|$ where (s_i) and (t_i) are the real components of s and t. Suppose that there exist positive nondecreasing functions $q(t)$ and $g(t)$, $t>0$, such that

$$\limsup_{\|t-s\|\to 0}\frac{E(X(s)-X(t))^2}{q^2(\|s-t\|)}<\infty\qquad \limsup_{t,s\to\tau}\frac{E(X(s)-X(t))^2}{g^2(\|s-t\|)}<1$$

and

$$\int_1^\infty q(e^{-y^2})\,dy<\infty\qquad \int_1^\infty g(e^{-y^2})\,dy<\infty.$$

Define

$$Q(h)=q(h)+(2+\sqrt{2})\int_1^\infty q(h2^{-y^2})\,dy<\infty,\qquad 0<h\leqq 1.$$

$$G(h)=g(h)+(2+\sqrt{2})\int_1^\infty g(h2^{-y^2})\,dy<\infty,\qquad 0<h\leqq 1.$$

$$Q^{-1}(x)=\sup(h\colon Q(h)\leqq x)$$

for $h>0$, let $B(h)=\{t\colon\ \|t-\tau\|\leqq h/2\}$ and define $\bar{\sigma}^2(h)=\max\{\sigma^2(t)\colon\ t\in[0,\ 1]^n\cap B'(h)\}$ where A' is the complement of A. Then we have the following result:

Theorem B ([3], Theorem 2.1). *Suppose that there exist functions g and q satisfying the conditions stated above. If, for every* $\varepsilon>0$,

$$\lim_{h\to 0}[Q^{-1}(G(h)/\varepsilon)]^{-n}\exp\left\{-\frac{\varepsilon^2}{2\sigma^4}\left[\frac{\sigma^2-\bar{\sigma}^2(h)}{G^2(h)}\right]\right\}=0$$

then

$$\lim_{u\to\infty} P(\max_{[0,1]^n} X(t)>u)/\psi(u/\sigma)=1$$

To prove Lemma 3 it is enough to consider the case $h=1$ and prove that $E(\gamma(t,t')-\gamma(s-s'))^2\leqq 4(\|(t,t')-(s,s')\|^{2\alpha})$ (see Example 4.1 in [3]).

Lemma 4. *Let* $\{X(t),\, t\geqq 0\}$ *be a* $FWP(\alpha)$ *with* $\alpha<1/2$ *and* $h>0$. *Then there exists a constant* C_2, *which may depend on* α *but is independent of h, such that*

$$\limsup_{u\to\infty}\frac{P\{\sup_{0\leqq t<t\leqq h}\gamma(t,t')\geqq\sigma(h)\,u\}}{u^{(1/\alpha)-2}P\{X(1)>u\}}\leqq C_2 \tag{2.2}$$

Proof. It is enough to consider the case $h=1$. We consider first the supremum of γ over the set $A=\{(t,t')\colon 0\leqq t<t'\leqq 1,\ t'-t\leqq 1/2\}$. For $\varepsilon>0$ and $m=[2/\varepsilon]$ we have that

$$\begin{aligned}P\{\sup_A\gamma(t,t')>u\}&\leqq\sum_{j=0}^{m}P\{\sup_{\substack{j\varepsilon/2\leqq t\leqq(j+1)\varepsilon/2\\ 0<t'-t\leqq 1/2}}\gamma(t,t')>u\}\\ &\leqq(m+1)\,P\{\sup_{0<t'-t\leqq(1+\varepsilon)/2}\gamma(t,t')>u\}\end{aligned}\tag{2.3}$$

and to obtain a bound for this probability we use Theorem 3.3 in [2] for the process γ over the square $R=\{0\leqq t\leqq(1+\varepsilon)/2,\ 0\leqq t'\leqq(1+\varepsilon)/2\}$. In our case $Q(x)=\mathcal{O}(x^\alpha)$ and $\sigma_R=\left(\frac{1+\varepsilon}{2}\right)^\alpha$, hence

$$\begin{aligned}(2.3)&\leqq(m+1)\,\mathrm{Const}(Q^{-1}(1/u))^{-2}\frac{\sigma_R}{u}\exp\left\{-\frac{u^2}{2\sigma_R^2}\right\}\\ &\leqq\mathrm{Const}\frac{1}{\varepsilon}u^{(2/\alpha)-1}\exp\left\{-\frac{u^2 2^{2\alpha}}{2(1+\varepsilon)^{2\alpha}}\right\}\end{aligned}$$

and if $\varepsilon<1$ this is $\mathcal{O}(\psi(u))$ as $u\to\infty$.

It remains to consider the process γ over the set $E=\{(t,t')\colon 0\leqq t\leqq 1/2,\ t+1/2<t'\leqq 1\}$. To do this we cover E by squares of side η with η satisfying $uQ(\eta)\leqq 1$, where Q is defined above. Since $Q(x)=\mathcal{O}(x^\alpha)$ in our case, we have $\eta=\mathcal{O}(u^{-1/\alpha})$. If u is large enough, the covering will be included in the set $S=\{(t,t')\colon 0\leqq t\leqq 2/3,\ t+1/3\leqq t'\leqq 1\}$ and $\sigma^2=\inf_S E(\gamma^2(t,t'))\geqq 1/3^{2\alpha}>1/3$. Therefore, the right hand side of (3.12) in [2] is uniformly bounded for all $u>u_0$.

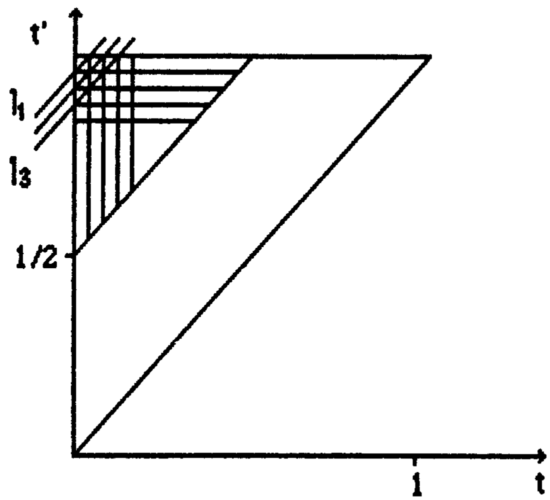

Fig. 1

We cover E by squares of sides parallel to the axes and length η starting at the right angled corner (see Fig. 1). Since X has stationary increments, the distribution of γ over squares of same size having diagonals over the same straight line is the same. Also, if we start counting from the right-angled corner, there are exactly i squares having diagonals over the straight line l_i of equation $t'=t+1-i\eta$, $i=1, \ldots, N$, where $N=[1/2\eta]+1$.

Let $E_i(\eta)=\{(t,t'): 0\leqq t\leqq \eta,\ 1-(i-1)\eta\leqq t'\leqq 1-i\eta\}$. Then, using Corollary 3.1 in [2], with $\sigma_{E_i(\eta)}=(1-(i-1)\eta)^\alpha$, we have

$$\begin{aligned} P\{\sup_E \gamma(t,t')>u\} &\leqq \text{Const} \sum_{i=0}^{N} P\{\sup_{E_i(\eta)} \gamma(t,t')>u\} \\ &\leqq \text{Const} \sum_{i=0}^{N} i\psi(u/\sigma_{E_i(\eta)}) \\ &\leqq \text{Const} \sum_{i=0}^{N} \frac{(i+1)(1+i\eta)^\alpha}{u} \exp\left\{-\frac{u^2}{2(1-i\eta)^{2\alpha}}\right\} \\ &\leqq \text{Const} \int_0^{1/\eta} \frac{(y+1)(1-y\eta)^\alpha}{u} \exp\left\{-\frac{u^2}{2(1-y\eta)^{2\alpha}}\right\} dy \end{aligned}$$

Using the method of Laplace to estimate this integral for large u, we get that it is asymptotically like $(\alpha\eta u^3)^{-1}\exp(-u^2/2)$ and since $\eta=\mathcal{O}(u^{-1/\alpha})$ we get the desired result. □

Next we consider the supremum of the process η over parallelograms.

Lemma 5. *Let $\{X(t), t \geqq 0\}$ be a $FWP(\alpha)$, $0<\alpha\leqq 1$. Then*

$$q(T, h, u) = P\{\sup_{\substack{0\leqq t\leqq T\\ 0<t'-t\leqq h}} (X(t') - X(t) \geqq \sigma(h)\, u\} \leqq C_\alpha \frac{T}{h} u^{(1/\alpha)-1} e^{-u^2/2} \tag{2.4}$$

for $u \geqq u_0$, where C_α is a constant which may depend on α but is independent of T and h.

Proof. The case $\alpha = 1/2$ was considered in [6] (see also [13]). Let $0<\alpha<1/2$, $0<\varepsilon\leqq 1$ and $m=[T/\varepsilon h]$, then

$$\begin{aligned} q(T, h, u) &\leqq \sum_{j=0}^{m} P\{\sup_{\substack{j\varepsilon h\leqq t\leqq (j+1)\varepsilon h\\ 0<t'-t\leqq h}} \gamma(t, t') \geqq \sigma(h)\, u\} \\ &\leqq (m+1)\, P\{\sup_{0\leqq t<t'\leqq h(1+\varepsilon)} \gamma(t, t') \geqq \sigma(h)\, u\} \\ &\leqq (m+1)\, P\left\{\sup_{0\leqq t<t'\leqq h(1+\varepsilon)} \frac{\gamma(t, t')}{\sigma(h(1+\varepsilon))} \geqq \frac{u}{\sigma(1+\varepsilon)}\right\} \end{aligned}$$

and by Lemma 4, for u sufficiently large this is bounded by

$$\begin{aligned} &\leqq \mathrm{Const}(m+1)\left(\frac{u}{\sigma(1+\varepsilon)}\right)^{(1/\alpha)-2} P\left(X(1) \geqq \frac{u}{\sigma(1+e)}\right) \\ &\leqq \mathrm{Const}\, \frac{T}{\varepsilon h} u^{(1/\alpha)-3} \exp\left\{-\frac{\mathrm{u}^2}{2(1-\varepsilon)^{2\alpha}}\right\} \end{aligned}$$

and choosing $\varepsilon = 1/u^2$ one obtains (2.4) by means of a simple calculation.

For the case $1/2<\alpha<1$ the proof is divided in two steps. We consider first the process $\Delta(t, t')$ over the set $B=\{(t, t')\colon 0\leqq t\leqq T,\ t+h(1+\varepsilon)\leqq t'\leqq t+h\}$ where $\varepsilon = 1/u^{1/\alpha}$. Lemma 2 shows that for $\mathbf{t}=(t, t')$ and $\mathbf{s}=(s, s')$ in B and u large we have

$$E(\Delta(t, t') - \Delta(s, s'))^2 \leqq \frac{4\,\|\mathbf{t}-\mathbf{s}\|^2}{h^{2\alpha}(1-\varepsilon)^{2\alpha}} \leqq \frac{4^{1+\alpha}}{h^{2\alpha}}\, \|\mathbf{t}-\mathbf{s}\|^2$$

Therefore, by Lemma 1

$$\begin{aligned} P\{\sup_B \gamma(t, t') > \sigma(h)\, u\} &\leqq P\{\sup_B \Delta(t, t') > u\} \\ &\leqq C_3 \frac{T}{h} u^{(1/\alpha)-1} \exp\left(-\frac{u^2}{2}\right) \end{aligned} \tag{2.5}$$

for u large and some constant C_3 which may depend on α. We still have to consider the supremum of γ over the set $D=\{(t, t')\colon 0\leqq t\leqq T,\ t<t'\leqq t+h(1-\varepsilon)\}$. Let $m=\left[\dfrac{Tu^{1/\alpha}}{h}\right]$, then

$$\begin{aligned} P\{\sup_D \gamma(t, t') > \sigma(h)\, u\} &\leqq \sum_{j=0}^{m} P\{\sup_{\substack{j\varepsilon h\leqq t\leqq (j+1)\varepsilon h\\ 0<t'-t\leqq h(1-\varepsilon)}} \gamma(t, t') > \sigma(h)\, u\} \\ &\leqq (m+1)\, P\{\sup_{0\leqq t<t'\leqq h} \gamma(t, t') > \sigma(h)\, u\} \end{aligned}$$

and by Lemma 3, for u sufficiently large this is bounded by

$$\leqq 2(m+1)\,P\{X(1)\geqq u\}$$
$$\leqq \text{Const}\,\frac{T}{h}\,u^{(1/\alpha)-1}\exp\left\{-\frac{u^2}{2}\right\} \tag{2.6}$$

Finally, combining (2.5) and (2.6) we obtain (2.4). □

Lemma 6 (Berman, Plackett, Slepian [1, 9, 14]). *Let $\{X_i, i=1, \ldots, n\}$ be centred, stationary Gaussian r.v.'s with $EX_i^2=1$ for all i and $EX_iX_j=r_{ij}$. Let $I(c,+1)=[c, \infty)$ and $I(c,-1)=(-\infty, c)$. If $c_i\in\mathbb{R}$, $i=1,\ldots,n$ denote by F_i the event $\{X_i\in I(c_i, \varepsilon_i)\}$ where ε_i is either $+1$ or -1. Let $K\subset(1,\ldots,n)$, then:*

i) $P\{\bigcap_{i\in K} F_i\}$ is an increasing function of r_{ij} if $\varepsilon_i\varepsilon_j=+1$, otherwise it is decreasing.

ii) If $\{K_l, l=1,\ldots,s)$ is a partition of K then

$$\left|P\{\bigcap_{i\in K} F_i\}-\prod_{l=1}^{m} P\{\bigcap_{i\in K_l} F_i\}\right|\leqq \sum_{i\leqq l<m\leqq s}\sum_{j\in K_l}\sum_{j\in K_m}|r_{ij}|\,g(c_i,c_j;r_{ij}^*)$$

where $g(x, y; r)$ is the standard bivariate Gaussian density with correlation r and r_{ij}^* is a number between 0 and r_{ij}.

The proof of the following version of the Borel-Cantelli lemma can be found in [12].

Lemma 7. *Let $(G_n, n\geqq 1)$ be a sequence of events. If*

i) $\sum_{n=0}^{\infty} P(G_n)=\infty$

ii) $$\liminf_{n\to\infty}\frac{\sum_{1\leqq i<k\leqq n}[P(G_i\cap G_k)-P(G_i)\,P(G_k)]}{\left(\sum_1^n P(G_i)\right)^2}=0 \tag{2.7}$$

then $P(G_n\ i.o.)=1$.

The following lemma is a consequence of the proof of Lemma 2.3 in [10], taking $a=1$.

Lemma 8 (Qualls-Watanabe). *Let $\{X(t), t\geqq 0\}$ be a real centred Gaussian process with $EX^2(t)=1$, which satisfies*

$$E(X(t)-X(s))^2\geqq C_4|t-s|^{2\alpha}$$

for $0\leqq s, t\leqq T$, $0<|t-s|<\delta_2$ where $0<\alpha\leqq 1$. Then

$$\liminf_{u\to\infty}\frac{P\{Z_u(T)>u\}}{T\psi(u)\,c(u)}\geqq C_4^{1/2\alpha}\,H_\alpha$$

where $Z_u(T) = \max_{0 \leq k \leq m} X(k/c(u))$, $m = [Tc(u)]$, $c(u) = u^{1/\alpha}$ *and* $0 < H_\alpha < \infty$ *is a constant which does not depend on* T.

3. Proof of Theorem 1

A. $I_\alpha(\phi) < \infty \Rightarrow \phi \in UUC(Y_5^*)$

The proof is similar to the convergent half of Theorem 1 in [6] and some details will be omitted. We may assume without loss of generality, that $1 \leqq \phi(t) \leqq t^{1/4}$ for all large t. Define the increasing sequence T_k, $k \geqq 0$ by

$$T_k = T_{k-1}(1 + \phi^{-2}(T_{k-1}))$$

where T_0 is chosen large enough so that $\phi(T_0) \geqq 1$. Define

$$B_k = \{ \sup_{T_{k-1} \leqq t \leqq T_k} \sup_{0 \leqq s \leqq a(T_k)} (X(t+s) - X(t)) \geqq \sigma(a(T_{k-1}))\, \phi(T_{k-1}) \}.$$

If T_0 is large we can use Lemma 5 to bound the probability of this event:

$$P(B_k) \leqq C_\alpha \frac{T_k - T_{k-1}}{a(T_k)} \left(\frac{a(T_{k-1})}{a(T_k)} \right)^{1-\alpha} \phi^{\frac{1}{\alpha}-1}(T_{k-1}) \exp\left\{ -\frac{1}{2} \left(\frac{a(T_{k-1})}{a(T_k)} \right)^{2\alpha} \phi^2(T_{k-1}) \right\}$$

and since $a(T_k) \leqq a(T_{k-1})(1 + \phi^{-2}(T_{k-1}))$ we obtain

$$\begin{aligned} P(B_k) &\leqq \text{Const} \frac{T_k - T_{k-1}}{a(T_k)} \phi^{\frac{1}{\alpha}-1}(T_{k-1}) \exp\left\{ -\frac{\phi^2(T_{k-1})}{2} \right\} \\ &\leqq \text{Const} \int_{T_{k-2}}^{T_{k-1}} \frac{(\phi(t))^{(1/\alpha)-1}}{a(t)} \exp\left\{ -\frac{\phi^2(t)}{2} \right\} dt \end{aligned}$$

Therefore $I_\alpha(\phi) < \infty \Rightarrow \sum_{n=0}^{\infty} P(B_n) < \infty$ and the Borel-Cantelli lemma implies by a standard argument that $\phi \in UUC(Y_5)$. A similar proof shows that $\phi \in \text{UUC}(Y_5^*)$.

B. $I_\alpha(\phi) = \infty \Rightarrow \phi \in ULC(Y_1)$

Let $\beta = \lim_{t \to \infty} \frac{a(t)}{t}$, if $\beta = 1$ then Theorem 5 in [15] implies the result. Assume $\beta < 1$ and define an increasing sequence $T_k \uparrow \infty$ by

$$0<\frac{a(T_0)}{T_0}<\beta'=\frac{1+\beta}{2}, \qquad \phi(T_0)>1$$

$$\xi(T_k)=T_k-a(T_k)=T_{k-1}, \quad k\geqq 1.$$

We define also

$$\delta_k=[\phi^{1/\alpha}(T_k)] \quad \delta_k'=[\lambda\delta_k] \qquad 0<\lambda<1;\ k\geqq 1$$

$$\tau_{k,j}=\frac{j\,a(T_{k-1})}{\delta_k} \quad t_{k,j}=T_{k-1}+\tau_{k,j} \quad j=1,\ldots,\delta_k';\ k\geqq 1$$

$$G_k=\{\max_{1\leqq j\leqq \delta_k'} Y_i(t_{k,j})>\phi(T_k)\}.$$

Suppose that s and t belong to the interval $[T_{k-1}, T_{k-1}+\lambda a(T_{k-1})]$ and $s\leqq t$. We shall show that there exists a constant $D_\alpha>0$ such that

$$E(Y_1(t)-Y_1(s))^2\geqq D_\alpha\frac{(t-s)^{2\alpha}}{a^{2\alpha}(T_{k-1})} \tag{2.8}$$

Start with

$$E(Y_1(t)-Y_1(s))^2 = \frac{2\sigma(a_t)\,\sigma(a_s)+\sigma^2(t-s+a_t-a_s)+\sigma^2(t-s)-\sigma^2(a_t+t-s)-\sigma^2(a_s-t+s)}{\sigma(a_t)\,\sigma(a_s)} \tag{2.9}$$

and consider

$$\sigma^2(a_t+t-s)+\sigma^2(a_s-t+s)=(a_t+t-s)^{2\alpha}+(a_s-(t-s))^{2\alpha}.$$

Using Taylor's theorem we get, for some $0<\theta<(t-s)$, that this is

$$=a_t^{2\alpha}+a_s^{2\alpha}+2\alpha(a_t^{2\alpha-1}-a_s^{2\alpha-1})(t-s)+\alpha(2\alpha-1)(t-s)^2((a_t+\theta)^{2\alpha-2} +(a_s-\theta)^{2\alpha-2}). \tag{2.10}$$

If $0<\alpha<1/2$ this expression is bounded above by $a_t^{2\alpha}+a_s^{2\alpha}$ and

$$\begin{aligned}(2.9)&\geqq\frac{(t-s+a_t-a_s)^{2\alpha}+(t-s)^{2\alpha}-(a_t^\alpha-a_s^\alpha)^2}{a_t^\alpha\, a_s^\alpha}\\ &\geqq\frac{(t-s+a_t-a_s)^{2\alpha}+(t-s)^{2\alpha}-(a_t-a_s)^{2\alpha}}{a_t^\alpha\, a_s^\alpha}\\ &\geqq\frac{(t-s)^{2\alpha}}{a_t^\alpha\, a_s^\alpha}\geqq\frac{(t-s)^{2\alpha}}{a^{2\alpha}(T_{k-1})}.\end{aligned}$$

If $1/2<\alpha<1$ then $(a_t^{2\alpha-1}-a_s^{2\alpha-1})\leqq(a_t-a_s)^{2\alpha-1}$; also $\theta<(t-s)<\lambda a(T_{k-1})\leqq\lambda a_s\leqq\lambda a_t$, and $a_t-a_s\leqq t-s$, hence

$$(2.10) \leqq a_t^{2\alpha}+a_s^{2\alpha}+2\alpha(t-s)^{2\alpha}+\alpha(2\alpha-1)(t-s)^{2\alpha}\left(\left(\frac{t-s}{a_t}\right)^{2-2\alpha}+\left(\frac{t-s}{(1-\lambda)\,a_s}\right)^{2-2\alpha}\right)$$

$$\leqq a_t^{2\alpha}+a_s^{2\alpha}+(t-s)^{2\alpha}\left(2\alpha+\alpha(2\alpha-1)(\lambda^{2-2\alpha}+\left(\frac{\lambda}{1-\lambda}\right)^{2-2\alpha}\right)\Bigg) \qquad (2.11)$$

On the other hand, since $1<2\alpha<2$, we have $(t-s+a_t-a_s)^{2\alpha} \geqq (t-s)^{2\alpha}+(a_t-a_s)^{2\alpha}$ and using this and (2.11) in (2.9) we get

$$(2.9) \geqq \frac{2a_t^\alpha a_s^\alpha+2(t-s)^{2\alpha}+(a_t-a_s)^{2\alpha}-a_t^{2\alpha}-a_s^{2\alpha}-(t-s)^{2\alpha}(2\alpha+\alpha(2\alpha-1)(\lambda^{2-2\alpha}(1+(1-\lambda)^{2\alpha-2})))}{a_t^\alpha a_s^\alpha}$$

but $(a_t-a_s)^{2\alpha} \geqq (a_t^\alpha-a_s^\alpha)^2=a_t^{2\alpha}+a_s^{2\alpha}-2a_t^\alpha a_s^\alpha$. Choosing $0<\lambda<1$ so that $D_\alpha=2-2\alpha-\alpha(2\alpha-1)\,\lambda^{2-2\alpha}(1+(1-\lambda)^{2\alpha-2})>0$ we get

$$E(Y_1(t)-Y_1(s))^2 \geqq D_\alpha \frac{(t-s)^{2\alpha}}{a^{2\alpha}(T_{k-1})}$$

and (2.8) is proved.

Now for $q \in [0, 1]$ define

$$V_k(q)=Y_1(T_{k-1}+q\lambda a(T_{k-1})).$$

Then (2.8) implies that for p and q in $[0, 1]$

$$E(V_k(p)-V_k(q))^2 \geqq D_\alpha \frac{(|p-q|\,\lambda a(T_{k-1}))^{2\alpha}}{a^{2\alpha}(T_{k-1})}=D_\alpha\,\lambda^{2\alpha}|p-q|^{2\alpha}.$$

On the other hand

$$P(G_k)=P\{\max_{1\leqq j\leqq \delta_k'} V_k(j/\delta_k)>\phi(T_k)\}$$

and using Lemma 8 we see that if T_0 is sufficiently large there exists a positive constant C_α' such that for all k we have

$$\begin{aligned} P(G_k) &\geqq C_\alpha'\,\delta_k\,\psi(\phi(T_k)) \\ &\geqq \mathrm{Const}\,\phi^{\frac{1}{\alpha}-1}(T_k)\exp\left\{-\frac{\phi^2(T_k)}{2}\right\} \\ &\geqq \mathrm{Const}\,\frac{T_k-T_{k-1}}{a(T_k)}\,\phi^{\frac{1}{\alpha}-1}(T_k)\exp\left\{-\frac{\phi^2(T_k)}{2}\right\} \\ &\geqq \int_{T_k}^{T_{k+1}} \frac{(\phi(t))^{(1/\alpha)-1}}{a(t)}\exp\left\{-\frac{\phi^2(t)}{2}\right\}dt \end{aligned} \qquad (2.12)$$

and $I_\alpha(\phi)=\infty \Rightarrow \sum_{n=0}^{\infty} P(G_n)=\infty$. To show that $P(G_k \text{ i.o.})=1$ we still have to prove (2.7). It is easy to see that

$$P(G_j \cap G_k)-P(G_j)\,P(G_k)=(P(G'_j \cap G'_k)-P(G'_j)\,P(G'_k)$$

where G' is the complement of G.

If $0<\alpha<1/2$ the variables $Y(t_{j,i})$ and $Y(t_{k,l})$ are negatively correlated if $j<k-1$, and in that case by Lemma 6, $P(G'_j \cap G'_k) \leqq P(G'_j)\,P(G'_k)$, which is sufficient for (2.7). If $1/2<\alpha<1$, using Lemma 6 we get

$$P(G_j \cap G_k)-P(G_j)\,P(G_k) \leqq \sum_{l=0}^{\delta'_j} \sum_{m=0}^{\delta'_k} \rho(t_{j,l};t_{k,m})\, g(\phi(T_j), \phi(T_k); \rho^*(t_{j,l};t_{k,m})) \tag{2.13}$$

where $\rho(t_{k,l};t_{k,m})=E(Y_1(t_{k,l})\,Y_1(t_{j,m}))$. This correlation can be written as

$$\rho(t_{j,l};t_{k,m})=\frac{1}{2Q^\alpha R^\alpha}(P^{2\alpha}+H(Q,R)),$$

where $P=\sum_{i=j}^{k-1} a(T_i)+\frac{m}{\delta_k} a(T_{k-1})-\frac{l}{\delta_j} a(T_{j-1})-a(t_{j,l})$, $Q=a(t_{j,l})$, $R=a(t_{k,m})$ and $H(U,V)=(P+U+V)^{2\alpha}-(P+U)^{2\alpha}-(P+V)^{2\alpha}$. By Taylor's theorem

$$H(Q,R)=-P^{2\alpha}+2\alpha(2\alpha-1)\,QRP^{2(\alpha-1)}+S$$

where $S=\frac{2\alpha(2\alpha-1)(2\alpha-2)}{3!}((Q+R)^3(P+\theta R+\theta Q)^{2\alpha-3}-Q^3(P+\theta Q)^{2\alpha-3}-R^3(P+\theta R)^{2\alpha-3})$ for some $0<\theta<1$. It is easy to see that

$$S \leqq \frac{2\alpha(2\alpha-1)(2\alpha-2)}{3!}\,Q^3(P+\theta Q)^{2\alpha-3}$$

and

$$H(Q,R) \leqq -P^{2\alpha}+3\alpha(2\alpha-1)\,QRP^{2\alpha-2}$$

whence

$$\begin{aligned}
\rho(t_{j,l};t_{k,m}) &\leqq \text{Const}\,\frac{QRP^{2\alpha-2}}{Q^\alpha R^\alpha} \\
&\leqq \text{Const}(a(t_{j,l})\,a(t_{k,m}))^{1-\alpha}\left(\sum_{i=j+1}^{k-1} a(T_i)\right)^{2\alpha-2} \\
&\leqq \text{Const}\left(\frac{a(T_j)\,a(T_k)}{\left(\sum_{i=j+1}^{k-1} a(T_i)\right)^2}\right)^{1-\alpha}
\end{aligned}$$

since $\beta<1$ we may assume, without loss of generality, that $a_1<1$, and then $T_k(1-a_1)\leqq T_{k-1}$ which implies $a(T_k)\leqq(1-a_1)^{-1}a(T_{k-1})$ and

$$\rho(t_{j,l};t_{k,m})\leqq \mathrm{Const}\left(\frac{a(T_j)}{\sum_{i=j+1}^{k-1} a(T_i)}\right)^{1-\alpha}$$

$$\leqq \mathrm{Const}(k-j-1)^{\alpha-1}\equiv\eta_{jk}$$

as long as $k>j+2$. We define

$$\rho_{jk}=\sup_{1\leqq l\leqq\delta'_j}\ \sup_{1\leqq k\leqq\delta'_k}\ \rho(t_{j,l};t_{k,m}).$$

Using this we see that (2.13) is bounded by

$$\frac{\delta'_j\,\delta'_k\,\eta_{jk}}{2\pi(1-\rho_{jk}^2)^{1/2}}\exp\left\{-\frac{\phi^2(T_j)+\phi^2(T_k)-2\hat\rho_{jk}\,\phi(T_j)\,\phi(T_k)}{2(1-\hat\rho_{jk}^2)}\right\}$$

where $\hat\rho_{jk}$ is the value of $\rho(t_{j,l};\ t_{k,m})$ which maximises the exponent. For our purpose it is sufficient to consider, for c fixed,

$$\sum_{k=c}^{n}\sum_{j=1}^{k-1}[P(G_j\cap G_k)-P(G_j)\,P(G_k)]$$

$$\leqq\left(\sum_{k=c}^{n}\sum_{j=1}^{k-v_k}+\sum_{k=c}^{n}\sum_{j=k-v_k}^{k-1}\right)\frac{\delta_j\,\delta_k\,\eta_{jk}}{2\pi(1-\rho_{jk}^2)^{1/2}}\exp\left\{-\frac{\phi^2(T_j)+\phi^2(T_k)-2\hat\rho_{jk}\,\phi(T_j)\,\phi(T_k)}{2(1-\hat\rho_{jk}^2)}\right\}$$

where $v_k=[(\phi(T_k))^{4/(1-\alpha)}]$. The first sum is bounded by

$$\sum_{k=c}^{n}\sum_{j=1}^{k-v_k}\frac{\eta_{jk}\,\phi(T_k)\,\phi(T_j)}{(1-\rho_{jk}^2)^{1/2}}\,\delta_k\,\psi(\phi(T_k))\,\delta_j\,\psi(\phi(T_j))\exp\{\phi^2(T_k)\,\eta_{jk}\}$$

and using (2.12) this is bounded by

$$\mathrm{Const}\sum_{k=c}^{n}\sum_{j=1}^{k-v_k}\frac{\eta_{jk}\,\phi^2(T_k)}{(1-\rho_{jk}^2)^{1/2}}\exp\{\phi^2(T_k)\,\eta_{jk}\}\,P(G_k)\,P(G_j)$$

but $k-j\geqq v_k$ implies $\eta_{jk}\,\phi^2(T_k)\leqq\mathrm{Const}\,v_k^{\alpha-1}\,\phi^2(T_k)\leqq\mathrm{Const}\,\phi^{-2}(T_k)$. Therefore, the sum above is bounded by

$$\text{Const}\,\phi^{-2}(T_c)\exp\{-\text{Const}\,\phi^{-2}(T_c)\}\sum_{k=c}^{n}\sum_{j=1}^{k-v_k}P(G_k)P(G_j)$$

and for c sufficiently large this is bounded by

$$\delta\left(\sum_{k=1}^{n}P(G_k)\right)^2.$$

The second sum is

$$\begin{aligned}&\sum_{k=c}^{n}\sum_{j=k-v_k}^{k-1}\frac{\delta_j\,\delta_k\,\eta_{jk}}{2\pi(1-\rho_{jk}^2)^{1/2}}\exp\left\{-\frac{\phi^2(T_j)+\phi^2(T_k)-2\hat{\rho}_{jk}\,\phi(T_j)\,\phi(T_k)}{2(1-\hat{\rho}_{jk}^2)}\right\}\\&=\sum_{k=c}^{n}\sum_{j=k-v_k}^{k-1}\frac{\eta_{jk}\,\phi^{\alpha}(T_k)\,\phi^{\frac{1}{\alpha}+1}(T_j)\,\psi(\phi(T_j))}{2\pi(1-\rho_{jk}^2)^{1/2}}\exp\left\{-\frac{(\phi(T_k)-\hat{\rho}_{jk}\,\phi(T_j))^2}{2(1-\hat{\rho}_{jk}^2)}\right\}\\&\leqq\text{Const}\sum_{k=c}^{n}\sum_{j=k-v_k}^{k-1}\phi^{\frac{1}{\alpha}+1}(T_k)\,\phi^{\frac{1}{\alpha}}(T_j)\,\psi(\phi(T_j))\exp\left\{-\phi^2(T_k)\left(\frac{1-\rho}{1+\rho}\right)\right\}\end{aligned}$$

where ρ is the largest covariance among the terms considered. Let $B=\left(\frac{1-\rho}{1+\rho}\right)^2>0$. Then if c is large enough this is

$$\begin{aligned}&\leqq\text{Const}\sum_{k=c}^{n}v_k\,\phi^{\frac{1}{\alpha}+1}(T_k)\,\phi^{\frac{1}{\alpha}}(T_{k-v_k})\,\psi(\phi(T_{k-v_k}))\exp\{-B\phi^2(T_k)\}\\&\leqq\text{Const}\,v_c\,\phi^{\frac{1}{\alpha}+1}(T_c)\exp\{-B\phi^2(T_c)\}\sum_{k=c-v_c}^{n}P(G_k)\\&\leqq\text{Const}\sum_{k=1}^{n}P(G_k).\end{aligned}$$

We have shown that given $\delta>0$, for c large

$$\sum_{k=c}^{n}\sum_{j=j}^{k-1}P(G_j\cap G_k)-P(G_j)P(G_k)\leqq\delta\left(\sum_{k=1}^{n}P(G_k)\right)^2+\text{Const}\sum_{k=1}^{n}P(G_k)$$

and this is enough to prove (2.7). □

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