Some Results about Smoothing Methods of Fourier Series (*) (**).

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Summary. – *In this paper we compare the Fourier polynomial's approximation, in $C(T^n)$ or in $L^p(T^n)$, with that one obtained by a class of smoothing methods, which naturally arise in solving ill posed problems. It is also given a sharp evaluation of the above approximation in the space* $\mathrm{Lip}(\alpha, C(T))$, $0 < \alpha < 1$.

1. – Introduction.

Let us consider an integrable, sufficiently smooth, function f defined on the N-dimensional torus. In order to obtain both a good graphic representation and a good L^2 approximation of f, some years ago M. FRONTINI and L. GOTUSSO ([4], [5]) approximated f in the case $N = 1, 2$ by trigonometric polynomials obtained with a technique similar to that one used by D. L. PHILLIPS [11] for smoothing the approximated solutions of an integral equation of the first kind.

This smoothing process was obtained by means of kernels of the form $\sum_n (1 - \sigma P(n))^{-1} \exp(2\pi i n t)$, where P is a suitable homogeneous polynomial of degree 4 and σ is a real positive smoothing parameter.

Many results were later obtained ([6], [3], [7]) on the subject, concerning the N-dimensional torus and general homogeneous polynomials of even degree k such that $P(x) > 0$ for every $x \in R^N$, $x \neq 0$.

Briefly, we recall some of the mentioned results.

If $N \geqq 1$, let Z^N be the lattice of integer points of R^N and $T^N = R^N/Z^N$ the N-dimensional torus. Let us name B, indifferently, the Lebesgue space $L^p(T^N)$, $1 \leqq p \leqq +\infty$, or the space of continuous functions $C(T^N)$ and denote their norm by $\| \ \|_B$.

If $f \in L^1(T^N)$ and $\sigma > 0$, let

$$f_\sigma \sim \sum_{n \in Z^N} \frac{\hat{f}(n)}{1 + \sigma P(n)} \exp(2\pi i n t), \qquad t \in T^N. \tag{1.1}$$

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Whenever $f \in B$, also $f_\sigma \in B$ and $\|f - f_\sigma\|_B \to 0$. Moreover, if k is big enough (e.g. $k > N$), the series (1.1) is absolutely and uniformly convergent to f_σ over T^N. Then in this case we can approximate every $f \in B$ as close as we want both by f_σ and by Fourier polynomials of f_σ, also in $L^1(T^N)$ and $C(T^N)$, where generally the approximation by Fourier polynomials fails.

In this paper we compare the approximation given in B by the above method and by more general other smoothing methods with that one given by Fourier polynomials.

2. – For every integer $m \geqq 0$ and for every real $\sigma \geqq 0$ let us set

$$P_{m,\sigma} = P_{m,\sigma}(f) = \sum_{|n| \leqq m} \frac{\hat{f}(n)}{1 + \sigma P(n)} \exp(2\pi i n t), \qquad t \in T^N. \tag{2.1}$$

In the Introduction we observed that if $B = L^1$ or $B = C$ there exist functions such that the inequality

$$\|f - P_{m,\sigma}\|_B \leqq \|f - P_{m,0}\|_B \tag{2.2}$$

is satisfied at least for sufficiently small σ and large m.

Nevertheless, this is not enough to ensure that $P_{m,\sigma}$ give an essentially better approximation of f than that one obtained by Fourier polynomials $P_{m,0}$. Indeed for every smoothing method, by using the properties of the lacunary Fourier series can be easily proved the following theorem.

THEOREM 1. – For every $\varepsilon > 0$ there exists a function $f \in B$ (which is not a trigonometric polynomial) with an absolutely convergent Fourier series such that every polynomial

$$Q(t) = \sum_{|n| \leqq m} \alpha_n \hat{f}(n) \exp(2\pi i n t), \qquad t \in T^N$$

satisfies the inequality

$$\|f - Q\|_B > \frac{K_B}{1 + \varepsilon} \|f - P_{m,0}\|_B \tag{2.3}$$

where

$$K_B = \begin{cases} \frac{1}{2} & \text{if } B = L^p(T^N),\ 1 \leqq p < 2 \\ p^{-\frac{1}{2}} & \text{if } B = L^p(T^N),\ 2 < p < +\infty \\ 1 & \text{if } B = C(T^N). \end{cases}$$

3. – If f is a trigonometric polynomial of degree $s \neq 0$, obviously (2.2) doesn't hold if $m \geqq s$. Moreover, in the case $B = C(T)$, for every function f of the form

$$f(t) = \sum_{n=0}^{\infty} a_n \cos 2\pi nt\,, \qquad t \in T$$

with $a_n > 0$ and $\sum_{n=0}^{+\infty} a_n < +\infty$, it can easily be seen that (2.2) doesn't hold whatever is $m > 0$ and σ.

The following theorem shows that the same happens in the case $B = L^1(T)$.

THEOREM 2. – There exist functions $f \in L^1(T)$ such that

$$\|f - P_{m,\sigma}\|_1 > \|f - P_{m,0}\|_1 \tag{3.1}$$

for every real $\sigma > 0$ and for every positive integer m.

This theorem is a particular case of the following more general result.

Let $M: (\boldsymbol{R}^+ \cup \{0\}) \times \boldsymbol{N} \to \boldsymbol{C}$ be such that $M(\sigma, 0) = 1$, $\forall \sigma > 0$ and $M(0, n) = 1$, $\forall n \in \boldsymbol{N}^+$.

If $f \in L^1(T)$ let us set

$$f_\sigma(t) = f_{\sigma,M}(t) \sim \sum_{n \in Z} M(\sigma, |n|)\hat{f}(n) \exp(2\pi i n t)\,,$$

$$Q_{m,\sigma}(t) = Q_{m,\sigma,M}(t) = \sum_{|n| \leqq m} M(\sigma, |n|)\hat{f}(n) \exp(2\pi i n t)\,.$$

THEOREM 3. – Suppose that

1) $M(\sigma, 1) \neq 1, \quad \forall \sigma > 0;$

2) $\operatorname*{Sup}_{\sigma > 0} \left|\dfrac{1 - M(\sigma, n)}{1 - M(\sigma, 1)}\right| < +\infty\,, \quad \forall n \in \boldsymbol{N}^+.$

Then there exist functions $f \in L^1(T)$ (which are not trigonometric polynomials) such that

$$\|f - Q_{m,\sigma}\|_1 > \|f - Q_{m,0}\|_1 \tag{3.2}$$

for every real $\sigma > 0$ and for every positive integer m.

REMARKS. – 1) Obviously, condition 1) cannot be relaxed if (3.2) has to be satisfied for every $\sigma > 0$ and $m > 0$. If instead of 1) we require $M(\sigma, n_0) \neq 1$, $\sigma > 0$ and we make the same assumption in 2), then the result holds for every $m > n_0$: Moreover, the proof of the theorem shows that the result is true also if 2) is verified only for $n = 2^k n_0$, $k = 1, 2, \dots$.

2) Theorem 3 can be applied, for instance, to the classical cases of Féjer and Poisson Kernels, whose Fourier transforms satisfy 1) and 2).

3) We can also apply Theorem 3 to the case $M(\sigma, n) = \hat{\mu}_\sigma(n)$ where $\{\mu(\sigma)\}_{\sigma>0}$ is a family of positive bounded measures which weak $*$ converges for $\sigma \to 0+$ to the unit mass measure $\delta(0)$. For instance, if $d\mu_\sigma(x) = (1/\sigma)\varphi(x/\sigma)\, dx$, where φ is a sufficiently smooth positive function, supported in a neigbourough of the origin, whose integral is one, then (2.2) holds for every sufficiently large m. Indeed, the hypotheses of Th. 3 in the weaker form of the remark 1 are satisfied.

Another case in this scheme is that of convolution semigroups, i.e. a family $\{\mu_\sigma\}_{\sigma>0}$ of positive bounded measures with: *a*) $\mu_\sigma(T) \leqq 1$, $\sigma > 0$; *b*) $\mu_\sigma * \mu_s = \mu_{\sigma+s}$ $\sigma, s > 0$; *c*) $\mu_\sigma \to \delta(0)$ for $\sigma \to 0^+$. (See e.g. [2], def. 8.1). From Th. 8.3 and 7.17 of [2] one can easily deduce that hypotheses of theorem 3 are satisfied, except for the trivial cases $\mu_\sigma = \delta(0)$, $\sigma > 0$.

4. – Now we come back to consider the smoothing method (1.1). We recall that if $f \in B$, for sufficiently large m, the polynomials $P_{m,\sigma}$ in (2.1) give us an as good approximation of f as we want. At present, we would like to give an estimate of such approximation, at least for some classes of functions.

An extimate of $\|f_\sigma - P_{m,\sigma}\|$ can be found for instance in [3], Th. 4 and [7], pp. 354-356. Here we obtain some evaluations of $\|f - f_\sigma\|$ for functions in Lipschitz classes.

A similar result for these classes was obtained in [9] and [14] in the case of Féjer sums.

We recall the definition of Lipschitz class. We say that $f \in B$ belongs to the Lipschitz class K lip (α, B), $0 < \alpha \leqq 1$ if we have

$$\|\Delta_u f\|_B \leqq K\|u\|^\alpha , \qquad \forall u \in T^N$$

where

$$(\Delta_u f)(t) = f(t + u) - f(t) .$$

THEOREM 4. – If $f \in K$ lip (α, B), $0 < \alpha \leqq 1$, we have

$$\|f - f_\sigma\|_B \leqq K C_{\alpha,N}\sigma^{\alpha/k} . \tag{4.1}$$

Moreover, if $N = 1$, there exists $M > 0$ such that

$$\sup_{f \in K \operatorname{lip}(\alpha, C(T))} \|f - f_\sigma\|_C > M\sigma^{\alpha/k} . \tag{4.2}$$

5. – In this section we give the proofs of the theorems.

THEOREM 1. – Let $\varepsilon > 0$ and $E \in Z^N$ a Sidon set, [13], such that for every absolutely convergent series

$$f(t) = \sum_{n \in E} a_n \exp(2\pi i n t) , \qquad t \in T^N$$

we have

$$\sum_{n\in E} |a_n| < (1+\varepsilon)\|f\|_\infty .$$

Such a set E may be, for instance, a lacunary set. (See e.g. [1], vol. 1, p. 179; vol. II, p. 246).

Therefore, if $1 \leqq p < 2$ we have

$$\|f - P_{m,0}\|_p \leqq \|f - P_{m,0}\|_2 \leqq \|f - Q\|_2 \leqq 2(1+\varepsilon)\|f - Q\|_p ;$$

if $2 < p < +\infty$, then

$$\|f - P_{m,0}\|_p \leqq \sqrt{p}(1+\varepsilon)\|f - P_{m,0}\|_2 \leqq \sqrt{p}(1+\varepsilon)\|f - Q\|_2 \leqq \sqrt{p}(1+\varepsilon)\|f - Q\|_p ;$$

if $p = \infty$ then

$$\|f - P_{m,0}\|_\infty \leqq \sum_{\substack{n\in E\\ |n|>m}} |\hat f(n)| \leqq \sum_{\substack{n\in E\\ |n|\leqq m}} |1-\alpha_n||\hat f(n)| + \sum_{\substack{n\in E\\ |n|>m}} |\hat f(n)| \leqq (1+\varepsilon)\|f - Q\|_\infty ,$$

q.e.d.

The proof shows that for $2 < p < +\infty$ (2.3) holds for every trigonometric polynomial of degree less than or equal to m.

THEOREM 3. – Let $\{a_n\}$ be a sequence of real positive numbers such that for every $n > 1$ we have

$$a_n < 2^{-(2n-1)} K_{2^{n-1}}^{-1} \tag{5.1}$$

where

$$K_n = \operatorname*{Sup}_{\sigma>0} \frac{1 - M(\sigma, n)}{1 - M(\sigma, 1)}$$

and moreover

$$a_n < 2^{-(2n-2m+1)} a_m , \qquad \forall m = 1, 2, \ldots, n-1 . \tag{5.2}$$

Let us consider the function

$$f(t) = \sum_{n=1}^{\infty} a_n \sin 2^n \pi t , \qquad t \in T .$$

For this kind of functions it suffices to consider $Q_{k\,\sigma}$ where $k = 2^n$, $n \in \boldsymbol{N}^+$. Let us set $Q_{2^n,\sigma} = R_{n,\sigma}$. Then for every $m > 0$ and $n = 1, 2, \ldots, m$

$$(f - R_{m,0})(t) = -(f - R_{m,0})(2^{-n} - t) ;$$

therefore $R_{m,0}$ satisfies the following conditions

$$\int_0^1 \sin 2^n\pi t \, \mathrm{sgn}\,\big((f - R_{m,0})(t)\big)\, dt = 0$$

for $n = 1, 2, \ldots, m$. This implies that $R_{m,0}$ is a best L^1 approximation of f in the class V_m of the polynomials of the form $\sum_{n=1}^{m} a_n \sin 2^n \pi t$ (1). Consequently

$$\|f - R_{m,\sigma}\|_1 \geqq \|f - R_{m,0}\|_1\,, \qquad \sigma > 0\,. \tag{5.3}$$

Because f is not a Chebychev set on $(0, 1)$, the polynomial of best approximation of f in V_m may not be unique; then we have to prove that in (5.3) the strict inequality holds.

To this aim let us consider

$$f - R_{m,\sigma} = \sum_{n=1}^{m} a_n\big(1 - M(\sigma, 2^{n-1})\big) \sin 2^n\pi t + \sum_{n>m} a_n \sin 2^n \pi t = \Sigma_1 + \Sigma_2\,. \tag{5.4}$$

We may always suppose $a_1 = 1$ and $1 - M(\sigma, 1) > 0$. By (5.1) we have

$$\begin{aligned}\Sigma_1 = \sum_{n=1}^{m} a_n\big(1 - M(\sigma, 2^{n-1})\big) 2^{n-1} \sin 2\pi t \prod_{s=1}^{t-1} \cos 2^s \pi t = \\ = \big(1 - M(\sigma, 1)\big) \sin 2\pi t \left\{1 + \sum_{n=2}^{m} \frac{1 - M(\sigma, 2^{n-1})}{1 - M(\sigma, 1)} 2^{n-1} a_n \cdot \prod_{s=1}^{n-1} \cos 2^s \pi t\right\} \geqq \\ \geq \big(1 - M(\sigma, 1)\big) \sin 2\pi t \left\{1 - \mathrm{sgn}\,(\sin 2\pi t) \sum_{n=2}^{m} 2^{-n}\right\} \geqq \\ \geqq \big(1 - M(\sigma, 1)\big) \sin 2\pi t \{1 - \tfrac{1}{2}\, \mathrm{sgn}\,(\sin 2\pi t)\}\,.\end{aligned}$$

From (5.2) we obtain

$$\begin{aligned}\Sigma_2 = \sin 2^{m+1}\pi t \left\{a_{m+1} + \sum_{n=2}^{m} 2^{n-1} a_{m+n} \prod_{s=1}^{n-1} \cos 2^{m+s}\pi t\right\} \geqq \\ \geqq a_{m+1} \sin 2^{m+1}\pi t \left\{1 - \mathrm{sgn}\,(\sin 2^{m+1}\pi t) \sum_{n=2}^{\infty} 2^{-n}\right\} \geqq \\ \geqq a_{m+1} \sin 2^{m+1}\pi t \{1 - \tfrac{1}{2}\, \mathrm{sgn}\,(\sin 2^{m+1}\pi t)\}\,.\end{aligned}$$

Let us set

$$\begin{aligned}\varphi_{m,\sigma}(t) = \big(1 - M(\sigma, 1)\big)\{1 - \tfrac{1}{2}\, \mathrm{sgn}\,(\sin 2\pi t)\} \sin 2\pi t + \\ + a_{m+1}\{1 - \tfrac{1}{2}\, \mathrm{sgn}\,(\sin 2^{m+1}\pi t)\} \sin 2^{m+1}\pi t\,.\end{aligned}$$

(1) See e.g. [12], p. 104, th. 4.2 or [8], p. 104, Cor. 1.5.

By (5.4) we have

$$f(t) - R_{m,\sigma}(t) \geqq \varphi_{m,\sigma}(t)\,, \qquad \forall t \in T\,.$$

Because

$$(f - R_{m,\sigma})(t) = -(f - R_{m,\sigma})(1-t)\,, \qquad \forall t \in T,\ \forall m > 0$$

the last inequality gives

$$\int_0^1 \sin 2\pi t \,\mathrm{sgn}\,\big(f(t) - R_{m,\sigma}(t)\big)\,dt \geqq 2\int_0^{\frac12} \sin 2\pi t\,\mathrm{sgn}\,\big(f(t) - R_{m,\sigma}(t)\big)\,dt \geqq \tag{5.5}$$
$$\geqq 2\int_0^{\frac12} \sin 2\pi t\,\mathrm{sgn}\,\varphi_{m,\sigma}(t)\,dt\,.$$

Now we check the sign of $\varphi_{m,\sigma}$ in $(0, \frac12)$.

For every k, $0 \leqq k \leqq 2^{m-1} - 1$ let

$$I_k = \left(\frac{k}{2^{m+1}}, \frac{k+1}{2^{m+1}}\right)$$

and let

$$I'_k = \left(\frac12 - \frac{k+1}{2^{m+1}}, \frac12 - \frac{k}{2^{m+1}}\right).$$

Let first consider an even k; in I_k, $\sin 2\pi t$ and $\sin 2^{m+1}\pi t$ are positive; therefore

$$\varphi_{m,\sigma}(t) > 0\,, \qquad \forall t \in I_k\,.$$

In I'_k the function $\sin 2\pi x$ has a positive minimum if $k > 0$; for $k = 0$ we have $\sin 2\pi x > 0$, $\forall x \in I'_k$, $x \neq 0$. On the contrary, $\sin 2^{m+1}\pi x$ is negative in the interior of I'_k and zero on the boundary. Then for σ small enough there exists $I''_{k,\sigma} \subsetneq I'_k$ such that

$$\varphi_{m,\sigma}(t) < 0\,, \quad \forall t \in I''_{k,\sigma}\,, \qquad \varphi_{m,\sigma}(t) > 0\,, \quad \forall t \in I'_k/I''_k\,.$$

Moreover, $I''_{k,\sigma} \uparrow I'_k$ if $\sigma \to 0^+$.

Then, by a simmetry argument, we easily obtain

$$\int_{I_k + I'_k} \sin 2\pi t\,\mathrm{sgn}\,\varphi_{m,\sigma}(t)\,dt > 0 \tag{5.6}$$

for every $\sigma > 0$ and even k, $0 \leqq k \leqq 2^{m-1} - 1$.

For odd k, analogous considerations prove that $\varphi_{m,\sigma}(t) > 0$, $\forall t \in I'_k$ and that for σ small enough there exists an interval $I''_{k,\sigma} \uparrow I_k$ for $\sigma \to 0_+$ such that $\varphi_{m,\sigma}(t) < 0$ in $I''_{k,\sigma}$ and $\varphi_{m,\sigma}(t) \geqq 0$ in $I_k/I''_{k,\sigma}$.

Therefore, (5.6) holds for every $\sigma > 0$ and for every k, $0 \leqq k \leqq 2^{m-1} - 1$.

Then

$$\int_0^{\frac{1}{2}} \sin 2\pi t \operatorname{sgn} \varphi_{m,\sigma}(t)\, dt > 0\,, \qquad \forall \sigma > 0$$

and by (5.5)

$$\int_0^1 \sin 2\pi t \operatorname{sgn} \big(f(t) - R_{m,\sigma}(t)\big)\, dt > 0\,, \qquad \forall \sigma > 0\,.$$

Consequently, $R_{m,\sigma}$ is not a best L^1-approximation of f in the class V_m (²) whatever is $\sigma > 0$. Then in (5.3) for every $\sigma > 0$ the strict inequality holds, q.e.d.

THEOREM 4. – Let $G \in L^1(R^N)$ be the function whose Fourier transform is $\hat{G} = (1 - P)^{-1}$. (See [3], th. 5.) Let us set

$$G_\sigma(x) = \sigma^{-N/k} G(\sigma^{-1/k} x)\,,$$
$$K_\sigma(x) = \sum_{n \in Z^N} G_\sigma(x + n)\,.$$

Let f^* the continued periodic function of f on R^N. Then

$$f_\sigma(x) = K_\sigma * f(x) = \int_{T^M} K_\sigma(u) f(x - u)\, du = \int_{R^N} G_\sigma(u) f^*(x - u)\, du = \int_{R^N} f^*(x + u) G_\sigma(-u)\, du\,.$$

Therefore

$$\|f - f_\sigma\|_B = \left\| \int_{R^N} \big(f^*(x + u) - f^*(x)\big) G_\sigma(-u)\, du \right\|_B \leqq \int_{R^N} \|\Delta_u f\|_B |G_\sigma(-u)|\, du \leqq$$
$$\leqq K \int_{R^N} \frac{\|u\|^\alpha}{\sigma^{N/k}} \left| G\left(\frac{-u}{\sigma^{1/k}}\right) \right| du = K \sigma^{\alpha/k} \int_{R^N} |x|^\alpha |G(-x)|\, dx\,.$$

The last integral exists ([3], th. 5); then (4.1) holds.

Let now be $N = 1$: in this case $P(x) = P_k(x) = x^k$, $k = 2, 4, \ldots$. For every $h = 1, 2, \ldots, k/2$, let us set

$$\varepsilon_h = \frac{(2h - 1)\pi}{k}\,, \qquad a_h = \sin \varepsilon_h\,, \qquad b_h = \cos \varepsilon_h\,.$$

We have ([10], p. 5):

$$G(x) = G_k(|x|) = \frac{2\pi}{k} \sum_{n=1}^{k/2} \sin\left(\varepsilon_h + 2\pi b_h |x|\right) \exp\left(-2\pi a_h |x|\right)\,;$$
$$K_\sigma(x) = K_{\sigma,k}(x) = \frac{2\pi}{k\sigma^{1/k}} \sum_{n \in Z} \sum_{h=1}^{k/2} \sin\left(\varepsilon_h + \frac{2\pi b_h}{\sigma^{1/k}} |x + n|\right) \exp\left(-\frac{2\pi a_h}{\sigma^{1/k}} |x + n|\right).$$

(²) See e.g. [12], p. 103, th. 4.2 or [8], p. 104, Cor. 1.5, Let us consider the function

$$f_\alpha(t) = \begin{cases} t^\alpha & 0 \leqq t < \frac{1}{2} \\ (1-t)^\alpha & \frac{1}{2} \leqq t < 1 . \end{cases}$$

Then for every α, $0 < \alpha \leqq 1$, $f_\alpha \in \mathrm{Lip}\,(\alpha, C(T))$ and we have

$$\|f_{\alpha,\sigma}\|_\infty = \|K_\sigma * f_\alpha\|_\infty \geqq |K_\sigma * f_\alpha(0)| =$$

$$= \frac{4\pi}{k\sigma^{1/k}} \left| \int_0^{+\infty} \sum_{h=1}^{k/2} \sin\left(\varepsilon_h + \frac{2\pi b_k}{\sigma^{1/k}} x\right) \exp\left(-\frac{2\pi a_h}{\sigma^{1/k}} x\right) f^*(x)\, dx \right| =$$

$$= \frac{4\pi}{k} \left| \int_0^{+\infty} \sum_{h=1}^{k/2} \sin\,(\varepsilon_h + 2\pi b_h x) \exp\,(-2\pi a_h x) f^*_\alpha(\sigma^{1/k} x)\, dx \right| \geqq$$

$$\geqq \frac{4\pi\sigma^{\alpha/k}}{k} \left| \int_0^{1/2\sigma^{1/k}} \sum_{h=1}^{k/2} x^\alpha \sin\,(\varepsilon^q + 2\pi b_h x) \exp\,(-2\pi a_h x)\, dx \right| -$$

$$- \frac{4\pi}{k} \left| \int_{1/2\sigma^{1/k}}^{\infty} \sum_{h=1}^{k/2} \sin\,(\varepsilon_h + 2\pi b_h x) \exp\,(-2\pi a_h x) f^*_\alpha(\sigma^{1/k} x)\, dx \right| \geqq \frac{4\pi\,\sigma^{\alpha/k}}{k} I_1 - \frac{4\pi}{k} I_2 .$$

We start evaluating I_1. From [10], p. 10 and p. 121 we obtain

$$\int_0^{+\infty} x^\alpha \sin\,(\varepsilon_h + 2\pi b_h x) \exp(-2\pi a_h x)\, dx = \frac{1}{(2\pi)^{\alpha+1}} \Gamma(\alpha+1) \sin\left((\alpha+1)\frac{\pi}{2} - \alpha\varepsilon_h\right).$$

Since $0 < \varepsilon_h < \pi$, we have $0 < (\alpha+1)(\pi/2) - \alpha\varepsilon_h < \pi$.
Then for δ sufficiently small, for every $\sigma < \tilde{\sigma}$ we have

$$I_1 = I_1(\sigma) > \delta > 0 .$$

On the other hand we have

$$I_2 < \int_{1/2\sigma^{1/k}}^{+\infty} \sigma^{-\alpha/k} \sum_{h=1}^{k/2} \exp\,(-2\pi a_h x)\, dx = \frac{\sigma^{-\alpha/k}}{2} \sum_{h=1}^{k/2} \frac{1}{a_h} \exp\left(\frac{-2a_h\pi^2}{\sigma^{1/k}}\right) = o(1) \quad \text{for } \sigma \to 0^+ .$$

Therefore, there does exist $M > 0$ such that for every $\sigma < \sigma_0(k)$ (4.2) holds.

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