Probab. Th. Rel. Fields 80, 347–364 (1989)

Probability Theory and Related Fields

Brownian Motions on Infinite Dimensional Quadric Hypersurfaces

In the memory of my friend Ichiro Enomoto

Yoshihei Hasegawa

Department of Mathematics, Nagoya Institute of Technology, Gokiso, Showa-ku, Nagoya, 466, Japan

Summary. A potential theory on an infinite dimensional quadric hypersurface S is developed following Lévy's limiting procedure. For a given real sequence $\{\lambda_n\}_{n=1}^{\infty}$ a quadratic form $h(x)$ on an infinite dimensional real sequence space $\mathbf{E}$ is defined by $h(x) := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \lambda_n x_n^2$, $x=(x_1, x_2, \ldots) \in \mathbf{E}$ and a quadric hypersurface S is defined by $S := \{x \in \mathbf{E};\ h(x)=c\}$, and the Laplacian $\bar{\Delta}_\infty$ on S is introduced by the limiting procedure. Instead of a direct use of $\bar{\Delta}_\infty$, the Brownian motion $\xi(t)=(\xi_1(t), \xi_2(t), \ldots)$, the diffusion process $(\xi(t), P^x)$ on S with the generator $\bar{\Delta}_\infty/2$ is constructed by solving a system of stochastic differential equations according to $\bar{\Delta}_\infty$. The law of large numbers for $X_n(t) := (\lambda_n, \xi_n(t))$ is proved, and ergodic properties are discussed.

0. Introduction

Paul Lévy initiated a potential theory on an infinite dimensional space in his book [5]. He gave an idea there to construct such objects as an infinite dimensional Laplacian and harmonic functions by a limiting procedure from the corresponding objects in $\mathbf{R}^N$, as $N \to \infty$. His potential theory has, however, peculiar phenomena; harmonic functions e.g., can be discontinuous ([5], pp. 305–306).

In the previous papers [2–4], the author intended to give a rigorous formulation of some aspects of Lévy's potential theory along Lévy's limiting procedure with the aid of an infinite dimensional Brownian motion $B(t, \omega) := (b_1(t, \omega), b_2(t, \omega), \ldots)$ on an infinite dimensional real sequence space $\mathbf{E}$, where $\{b_n(t, \omega)\}_{n=1}^{\infty}$ are mutually independent 1-dimensional Brownian motions. Actually, in those papers, Lévy's infinite dimensional Laplacian is thought of as twice the infinitesimal generator of the $B(t, \omega)$, and therefore harmonic functions in Lévy's sense can be interpreted by the $B(t, \omega)$.

Here we shall develop a potential theory on an infinite dimensional real hypersurface S of a diagonal quadratic form, as the quadric hypersurface S

seems to be the most important and accessible curved submanifold of $\mathbf{E}$. First we shall introduce an infinite dimensional formal Laplacian $\bar{\Delta}'_\infty$ on S by a limiting procedure ($N\to\infty$) from the finite dimensional Laplacian $\bar{\Delta}_N$ on the corresponding finite dimensional quadric hypersurface S_N. Next we shall construct an infinite dimensional Brownian motion $\xi(t)$ on S having $\bar{\Delta}'_\infty/2$ as formal infinitesimal generator. Then we shall define the Laplacian $\bar{\Delta}_\infty$ on S as twice the infinitesimal generator of $\xi(t)$, and develop the potential theory on S with the aid of $\xi(t)$.

Therefore, in this paper we shall construct Brownian motions $\xi(t)$ on infinite dimensional quadric hypersurfaces S and shall study their laws of large numbers and ergodic properties.

More precisely, we shall introduce a real sequence space $\mathbf{E}$ with the topology by semi-metrics $\{d_N; 1\leqq N\leqq\infty\}$, $d_N(x,y):=\left(\frac{1}{N}\sum_{n=1}^{N}(x_n-y_n)^2\right)^{1/2}$, $d_\infty(x,y):=\limsup_{N\to\infty} d_N(x,y)$ for $x=(x_1,x_2,\ldots)$, $y=(y_1,y_2,\ldots)\in\mathbf{E}$, and with the cylindrical σ-algebra $\mathscr{E}$. Next with the aid of a real bounded fixed sequence $\Lambda:=\{\lambda_n\}_{n=1}^{\infty}$ ($|\lambda_n|\leqq A$, $n\geqq 1$) such that $\frac{1}{N}\sum_{n=1}^{N}\delta_{\lambda_n}(d\sigma)$ converges weakly to a probability measure $\gamma(d\sigma)$ as $N\to\infty$, we define $\mathbf{E}(\Lambda)$ as the subset of $\mathbf{E}$ consisting of all points $x=(x_1,x_2,\ldots)\in\mathbf{E}$ such that $\frac{1}{N}\sum_{n=1}^{N}\delta_{(\lambda_n,x_n)}(d\sigma\times d\eta)$ converges weakly to a probability measure $\pi_x(d\sigma\times d\eta)$ as $N\to\infty$. (Here δ_a stands for the measure having mass one at a.) Now we define a diagonal quadratic form $h(x)$ as follows:

$$h(x):=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda_n x_n^2 \qquad \text{for } x=(x_1,x_2,\ldots)\in\mathbf{E}(\Lambda). \tag{0.1}$$

Then the infinite dimensional quadric hypersurface $S=S_c$ ($c\in\mathbf{R}$) is given by

$$S:=\{x\in\mathbf{E}(\Lambda);\, h(x)=c\}. \tag{0.2}$$

Now our next task is to construct the formal Laplacian $\bar{\Delta}'_\infty$ on S by the limiting procedure in the same way as [2–4]. The counterpart of $h(x)$ in $\mathbf{R}^N$ is considered to be $h_N(x):=\frac{1}{N}\sum_{n=1}^{N}\lambda_n x_n^2$, $x=(x_1,\ldots,x_N)\in\mathbf{R}^N$ and the Riemannian metric of $\mathbf{R}^N$ to be $ds_N^2:=\frac{1}{N}\sum_{n=1}^{N}dx_n^2$. Hence the Laplacian $\bar{\Delta}_N$ on the quadric hypersurface $S_{N,c}:=\{x\in\mathbf{R}^N;\, h_N(x)=c\}$ is given by

$$\bar{\Delta}_N:=N(\partial^2/\partial x_1^2+\ldots+\partial^2/\partial x_N^2)-(N-1)K_N\,\partial/\partial v_N-\partial^2/\partial v_N^2,$$

where $\partial/\partial v_N$ denotes the outer normal differentiation of $S_{N,c}$ and K_N is the mean curvature of $S_{N,c}$. Therefore, by the limiting procedure $\bar{\Delta}'_\infty := \lim_{N\to\infty} \bar{\Delta}_N/N$, the formal Laplacian $\bar{\Delta}'_\infty$ on S is defined by

$$\bar{\Delta}'_\infty := \sum_{n=1}^{\infty} \partial^2/\partial x_n^2 - (\lambda/v(x)) \sum_{n=1}^{\infty} \lambda_n x_n \partial/\partial x_n, \tag{0.3}$$

where

$$v(x) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda_n^2 x_n^2, \qquad x=(x_1, x_2, \ldots)\in S, \tag{0.4}$$

$$\lambda := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda_n.$$

Consequently the Brownian motion $\xi(t,\omega) := (\xi_1(t,\omega), \xi_2(t,\omega), \ldots)$, the conservative diffusion process on S with the formal generator $\bar{\Delta}'_\infty/2$, will be given as solution of the following system of stochastic differential equations:

$$d\xi_n(t,\omega) = dw_n(t,\omega) - \lambda \lambda_n \xi_n(t,\omega)/(2v(\xi(t,\omega)))\,dt, \qquad (n \geqq 1) \tag{0.5}$$

where $\{w_n(t)\}_{n=1}^\infty$ are mutually independent 1-dimensional standard Wiener processes. Then our first result is the following

Theorem A. *The solution $\xi(t)$ of (0.5) exists and the pathwise uniqueness of solutions of (0.5) holds. The Brownian motion $(\xi(t), P^x)$ on S exists and $\{\xi_n(t)\}_{n=1}^\infty$ are mutually independent $(P^x)_{x\in S}$.*

Now we shall introduce a time-inhomogeneous $[-A, A] \times \mathbf{R}$-valued diffusion process $(X(t,\omega), P^{(\sigma,\eta)})$. Given x on the surface S and the Brownian motion $\xi(t,\omega)$ starting at x, we can define a deterministic positive continuous function $\tilde{v}(t,x)$ by

$$\tilde{v}(t,x) := v(\xi(t,\omega)) \qquad \text{for any } t \geqq 0 \text{ a.s. } (P^x). \tag{0.6}$$

We define $(X(t,\omega), P^{(\sigma,\eta)})$ as the diffusion process with the infinitesimal generator $L(t)$, $(t \geqq 0)$:

$$L(t) := (1/2)\,\partial^2/\partial\eta^2 - (\lambda\sigma\eta)/(2\tilde{v}(t,\chi))\,\partial/\partial\eta. \tag{0.7}$$

The superscript (σ,η) denotes conditioning that $X(0,\omega) = (\sigma,\eta)$ a.s. $P^{(\sigma,\eta)}$.

Our law of large numbers can be stated as follows:

Theorem B. *Set*

$$X_n(t,\omega) := (\lambda_n, \xi_n(t,\omega)), \qquad (n \geqq 1). \tag{0.8}$$

Then $\frac{1}{N}\sum_{n=1}^{N} \delta_{X_n(\cdot,\omega)}$ *converges weakly to a probability measure Π^x on $C([0,\infty) \to [-A, A] \times \mathbf{R})$ as $N\to\infty$ P^x-almost surely, $(x\in S)$, where*

$$\int \phi(y)\,\Pi^x(dy) := \int E^{(\sigma,\eta)}[\phi(X(\cdot))]\,\pi_x(d\sigma \times d\eta) \tag{0.9}$$

for any bounded continuous function $\phi(y)$ on $C([0,\infty) \to [-A, A] \times \mathbf{R})$.

Now we shall describe the ergodic properties of the Brownian motion $(\xi(t), P^x)$ on the S.

Theorem C. (1) *If* $\gamma((-\infty, 0))>0$, $\xi(t)$ *is transient and has no* σ*-finite invariant measure.*

(2) $\xi(t)$ *has an invariant probability measure* μ, *if and only if* $\lambda_n>0$ *for all* $n\geqq 1$, *and in this case* μ *is unique and* $\lim_{t\to\infty} E^x[\phi(\xi(t))]=\int\phi(y)\mu(dy)$ *for any* $x\in S$ *and any bounded continuous measurable function* ϕ *on* S.

1. Infinite Dimensional Quadric Hypersurfaces

We shall begin with some definitions. Throughout this paper, δ_x denotes the measure having mass one at x.

Definition 1.1. The space $\mathbf{E}$ consists of all sequences $x=(x_1, x_2, \dots)\in\mathbf{R}^\infty$ such that $\sup_N \frac{1}{N}\sum_{n=1}^{N}\exp(\beta|x_n|)<\infty$ for any $\beta>0$ and that the probability measures $\frac{1}{N}\sum_{n=1}^{N}\delta_{x_n}(d\eta)$ on $\mathbf{R}$ converge weakly to a probability measure $\hat{\pi}_x(d\eta)$ on $\mathbf{R}$ as $N\to\infty$. The space $\mathbf{E}$ is endowed with the topology by the semi-metrics $\{d_N(x, y); 1\leqq N\leqq\infty\}$:

$$d_N(x, y):=\left(\frac{1}{N}\sum_{n=1}^{N}(x_n-y_n)^2\right)^{1/2}, \qquad d_\infty(x, y):=\limsup_{N\to\infty} d_N(x, y) \tag{1.1}$$

for $x=(x_1, x_2, \dots)$, $y=(y_1, y_2, \dots)\in\mathbf{E}$, and is equipped with the cylindrical σ-algebra $\mathscr{E}$.

Throughout this paper, we shall fix a real bounded sequence $\Lambda:=\{\lambda_n\}_{n=1}^{\infty}$ ($|\lambda_n|\leqq A$, $n\geqq 1$) such that the probability measures $\frac{1}{N}\sum_{n=1}^{N}\delta_{\lambda_n}(d\sigma)$ on $[-A, A]$ converge weakly to a probability measure $\gamma(d\sigma)$ on $[-A, A]$ with $\gamma(\{0\})=0$ as $N\to\infty$, and we assume $\lambda:=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda_n=\int\sigma\gamma(d\sigma)>0$ and call the γ the spectral measure of the Λ.

Definition 1.2. The space $\mathbf{E}(\Lambda)$ consists of all points $x=(x_1, x_2, \dots)\in\mathbf{E}$ such that the probability measures $\frac{1}{N}\sum_{n=1}^{N}\delta_{(\lambda_n, x_n)}(d\sigma\times d\eta)$ on $[-A, A]\times\mathbf{R}$ converge weakly to a probability measure $\pi_x(d\sigma\times d\eta)$ on $[-A, A]\times\mathbf{R}$ as $N\to\infty$.

Now the first assertion is the following

Proposition 1.1.

(I) $\mathbf{E}(\Lambda)$ *is a* d_∞*-closed measurable subset of the* $\mathbf{E}$.

(II) $\pi_x(d\sigma\times d\eta)$ *is weakly* d_∞*-continuous on* $\mathbf{E}(\Lambda)$, *as a function of* x.

(III) $\pi_x(B)$ *is* $\mathscr{E}$*-measurable in* x *for any* $B\in\mathscr{B}(\mathbf{R}^2)$.

Proof. It holds that

$$\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\phi(\lambda_n, x_n)-\frac{1}{N}\sum_{n=1}^{N}\phi(\lambda_n, y_n)\right|\leqq\left\|\frac{\partial\phi}{\partial\eta}\right\|_\infty d_\infty(x, y)$$

for any $x=(x_1, x_2, \dots)$, $y=(y_1, y_2, \dots)\in\mathbf{E}$ and $\phi(\sigma, \eta)\in C_0^\infty(\mathbf{R}^2)$, the space of real-valued C^∞-functions on $\mathbf{R}^2$ with compact supports. Here $\|\phi\|_\infty$ denotes the supremum norm of a function ϕ. Observing $\hat{\pi}_a(\mathbf{R})=1$ for any $a\in\mathbf{E}$, we have therefore the d_∞-closedness of $\mathbf{E}(\Lambda)$. The other assertions are obvious. □

Definition 1.3. An infinite dimensional diagonal quadratic form $h(x)$ is defined by

$$h(x):=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda_n x_n^2 \qquad \text{for } x=(x_1, x_2, \dots)\in\mathbf{E}(\Lambda); \tag{1.2}$$

the set $S=S_c$, $c\in\mathbf{R}$, defined by

$$S:=\{x\in\mathbf{E}(\Lambda);\ h(x)=c\} \tag{1.3}$$

is called an infinite dimensional quadric hypersurface or simply a quadric hypersurface.

Then S is a d_∞-closed measurable subset of $\mathbf{E}(\Lambda)$.

Now we introduce another measure $\rho(d\sigma, x)$, which is repeatedly used later.

Definition 1.4. For each $x=(x_1, x_2, \dots)\in\mathbf{E}(\Lambda)$, we put

$$\rho(d\sigma, x):=\int\eta^2\,\pi_x(d\sigma\times d\eta), \tag{1.4}$$

$$v(x):=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda_n^2 x_n^2. \tag{1.5}$$

2. Laws of Large Numbers

In this section, we shall consider laws of large numbers for mutually independent continuous processes $\{\xi_n(t, \omega)\}_{n=1}^{\infty}$, which are obtained from the stochastic differential equations (0.5) by substituting a given continuous function $\hat{v}(t)>0$ for the factor $v(\xi(t, \omega))$ of (0.5).

We begin with the following

Proposition 2.1. *Put* $\mathbf{T}:=[0, T]^{\otimes p}=\{t=(t_1, \dots, t_p);\ 0\leqq t_i\leqq T,\ 1\leqq i\leqq p\}$, $(T>0)$ *and assume that a family of random fields* $\{X_n(t, \omega);\ t\in\mathbf{T}\}_{n=1}^{\infty}$ *on a probability space* $(\Omega, \mathscr{F}, P)$ *satisfies the following conditions.*

(1) $\{X_n(t, \omega)\}_{n=1}^{\infty}$ *are mutually independent random fields.*

(2) $\sup_N\frac{1}{N}\sum_{n=1}^{N}E[\|X_n\|_T^r]<\infty$ *for some* $r>1$,

where $\|x\|_T := \sup_{t\in\mathbf{T}} |x(t)|$ *for* $x \in C(\mathbf{T}\to\mathbf{R})$.

(3) $E[X_n(\cdot,\omega)]=0$, $(n\geqq 1)$ *in the Bochner integral sense in* $C(\mathbf{T}\to\mathbf{R})$.

(4) *The family* $\{X_n(t,\omega);\, t\in\mathbf{T}\}_{n=1}^{\infty}$ *is uniformly tight in* $C(\mathbf{T}\to\mathbf{R})$.

Then the sample path of $\frac{1}{N}\sum_{n=1}^{N} X_n(t,\omega)$ *converges to zero uniformly on* $\mathbf{T}$ *as* $N\to\infty$ *almost surely*(P).

Proof. See [1] for the proof. □

Now we shall fix a point $x=(x_1,x_2,\ldots)\in\mathbf{E}(A)$ with $v(x)>0$ and a continuous function $\hat{v}(t)>0$ on $[0,\infty)$, and define a sequence of processes $\tilde{X}_n(t,\omega) := (\lambda_n, \tilde{\xi}_n(t,\omega))$, $(n\geqq 1)$ by

$$\tilde{\xi}_n(t,\omega):=\exp(-\lambda\lambda_n\hat{u}(t)/2)\left(x_n+\int_0^t \exp(\lambda\lambda_n\hat{u}(s)/2)\,dw_n(s,w)\right), \tag{2.1}$$

where $\{w_n(t)\}_{n=1}^{\infty}$ are mutually independent 1-dimensional standard Wiener processes on a probability space $(\Omega,\mathscr{F},P)$ and

$$\hat{u}(t):=\int_0^t 1/\hat{v}(s)\,ds. \tag{2.2}$$

Next we shall introduce another diffusion process $(X(t,\omega), P^{(\sigma,\eta)})$ with state space $[-A,A]\times\mathbf{R}$ and generator $L(t)$, $(t\geqq 0)$:

$$X(t,\omega):=(\tilde{\sigma}(\omega),\tilde{\zeta}(t,\omega))$$

and

$$L(t):=(1/2)\,\partial^2/\partial\eta^2-(\lambda\sigma\eta)/(2\hat{v}(t))\,\partial/\partial\eta. \tag{2.3}$$

We shall fix $T>0$, $\phi\in C_0^\infty(\mathbf{R}^{2p})$, and set

$$Z_n(t,\omega):=\phi(\tilde{X}_n(t_1,\omega),\ldots,\tilde{X}_n(t_p,\omega))-E[\phi(\tilde{X}_n(t_1),\ldots,\tilde{X}_n(t_p))]$$

for $t=(t_1,\ldots,t_p)\in\mathbf{T}=[0,T]^{\otimes p}$.

Then we have

Lemma 2.2. *For an integer* r *with* $r/2\geqq p+1$, *there exists a constant* $c=c(p,r,\phi,T)$[1] *such that* $(n\geqq 1)$

$$E[|Z_n(t)-Z_n(t')|^r]\leqq c\sum_{i=1}^{p}|t_i-t_i'|^{r/2} \tag{2.4}$$

for any $t=(t_1,\ldots,t_p)$, $t'=(t_1',\ldots,t_p')\in\mathbf{T}$,

$$E[|Z_n(0)|^r]\leqq(2\,\|\phi\|_\infty)^r.$$

---

[1] $c(z)$ denotes a positive constant depending only on z in this paper

Proof. It holds that for $t>s\geqq 0$

$$\tilde{\xi}_n(t)-\tilde{\xi}_n(s)=(\exp(-\lambda\lambda_n(\hat{u}(t)-\hat{u}(s))/2)-1)\,\tilde{\xi}_n(s) + \exp(-\lambda\lambda_n\hat{u}(t)/2)\int_s^t \exp(\lambda\lambda_n\hat{u}(\tau)/2)\,dw_n(\tau,\omega),$$

$$\tilde{\xi}_n(t)-\tilde{\xi}_n(s)=(1-\exp(\lambda\lambda_n(\hat{u}(t)-\hat{u}(s))/2))\,\tilde{\xi}_n(t) + \exp(-\lambda\lambda_n\hat{u}(s)/2)\int_s^t \exp(\lambda\lambda_n\hat{u}(\tau)/2)\,dw_n(\tau,\omega).$$

Hence by Jensen's inequality and Burkholder's one we have the following estimate for $t>s\geqq 0$ and $R>0$:

$$\begin{aligned} &E[|\tilde{\xi}_n(t)-\tilde{\xi}_n(s)|^r;\ \tilde{\xi}_n(t)\ \text{or}\ \tilde{\xi}_n(s)\in[-R,R]] \\ &\leqq 2^r(\exp(\lambda A(\hat{u}(t)-\hat{u}(s))/2)-1)^r R^r \\ &\quad + c(r)\exp(\lambda A\,\hat{u}(t)\,r/2)\left(\int_s^t \exp(\lambda A\,\hat{u}(\tau))\,d\tau\right)^{r/2}, \end{aligned}$$

where $\operatorname{supp}\phi\subset[-R,R]^{\otimes 2p}$. □

Therefore we have

Proposition 2.3.

$$\frac{1}{N}\sum_{n=1}^{N}\delta_{\tilde{X}_n(t_1,\omega)}(d\sigma_1\times d\eta_1)\dots\delta_{\tilde{X}_n(t_p,\omega)}(d\sigma_p\times d\eta_p) \xrightarrow[N\to\infty]{} \int P^{(\sigma,\eta)}(X(t_i)\in d\sigma_i\times d\eta_i,\ i=1,\dots,p)\,\pi_x(d\sigma\times d\eta) \tag{2.5}$$

weakly in $\mathscr{P}(\mathbf{R}^{2p})$ *for any* $(t_1,\dots,t_p)\in[0,\infty)^{\otimes p}$ a.s. (P), *where* $\mathscr{P}(\mathbf{R}^{2p})$ *stands for the space of probability measures on* $\mathbf{R}^{2p}$.

Proof. Lemma 2.2 shows the uniform tightness of the family $\{Z_n(t,\omega);\ t\in\mathbf{T}\}_{n=1}^{\infty}$ for the $\phi\in C_0^{\infty}(\mathbf{R}^{2p})$ by the Totoki-Kolmogorov criterion. Hence by Proposition 2.1, the sample path of $\frac{1}{N}\sum_{n=1}^{N}Z_n(t,\omega)$ converges to zero uniformly on $\mathbf{T}$ as $N\to\infty$ almost surely(P). Since

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}E[\phi(\tilde{X}_n(t_1),\dots,\tilde{X}_n(t_p))] = \int E^{(\sigma,\eta)}[\phi(X(t_1),\dots,X(t_p))]\,\pi_x(d\sigma\times d\eta)$$

for any $(t_1,\dots,t_p)\in[0,\infty)^{\otimes p}$, the proof is completed. □

Consequently we have

Theorem 2.4. *For* a.e. $\omega(P)$, *the probability measures* $\frac{1}{N}\sum_{n=1}^{N}\delta_{\tilde{X}_n(\cdot,\omega)}$ *on* $C([0,\infty)\to[-A,A]\times\mathbf{R})$ *converge weakly to a probability measure* Π^x *on* $C([0,\infty)\to[-A,A]\times\mathbf{R})$ *as* $N\to\infty$, *where the* Π^x *is defined as follows:*

$$\int f(y)\Pi^x(dy):=\int E^{(\sigma,\eta)}[f(X(\cdot))]\,\pi_x(d\sigma\times d\eta) \tag{2.6}$$

for bounded continuous functions f *on* $C([0,\infty)\to[-A,A]\times\mathbf{R})$.

Proof. With the aid of Proposition 2.3, it is sufficient for the proof to show for a.e. $\omega(P)$

$$\lim_{C\uparrow\infty}\inf_N\frac{1}{N}\sum_{n=1}^{N}\chi_{[0,C]}{}^2(|\tilde{X}_n(0,\omega)|)=1,$$

$$\lim_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sup_{\substack{|s-t|\leqq\delta\\ s,t\leqq T}}\chi_{[\rho,\infty)}(|\tilde{\xi}_n(t,\omega)-\tilde{\xi}_n(s,\omega)|)=0$$

for any $\rho, T>0$.

To this end, we choose $L_1, L_2 (L_1<L_2)$ for a given $\varepsilon>0$ so that $\hat{\pi}_x([L_1,L_2])>1-\varepsilon$ and $[L_1,L_2]$ is a $\hat{\pi}_x$-continuity interval. Then the mutual independence of $\{\tilde{\xi}_n(t,\omega)\}_{n=1}^{\infty}$ shows that for a.e. $\omega(P)$

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{[\rho,\infty)}(\sup_{\substack{|s-t|\leqq\delta\\ s,t\leqq T}}|\tilde{\xi}_n(t,\omega)-\tilde{\xi}_n(s,\omega)|)$$

$$=\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}P(\sup_{\substack{|s-t|\leqq\delta\\ s,t\leqq T}}|\tilde{\xi}_n(t)-\tilde{\xi}_n(s)|\geqq\rho)$$

$$\leqq 2\varepsilon \qquad \text{for sufficiently small } \delta>0,$$

because $\{\tilde{\xi}_n(t,\omega);\ L_1\leqq x_n\leqq L_2\}$ is uniformly tight. □

We have the following proposition concerning the growth of test functions.

Proposition 2.5 *For any* $\beta>0$,

$$\sup_N\frac{1}{N}\sum_{n=1}^{N}\exp\left(\beta\sum_{k=1}^{p}|\tilde{X}_n(t_k,\omega)|\right)<\infty \tag{2.7}$$

for $t_1,\dots,t_p\geqq 0$ a.s. (P).

Proof. The proof is immediate. □

Consequently Theorem 2.4 and Proposition 2.5 give

---

2 χ_D denotes the indicator function of the set D

Theorem 2.6. *Set*

$$\tilde{\xi}(t,\omega):=(\tilde{\xi}_1(t,\omega),\tilde{\xi}_2(t,\omega),\dots)\in\mathbf{R}^\infty. \tag{2.8}$$

Then we have the following assertions.

$$\pi_{\tilde{\xi}(t,\omega)}(d\sigma\times d\eta)=\int P^{(\sigma',\eta')}(X(t,\omega)\in d\sigma\times d\eta)\ \ \pi_x(d\sigma'\times d\eta') \tag{2.9}$$

for any $t\geqq 0$ a.s. (P).

$$\rho(d\sigma,\tilde{\xi}(t,\omega))=\hat{\rho}_t(d\sigma,x)\quad \text{for any } t\geqq 0 \text{ a.s. } (P), \tag{2.10}$$

where

$$\hat{\rho}_t(d\sigma,x):=\exp(-\lambda\sigma\hat{u}(t))\left(\rho(d\sigma,x)+\int_0^t\exp(\lambda\sigma\hat{u}(s))ds\,\gamma(d\sigma)\right). \tag{2.11}$$

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(\tilde{\xi}_n(t,\omega)-\tilde{\xi}_n(s,\omega))^2 &=\int\exp(-\lambda\sigma\hat{u}(t))\int_s^t\exp(\lambda\sigma\hat{u}(\tau))\,d\tau\,\gamma(d\sigma)\\ &\quad+\int(\exp(-\lambda\sigma(\hat{u}(t)-\hat{u}(s))/2)-1)^2\,\hat{\rho}_s(d\sigma,x)\end{aligned} \tag{2.12}$$

for any $t\geqq s\geqq 0$ a.s. (P).

3. A Nonlinear Integral Equation

Let $\mathcal{M}$ stand for the family of finite measures ρ absolutely continuous with respect to γ such that $\int\sigma^2\gamma(d\sigma)>0$, equipped with the weak convergence topology and the topological σ-algebra.

Given $\rho\in\mathcal{M}$, we shall show in this section the existence and the uniqueness of solutions $u(t)$ of the following nonlinear integral equation:

$$\int_I\sigma\exp(-\lambda\sigma u(t))\left(\rho(d\sigma)+\int_0^t\exp(\lambda\sigma u(s))\,ds\,\gamma(d\sigma)\right)=\int_I\sigma\rho(d\sigma), \tag{3.1}$$

where $I=[-A,A]$.

First we notice the following propositions.

Proposition 3.1. *For the* $\rho(d\sigma)$, *the following two integral equations are mutually equivalent in* $C([0,\infty))$:

$$\int_I\sigma\exp(-\lambda\sigma u)\left(\rho(d\sigma)+\int_0^u\exp(\lambda\sigma\eta)\,w(\eta)\,d\eta\,\gamma(d\sigma)\right)=\int_I\sigma\rho(d\sigma), \tag{3.2}$$

$$w(u)-\int_0^u K(u-\eta)\,w(\eta)\,d\eta=\int_I\sigma^2\exp(-\lambda\sigma u)\,\rho(d\sigma), \tag{3.3}$$

where

$$K(u) := \int_I \sigma^2 \exp(-\lambda \sigma u)\, \gamma(d\sigma). \tag{3.4}$$

Proof. The proof is obvious. □

Proposition 3.2. *The solution* $w(u)$ *of the Eq.* (3.3) *in* $C([0, \infty))$ *uniquely exists. It is a continuous, strictly positive global solution given by*

$$w(u) = \int_I W(u, \sigma)\, \sigma^2\, \gamma(d\sigma), \tag{3.5}$$

where

$$W(u, \sigma) := \exp(-\lambda \sigma u) + \int_0^u \Gamma(u-\eta) \exp(-\lambda \sigma \eta)\, d\eta \tag{3.6}$$

with the resolvent kernel $\Gamma(u-\eta)$ *of* (3.3).

Proof. The proof is immediate. □

Concerning the estimate of the $w(u)$, we have

Proposition 3.3. (I) *Assume* $\gamma([-A, 0)) > 0$. *Then*

$$c_1(\exp(c_1 u))/(c_1 + \lambda A) \int_I \sigma^2 \rho(d\sigma) \le w(u) \le 2 \exp((\lambda A + \gamma_2) u) \int_I \sigma^2 \rho(d\sigma), \tag{3.7}$$

where

$$c_1 := \int_{-A}^{0} \sigma^2 \gamma(d\sigma), \qquad \gamma_2 := \int_I \sigma^2 \gamma(d\sigma).$$

(II) *Assume* $\gamma([-A, 0]) = 0$. *Then*

$$c_2 \int_0^A \sigma^2 \rho(d\sigma) \le w(u) \le A \int_0^A \sigma \rho(d\sigma), \tag{3.8}$$

where c_2 *is a strictly positive constant depending only on* γ.

Proof. Assume $\gamma([-A, 0)) > 0$. Then

$$c_1 \exp(c_1(u-\eta)) \le \Gamma(u-\eta) \le \gamma_2 \exp((\lambda A + \gamma_2)(u-\eta)).$$

Hence we have

$$W(u, \sigma) \ge (c_1/(c_1 + \lambda \sigma)) \exp(c_1 u) + (\lambda \sigma/(c_1 + \lambda \sigma)) \exp(-\lambda \sigma u) \quad \text{for } \sigma \ge 0,$$

$$W(u, \sigma) \ge \exp(c_1 u) \quad \text{for } \sigma \le 0,$$

$$W(u, \sigma) \le 2 \exp((\lambda A + \gamma_2) u) \quad \text{for } \sigma \in [-A, A].$$

Therefore we have (3.7) because of (3.5).

Next assume $\gamma([-A, 0])=0$. Denoting by $W_1(u)$ the unique global solution of the following equation in $C([0, \infty))$, we have $W(u, \sigma) \geqq W_1(u)$ for any $\sigma \geqq 0$:

$$W_1(u) - \int_0^u K(u-\eta)\, W_1(\eta)\, d\eta = \exp(-\lambda A u).$$

Furthermore,

$$\int_0^\infty \exp(-p u)\, W_1(u)\, du/(\lambda/(p A)) \to 1 \qquad \text{as } p \downarrow 0.$$

Hence $c_2 := \inf_{u \geqq 0} W_1(u) > 0$, which proves the left-hand side of (3.8). An easy combination of (3.2), (3.3) yields the right-hand side of (3.8). □

Now we shall construct a global solution of (3.1) for $\rho \in \mathscr{M}$. Denote by $\hat{C}_0^1([0, T))$, $(0 < T \leqq \infty)$ the family of $f(t) \in C^1([0, T))$ with $f(0)=0$. ($f'(0)$ means the right derivative at $t=0$.) Then we can state the following

Theorem 3.4. *The Eq.* (3.1), *($\rho \in \mathscr{M}$) has a unique solution $u(t)=u(t, \rho)$ in $\hat{C}_0^1([0, T))$ for any $T \in (0, \infty]$, and the mapping $(t, \rho) \to u(t, \rho)$ is continuous on $[0, \infty) \times \mathscr{M}$.*

Proof. For the solution $w(u)$ on $[0, U)$ of (3.2), $(0 < U \leqq \infty)$, we put

$$T := \lim_{u \uparrow U} \int_0^u w(\eta)\, d\eta, \tag{3.9}$$

and define a function $u(t)$ on $[0, T)$ as follows:

$$\int_0^{u(t)} w(\eta)\, d\eta = t. \tag{3.10}$$

Then it is easily seen that $u(t) \in \hat{C}_0^1([0, T))$ and

$$u'(t) = 1/w(u(t)) \qquad \text{on } [0, T), \tag{3.11}$$

and therefore $u(t)=u(t, \rho)$ is a solution of (3.1) on $[0, T)$.

Since $\int_0^\infty w(\eta)\, d\eta = \infty$ by Proposition 3.3, we have a global solution $u(t, \rho)$ of (3.1).

To show uniqueness of the solutions of (3.1), let $u(t) \in \hat{C}_0^1([0, T))$, $(0 < T \leqq \infty)$ be a solution of (3.1). Then we have

$$u'(t) = 1/\tilde{v}(t, \rho) \qquad \text{on } [0, T), \tag{3.12}$$

where

$$\tilde{v}(t, \rho) := \int_I \sigma^2 \exp(-\lambda \sigma u(t)) \cdot \left(\rho(d\sigma) + \int_0^t \exp(\lambda \sigma u(s))\, ds\, \gamma(d\sigma)\right) > 0. \tag{3.13}$$

Hence by putting

$$w(u(t))=\tilde{v}(t,\rho), \tag{3.14}$$

we have the unique solution $w(u)$ of (3.2) on $[0, U), (U:=\lim_{t\uparrow T} u(t))$. Furthermore we construct a solution $\tilde{u}(t)$ of (3.1) on $[0, T'), \left(T':=\lim_{u\uparrow U} \int_0^u w(\eta)\,d\eta\right)$ by (3.10) with the aid of $w(u)$. Then it can be easily seen that $T=T'$ and $u(t)=\tilde{u}(t)$ on $[0, T)$, i.e. uniqueness of the solutions of (3.1).

Now we shall show the (t, ρ)-continuity of $u(t, \rho)$. Assume that there exist sequences $t_n \geqq 0$, $\rho_n \in \mathscr{M}$ $(n \geqq 1)$ and $t_0 \geqq 0$, $\rho_0 \in \mathscr{M}$, $\varepsilon>0$ such that $\lim_{n\uparrow\infty} t_n=t_0$, $\lim_{n\uparrow\infty} \rho_n=\rho_0$ weakly, and $|u(t_n, \rho_n)-u(t_0, \rho_0)| \geqq \varepsilon$ for $n \geqq 1$. Here $\{u(t_n, \rho_n)\}_{n=1}^{\infty}$ are bounded by Proposition 3.3 through (3.12), (3.14). Therefore by Proposition 3.3 again the dominated convergence theorem can be applied to show

$$\liminf_{n\to\infty} \left| \int_0^{u(t_n,\rho_n)} w(\eta, \rho_n)\,d\eta - \int_0^{u(t_0,\rho_0)} w(\eta, \rho_0)\,d\eta \right| \geqq C\varepsilon \int \sigma^2 \rho_0(d\sigma),$$

where $C=c_2$ in the case $\gamma([-A, 0))=0$ and $C=c_1/(c_1+\lambda A)$ in the case $\gamma([-A, 0))>0$. This induces an obvious contradiction through (3.10). □

Also $\tilde{v}(t, \rho)$ depends in a continuous way on (t, ρ), because of (3.13).

Now put

$$u(t, x):=u(t, \rho(\cdot, x)), \tag{3.15}$$

$$\tilde{v}(t, x):=\tilde{v}(t, \rho(\cdot, x)), \tag{3.16}$$

for $x \in \mathbf{E}(\Lambda)$ with $v(x)>0$, abusing slightly notations.

Then we have the following

Corollary 3.5. *The mappings $(t, x) \to u(t, x)$ and $\tilde{v}(t, x)$ are measurable and continuous on the set $[0, \infty) \times \{x \in \mathbf{E}(\Lambda);\ v(x)>0\}$, with respect to the d_∞-semi-metric.*

4. Brownian Motion on the Quadric Hypersurface S

In this section, by making use of the preceding results, we shall construct Brownian motion $(\xi(t, \omega), P^x)$ on the quadric hypersurface $S=S_c (c \neq 0)$ or $\mathring{S}_0=\{x \in S_0;\ v(x)>0\}$.

First we shall show the existence of a solution $\zeta(t, \omega):=(\zeta_1(t, \omega), \zeta_2(t, \omega), \ldots)$ of the following system of stochastic differential equations:

$$\zeta_n(t, \omega)=\zeta_n(0, \omega)+w_n(t, \omega) - \int_0^t \lambda\lambda_n \zeta_n(s, \omega)/(2v(\zeta(s, \omega)))\,ds, \quad (n \geqq 1) \tag{4.1}$$

on a complete probability space $(\Omega, \mathscr{G}, P; \mathscr{G}_t)$.

Let $W(t,\omega)=(w_1(t), w_2(t,\omega), \ldots)$ be the sequence of mutually independent 1-dimensional $\mathscr{G}_t$-adapted standard Wiener processes satisfying

$$E[W(t+h)-W(t)|\mathscr{G}_t]=0 \qquad \text{for any } t, h \geqq 0.$$

Now we are in a position to state

Definition 4.1. A process $\zeta(t,\omega)$ defined on the complete probability space $(\Omega, \mathscr{G}, P; \mathscr{G}_t)$ is called a solution of (4.1), if the following conditions (i), (ii) are satisfied.

(i) $\zeta(t)=(\zeta_1(t), \zeta_2(t), \ldots)$ is a $\mathscr{G}_t$-adapted conservative continuous process on S.

(ii) $(\zeta(t))_{t\geqq 0}$ satisfies (4.1) with probability one.

Now we have

Theorem 4.1. *We are given a sequence $W(t)=(w_1(t), w_2(t), \ldots)$ of mutually independent 1-dimensional $\mathscr{G}_t$-adapted standard Wiener processes on the complete probability space $(\Omega, \mathscr{G}, P; \mathscr{G}_t)$. Next put for any $x=(x_1, x_2, \ldots)\in S$*

$$\xi^x(t,\omega):=(\xi_1^x(t,\omega), \xi_2^x(t,\omega), \ldots), \tag{4.2}$$

$$\xi_n^x(t,\omega):=\exp(-\lambda\lambda_n u(t,x)/2) \cdot \left(x_n + \int_0^t \exp(\lambda\lambda_n u(s,x)/2)\, dw_n(s,\omega)\right), \tag{4.3}$$

where $u(t,x)$ is the global solution in $\hat{C}_0^1([0,T))$ of (3.1) with $\rho=\rho(d\sigma,x)$. Then the process $\xi^x(t)$ is a solution of (4.1) with $\xi^x(0)=x$ a.s..

Proof. Applying Theorem 2.6 to $\xi^x(t,\omega)$, we have

$$\tilde{v}(t,x)=v(\xi^x(t,\omega)) \qquad \text{for any } t\geqq 0 \text{ a.s.}, \tag{4.4}$$

where

$$\tilde{v}(t,x)=\int_{-A}^{A} \sigma^2 \hat{\rho}_t(d\sigma,x). \tag{4.5}$$

Hence by (3.12), Ito's formula shows that $\{\xi_n^x(t)\}_{n=1}^{\infty}$ satisfies (4.1). Appealing to Theorem 2.6 again, we can see that $\xi^x(t,\omega)$ is a continuous $\mathscr{G}_t$-adapted process on S. □

Next we shall show the pathwise uniqueness of solutions for (4.1).

Lemma 4.2. *Put for $x=(x_1, x_2, \ldots)\in S$,*

$$b(x):=(\lambda_1 x_1, \lambda_2 x_2, \ldots)/v(x)\in \mathbf{R}^\infty. \tag{4.6}$$

Then for $x, y\in S$

$$d_\infty(b(x), b(y))\leqq (A/v(x))\,(2+\sqrt{v(x)/v(y)})\, d_\infty(x,y). \tag{4.7}$$

Proof. The proof is immediate. □

Theorem 4.3. *Fix* $x \in S$. *Let* $\zeta(t,\omega) := (\zeta_1(t,\omega), \zeta_2(t,\omega), \ldots)$ *be a solution of* (4.1) *with* $\zeta(0) = x$ a.s. *on the complete probability space* $(\Omega, \mathscr{G}, P; \mathscr{G}_t)$ *and let* $\xi^x(t)$ *be the solution of* (4.1) *on the* $(\Omega, \mathscr{G}, P; \mathscr{G}_t)$. *Then we have*

$$\zeta(t,\omega) = \xi^x(t,\omega) \qquad \textit{for any } t \geqq 0 \text{ a.s. } (P). \tag{4.8}$$

Proof. By the continuity of $\zeta(t,\omega)$, $\xi^x(t,\omega)$ on S, there exists almost surely a constant $c(x, T, \omega) > 0$ for any $T > 0$ such that

$$d_\infty(\xi^x(t,\omega), \zeta(t,\omega)) \leqq c(x, T, \omega) \int_0^t d_\infty(\xi^x(s,\omega), \zeta(s,\omega))\, ds, \qquad t \leqq T$$

with the aid of (4.1) and Lemma 4.2. Hence we have $d_\infty(\xi^x(t,\omega), \zeta(t,\omega)) = 0$ a.s. by Grownwall inequality. Consequently $v(\zeta(t,\omega)) = v(\xi^x(t,\omega)) = \tilde{v}(t,x)$ a.s., which shows (4.8) through (4.1). □

Now we put for a bounded measurable function f on S

$$\int_S f(z)\, p_t(x, dz) := E[f(\xi^x(t))]. \tag{4.9}$$

Then we have

Proposition 4.4. *For a bounded measurable function* f *on* S *and* $s, t \geqq 0$,

$$E[f(\xi^x(s+t)) \mid \mathscr{G}_s](\omega) = \int f(z)\, p_t(y, dz) \text{ a.s.}, \tag{4.10}$$

where $y := \xi^x(s,\omega)$.

Proof. By Theorem 4.3, we have

$$\xi_n^x(s+t) = \exp(-\lambda\lambda_n u(t, \xi^x(s))/2)\left(\xi_n^x(s,\omega) + \int_0^t \exp(\lambda\lambda_n u(\tau, \xi^x(s))/2)\, dw_n(s+\tau)\right), \tag{4.11}$$

$t \geqq 0$ a.s. for any $s \geqq 0$, which shows (4.10). □

Now the law P^x, $(x \in S)$ on $\mathscr{C} := C([0,\infty) \to S)$ induced by the solution $\xi^x(t)$ with $\xi^x(0) = x$ a.s. is well defined. Then by putting

$$\xi(t, w) := w(t) \qquad \text{for } w \in \mathscr{C}, \tag{4.12}$$

$$\mathscr{F}_t^0 := \sigma(\xi(s); s \leqq t), \qquad \mathscr{F}^0 := \sigma(\xi(s); s < \infty) \tag{4.13}$$

and denoting by $\mathscr{F}_t$, $\mathscr{F}$ the completion of $\mathscr{F}_t^0$, $\mathscr{F}^0$ as usual, we can see

Theorem 4.5. $(\mathscr{C}, \mathscr{F}, \mathscr{F}_t, \xi(t), P^x)$ *with the state space* S *is a diffusion process with the Feller property: if* $f(x)$ *is a bounded continuous measurable function on* S, *so is* $E^x[f(\xi(t))]$, $(t \geqq 0$; *fixed*).

Proof. The diffusion property is easily seen by Proposition 4.4. Hence we have only to show the Feller property. Let $\{X_n(\omega)\}_{n=1}^{\infty}$ be mutually independent random variables on the probability space $(\Omega, \mathscr{G}, P)$ with law $N(0,1)$ and put

$$Y(x,\omega) := (Y_1(x,\omega), Y_2(x,\omega), \ldots)$$

$$Y_n(x,\omega) := \exp(-\lambda \lambda_n u(t,x)/2)\left(x_n + \sqrt{\int_0^t \exp(\lambda \lambda_n u(s,x))\, ds}\, X_n(\omega)\right)$$

for $x=(x_1, x_2, \ldots) \in S$ and a fixed $t \geqq 0$. Then the law of $(\xi^x(t), P)$ is identical with the one of $(Y(x), P)$.

Now we are given $\{x_k\}_{k=1}^{\infty} \subset S$ such that x_k converges to a point $a \in S$ as $k \to \infty$. Then applying the strong law of large numbers to the independent random variables $\{Y_n(x_k) - Y_n(a)\}_{n=1}^{\infty}$, we can see that $Y(x_k)$ converges to $Y(a)$ as $k \to \infty$ a.s.. Hence by the dominated convergence theorem, we have

$$\lim_{k \to \infty} E[f(Y(x_k))] = E[f(Y(a))]. \quad \square$$

Definition 4.2. The diffusion process $(\mathscr{C}, \mathscr{F}, \mathscr{F}_t, \xi(t,\omega), P^x)$ with the state space S is called the Brownian motion on S.

5. Ergodic Properties of the Brownian Motion $\xi(t, \omega)$

In this section, we shall study the ergodic properties of the Brownian motion $\xi(t,\omega) = (\xi_1(t,\omega), \xi_2(t,\omega), \ldots)$ on the quadric hypersurface $S = S_c$ $(c \neq 0)$ or $\mathring{S}_0$.

We shall begin with

Proposition 5.1. (i) *Assume* $\gamma([-A, 0)) > 0$. *Then we have*

$$t \int_{-A}^{0} \sigma^2 \gamma(d\sigma) \leqq \tilde{v}(t,x) \leqq \tilde{v}(0,x) + \left(\lambda A + \int_{-A}^{A} \sigma^2 \gamma(d\sigma)\right) t \qquad \textit{for any } t \geqq 0. \quad (5.1)$$

(ii) *Assume* $\gamma((0, A]) = 1$. *Then*

$$\lim_{t \to \infty} \tilde{v}(t,x) = \lambda c \qquad \textit{for } x \in S_c, (c > 0). \quad (5.1)$$

Proof. Assume $\gamma([-A, 0)) > 0$ and put

$$\tilde{v}_+(t,x) := \int_0^A \sigma^2 \exp(-\lambda \sigma u(t,x)) \left(\rho(d\sigma, x) + \int_0^t \exp(\lambda \sigma u(s,x))\, ds\, \gamma(d\sigma)\right),$$

$$\tilde{v}_-(t,x) := \int_{-A}^{0} \sigma^2 \exp(-\lambda \sigma u(t,x)) \left(\rho(d\sigma, x) + \int_0^t \exp(\lambda \sigma u(s,x))\, ds\, \gamma(d\sigma)\right).$$

Then we have the estimate (i) from the following ones:

$$\int_{-A}^{0} \sigma^2 \gamma(d\sigma) \leqq \tilde{v}'_-(t,x) \leqq \lambda A + \int_{-A}^{0} \sigma^2 \gamma(d\sigma), \qquad \tilde{v}'_+(t,x) \leqq \int_{0}^{A} \sigma^2 \gamma(d\sigma).$$

Next assume $\gamma([-A,0))=0$. An application of the Laplace transform to the both sides of (3.3) with $\rho=\rho(\cdot,x)$ shows $\lim_{u\to\infty} w(u)=\lambda c$, which yields (5.2) through (3.14). □

Furthermore, in the case $\gamma((0,A])=1$, we set

$$\tilde{S}_c := \{x \in S_c;\ \rho(d\sigma,x)=(c/\sigma)\gamma(d\sigma),\ \sigma>0\}, \qquad (c>0). \tag{5.3}$$

Then we have

Proposition 5.2. *Assume* $\gamma((0,A])=1$.

(i) $\xi(t,\omega)$ *is a conservative diffusion process on the* d_∞*-closed measurable subset* $\tilde{S}_c$.

(ii) $\xi(t,\omega)\notin\tilde{S}_c$ *for any* $t\geqq 0$ a.s. (P^x), *if* $x\in S_c$, $x\notin\tilde{S}_c$.

Proof. Notice that $\rho(d\sigma,x)=(c/\sigma)\gamma(d\sigma)$, $(\sigma>0)$ is equivalent to $\tilde{v}(t,x)=\lambda c$ for any $t\geqq 0$. □

Now we proceed to study the ergodic properties of the Brownian motion $\xi(t)$ on S_c.

Proposition 5.3. (i) *Assume* $\gamma([-A,0))>0$. *Then* $\xi(t)$ *has no* σ*-finite invariant measure on* S_c.

(ii) *Assume* $\gamma((0,A])=1$ *and there is one* $\lambda_n\leqq 0$ *at least. Then* $\xi(t)$ *on* S_c *has no invariant probability measure.*

Proof. Use Proposition 5.1 (i) in the case (i). □

Remark. Assume $\gamma((0,A])=1$ only. Then there exists a probability measure ν on S_c such that

$$\int_{S_c} f(x)\,\nu(dx) = \int_{S_c} E^x[f(\xi(t))]\,\nu(dx), \qquad t\geqq 0 \tag{5.4}$$

holds for any d_∞-continuous bounded measurable function $f(x)$ on S_c.

Next we shall introduce the following condition:

$$\lambda_n>0,\ (n\geqq 1) \quad \text{and}$$
$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\exp(\beta/\lambda_n) = \int_{0}^{A}\exp(\beta/\sigma)\,\gamma(d\sigma) \qquad \text{for any } \beta>0. \tag{5.5}$$

Definition 5.1. Under the condition (5.5), we denote by μ the induced measure by $\tilde{\zeta}(\omega)=(\tilde{\zeta}_1(\omega),\tilde{\zeta}_2(\omega),\dots)$ on $\mathbf{R}^\infty$, where $\{\tilde{\zeta}_n(\omega)\}_{n=1}^\infty$ are mutually independent random variables on the $(\Omega,\mathscr{G},P)$ with law $N(0,c/\lambda_n)$ respectively.

Then we have

Proposition 5.4. *Assume the condition* (5.5). *Then* μ *is an invariant probability measure of* $\xi(t)$ *and* $\operatorname{supp}(\mu) \subset \tilde{S}_c$.

Proof. In fact

$$\pi_{\tilde{\zeta}(\omega)}(d\sigma \times d\eta) = \frac{1}{(2\pi c/\sigma)^{1/2}} \exp(-\eta^2/(2c/\sigma))\, d\eta\, \gamma(d\sigma) \quad \text{a.s.}\,(P). \quad \square \tag{5.6}$$

Theorem 5.5. *Assume the condition* (5.5). *Then*

$$\lim_{t\to\infty} E^x[\phi(\xi(t))] = \int \phi(y)\, \mu(dy), \qquad x \in S_c \tag{5.7}$$

for any bounded continuous measurable function ϕ *on* S_c.

Proof. First observe that

$$E^x[\phi(\xi(t))] = E[\phi(\hat{\xi}(t))], \qquad (x = (x_1, x_2, \ldots),\ t \geqq 0), \tag{5.8}$$

where $\hat{\xi}(t, \omega) = (\hat{\xi}_1(t, \omega), \hat{\xi}_2(t, \omega), \ldots)$ is given by

$$\hat{\xi}_n(t, \omega) := \exp(-\lambda \lambda_n u(t)/2) \left(x_n + \sqrt{\lambda_n/c \int_0^t \exp(\lambda \lambda_n u(s))\, ds\, \tilde{\zeta}_n(\omega)} \right). \tag{5.9}$$

Next we define $\tilde{\Phi}(\sigma, t)$, $(\sigma, t \geqq 0)$ by

$$\tilde{\Phi}(\sigma, t) := \left(\sqrt{(\sigma/c) \exp(-\lambda \sigma u(t)) \int_0^t \exp(\lambda \sigma u(s))\, ds} - 1 \right)^2.$$

Then by $\tilde{v}(t, x) \leqq cA$, we have $\tilde{\Phi}(\sigma, t) \leqq (\sqrt{A/\lambda} + 1)^2$.

Now Kolmogorov's law of large numbers shows that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $[0, \delta)$ is a γ-continuity set and

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{\substack{1 \leqq n \leqq N \\ \lambda_n \leqq \delta}} \tilde{\Phi}(\lambda_n, t)\, \tilde{\zeta}_n^2(\omega) \leqq (\sqrt{A/\lambda} + 1)^2 \int_0^\delta c/\sigma\, \gamma(d\sigma) < \varepsilon$$

for any $t \geqq 0$ a.s. (P). Next Proposition 2.1 can be applied to show

$$\frac{1}{N} \sum_{\substack{1 \leqq n \leqq N \\ \lambda_n > \delta}} \tilde{\Phi}(\lambda_n, t)\, (\tilde{\zeta}_n^2(\omega) - c/\lambda_n)$$

converges to zero uniformly on any compact set of t as $N \to \infty$ a.s. (P), and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{1 \leqq n \leqq N \\ \lambda_n > \delta}} \tilde{\Phi}(\lambda_n, t) = \int_\delta^A \tilde{\Phi}(\sigma, t)\, c/\sigma\, \gamma(d\sigma) \to 0 \qquad \text{as } t \to \infty.$$

Hence we have

$$\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\tilde{\Phi}(\lambda_n,t)\tilde{\zeta}_n^2(\omega)=0 \quad \text{a.s.}\,(P),$$

which shows

$$\lim_{t\to\infty} d_\infty(\hat{\xi}(t,\omega),\tilde{\zeta}(\omega))=0 \quad \text{a.s.}\,(P). \tag{5.10}$$

Furthermore it is immediate that $d_N(\hat{\xi}(t,\omega),\tilde{\zeta}(\omega))\to 0$ as $t\to\infty$ a.s. (P), $(N<\infty)$. Therefore the dominated convergence theorem completes the proof. □

Consequently we have the following

Corollary 5.6. (i) *Under the condition* (5.5), *the* $\xi(t)$ *on* S_c *has a unique invariant probability measure.*

(ii) *Assume that the Brownian motion* $\xi(t,\omega)$ *on* S_c *has the standard Gaussian white noise as its invariant measure. Then* $S_c=\{x\in\mathbf{E}(A);\, d_\infty(x,0)=1\}$.

Finally it should be noted that the invariant probability measure μ of $\xi(t,\omega)$ on S_c is supported by the restricted part $\tilde{S}_c$ of S_c, if it exists. This is just in concordance with P. Lévy's observation [5], because the invariant probability measure of the Brownian motion $\xi(t)$ on S_c can be thought of as the area of the hypersurface S_c.

Acknowledgement. The author thanks Professors Tadahisa Funaki and Izumi Kubo for their invaluable advice. Proofs of Theorem 4.3 and Proposition 4.4 are due, in part to Professor T. Funaki.

References

1. Daffer, P.Z., Taylor, R.L.: Laws of large numbers for $D[0,1]$. Ann. Probab. **7**, 85–95 (1979)
2. Hasegawa, Y.: Lévy's functional analysis in terms of an infinite dimensional Brownian motion. I. Osaka J. Math. **19**, 405–428 (1982)
3. Hasegawa, Y.: Lévy's functional analysis in terms of an infinite dimensional Brownian motion. II. Osaka J. Math. **19**, 549–570 (1982)
4. Hasegawa, Y.: Lévy's functional analysis in terms of an infinite dimensional Brownian motion. III. Nagoya Math. J. **90**, 155–173 (1983)
5. Lévy, P.: Problèmes concrets d'analyse fonctionnelle. Paris: Gauthier-Villars 1951

Received November 5, 1986; in revised form July 4, 1988