Everywhere-Regularity for Some Quasilinear Systems with a Lack of Ellipticity (*) (**).

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Summary. – *It is shown that the derivatives of the solutions of certain quasilinear degenerate elliptic systems are Hölder-continuous, everywhere, in the interior of the domain. This work generalizes a result of K. Uhlenbeck* [34].

1. – Introduction and main result.

In this paper, we show the $C^{1,\mu}$-everywhere-regularity of vector valued functions $u \in H^{1,p}(\Omega) = H^{1,p}(\Omega; \boldsymbol{R}^n)$ $(p \in (1, \infty))$ solving quasilinear elliptic systems of the type

$$(1.1) \qquad \int_\Omega A(x, q(\nabla u)) \cdot b(\nabla u, \nabla \varphi)\, dx = \int_\Omega \sum_{i=1}^N a^i(x, u, \nabla u) \cdot \varphi^i \, dx\,, \qquad \forall \varphi \in C_c^\infty(\Omega)\,.$$

Here, Ω is an open subset of $\boldsymbol{R}^n$, b is a symmetric and positive definite bilinear form with smooth coefficients depending on $x \in \Omega$ and

$$q(\eta) = b(\eta, \eta)$$

is the corresponding quadratic form.

This work generalizes a result of K. UHLENBECK [34] (1977). Her paper contained two novelties. The first one was that she treated a lack of ellipticity arising in connection with variational integrals of the form

$$(1.2) \qquad \int_\Omega |\nabla u|^p \, dx$$

for $p > 2$. The second one is that she obtained an everywhere-regularity result.

Her work originated an extensive study of quasilinear elliptic *equations* having a *lack of ellipticity* like the Euler-equations of the variational integral (1.2), for $p > 1$, especially in the United States. L. C. EVANS [8] (1981) and J. LEWIS [24] (1981)

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showed the $C^{1,\mu}$-regularity for rather special equations. In the same year, F. DE THELIN [6] obtained an $H^{2,p}$-regularity result, also for a special class of such equations. Finally, at the beginning of this year (1982), independently of each other, E. DI BENEDETTO [7] and the author [32] proved the $C^{1,\mu}$-regularity of the solutions of rather general quasilinear equations which are allowed to have such a lack of ellipticity. It should not be forgotten, however, that N. URALTSEVA [35] obtained EVANS' result already in 1968.

For such equations and systems, $C^{1,\mu}$-regularity is optimal. Namely, there are scalar functions which minimize the variational integral (1.2), for $p > 2$, and which do not belong to $C^{1,\mu}$, if $\mu \in (0, 1)$ is chosen sufficiently close to one (cf. [32]).

In contrast to equations, everywhere-regularity cannot be obtained for *general elliptic quasilinear systems*. The counter-example of GIUSTI and MIRANDA [16] (1968) shows that it is generally impossible to obtain C^{μ}-everywhere-regularity for homogenous quasilinear systems with analytic coefficients satisfying the usual ellipticity and growth conditions. Moreover, NEČAS [30] (1977) presents a vector valued function $u \in H^{1,\infty}$ which does not belong to C^1 and which minimizes the variational integral

$$\int_{\Omega} F(\nabla u)\, dx \,,$$

where F is analytic and satisfies the usual ellipticity and growth conditions, too.

Nevertheless, one can obtain *almost-everywhere-regularity* for rather general quasilinear elliptic systems. In 1968, this was proven by C. B. MORREY [28] and GIUSTI-MIRANDA [17]. These authors used a stretching argument which can be found already in the works of DE GIORGI [5] (1960) and ALMGREN [1] (1968). Another approach is due to GIAQUINTA-GIUSTI [11] (1968) and GIAQUINTA-MODICA [15] (1979). In this one, inverse Hölder-inequalities play an important part. All these results can be summarized as follows. A weak solution is regular near a point x_0, if its values are concentrated around some point u_0, in a neighborhood of the point x_0. One obtains almost-everywhere-regularity, because this smallness condition is satisfied almost everywhere.

In 1977, K. UHLENBECK [34] obtained *everywhere-$C^{1,\mu}$-regularity* for quasilinear elliptic systems of the type

$$\operatorname{div} \{A(|\nabla u|^2)\cdot \nabla u\} = 0 \,. \tag{1.3}$$

Her important observation was that $|\nabla u|^2$ satisfies an inequality of the type

$$\lambda \cdot |\nabla^2 u|^2 - \sum_{\alpha,\beta=1}^{n} \frac{d}{dx_\alpha}\left\{b_{\alpha\beta}\cdot \frac{d}{dx_\beta}|\nabla u|^2\right\} \leqq 0 \,, \tag{1.4}$$

where the matrix $(b_{\alpha\beta})$ is uniformly bounded and uniformly positive definite in Ω.

With the aid of (1.4), she showed that

$$\sup_{B_R} |\nabla u|$$

becomes smaller, if one shrinks the radius R of the ball B_R down, unless the smallness condition for regularity is satisfied.

In 1979, P. A. IVERT [22] generalized UHLENBECK's result to quasilinear elliptic systems of the type

$$\sum_{\alpha,\beta=1}^{n} \frac{d}{dx_\alpha} \{a_{\alpha\beta}(x, u, |\nabla u|^2) \cdot u_{x\beta}\} = f(x, u, \nabla u) . \tag{1.5}$$

He excluded, however, any degeneration of ellipticity.

Similarly as K. UHLENBECK and P. A. IVERT, we derive that $q(\nabla u)$ is *quasisubharmonic*, i.e. it satisfies an inequality of the type

$$\begin{aligned} &\int_\Omega A(x, q(\nabla u)) \cdot |\nabla^2 u|^2 \cdot G(q(\nabla u)) \cdot \psi^2 \, dx + \\ &+ \int_\Omega A(x, q(\nabla u)) \cdot |\nabla q(\nabla u)|^2 \cdot G'(q(\nabla u)) \cdot \psi^2 \, dx \leqq \\ &\leqq c \cdot \int_\Omega A(x, q(\nabla u)) \cdot |\nabla q(\nabla u)| \cdot G(q(\nabla u)) \cdot |\nabla \psi| \cdot |\psi| \, dx + \text{lower order terms}, \end{aligned} \tag{1.6}$$

for all $\psi \in C_c^\infty(\Omega)$ and all smooth, nondecreasing and nonnegative functions G. Only from this inequality, we derive a *Strong Maximum Principle*. Apart from perturbation terms, it states that

$$\sup_{B_R} q(\nabla u)$$

becomes smaller, if one shrinks the radius R of the ball B_R down, unless u is highly concentrated around some point which is just the smallness condition for regularity.

In contrast to K. UHLENBECK and P. A. IVERT, we can apply our methods to more general systems. We also admit a lack of ellipticity arising in connection with variational integrals of the form (1.2), for $p < 2$.

Similar notions of quasisubharmonicity have been exploited also in other connections to obtain regularity results. M. MEIER [26] introduced a condition which implies that $|u|^2$ satisfies an inequality which is slightly weaker than (1.6). He used it for L^∞-bounds for the solution itself. L. CAFFARELLI [2] realized that, for solutions u of certain semilinear systems studied by GIAQUINTA, HILDEBRANDT, v. WAHL, WIDMAN and WIEGNER (cf. [13], [21], [36], [37]), the functions $|u - u_0|^2$ are subharmonic, where u_0 belongs to a certain set of « test-vectors ». This observation enabled him to give a simpler proof of the C^μ-regularity result of HILDEBRANDT-WIDMAN [21]. Having finished the first version of this paper, the author [33] derived a Strong Maximum Principle for harmonic mappings from an inequality similar to (1.6). Thus he could simplify the proofs of HILDEBRANDT-JOST-WIDMAN [19] or GIAQUINTA-HILDEBRANDT [13].

Now, we come to an exact formulation of our *assumptions*. The bilinear form b has the form

$$b(\eta, \eta') = \sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} \gamma_{\alpha\beta}(x)\cdot g^{ij}(x)\cdot \eta^i_\alpha \cdot \eta'^j_\beta . \tag{1.7}$$

For some positive constants λ and Λ,

$$q(\eta) \geqq \beta\cdot|\eta|^2 , \quad \forall \eta \in R^{n\cdot N},\ \forall x \in \Omega , \tag{1.8}$$

$$\sum_{\alpha,\beta=1}^{n} |\gamma_{\alpha\beta}|_{C^1(\bar\Omega)} + \sum_{i,j=1}^{N} |g^{ij}|_{C^1(\bar\Omega)} \leqq \Lambda . \tag{1.9}$$

The function A satisfies

$$\lambda\cdot(\varkappa + t)^{p-2} \leqslant A(x, t^2) \leqq \Lambda\cdot(\varkappa + t)^{p-2} , \tag{1.10}$$

$$(\lambda - \tfrac{1}{2})\cdot A(x, t) \leqq \frac{\partial A}{\partial t}(x, t)\cdot t \leqq \Lambda\cdot A(x, t) , \tag{1.11}$$

$$\left|\frac{\partial^2 A}{\partial t^2}(x, t)\right|\cdot t^2 + \sum_{\alpha=1}^{n} \left|\frac{\partial A}{\partial x_\alpha}(x, t)\right| \leqq \Lambda\cdot A(x, t) , \tag{1.12}$$

for some $\varkappa \in [0, 1]$, some $p \in (1, \infty)$, all $t > 0$ and all $x \in \Omega$. The right sides a^i of (1.1) are Carathéodory-functions, i.e., they are continous in u and ∇u and measurable in x. In addition to that,

$$\sum_{i=1}^{N} |a^i(x, v, \eta)| \leqq \Lambda\cdot(1 + |\eta|)^{p-1} , \tag{1.13}$$

for all $x \in \Omega$, $v \in \boldsymbol{R}^n$ and all $\eta \in \boldsymbol{R}^{n\cdot N}$.

THEOREM. – Let B_R be a ball with radius $R \in (0, 1]$ such that $B_{3R} \subset \Omega$. Then, there are positive constants c and μ which depend only on n, N, p, λ and Λ such that

$$M^p = \operatorname*{ess\,sup}_{B_{2R}} |\nabla u|^p \leqq c\cdot R^{-n}\cdot \int_{B_{3R}} (1 + |\nabla u|)^p\, dx , \tag{1.14}$$

$$|\nabla u(x) - \nabla u(x')| \leqq c\cdot(1 + M)\cdot R^{-\mu}\cdot|x - x'|^\mu , \tag{1.15}$$

for all solutions $u \in H^{1,p}(\Omega)$ of (1.1) and all $x, x' \in B_R$.

With respect to applications, for example to harmonic mappings, the condition

(1.13) is « unnatural ». It should be replaced by

$$\sum_{i=1}^{N} |a^i(x, \nu, \eta)| \leqq \Lambda\cdot(1 + |\eta|)^p . \tag{1.13'}$$

Moreover, the coefficients $\gamma_{\alpha\beta}$ and g^{ij} should depend also on the solution u. Thus, (1.9) should be replaced by

$$\sum_{\alpha,\beta=1}^{n} |\gamma_{\alpha\beta}|_{C^1(\overline{\Omega\times\boldsymbol{R}^n})} + \sum_{i,j=1}^{N} |g^{ij}|_{C^1(\overline{\Omega\times\boldsymbol{R}^n})} \leqq \Lambda . \tag{1.9'}$$

The counterexamples of HILDEBRANDT-KAUL-WIDMAN [20] and M. STRUWE [31] show, however, that the conditions (1.7)-(1.8), (1.9′), (1.10)-(1.12) and (1.13′) are not sufficient for everywhere-regularity. Nevertheless, there are a lot of cases in which one can show the C^μ-regularity of the weak solutions. The case $p > n$ is trivial, by Sobolev's imbedding theorem. In the case $p = n$, the C^μ-regularity of minima follows from the work [12] of GIAQUINTA-GIUSTI. For the case $p = 2$, there is a lot of literature, for example, the works [13], [21], [36] and [37] cited above in which the C^μ-regularity is proven under certain « smallness conditions ». We think that our proofs can be used in those cases, too, in order to show the $C^{1,\mu}$-regularity of the solutions. Some technical additions, however, are necessary.

2. – Preliminaries.

Here and in the following, we want to rewrite the system (1.1) in a more general form. For this, we introduce the notation

$$a^i_\alpha(x, \eta) = A\big(x, q(\eta)\big)\cdot \sum_{\beta=1}^{n}\sum_{j=1}^{N} \gamma_{\alpha\beta}(x)\cdot g^{ij}(x)\cdot \eta^j_\beta . \tag{2.1}$$

Then, (1.1) is equivalent to

$$\int_\Omega \sum_{\alpha=1}^{n}\sum_{i=1}^{N} a^i_\alpha(x, \nabla u)\cdot \varphi^i_{x_\alpha}\, dx = \int_\Omega \sum_{i=1}^{N} a^i(x, u, \nabla u)\cdot \varphi^i\, dx , \qquad \forall \varphi \in C^\infty_c(\Omega) . \tag{2.2}$$

From (1.7)-(1.11), one easily derives that there are positive constants λ' and Λ' which depend only on n, N, p, λ and Λ such that

$$\sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{N} \frac{\partial a^i_\alpha}{\partial \eta^j_\beta}(x, \eta)\cdot \xi^j\cdot \xi^i \geqq \lambda'\cdot(\varkappa + |\eta|)^{p-2}\cdot |\xi|^2 , \tag{2.3}$$

$$\sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{N} \left|\frac{\partial a^i_\alpha}{\partial \eta^j_\beta}(x, \eta)\right| \leqq \Lambda'\cdot(\varkappa + |\eta|)^{p-2} , \tag{2.4}$$

$$\sum_{\alpha,\beta=1}^{n}\sum_{i=1}^{N} \left|\frac{\partial a^i_\alpha}{\partial x^i_\beta}(x, \eta)\right| \leqq \Lambda'\cdot(\varkappa + |\eta|)^{p-2}\cdot|\eta| , \tag{2.5}$$

for all $x \in \Omega$, all $\eta \in \mathbf{R}^{n\cdot N}\setminus\{0\}$ and all $\xi \in \mathbf{R}^{n\cdot N}$.

LEMMA 2.1. – We can choose Λ' in such a way that

$$\sum_{\alpha=1}^{n}\sum_{i=1}^{N}|a_\alpha^i(x,\eta)-a_\alpha^i(x',\eta)| \leqq \Lambda'\cdot(\varkappa+|\eta|)^{p-2}\cdot|\eta|\cdot|x-x'|\,, \tag{2.6}$$

for all $\eta \in \mathbf{R}^{n\cdot N}$ and all $x, x' \in \Omega$ satisfying

$$|x-x'| \leqq \operatorname{dist}(x, \partial\Omega)\,.$$

Moreover, for every $\delta > 0$, there is a constant Λ'_δ depending only on $n, N, p, \lambda, \Lambda$ and δ such that

$$\sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{N}\left|\frac{\partial a_\alpha^i}{\partial\eta_\beta^j}(x,\eta)-\int_0^1\frac{\partial a_\alpha^i}{\partial\eta_\beta^j}(x, t\cdot\eta+(1-t)\cdot\eta')\,dt\right| \leqq \\ \leqq \Lambda'_\delta\cdot(\varkappa+|\eta|)^{p-2}\cdot|\eta|^{-1}\cdot|\eta-\eta'|\,, \tag{2.7}$$

for all $x \in \Omega$ and all $\eta, \eta' \in \mathbf{R}^{n}\setminus\{0\}$ satisfying

$$|\eta| \geqq \delta\cdot|\eta'|\,.$$

LEMMA 2.2. – We can choose λ' and Λ' in such a way that

$$\sum_{\alpha,\beta,\varrho,\sigma=1}^{n}\sum_{i,j=1}^{N}\frac{\partial a_\alpha^i}{\partial\eta_\beta^j}(x,\eta)\cdot\xi_{\beta\varrho}^j\cdot\gamma_{\varrho\sigma}(x)\cdot\xi_{\alpha\sigma}^i \geqq \lambda'\cdot(\varkappa+|\eta|)^{p-2}\cdot|\xi|^2\,, \tag{2.8}$$

$$\sum_{\alpha,\beta,\varrho,\sigma=1}^{n}\sum_{i,j=1}^{N}\frac{\partial a_\alpha^i}{\partial\eta_\beta^j}(x,\eta)\cdot\xi_{\beta\varrho}^j\cdot\gamma_{\varrho\sigma}(x)\cdot\eta_\sigma^i\cdot b(\eta,\xi_\alpha) \geqq \lambda'\cdot(\varkappa+|\eta|)^{p-2}\cdot\sum_{\alpha=1}^{n}(b(\eta,\xi_\alpha))^2\,, \tag{2.9}$$

$$\sum_{\alpha=1}^{n}\left|\sum_{\beta,\varrho,\sigma=1}^{n}\sum_{i,j=1}^{N}\frac{\partial a_\alpha^i}{\partial\eta_\beta^j}(x,\eta)\cdot\xi_{\beta\varrho}^j\cdot\gamma_{\varrho\sigma}(x)\cdot\eta_\sigma^i\right| \leqq \Lambda'\cdot(\varkappa+|\eta|)^{p-2}\cdot\left\{\sum_{\alpha=1}^{n}(b(\eta,\xi_\alpha))^2\right\}^{\frac12}\,, \tag{2.10}$$

for all $x \in \Omega$, all $\eta \in \mathbf{R}^{n\cdot N}\setminus\{0\}$ and all $\xi = (\xi_1, \ldots, \xi_n) \in R^{n\cdot n\cdot N}$ satisfying

$$\xi_{\alpha\beta}^i = \xi_{\beta\alpha}^i\,.$$

LEMMA 2.3. – We can choose λ' and Λ' in such a way that

$$\sum_{\alpha=1}^{n}\sum_{i=1}^{N}\{a_\alpha^i(x,\eta)-a_\alpha^i(x,\eta')\}\cdot(\eta_\alpha^i-\eta_\alpha'^i) \geqq \lambda'\cdot(\varkappa+|\eta|+|\eta'|)^{p-2}\cdot|\eta-\eta'|^2\,, \tag{2.11}$$

$$\sum_{\alpha=1}^{n}\sum_{i=1}^{N}|a_\alpha^i(x,\eta)-a_\alpha^i(x,\eta')| \leqq (\varkappa+|\eta|+|\eta'|)^{p-2}\cdot|\eta-\eta'|\,, \tag{2.12}$$

for all $x \in \Omega$ and all $\eta, \eta' \in \mathbf{R}^{n\cdot N}$. Moreover,

$$\sum_{\alpha=1}^{n} \sum_{i=1}^{N} \{a_\alpha^i(x, \eta) - a_\alpha^i(x, \eta')\} \cdot (\eta_\alpha^i - \eta_\alpha'^i) \geqq \lambda' \cdot |\eta - \eta'|^p , \tag{2.13}$$

provided that

$$p \geqq 2 . \tag{2.14}$$

PROOF OF LEMMA 2.1. – The inequality (2.6) is an easy consequence of (2.5). In order to prove (2.7), we pick a $\delta > 0$. By c, we denote a generic constant depending only on $n, N, p, \lambda, \Lambda$ and δ. We split the proof by considering two cases.

1-st *Case.* – $|\eta - \eta'| \geqq \frac{1}{2} \cdot |\eta|$.

In this case,

$$\left|\frac{\partial a_\alpha^i}{\partial \eta_\beta^j}(x, \eta) - \int_0^1 \frac{\partial a_\alpha^i}{\partial \eta_\beta^j}(x, t\cdot\eta + (1-t)\cdot\eta')\, dt\right| \leqq \tag{2.15}$$

$$\leqq \left|\frac{\partial a_\alpha^i}{\partial \eta_\beta^j}(x, \eta)\right| + \int_0^1 \left|\frac{\partial a_\alpha^i}{\partial \eta_\beta^j}(x, t\cdot\eta + (1-t)\cdot\eta')\right| dt \leqq$$

$$\leqq c\cdot(\varkappa + |\eta|)^{p-2} \leqq c\cdot(\varkappa + |\eta|)^{p-2}\cdot|\eta|^{-1}\cdot|\eta - \eta'| .$$

2-nd *Case.* – $|\eta - \eta'| \leqq \frac{1}{2} \cdot |\eta|$.

In this case,

$$\left|\frac{\partial a_\alpha^i}{\partial \eta_\beta^j}(x, \eta) - \int_0^1 \frac{\partial a_\alpha^i}{\partial \eta_\beta^j}(x, t\cdot\eta + (1-t)\cdot\eta')\, dt\right| = \tag{2.16}$$

$$= \left|\int_0^1\!\!\int_0^1 \sum_{\gamma=1}^{n} \sum_{k=1}^{N} \frac{\partial^2 a_\alpha^i}{\partial_\beta^j\, \partial \eta_\gamma^k}(x, s\cdot\eta + (1-s)\cdot t\cdot\eta + (1-s)\cdot(1-t)\cdot\eta')\cdot(1-t)\cdot(\eta_\gamma^k - \eta_\gamma'^k)\, ds\, dt\right| \leqq$$

$$\leqq c\cdot(\varkappa + |\eta|)^{p-2}\cdot|\eta|^{-1}\cdot|\eta - \eta'| .$$

This obviously proves (2.7).

PROOF OF LEMMA 2.2. – There is a uniquely determined symmetric matrix $R = (r_{\alpha\beta})$ such that

$$\gamma_{\varrho\sigma} = \sum_{\nu=1}^{n} r_{\varrho\nu}\cdot r_{\nu\sigma} .$$

Setting

$$\xi_{\beta\nu}'^{j} = \sum_{\varrho=1}^{n} r_{\varrho\nu}\cdot \xi_{\beta\varrho}^j ,$$

we obtain that

$$\sum_{\alpha,\beta,\varrho,\nu=1}^{n}\sum_{i,j=1}^{N}\frac{\partial a_\alpha^i}{\partial \eta_\beta^j}\cdot\xi_{\beta\varrho}^j\cdot\gamma_{\varrho\sigma}\cdot\xi_{\alpha\sigma}^i=\sum_{\alpha,\beta,\nu=1}^{n}\sum_{i,j=1}^{N}\frac{\partial a_\alpha^i}{\partial \eta_\beta^j}\cdot\xi_{\beta\nu}^{\prime j}\cdot\xi_{\nu\alpha}^{\prime i}\,. \tag{2.17}$$

The inequality (2.8) follows from (2.3), (2.17) and the fact that

$$c^{-1}\cdot|z|\leqq|Rz|\leqq c\cdot|z|\,,\qquad\forall z\in R^n\,,$$

for some $c>0$ depending only on n, β and Λ.

In order to derive (2.9) and (2.10), we use the identity

$$\begin{aligned}\sum_{\beta,\varrho,\sigma=1}^{n}\sum_{i,j=1}^{N}\frac{\partial a_\alpha^i}{\partial \eta_\beta^j}\cdot\xi_{\beta\varrho}^j\cdot\gamma_{\varrho\sigma}\cdot\eta_\sigma^i&=\Lambda\cdot\sum_{\beta=1}^{n}\gamma_{\alpha\beta}\cdot b(\eta,\xi_\beta)+\\&+2\cdot\frac{\partial\Lambda}{\partial t}\cdot\sum_{\beta,\varrho,\sigma=1}^{n}\sum_{i,j=1}^{N}b(\eta,\xi_\varrho)\cdot\gamma_{\varrho\sigma}\cdot\eta_\sigma^i\cdot g^{ij}\cdot\eta_\beta^j\cdot\gamma_{\alpha\beta}\,.\end{aligned} \tag{2.18}$$

This and (1.12) show that (2.10) is true. For (2.9), we remark that there is a matrix $S=(s^{ij})$ such that, for

$$\eta'_\alpha=S\eta_\alpha\,,$$

the following is true.

$$\begin{aligned}0&\leqq\sum_{\alpha,\beta,\varrho,\sigma=1}^{n}\sum_{i,j=1}^{N}b(\eta,\xi_\varrho)\cdot\gamma_{\varrho\sigma}\cdot\eta_\sigma^i\cdot g^{ij}\cdot\eta_\beta^j\cdot\gamma_{\alpha\beta}\cdot b(\eta,\xi_\alpha)=\sum_{i=1}^{N}\Big\{\sum_{\alpha,\beta=1}^{n}\gamma_{\alpha\beta}\cdot\eta_\beta^{\prime j}\cdot b(\eta,\xi_\alpha)\Big\}^2\leqq\\&\leqq\Big\{\sum_{i=1}^{N}\sum_{\alpha,\beta=1}^{n}\gamma_{\alpha\beta}\cdot\eta_\alpha^{\prime i}\cdot\eta_\beta^{\prime j}\Big\}\cdot\Big\{\sum_{\alpha,\beta=1}^{n}\gamma_{\alpha\beta}\cdot b(\eta,\xi_\alpha)\cdot b(\eta,\xi_\beta)\Big\}=q(\eta)\cdot\sum_{\alpha,\beta=1}^{n}\gamma_{\alpha\beta}\cdot b(\eta,\xi_\alpha)\cdot b(\eta,\xi_\beta)\,.\end{aligned} \tag{2.19}$$

Now, we can conclude the proof of Lemma 2.2 stating that (2.9) follows from (1.12), (2.18) and (2.19).

PROOF OF LEMMA 2.3. – By (2.4), there is a constant c depending only on n, N, p, λ and Λ such that

$$|a_\alpha^i(x,\eta)-a_\alpha^i(x,\eta')|\leqq c\cdot\int_0^1\{\varkappa+t\cdot(|\eta|+|\eta'|)\}^{p-2}\cdot|\eta-\eta'|\,dt\,.$$

This obviously implies that (2.12) is true.

For (2.11) and (2.13), we may suppose that $|\eta'|\geqq|\eta|$. This implies that, for all $t\in[0,\frac{1}{4}]$,

$$\tfrac{1}{4}\cdot|\eta-\eta'|\leqq\varkappa+|\eta'+t\cdot(\eta-\eta')|\leqq\varkappa+|\eta|+|\eta'|\leqq4\cdot(\varkappa+|\eta'+t\cdot(\eta-\eta')|)\,. \tag{2.20}$$

The estimates (2.11) and (2.13) follow from (2.3), (2.20) and the inequality

$$\sum_{\alpha=1}^{n}\sum_{i=1}^{N}\{a_\alpha^i(x,\eta)-a_\alpha^i(x,\eta')\}\cdot(\eta_\alpha^i-\eta_\alpha'^i)\geqq$$
$$\geqq\int_0^{1/4}\sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{N}\frac{\partial a_\alpha^i}{\partial\eta_\beta^j}(x,\eta'+t\cdot(\eta-\eta'))\cdot(\eta_\alpha^i-\eta_\alpha'^i)\cdot(\eta_\beta^j-\eta_\beta'^j)\,dt\,.$$

3. – Proof of the Theorem.

The statements of the theorem follow directly from Proposition 5.1 and 6.1, unless

$$p>2\,. \tag{3.1}$$

If (3.1) holds, we cannot prove the estimates (1.14) and (1.15), directly. In this case, however, the theorem can be easily derived from the next two lemmas.

In the following, B_R stands for a ball with radius R satisfying

$$B_{3R}\subset\Omega\,. \tag{3.2}$$

LEMMA 3.1. – *Local Existence of Smooth Solutions.*

There is an $R_0>0$ depending only on n, N, p, λ and Λ such that the following is true. If

$$R\leqq R_0\,, \tag{3.3}$$

then, for any $g\in H^{1,p}(B_{3R})$, there exists a solution $u\in H^{1,p}(B_{3R})$ of the system (1.1) (or (2.2)) satisfying

$$u=g\,,\qquad \text{on } \partial B_{3R}\,.$$

Moreover, the regularity result of the theorem holds for u.

For the next lemma, let u be an arbitrary $H^{1,p}(\Omega)$-solution of the system (1.1) (or (2.2)). We set

$$\tilde a^i(x,\eta)=\begin{cases}\;\;\,2\cdot\Lambda\cdot(1+|\eta|)^{p-1}\,, & \text{if } a^i(x,u,\nabla u)\geqq\;\;\,2\cdot\Lambda\cdot(1+|\eta|)^{p-1}\,,\\ -2\cdot\Lambda\cdot(1+|\eta|)^{p-1}\,, & \text{if } a^i(x,u,\nabla u)\leqq-2\cdot\Lambda\cdot(1+|\eta|)^{p-1}\,,\\ a^i(x,u,\nabla u)\,, & \text{otherwise}\,.\end{cases}$$

By Lemma 3.1, there is an $R_1>0$ depending only on n, N, p, λ and Λ such that there is a function $u \in H^{1,p}(B_{3R})$ satisfying the regularity result of the theorem and

$$\int_{B_{3R}} \sum_{\alpha=1}^{n} \sum_{i=1}^{N} a^i(x, \nabla\tilde{u}) \cdot \varphi^i_{x_\alpha}\, dx = \int_{B_{3R}} \sum_{i=1}^{N} \tilde{a}^i(x, \nabla\tilde{u}) \cdot \varphi^i\, dx\,, \quad \forall \varphi \in C^\infty_c(B_{3R})\,, \tag{3.5}$$

$$\tilde{u} = u\,, \quad \text{on } \partial B_{3R}\,, \tag{3.6}$$

provided that

$$R \leqq R_1\,. \tag{3.7}$$

LEMMA 3.2. – *Local Uniquess.*

Suppose that (3.1) holds. Then there is an $R_2 \in (0, R_1]$ depending only on n, N, p, λ and Λ such that

$$\tilde{u} = u\,, \quad \text{in } B_{3R}\,, \tag{3.8}$$

provided that

$$R \leqq R_2\,. \tag{3.9}$$

PROOF OF LEMMA 3.1. – For $\varepsilon, h \in (0, 1]$, we set

$$\nabla_h v(x) = h^{-1} \cdot \begin{cases} \big(v(x + he_\alpha) - v(x)\big)_{\alpha=1,2,\dots,n}\,, & \text{if } \operatorname{dist}(x, \partial B_{3R}) > h\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

where $e_1, e_2, \dots, e_n$ are the unit vectors of $\boldsymbol{R}^n$. Moreover,

$$a^i_{h,\varepsilon}(x, v) = \begin{cases} 1/\varepsilon\,, & \text{if } a^i(x, v, \nabla_h v) \geqq 1/\varepsilon\,, \\ -1/\varepsilon\,, & \text{if } a^i(x, v, \nabla_h v) \leqq -1/\varepsilon\,, \\ a^i(x, v, \nabla_h v)\,, & \text{otherwise}\,, \end{cases}$$

$$a^i_\varepsilon(x, \nu, \eta) = \begin{cases} 1/\varepsilon\,, & \text{if } a^i(x, \nu, \eta) \geqq 1/\varepsilon\,, \\ -1/\varepsilon\,, & \text{if } a^i(x, \nu, \eta) \leqq -1/\varepsilon\,, \\ a^i(x, \nu, \eta)\,, & \text{otherwise}\,, \end{cases}$$

$$a^i_{\varepsilon,\alpha}(x, \eta) = A\big(x, q(\eta) + \varepsilon\big) \cdot \sum_{\beta=1}^{n} \sum_{j=1}^{N} \gamma_{\alpha\beta}(x) \cdot g^{ij}(x) \cdot \eta^j_\beta\,.$$

We note that the coefficients $a^i_{\varepsilon,\alpha}$, a^i_ε and $a^i_{h,\varepsilon}$ satisfy the inequalities of Section 2 with the same constants λ', Λ' and Λ'_δ as the coefficients a^i_α and a^i. The constant $\varkappa$, however, has to be replaced by

$$\varkappa_\varepsilon = \varkappa + \varepsilon > 0 .$$

From the theory of monotone operators (cf. [18], [25]), we know that there exists a function $u_{h,\varepsilon} \in H^{1,p}(B_{3R})$ satisfying

$$\int_{B_{3R}} \sum_{\alpha=1}^{n} \sum_{i=1}^{N} a^i_{\varepsilon,\alpha}(x, \nabla u_{h,\varepsilon}) \cdot \varphi^i_{x_\alpha}\, dx = \int_{B_{3R}} \sum_{i=1}^{N} a^i_{h,\varepsilon}(x, u_{h,\varepsilon}) \cdot \varphi^i\, dx , \qquad \forall \varphi \in C^\infty_c(B_{3R}) ,$$

$$u_{h,\varepsilon} = g , \qquad \text{on } \partial B_{3R} .$$

As the L^∞-norm of the $a^i_{h,\varepsilon}$ is bounded uniformly, with respect to h, one can easily show that this is true also for the $H^{1,p}_0(B_{3R})$-norm of the $u_{h,\varepsilon}$. As $\varkappa_\varepsilon > 0$, we can apply Proposition 5.1 and 6.1, to $u_{h,\varepsilon}$. This gives local $C^{1,\mu}$-estimates for the $u_{h,\varepsilon}$ which are independent of h. Thus, we may let h tend to zero in order to obtain a function $u \in H^{1,p}(B_{3R})$ with Hölder-continous derivatives satisfying

$$\int_{B_{3R}} \sum_{\alpha=1}^{N} \sum_{i=1}^{N} a^i_{\varepsilon,\alpha}(x, \nabla u_\varepsilon) \cdot \varphi^i_{x_\alpha}\, dx = \int_{B_{3R}} \sum_{i=1}^{N} a^i_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \varphi^i\, dx , \qquad \forall \varphi \in C^\infty_c(B_{3R}) , \tag{3.10}$$

$$u_\varepsilon = g , \qquad \text{on } \partial B_{3R} . \tag{3.11}$$

Now, it is easy to see that Lemma 3.1 is true, if we can bound the $H^{1,p}_0(B_{3R})$-norm of u_ε uniformly with respect to ε. For this, we insert

$$\varphi = u_\varepsilon - g$$

into (3.10) and use Lemma 2.3. This gives that

$$\int_{B_{3R}} (1 + |\nabla u_\varepsilon|)^p\, dx \leqq c \cdot \int_{B_{3R}} (1 + |\nabla u_\varepsilon|)^{p-1} \cdot \{(1 + |\nabla g|) + |u_\varepsilon - g|\}\, dx , \tag{3.12}$$

for some constant c depending only on n, N, p, λ and Λ. By Sobolev's imbedding theorem, there is a c' depending only on n, N and p such that

$$\int_{B_{3R}} |u_\varepsilon - g|^p\, dx \leqq c' \cdot R^p \cdot \int_{B_{3R}} (1 + |\nabla u_\varepsilon|)^p + (1 + |\nabla g|)^p\, dx . \tag{3.13}$$

With the aid of Young's inequality, the estimates (3.12) and (3.13) can be combined to the desired bound for the $H^{1,p}_0(B_{3R})$-norm of the u_ε, if R is « sufficiently small ».

PROOF OF LEMMA 3.2. – We note that

$$\tilde{a}^i(x, \nabla u) = a^i(x, u, \nabla u) , \qquad \text{if } |\nabla u| \leqq 1 + 2 \cdot |\nabla u| ,$$

and insert $(u - u)$ into (2.2) and (3.5). Thus we can use (1.13), (3.8) and Lemma 2.3 to obtain that

$$\int_{B_{3R}} |\nabla u - \nabla \tilde{u}|^p \, dx \leq c \cdot \int_{B_{3R}} |\nabla u - \nabla u|^{p-1} \cdot |u - u| \, dx \,, \tag{3.14}$$

for some $c > 0$ depending only on n, N, p, λ and Λ. Moreover, by Sobolev's imbedding theorem, there is another constant c' depending only on n, N and p such that

$$\int_{B_{3R}} |u - u|^p \, dx \leq c' \cdot R^p \cdot \int_{B_{3R}} |\nabla u - \nabla u|^p \, dx \,. \tag{3.15}$$

With the aid of Young's inequality, the estimates (3.14) and (3.15) can be combined to

$$\int_{B_{3R}} |\nabla u - \nabla \tilde{u}|^p \, dx \leq 0 \,,$$

provided that R is «sufficiently small». This obviously implies the uniquess result of Lemma 3.2.

4. – Integrability of the second derivatives and quasisubharmonicity.

In this section, u stands for an arbitrary weak solution of (1.1) (or (2.2)). Here and in the following, we suppose that

$$\varkappa > 0 \,, \quad \text{if } p > 2 \,, \tag{4.1}$$

$$a^i(x, u, \nabla u) \in L^\infty(\Omega) \,, \quad \text{for } i = 1, 2, \ldots, N, \text{ if } p > 2 \,. \tag{4.2}$$

Lemma 4.1. – *Integrability of the Second Derivatives.*

The second derivatives of u are measurable functions and

$$(1 + |\nabla u|)^{(p-2)/2} \cdot \nabla^2 u \in L^2_{\text{loc}}(\Omega) \,. \tag{4.3}$$

Lemma 4.2. – *Quasisubharmonicity.*

There is a constant c depending only on n, N, p, λ and Λ such that, for every nonnegative function $\psi \in C_c^\infty(\Omega)$ and every piecewise C^1-function $G\colon R \to R$ satisfying

$$G(t) \geq 0 \,, \quad \forall t \in \boldsymbol{R} \,, \tag{4.4}$$

$$G'(t) \geq 0 \,, \quad \forall t \in \boldsymbol{R} \,, \tag{4.5}$$

$$G'(t) = 0 \,, \quad \forall t \geq t_0 \,, \tag{4.6}$$

for some $t_0 > 0$, the following estimate holds.

$$\begin{aligned}(4.7)\qquad &\int_\Omega (\varkappa + |\nabla u|)^{p-2}\cdot|\nabla^2 u|^2\cdot G(q(\nabla u))\cdot\psi^2\,dx + \\ &+ \int_\Omega (\varkappa + |\nabla u|)^{p-2}\cdot|\nabla q(\nabla u)|^2\cdot G'(q(\nabla u))\cdot\psi^2\,dx \leqq \\ &\leqq c\cdot\int_\Omega (\varkappa + |\nabla u|)^{p-2}\cdot|\nabla q(\nabla u)|\cdot G(q(\nabla u))\cdot|\nabla\psi|\cdot\psi\,dx + \\ &+ c\cdot\int_\Omega (1 + |\nabla u|)^{p}\cdot G(q(\nabla u))\cdot(\psi + |\nabla\psi|)\cdot\psi\,dx + \\ &+ c\cdot\int_\Omega (1 + |\nabla u|)^{p+2}\cdot G'(q(\nabla u))\cdot\psi^2\,dx + \\ &+ c\cdot\int_\Omega (1 + |\nabla u|)^{p-1}\cdot|\nabla^2 u|\cdot G(q(\nabla u))\cdot\psi^2\,dx + \\ &+ c\cdot\int_\Omega (1 + |\nabla u|)^{p}\cdot|\nabla q(\nabla u)|\cdot G'(q(\nabla u))\cdot\psi^2\,dx\,.\end{aligned}$$

PROOF OF LEMMA 4.1. – Already in the classical book of MORREY [27], it is proven that (4.1), (4.2) and (2.3)-(2.5) imply (4.3), in the case $p > 2$. (In [27], some differentiability of the right sides a^i is required. But, using the boundedness of the a^i, it is easily verified that the differentiability is superflous.) In contrast to the case $p > 2$, the statement (4.3) must be proven, if $p \leqq 2$.

For this, we set

$$\begin{aligned} h &= h\cdot e_\varrho\,, \\ v_{h_\varrho}(x) &= h^{-1}\cdot\{v(x + h_\varrho) - v(x)\}\,,\end{aligned}$$

where e_ϱ is the ϱ-th unit vector of $\boldsymbol{R}^n$ and $h \neq 0$. By ψ, we denote an arbitrary non-negative function belonging to $C_c^\infty(\Omega)$. We choose an $h_0 > 0$ such that we may insert

$$\varphi = \{u_{h_\varrho}\cdot\psi^2\}_{-h_\varrho}$$

into (2.2), for all $h \in (0, h_0]$. Thus, we obtain that

$$\begin{aligned}(4.8)\qquad & c\cdot\int_\Omega (1 + |\nabla u(x)| + |\nabla u(x + h_\varrho)|)^{p-2}\cdot|\nabla u_{h_\varrho}(x)|^2\cdot\psi^2(x)\,dx \leqq \\ &\leqq \int_0^1\!\!\int_\Omega \sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{N}\frac{\partial a_\alpha^i}{\partial\eta_\beta^j}\big(x + h_\varrho, t\cdot\nabla u(x + h_\varrho) + (1-t)\cdot\nabla u(x)\big)\cdot u^j_{h_\varrho,x_\beta}(x)\cdot u^i_{h_\varrho,x_\alpha}(x)\cdot\psi^2(x)\,dx\,dt = \\ &= \int_0^1\!\!\int_\Omega \sum_{\alpha=1}^{n}\sum_{i=1}^{N} a_\alpha^i(x + t\cdot h_\varrho, \nabla u(x + t\cdot h_\varrho))\cdot 2\cdot\frac{d}{dx_\varrho}\{u^i_{h_\varrho}(x)\cdot\psi(x)\cdot\psi_{x_\alpha}(x)\}\,dx\,dt -\end{aligned}$$

$$-\int_0^1\int_\Omega \sum_{\alpha=1}^n \sum_{i=1}^N \frac{\partial a_\alpha^i}{\partial x_\varrho}\left(x + t\cdot h_\varrho, \nabla u(x)\right)\cdot u^i_{h_\varrho, x_\alpha}(x)\cdot \psi^2(x)\, dx\, dt -$$

$$-\int_0^1\int_\Omega \sum_{i=1}^N a^i\left(x + t\cdot h_\varrho, u(x + t\cdot h_\varrho), \nabla u(x + t\cdot h_\varrho)\right)\cdot \frac{d}{dx_\varrho}\{u^i_{h_\varrho}(x)\cdot \psi^2(x)\}\, dx\, dt\,,$$

for some $c > 0$ independent of h. Moreover, by Young's inequality,

$$\int_\Omega |\nabla u_{h_\varrho}(x)|^p\cdot \psi^p(x)\, dx \leqq (2-p)/2\cdot \int_\Omega \left(1 + |\nabla u(x)| + |\nabla u(x + h_\varrho)|\right)^p\cdot \psi^{2p/(2-p)}(x)\, dx + \tag{4.9}$$
$$+ p/2\cdot \int_\Omega \left(1 + |\nabla u(x)| + |\nabla u(x + h)|\right)^{p-2}\cdot |\nabla u_{h_\varrho}(x)|^2\cdot \psi^2(x)\, dx\,.$$

With the aid of (1.13), (2.3)-(2.5), (2.12) and Young's inequality, one can combine (4.8) and (4.9) to a uniform estimate for the quantity

$$\int_\Omega |\nabla u_{h_\varrho}(x)|^p\cdot \psi^p(x) + \left(1 + |\nabla u(x)| + |\nabla u(x + h_\varrho)|\right)^{p-2}\cdot |\nabla u_{h_\varrho}(x)|^2\cdot \psi^2(x)\, dx\,.$$

This proves that Lemma 4.1 is true, also in the case $p \leqq 2$.

PROOF OF LEMMA 4.2. – We use the same notations as in the proof of Lemma 4.1 and insert

$$\varphi(x) = -\sum_{\varrho,\sigma=1}^n \{\gamma_{\varrho\sigma}\cdot u_{h_\sigma}\cdot G\left(q(\nabla_h u)\right)\cdot \psi^2\}_{-h_\varrho}$$

into (2.2), where $\nabla_h u$ is defined correspondingly to ∇u. By (4.3), we may let h tend to zero in order to obtain that

$$\sum_{\alpha,\beta,\varrho,\sigma=1}^n \sum_{i,j=1}^N \int_\Omega \frac{\partial a_\alpha^i}{\partial \eta_\beta^j}\cdot u^j_{x_\varrho x_\beta}\cdot \gamma_{\varrho\sigma}\cdot \psi^2\cdot \{u^i_{x_\sigma x_\alpha}\cdot G\left(q(\nabla u)\right) + 2\cdot u^i_{x_\sigma}\cdot b(\nabla u, \nabla u_{x_\alpha})\cdot G'\left(q(\nabla u)\right)\}\, dx \leqq$$

$$\leqq \sum_{\alpha,\varrho,\sigma=1}^n \sum_{i=1}^N \int_\Omega a_\alpha^i\cdot \frac{d}{dx_\varrho}\left\{\frac{d\gamma_{\varrho\sigma}}{dx_\alpha}\cdot u^i_{x_\sigma}\cdot G\left(q(\nabla u)\right)\cdot \psi^2 + \right.$$

$$\left. + \gamma_{\varrho\sigma}\cdot u_{x_\sigma}\cdot \frac{\partial q}{\partial x_\alpha}(x, \nabla u)\cdot G'\left(q(\nabla u)\right)\cdot \psi^2 + 2\cdot \gamma_{\varrho\sigma}\cdot u_{x_\sigma}\cdot G\left(q(\nabla u)\right)\cdot \psi\cdot \psi_{x_\alpha}\right\} dx -$$

$$-\sum_{\alpha,\varrho,\sigma=1}^n \sum_{i=1}^N \int_\Omega \frac{\partial a_\alpha^i}{\partial x_\varrho}\cdot \gamma_{\varrho\sigma}\cdot \psi^2\cdot \{u^i_{x_\sigma x_\alpha}\cdot G\left(q(\nabla u)\right) + 2\cdot u^i_{x_\sigma}\cdot b(\nabla u, \nabla u_{x_\alpha})\cdot G'\left(q(\nabla u)\right)\}\, dx -$$

$$-\sum_{\varrho,\sigma=1}^n \sum_{i=1}^N \int_\Omega a^i(x, u, \nabla u)\cdot \frac{d}{dx_\varrho}\{\gamma_{\varrho\sigma}\cdot u^i_{x_\sigma}\cdot G\left(q(\nabla u)\right)\cdot \psi^2\}\, dx =$$

$$= -\sum_{\alpha,\beta,\varrho,\sigma=1}^{n}\sum_{i,j=1}^{N}\int_{\Omega}\frac{\partial a_\alpha^i}{\partial\eta_\beta^j}\cdot u_{x_\varrho x_\beta}^j\cdot u_{x_\sigma}^i\cdot\left\{\frac{d\gamma_{\varrho\sigma}}{dx_\alpha}\cdot G(q(\nabla u))\cdot\psi^2 + \right.$$

$$\left. + \gamma_{\varrho\sigma}\cdot\frac{\partial q}{\partial x_\alpha}(x,\nabla u)\cdot G'(q(\nabla u))\cdot\psi^2 + 2\cdot\gamma_{\varrho\sigma}\cdot G(q(\nabla u))\cdot\psi\cdot\psi_{x_\alpha}\right\}dx -$$

$$-\sum_{\alpha,\varrho,\sigma=1}^{n}\sum_{i=1}^{N}\int_{\Omega}\frac{\partial a_\alpha^i}{\partial x_\varrho}\cdot\frac{d}{dx_\alpha}\{\gamma_{\varrho\sigma}\cdot u_{x_\sigma}^i\cdot G(q(\nabla u))\cdot\psi^2\}\,dx -$$

$$-\sum_{\varrho,\sigma=1}^{n}\sum_{i=1}^{N}\int_{\Omega}a^i(x,u,\nabla u)\cdot\frac{d}{dx_\varrho}\{\gamma_{\varrho\sigma}\cdot u_{x_\sigma}^i\cdot G(q(\nabla u))\cdot\psi^2\}\,dx\,.$$

This, the ellipticity and growth conditions (1.13), (2.3)-(2.5) and Lemma 2.2 imply (4.7).

5. – Boundedness of the gradient.

In this section, u stands for an arbitrary weak solution of (1.1) (or (2.2)). We suppose that (4.1) and (4.2) hold, because we will make use of Lemma 4.1. In the following, B_R stands for a ball with radius $R\in(0,1]$ satisfying

$$B_{3R}\subset\Omega\,.$$

PROPOSITION 5.1. – *Boundedness of the Gradient.*

There is a constant c depending only on n, N, p, λ and Λ such that

$$\operatorname*{ess\,sup}_{B_{2R}}|\nabla u|^p \leq c\cdot R^{-n}\cdot\int_{B_{3R}}(1+|\nabla u|)^p\,dx\,. \tag{5.1}$$

PROOF. – We set

$$\begin{aligned} w &= 1+\max\{0, q(\nabla u)-1\}\,,\\ R_i &= 2\cdot R + 2^{-i}\cdot R\,,\\ B_i &= B_{R_i}\,,\\ s_i &= p\cdot n^i/(n-1)^i\,, \end{aligned}$$

for $i\in N\cup\{0\}$. By c, we denote a generic constant which may depend only on n, N, p, λ and Λ.

The idea of this proof is due to J. MOSER [21]. Namely, the boundedness of the gradiente follows from a simple recursion formula for the integrals

$$\int_{B_i}w^{s_i}\,dx\,. \tag{5.2}$$

In order to establish this one, we assume that the integral (5.2) is bounded, for some $i \in \boldsymbol{N} \cup \{0\}$. We pick a $t_0 > 0$ and set

$$G(t) = \begin{cases} 0\,, & \text{for } t \leqq 1\,, \\ t^{s_i - p/2} - 1\,, & \text{for } t \in [1, t_0 + 1]\,, \\ t_0^{s_i - p/2}\,, & \text{for } t \geqq t_0 + 1\,, \end{cases}$$

if $i \in \boldsymbol{N}$, and

$$G(t) = \begin{cases} 0\,, & \text{for } t \leqq 1\,, \\ t - 1\,, & \text{for } t \in [1, 2]\,, \\ 1\,, & \text{for } t \geqq 2\,, \end{cases}$$

if $i = 0$. Moreover, we choose a nonnegative function $\psi \in C_c^\infty(\Omega)$ satisfying

$$\begin{aligned} \psi &= 1\,, && \text{in } B_{i+1} \\ \operatorname{supp} \psi &\subset B_{i+1}\,, \\ |\nabla\psi| &\leqq c \cdot R^{-1} \cdot 2^i\,, && \text{in } \boldsymbol{R}^n\,. \end{aligned}$$

We note that

$$\begin{aligned} (\varkappa + |\nabla u|)^{p-2} &\approx w^{(p-2)/2}\,, \\ (1 + |\nabla u|)^2 &\approx |\nabla u|^2 \approx w\,, \\ G(q(\nabla u)) &= G(w)\,, \end{aligned}$$

whenever $G(q(\nabla u)) \neq 0$, and that

$$\int_\Omega w^{(p-4)/2} \cdot |\nabla w|^2 \cdot G(q(\nabla u)) \cdot \psi^2 \, dx \leqq c \cdot \int_\Omega w^{(p-2)/2} \cdot |\nabla^2 u|^2 \cdot G(q(\nabla u)) \cdot \psi^2 \, dx\,.$$

Thus, we can use (4.7) and Young's inequality, to obtain that

$$\begin{aligned} \int_\Omega w^{(p-4)/2} \cdot |\nabla w|^2 \cdot G(w) \cdot \psi^2 \, dx + \int_\Omega w^{(p-2)/2} \cdot |\nabla w|^2 \cdot G'(w) \cdot \psi^2 \, dx \leqq \\ \leqq c \cdot R^{-2} \cdot 2^{2i} \cdot \int_{B_i} w^{p/2} \cdot G(w) \, dx + c \cdot R^{-2} \cdot 2^{2i} \cdot \int_{B_i} w^{(p+2)/2} \cdot G'(w) \, dx\,. \end{aligned}$$

The right hand side of this inequality can be bounded independently of $t_0 > 0$, in terms of the integral (5.2). Therefore, we may let t_0 tend to infinity. This gives the estimate

$$\int_\Omega |\nabla(w^{s_i/2}\cdot\psi)|^2\,dx \leqq c\cdot s_i\cdot R^{-2}\cdot 2^{2i}\cdot\int_{B_i} w^{s_i}\,dx\,. \tag{5.3}$$

From (5.3) and Sobolev's imbedding theorem, one derives the desired recursion formula

$$\Big\{R^{-n}\cdot\int_{B_{i+1}} w^{s_{i+1}}\,dx\Big\}^{1/s_{i+1}} \leqq \{2^{2i}\cdot s_i\cdot c\}^{1/s_i}\times\Big\{R^{-n}\cdot\int_{B_i} w^{s_i}\,dx\Big\}^{1/s_i}.$$

This implies that, for all $i\in \boldsymbol{N}$,

$$\Big\{R^{-n}\cdot\int_{B_i} w^{s_i}\,dx\Big\}^{1/s_i} \leqq c\cdot\Big\{R^{-n}\cdot\int_{B_{2R}} w^{p}\,dx\Big\}^{1/p},$$

which proves (5.1).

6. – Hölder-continuity of the gradient.

In this section, u stands for an arbitrary weak solution of (1.1) (or (2.2)). We suppose that (4.1) and (4.2) hold, because we will make use of Lemma 4.1. In the following, B_R stands for a ball with radius $R\in(0,1]$ satisfying

$$B_{3R}\subset\Omega\,. \tag{6.1}$$

In addition to that, we introduce the notations

$$\begin{aligned} U_r &= (\text{meas } B_r)^{-1}\cdot\int_{B_r}\nabla u\,dx\,,\\ Q_r^2 &= \operatorname*{ess\,sup}_{B_r} q(\nabla u)\,,\\ \mu' &= 1/(8\cdot p)\,. \end{aligned}$$

PROPOSITION 6.1. – *Hölder-Continuity of the Gradient.*

There are positive constants c and μ which both depend only on n, N, p, λ and Λ such that

$$|\nabla u(x)-\nabla u(x_0)|\leqq c\cdot(1+Q_{2R})\cdot R^{-\mu}\cdot|x-x_0|^\mu\,, \tag{6.2}$$

for all $x\in B_R$, where x_0 is the center of B_R.

Proposition 6.1 is an easy consequence of the following two propositions.

PROPOSITION 6.2. – There are positive constants c, μ and ε depending only on n, N, p, λ and Λ such that the inequalities

$$\int_{B_{R/4}} |\nabla u - U_{R/4}|^2\, dx \leqq \varepsilon\cdot Q_R^2\cdot R^n\,, \tag{6.3}$$

$$q(U_{R/4}) \geqq \tfrac{1}{2}\cdot Q_R^2\,, \quad \text{in } B_R\,, \tag{6.4}$$

$$Q_R \geqq R^{\mu'}\,, \tag{6.5}$$

imply that

$$|\nabla u(x) - \nabla u(x_0)| \leqq c\cdot Q_R\cdot R^{-\mu}\cdot |x - x_0|^\mu\,, \tag{6.6}$$

for all $x \in B_R$, where x_0 is the center of B_R.

PROPOSITION 6.3. – *Strong Maximum Principle.*

Choose an $\varepsilon > 0$. Then, there is a positive constant δ depending only on $n, N, p, \lambda, \Lambda$ and ε such that (6.3)-(6.5) are true or that

$$Q_{R/4}^2 \leqq (1-\delta)\cdot Q_R^2 + \delta^{-1}\cdot(1+Q_R^2)\cdot R^{\mu'}\,. \tag{6.7}$$

LEMMA 6.1. – *Inverse Hölder Inequality.*

Suppose that

$$Q_R \geqq \tfrac{1}{4}\cdot R^{\mu'} \tag{6.8}$$

and let $V \in \boldsymbol{R}^{n\cdot N}$ satisfy

$$2\cdot Q_R^2 \geqq q(V) \geqq \tfrac{1}{4}\cdot Q_R^2\,, \quad \text{in } B_R\,. \tag{6.9}$$

Then, there is an exponent $s > 2$ and a constant c which both depend only on n, N, p, λ and Λ such that

$$\int_{B_{R/2}} |\nabla u - V|^s\, dx \leqq c\cdot R^{n(2-s)/2}\cdot\left\{\int_{B_R}|\nabla u - V|^2\, dx\right\}^{s/2} + c\cdot Q_R^{s/2}\cdot R^{n+s/2}\,. \tag{6.10}$$

LEMMA 6.2. – *Recursion Formula.*

Suppose that (6.8) holds and let V and s be as in Lemma 6.1. Then, there is a constant c depending only on n, N, p, λ and Λ such that the following is true. For each $\sigma \in (0, \frac{1}{4}]$, there is a $V_\sigma \in \boldsymbol{R}^{n\cdot N}$ satisfying

$$|V_\sigma - V|^2 \leqq c\cdot R^{-n}\cdot\int_{B_R}|\nabla u - V|^2\, dx\,, \tag{6.11}$$

$$(6.12)\qquad \int_{B_{\sigma R}} |\nabla u - V_\sigma|^2\, dx \leq c\cdot\sigma^{n+2}\cdot\int_{B_R} |\nabla u - V|^2\, dx + \\ + c\cdot Q_R^{2(s-2)/s}\cdot\left\{R^{-n}\cdot\int_{B_R}|\nabla u - V|^2\, dx\right\}^{(s-2)/s}\cdot\int_{B_R}|\nabla u - V|^2\, dx + c\cdot Q_R^2\cdot R^{n+1}\,.$$

PROOF OF PROPOSITION 6.2. – With the aid of Lemma 6.2, we can determine a $\sigma\in(0,\frac{1}{4}]$, an $\varepsilon'>0$, an $R^*>0$ and a $c^*>0$ only in dependence on n, N, p, λ and Λ such that the following is true, for

$$R_i = \sigma^i\cdot R/4\,,$$

$$B_i = B_{R_i}\,.$$

There are constant vectors $W_0, W_1, \ldots, W_k$ such that

$$W_0 = U_{R/4}\,,$$

$$(6.13)\qquad R_{i+1}^{-n}\cdot\int_{B_{i+1}}|\nabla u - W_{i+1}|^2\, dx \leq \tfrac{1}{4}\cdot\varepsilon'\cdot Q_{R_i}^2\cdot R_i^{n+1/2} + \tfrac{1}{4}\cdot R_i^{-n}\cdot\int_{B_i}|\nabla u - W_i|^2\, dx\,,$$

$$(6.14)\qquad |W_{i+1} - W_i|^2 \leq c^*\cdot R_i^{-n}\cdot\int_{B_i}|\nabla u - W_i|^2\, dx\,,$$

for $i = 0, 1, \ldots, k$, provided that

$$R \leq R^*\,,$$

$$(6.15)\qquad Q_{R_i} \geq R_i^{\mu'}/4\,,$$

$$(6.16)\qquad R_i^{-n}\cdot\int_{B_i}|\nabla u - W_i|^2\, dx \leq \varepsilon'\cdot Q_{R_i}^2\,,$$

$$(6.17)\qquad 2\cdot Q_{R_i}^2 \geq q(W_i) \geq \tfrac{1}{4}\cdot Q_{R_i}^2\,,\quad \text{in } B_i\,,$$

for $i = 0, 1, \ldots, k$. Now, it is easy to see that one can find an $\varepsilon > 0$ depending only on n, N, p, λ and Λ such that (6.15)-(6.17) remain valid, for all $i\in\mathbf{N}$, if (6.3)-(6.5) hold. The inequality (6.13) shows that there are positive constants c' and μ which depend only on n, N, p, λ and Λ such that

$$(6.18)\qquad \int_{B_r}|\nabla u - U_r|^2\, dx \leq c' Q_R^2\cdot r^{2\mu+n}\,,$$

for all $r\in(0,R]$, provided that (6.3)-(6.5) hold, for the ε determined above. The constants appearing in (6.18) do not depend on the center of the ball B_R. Therefore, the conclusion of Proposition 6.2 follows from the above considerations and a vell known criterion for Hölder-continous functions due to S. CAMPANATO [3].

In the following proofs, we will often use the inequality

$$(1 + Q_R)^{p-1} \leqq 2^{p+2} \cdot R^{-\frac{1}{2}} \cdot (\varkappa + Q_R)^{p-2} \cdot Q_R \tag{6.19}$$

which is an easy consequence of (6.8).

PROOF OF LEMMA 6.1. – Let C_r be an arbitrary cube with side length $r > 0$ such that

$$C_{2r} \subset B_R .$$

In the following, c stands for a positive generic constant depending only on n, N, p, λ and Λ.

We choose a nonnegative function $\psi \in C_c^\infty(\boldsymbol{R}^n)$ satisfying

$$\psi = 1 , \qquad \text{in } C_r ,$$

$$\operatorname{supp} \psi \subset C_{2r} ,$$

$$|\nabla \psi| \leqq c \cdot r^{-1} , \qquad \text{in } \boldsymbol{R}^n ,$$

and set

$$w_0 = (\text{meas } C_{2r})^{-1} \cdot \int_{C_{2r}} u - \sum_{\alpha=1}^{n} V_\alpha \cdot x_\alpha \, dx ,$$

$$w(x) = u(x) - \sum_{\alpha=1}^{n} V_\alpha \cdot x_\alpha - w_0 ,$$

where $V = (V_1, V_2, \ldots, V_n)$. We insert

$$\varphi = w \cdot \psi^2$$

into (2.2). This gives that

$$\int_\Omega \sum_{\alpha=1}^{n} \sum_{i=1}^{N} \{a_\alpha^i(x, \nabla u) - a_\alpha^i(x, V)\} \cdot w_{x_\alpha}^i \cdot \psi^2 \, dx =$$

$$= -2 \cdot \int_\Omega \sum_{\alpha=1}^{n} \sum_{i=1}^{N} \{a_\alpha^i(x, \nabla u) - a_\alpha^i(x, V)\} \cdot w_{x_\alpha} \cdot \psi \cdot \psi_{x_\alpha} \, dx + \int_\Omega \sum_{i=1}^{N} a^i(x, u, \nabla u) \cdot w^i \cdot \psi^2 \, dx .$$

With the aid of (1.13), (6.9), (6.19), Lemma 3.3 and Young's inequality, we obtain that

$$\int_{C_r} |\nabla w|^2 \, dx \leqq c \cdot r^{-2} \cdot \int_{C_{2r}} |w|^2 \, dx + c \cdot Q_R^2 \cdot r^{n+1} .$$

To this, we apply Sobolev's imbedding theorem. This gives that

$$\int_{C_r}|\nabla w|^2\,dx \leq c\cdot r^{-n/(n-1)}\cdot\left\{\int_{C_{2r}}|\nabla w|^{2(n-1)/n}\,dx\right\}^{n/(n-1)} + c\cdot Q_R^2\cdot r^{n+1}\,.$$

Now, we can make use of the local version of « Gehring's Lemma » [7] which is due to GIAQUINTA-MODICA [15]. It states that

$$\int_{C_r}|\nabla w|^s\,dx \leq c\cdot r^{n(2-s)/2}\cdot\left\{\int_{C_{2r}}|\nabla w|^2\,dx\right\}^{s/2} + c\cdot Q_R^s\cdot r^{n+s/2}\,,$$

for some $s>2$ depending only on n, N, p, λ and Λ. This obviously proves the conclusion of Lemma 6.1.

PROOF OF LEMMA 6.2. – By c, we denote a generic constant depending only on n, N, p, λ and Λ. In the following, v is the $H^{1,2}$-solution of

$$\sum_{\alpha,\beta=1}^{n}\sum_{j=1}^{N}\frac{d}{dx_\alpha}\left\{\frac{\partial a_\alpha^i}{\partial\eta_\beta^j}(x_0, V)\cdot v_{x_\beta}^j\right\} = 0\,,\quad \text{in } B_{R/2}\,, \tag{6.20}$$

for $i = 1, 2, \ldots, N$ and

$$v = u\,,\quad \text{on } \partial B_{R/2}\,, \tag{6.21}$$

where x_0 is the center of B_R. We pick a $\sigma\in(0,\frac{1}{4}]$ and set

$$V_\sigma = (\text{meas } B_{\sigma R})^{-1}\cdot\int_{B_{\sigma R}}\nabla v\,dx\,.$$

In order to prove the recursion formula (6.11), we use the function

$$\tilde{w}(x) = v(x) - \sum_{\alpha=1}^{n} V_\alpha\cdot x_\alpha\,.$$

The definitions imply that $\tilde{w}$ solves the system (6.20) and that

$$\tilde{w}(x) = u(x) - \sum_{\alpha=1}^{n} V_\alpha\cdot x_\alpha\,,\quad \text{on } \partial B_{R/2}\,.$$

We note that, by (6.9) and the ellipticity and growth conditions (2.3) and (2.4), the system (6.20) is strongly elliptic. Moreover, its coefficients are constants. Therefore, we can apply some well known results on linear elliptic systems to obtain that

$$|V_\sigma - V| \leqq \operatorname*{ess\,sup}_{B_{\sigma R}} |\nabla\tilde{w}| \leqq c\cdot R^{-n}\cdot \int_{B_{R/2}} |\nabla\tilde{w}|^2\,dx \leqq c\cdot R^{-n}\cdot \int_{B_{R/2}} |\nabla u - V|^2\,dx\,.$$

This inequality obviously implies (6.11).

In order to prove the recursion formula (6.12), we use the function

$$w = u - v\,.$$

From the differential equation (2.2), one derives that

$$0 = \int_\Omega \sum_{i=1}^N a^i(x, u, \nabla u)\cdot w^i\,dx - \int_\Omega \sum_{\alpha=1}^n \sum_{i=1}^N \{a^i_\alpha(x, \nabla u) - a^i_\alpha(x_0, \nabla u)\}\cdot w^i_{x_\alpha}\,dx -$$
$$- \int_\Omega\int_0^1 \sum_{\alpha,\beta=1}^n \sum_{i,j=1}^N \frac{\partial a^i_\alpha}{\partial \eta^j_\beta}\big(x_0, t\cdot\nabla u + (1-t)\cdot V\big)\cdot(u^j_{x_\beta} - V^j_\beta)\cdot w^i_{x_\alpha}\,dx\,,$$

This, (6.20) and (6.21) imply that

$$\int_\Omega \sum_{\alpha,\beta=1}^n \sum_{i,j=1}^N \frac{\partial a^i_\alpha}{\partial \eta^j_\beta}(x_0, V)\cdot w^j_{x_\beta}\cdot w^i_{x_\alpha}\,dx =$$
$$= \int_\Omega \sum_{i=1}^N a^i(x, u, \nabla u)\cdot w^i\,dx - \int_\Omega \sum_{\alpha=1}^n \sum_{i=1}^N \{a^i_\alpha(x, \nabla u) - a^i_\alpha(x_0, \nabla u)\}\cdot w^i_{x_\alpha}\,dx +$$
$$+ \int_\Omega\int_0^1 \sum_{\alpha,\beta=1}^n \sum_{i,j=1}^N \left\{\frac{\partial a^i_\alpha}{\partial \eta^j_\beta}(x_0, V) - \frac{\partial a^i_\alpha}{\partial \eta^j_\beta}\big(x_0, t\cdot\nabla u + (1-t)\cdot V\big)\right\}\cdot(u^j_{x_\beta} - V^j_\beta)\times w^i_{x_\alpha}\,dx\,.$$

To this, we apply (1.13), (2.3), (6.19), Lemma 2.1 and 2.3, Sobolev's imbedding theorem and Young's inequality, to obtain that

$$\int_{B_{R/2}} |\nabla w|^2\,dx \leqq c\cdot Q_R^{-2}\cdot \int_{B_{R/2}} |\nabla u - V|^4\,dx + c\cdot Q_R^2\cdot R^{n+1}\,.$$

With the aid of Lemma 6.1 and Hölder's inequality, we derive from the last inequality that

$$\text{(6.22)}\qquad \int_{B_{R/2}} |\nabla w|^2\,dx \leqq c\cdot Q_R^{2(2-s)/s}\cdot \left\{R^{-n}\cdot\int_{B_R} |\nabla u - V|^2\,dx\right\}^{(s-2)/s}\cdot \int_{B_R} |\nabla u - V|^2\,dx + c\cdot Q_R^2\cdot R^{n+1}\,.$$

Already at the beginning of this proof, we noted that (6.20) is a strongly elliptic system with constant coefficients. Therefore, v satisfies an estimate due to S. CAMPANATO [3], namely

$$\int_{B_R}|\nabla v - V_\sigma|^2\,dx \leqq c\cdot\sigma^{n+2}\cdot\int_{B_{R/2}}|\nabla v - V|^2\,dx \leqq c\cdot\sigma^{n+2}\cdot\int_{B_R}|\nabla u - V|^2\,dx\,. \tag{6.23}$$

By means of the inequality

$$|\nabla u - V_\sigma|^2 \leqq 2\cdot|\nabla w|^2 + 2\cdot|\nabla v - V_\sigma|^2\,,$$

the estimates (6.22) and (6.23) can be combined to (6.12). For the proof of Proposition 5.3, we need the following lemma.

LEMMA 6.3. – Suppose that (6.5) holds. Then, there is a constant c depending only on n, N, p, λ and Λ such that

$$\begin{aligned}\int_{A_{k,r}} |\nabla q(\nabla u)|^2 + |\nabla^2 u|^2\cdot\{q(\nabla u) - k\}\,dx \leqq \\ \leqq c\cdot(r' - r)^{-2}\cdot(Q_R^2 - k)^2\cdot\text{meas } A_{k,r'} + c\cdot R^{-1}\cdot(1 + Q_R^2)^2\cdot\text{meas } A_{k,r'}\,,\end{aligned} \tag{6.24}$$

for all

$$0 \leqq k \leqq Q_R^2\,,$$
$$0 < r < r' \leqq R\,,$$

where

$$A_{k,r} = \{x \in B_r | q(\nabla u) > k\}\,.$$

PROOF OF LEMMA 6.3. – We choose a nonnegative function $\psi \in C_c^\infty(\boldsymbol{R}^n)$ satisfying

$$\begin{aligned} \psi &= 1\,, && \text{in } B_r\,,\\ \psi &= 0\,, && \text{outside } B_{r'}\,,\\ |\nabla\psi| &\leqq c'\cdot(r' - r)^{-1}\,, && \text{in } \boldsymbol{R}^n\,,\end{aligned}$$

for some constant c' depending only on n. Moreover, we set

$$G(t) = \max\,(t - k,\, 0)\,.$$

Inserting these functions ψ and G into (4.7), one easily obtains (6.24), with the aid of (6.19) and Young's inequality.

PROOF OF PROPOSITION 6.3. – During this proof, c stands for a generic constant depending only on n, N, p, λ and Λ. We will suppose that (6.5) holds and that

$$R \leqq R_0 , \tag{6.25}$$

for some $R_0 > 0$ which depends only on $n, N, p, \lambda, \Lambda$ and ε and which will be determined later on. Namely, in the other cases, there is nothing to show.

Let us consider the inequality

$$\text{meas } A_{(1-\sigma)Q_R^2, R/2} \geqq (1-\sigma)\cdot \text{meas } B_{R/2} , \tag{6.26}$$

where $\sigma \in (0, \frac{1}{4}]$ and where $A_{k,r}$ is defined as in Lemma 6.3. From Lemma 6.3 and Proposition 4 of [32], we know that (6.7) holds, for some $\delta > 0$ depending only on $n, N, p, \lambda, \Lambda$ and σ, if (6.26) is wrong.

Hence, we may suppose that, in addition to (6.5) and (6.25), the inequality (6.26) holds, for some $\sigma \in (0, \frac{1}{4}]$ which depends only on $n, N, p, \lambda, \Lambda$ and ε and which we will determine later on.

Lemma 6.3 and (6.5) imply that

$$\int_{A_{(1-\varrho)Q_R^2, R/4}} |\nabla^2 u|^2\, dx \leqq c\cdot \varrho \cdot Q_R^2 \cdot R^{n-2} + c\cdot Q_R^2 \cdot R^{n-\frac{3}{2}} , \tag{6.27}$$

for all $\varrho \in (0, \frac{1}{4}]$. By (1.8) and (1.9), it is easy to construct a function V such that

$$\begin{aligned} V &= \nabla u , && \text{if } q(\nabla u) \geq 7/8 \cdot Q_R^2 , \\ V &= 0 , && \text{if } q(\nabla u) \leq 3/4 \cdot Q_R^2 , \\ |\nabla V|^2 &\leqq c\cdot |\nabla^2 u|^2 , \end{aligned}$$

in B_R, provided that R is sufficiently small. We set

$$\begin{aligned} V_0 &= (\text{meas } B_{R/4})^{-1} \cdot \int_{B_{R/4}} V\, dx , \\ A_1 &= A_{3Q_R^2/4, R/4} \setminus A_{(1-\sigma)Q_R^2, R/4} , \\ A_2 &= A_{(1-\sigma)Q_R^2, R/4} . \end{aligned}$$

From (6.26), (6.27), Sobolev's imbedding theorem and the properties of V, we derive that

$$\begin{aligned} \int_{B_{R/4}} |V - V_0|^2\, dx &\leqq c\cdot R^{2-n/(n-1)} \cdot \left\{ \int_{A_1 \cup A_2} |\nabla V|^{2(n-1)/n}\, dx \right\}^{n/(n-1)} \leqq \\ &\leqq c\cdot R^{2-n/(n-1)} \cdot (\text{meas } A_1)^{1/(n-1)} \cdot \int_{A_1} |\nabla^2 u|^2\, dx + c\cdot R^2 \cdot \int_{A_2} |\nabla^2 u|^2\, dx \leqq \\ &\leqq c\cdot \sigma^{1/(n-1)} \cdot Q_R^2 \cdot R^n + c\cdot Q_R^2 \cdot R^{n+\frac{1}{2}} . \end{aligned}$$

This, (6.26) and the properties of V imply that

$$(6.28)\qquad \int_{B_{R/4}} |\nabla u - U_{R/4}|^2\,dx \leqq \int_{B_{R/4}} |\nabla u - V_0|^2\,dx \leqq$$
$$\leqq 2\cdot\int_{B_{R/4}} |\nabla u - V|^2 + |V - V_0|^2\,dx \leqq c\cdot\sigma^{1/(n-1)}\cdot Q_R^2\cdot R^n + c\cdot Q_R^2\cdot R^{n+\frac{1}{2}}\,.$$

Moreover, from (1.8), (1.9), (6.26) and (6.28), one derives that

$$(6.29)\qquad \sqrt{q(x, U_{R/4})} \geqq (1 - c\cdot R)\cdot(\text{meas}\,B_{R/4})^{-1}\cdot\int_{B_{R/4}}\sqrt{q(y, U_{R/4})}\,dy \geqq$$
$$\geqq (1 - c\cdot R)\cdot(\text{meas}\,B_{R/4})^{-1}\cdot\int_{B_{R/4}}\sqrt{q(\nabla u)} - c\cdot|\nabla u - U_{R/4}|\,dy \geqq$$
$$\geqq (1 - c\cdot R)\cdot Q_R\cdot\{(1-\sigma)^{\frac{3}{2}} - c\cdot\sigma^{1/(2(n-1))} - c\cdot R^{\frac{1}{4}}\}\,,$$

for all $x \in B_R$, if R is sufficiently small. The inequalities (6.28) and (6.29) imply that (6.3) and (6.4) are true, if (6.25) and (6.26) hold, for some $R_0 > 0$ and some $\sigma > 0$ which depend only on $n, N, p, \lambda, \Lambda$ and ε. Moreover, (6.5) holds, by assumption. Hence, we can conclude the proof, because we discussed the other cases, already at the beginning of the proof.

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