On the Γ-Convergence for Multiple Integrals Depending on Vector Valued Functions (*).

LUCIANA NANIA

Sunto. – *Si prendono in esame successioni di integrali del tipo* $F_h = \int_\Omega f_h(x, Du)\,dx$, *ove* u *è una funzione vettoriale, e si stabilisce una condizione sufficiente affinchè per la successione* F_h *la* Γ*-convergenza nella topologia* L^p, $1 \leqslant p < \infty$, *sia equivalente alla* Γ*-convergenza nella topologia* L^∞. *Si fa inoltre un'applicazione dei risultati ottenuti dimostrando un teorema di convergenza delle soluzioni di disequazioni variazionali relative a vincoli di tipo ostacolo.*

1. – Introduction.

The aim of this paper is to show the results we have obtained on the Γ-convergence, in different topologies, of sequence of functionals

(1.1)
$$F_h(\Omega, u) = \int_\Omega f_h(x, Du)\,dx$$

where Ω is an open bounded set in R^n, with a smooth boundary, u is a vector of $H^{1,p}(\Omega, R^N)$, $p > 1$, and Du denotes the gradient of u, i.e. the vector $(D_\alpha u^i)$, $\alpha = 1, \ldots, n$, $i = 1, \ldots, N$, $D_\alpha = \delta/\delta x_\alpha$.

We shall suppose that:

(1.2) for any h, f_h: $\Omega \times R^{nN} \to R$ is a mesurable function in x for each $\xi \in R^{nN}$ and convex in ξ for almost every $x \in \Omega$;

(1.3) there exists a real number $\Lambda \geqslant 1$ such that:

$$|\xi|^p \leqslant f_h(x, \xi) \leqslant \Lambda(1 + |\xi|^p)\ .$$

For example, under our hypotheses we have, for $p = 2$, the quadratic functionals:

(1.4)
$$F_h(\Omega, u) = \int_\Omega \sum_{i,j,\alpha\beta} A^{\alpha\beta}_{ij,h}(x)\, D_\alpha u^i D_\beta u^j\,dx$$

(*) Entrata in Redazione il 12 giugno 1982; copia revisionata ricevuta il 7 gennaio 1983.

where $A^{\alpha\beta}_{ij,h}(x)$ $(A^{\alpha\beta}_{ij,h} = A^{\beta\alpha}_{ji,h})$ are bounded continuous functions and

$$|\xi|^2 \leqslant \sum A^{\alpha\beta}_{ij,h}(x)\,\xi^j_\beta\,\xi^i_\alpha \leqslant \Lambda|\xi|^2 .$$

In the general case of functionals (1.1) and under assumptions (1.2) and (1.3), CARBONE and SBORDONE [5]—(generalizing the results of DE GIORGI [8], SBORDONE [17] and also, for the quadratic case, SPAGNOLO [18], TARTAR [19], MARCELLINI-SBORDONE [14], MARCELLINI [13], ZHIKOV-KOZLOV-OLEINIK-KHA T'EN NGOAN [20])—have proved the existence of a subsequence Γ-converging (see Definition 2.1) in the strong topology of L^p and in the weak topology of $H^{1,p}$ to an integral:

$$F(\Omega, u) = \int_\Omega f(x, Du)\, dx \tag{1.5}$$

with f verifying hypotheses of type (1.2) and (1.3). To simplify, we shall still denote this subsequence F_h; then we have, for any $u \in C^1(\Omega, R^N)$:

$$\begin{aligned} F(\Omega, u) &= \Gamma^-(s - L^p(\Omega, R^N)) \lim_h F_h(\Omega, u) = \\ &= \Gamma^-(w - H^{1,p}(\Omega, R^N)) \lim_h F_h(\Omega, u) = \Gamma^-(w - H^{1,p}_0(\Omega, R^N)) \lim_h F_h(\Omega, u) . \end{aligned} \tag{1.6}$$

In general, we cannot assert that the same result holds in L^∞ too.

In the scalar case $N = 1$, DE GIORGI has proved the equivalence of the Γ-convergence in the strong topology of L^1 and L^∞ for integrals of the type:

$$\int_\Omega f(x, u, Du)\, dx . \tag{1.7}$$

But, the general conditions assumed by DE GIORGI do not enable us to obtain the result in the vector case $N > 1$.

We note that, in particular, for a single functional (i.e., if $f_h = f$), the integral (1.5) is Γ-converging to itself with respect to a topology *iff* it is a lower-semicontinuous function in the same topology.

Therefore, from De Giorgi's theorem we deduce also that L^1 lower-semicontinuity of F is equivalent, in the scalar case, to L^∞ lower-semicontinuity.

On the contrary, there exists a counterexample due to EISEN [10] (which we shall refer to in § 4) to the $L^p(\Omega, R^N)$ lower-semicontinuity of an integral of the kind (1.7) which is $C^0(\Omega, R^N)$ lower-semicontinuous.

Therefore, when $N > 1$, we cannot expect in general that from Γ-convergence in L^p we get Γ-convergence on L^∞.

The main purpose of this paper is to prove that, under hypotheses (1.2), (1.3) and assuming furthermore:

$$(1.8)\quad \begin{cases} f(x,\xi) \text{ is strictly convex in } \xi \text{ for almost any } x\in\Omega \text{ and the gradient} \\ f_\xi(x,\xi) \text{ exists, is a Caratheodory function and is bounded for } \xi \text{ bounded}, \end{cases}$$

for any $u\in C^1(\bar{\Omega}, R^N)$, we have:

$$(1.9)\qquad \Gamma^-(L^p(\Omega, R^N))\lim_h F_h(\Omega, u) = \Gamma^-(L^\infty(\Omega, R^N)\lim_h F_h(\Omega, u)\,.$$

In order to prove (1.9), in section 3 we show, by using some recent Meyer's type [15] regularity results of GIAQUINTA and GIUSTI [11], that, for any $u\in C^1(\bar{\Omega}, R^N)$, there exists a positive number ε such that:

$$F(\Omega, u) = \Gamma^-(w - H^{1,p+\varepsilon}(\Omega, R^N))\lim_h F_h(\Omega, u)\,.$$

In the last section, we make an application of the results obtained, showing the convergence of solutions of some variational inequalities.

To this purpose, we point out that results of this type have been obtained previously, in the case $N=1$ (BENSOUSSAN-LIONS-PAPANICOLAOU [2], BOCCARDO-MARCELLINI [3], MARCELLINI-SBORDONE [14], BOCCARDO-MURAT [4], ATTOUCH-PICARD [1]), sometimes by means of different methods: for example, the Hölder-continuity theorem for solutions of uniformly elliptic equations has been used (see BOCCARDO-MARCELLINI [3]), but it is known that such a Hölder-continuity does not hold for solutions of elliptic systems [12].

I wish to thank Prof. P. MARCELLINI for having suggested to me this hint of research and for his very useful advices.

2. – Preliminary recalls.

DEFINITION 2.1 [9]. – Let (V,τ) be a topological space, τ a metrizable topology, $F_h\colon V\to R\cup\{+\infty, -\infty\}$ a sequence of functionals; the functional $F\colon V\to R\cup \cup\{+\infty,-\infty\}$ is the $\Gamma^-(\tau)$ limit of F_h on V

$$F(u) = \Gamma^-(\tau)\lim_h F_h(u) \qquad \forall u\in V$$

iff

i) $\forall u_h,\ u\in V\colon\ u_h \xrightarrow{\tau} u \Rightarrow F(u) \leqslant \liminf_h F_h(u_h)$,

ii) $\forall u\in V\ \exists u_h\in V\colon u_h \xrightarrow{\tau} u$ and $F(u) = \lim_h F_h(u_h)$.

Under suitable assumptions, the $\Gamma^-(\tau)$ convergence implies the convergence of minima points and values of F_h (see DE GIORGI-FRANZONI [9], § 2). Under our hypotheses of convexity and equiboundedness (1.2), (1.3) for functionals (1.1), the Γ-convergence in the weak topology of $H^{1,p}$ can be characterized in terms of convergence of minima points and values of the integrals F_h (see BOCCARDO-MARCELLINI [3], theorems 2.5 and 2.7).

THEOREM 2.1 ([3], theorem 2.7). – Let F_h, F satisfy (1.2) and (1.3), then $F = \Gamma^-(w - H_0^{1,p}) \lim F_h$ on $H_0^{1,p}$ *iff* $\forall v^* \in H^{-1,p'}\, u_h(v^*) \to u(v^*)$ in $w - H_0^{1,p}$, where $u_h(v^*)$ (resp. $u(v^*)$) is the minimum point in $H_0^{1,p}$ of $v \to F_h(v) - \langle v^*, v\rangle$ (resp. $F(v) - \langle v^*, v\rangle$).

Thereafter, we shall use the following:

THEOREM 2.2 ([3], theorem 2.7). – Let F_h satisfy (1.2) and (1.3), $F\colon H_0^{1,p} \to [0, \infty]$ and let K_0 be a dense subset of $H_0^{1,p}$. If i) holds (with τ weak topology of $H_0^{1,p}$) and if ii) holds for every $u \in K_0$, then $F = \Gamma^-(w - H_0^{1,p}) \lim_h F_h$ on $H_0^{1,p}$.

3. – Γ-convergence in the weak topology of $H^{1,p+\varepsilon}(\Omega, R^N)$.

Let's consider the integrals F_h and F as defined in the Introduction, under convexity and boundedness assumptions (1.2), (1.3) and (1.8). For any h, we set

$$(3.1)\qquad \begin{aligned} g_h(x, \xi) &= \begin{cases} f_h(x, \xi) & x \in \Omega\,,\ \xi \in R^{nN} \\ |\xi|^p & x \in R^n - \Omega\,,\ \xi \in R^{nN}\,; \end{cases} \\ g(x, \xi) &= \begin{cases} f(x, \xi) & x \in \Omega\,,\ \xi \in R^{nN} \\ |\xi|^p & x \in R^n - \Omega\,,\ \xi \in R^{nN}\,. \end{cases} \end{aligned}$$

Let Ω' be an open bounded set in R^n compactly containing Ω; then, let's consider the sequence:

$$(3.3)\qquad G_h(\Omega', u) = \int_{\Omega'} g_h(x, Du)\, dx\,.$$

LEMMA 3.1. – *For any* $u \in C^1(\bar{\Omega}, R^N)$,

$$(3.4)\qquad G(\Omega', u) = \int_{\Omega'} g(x, Du)\, dx = \Gamma^-(w - H^{1,p}(\Omega', R^N)) \lim_h G_h(\Omega', u)\,.$$

PROOF. – We begin by observing that i) of Definition 2.1 follows in Ω from (1.6) and in $\Omega' - \Omega$ from lower-semicontinuity of the integral

$$\int_{\Omega'-\Omega} |Du|^p\, dx\,.$$

To check ii), we use Theorem 2.1 of [5] (with the assumption of equiboundedness (1.3) the result of Carbone and Sbordone holds in the strong topology of L^p, in addition to the L^∞ topology, therein considered). Consequently, there exists a sequence u_h, $u_h \to u$ in $L^p(\Omega, R^N)$ and $\operatorname{spt}(u_h - u) \subset \Omega$, such that

$$\lim_h \int_\Omega f_h(x, Du_h)\, dx = \int_\Omega f(x, Du)\, dx .$$

By setting $u_h = u$ in $\Omega' - \Omega$, we obtain the assertion.

Now we can prove the following

THEOREM 3.1. – *For any* $u \in C^1(\bar{\Omega}, R^N)$, *if* (1.8) *holds, there exists a positive number* ε *such that:*

$$F(\Omega, u) = \Gamma^-(w - H^{1,p+\varepsilon}(\Omega, R^N)) \lim_h F_h(\Omega, u) .$$

PROOF. – Let $G(\Omega', v)$ be the functional defined by (3.4) and u a vector of $H^{1,p}(\Omega', R^N)$. Since f is convex, it follows that for almost every $x \in \Omega'$

$$g(x, \xi) - \sum_{i,\alpha} g_{\xi^i_\alpha}(x, \eta)\cdot \xi^i_\alpha \geqslant g(x, \eta) - \sum_{i,\alpha} g_{\xi^i_\alpha}(x, \eta)\cdot \eta^i_\alpha \qquad \forall \xi, \eta \in R^{nN} . \tag{3.5}$$

If $u \in C^1(\Omega', R^N)$, by setting $\xi = Dv$, $\eta = Du$ and integrating (3.5) over Ω', we have

$$\int_{\Omega'} \Big(g(x, Dv) - \sum_{i,\alpha} g_{\xi^i_\alpha}(x, Du)\cdot D_\alpha v^i\Big)\, dx \geqslant \int_{\Omega'} \Big(g(x, Du) - \sum_{i,\alpha} g_{\xi^i_\alpha}(x, Du)\cdot D_\alpha u^i\Big)\, dx . \tag{3.6}$$

Therefore, u minimizes on $H^{1,p}_0(\Omega', R^N) + u$ the functional

$$L(\Omega', v) = \int_{\Omega'} \Big(g(x, Dv) - \sum_{i,\alpha} g_{\xi^i_\alpha}(x, Du)\cdot D_\alpha v^i\Big)\, dx . \tag{3.7}$$

Such a minimum is unique, since f is strictly convex with respect to ξ. For any h, let u_h be the minimum point of the functional

$$L_h(\Omega', v) = \int_{\Omega'} \Big(g_h(x, Dv) - \sum_{i,\alpha} g_{\xi^i_\alpha}(x, Du)\cdot D_\alpha v^i\Big)\, dx . \tag{3.8}$$

We set $M = \max_{\bar{\Omega}} |g_\xi(x, Du(x))|$; if $k = \min\{\frac{1}{2}|Dv|^p - M|Dv|\}$, we have

$$|Dv|^p - M|Dv| \geqslant \tfrac{1}{2}|Dv|^p - k$$

and from this we get:

$$\tfrac{1}{2}|Dv|^p - k \leq g_h(x, Dv) - \sum_{i,\alpha} g_{\xi^i_\alpha}(x, Du)\cdot D_\alpha v^i \leq M + c_1|Dv|^p . \tag{3.9}$$

Consequently, on Ω' the assumptions of Theorem 4.1 of GIAQUINTA and GIUSTI [11] are checked, therefore $u_h \in H^{1,p+\varepsilon}_{\text{loc}}(\Omega', R^N)$ for some $\varepsilon > 0$; in addition, there exists a constant c_2 such that

$$\|1 + |Du_h|\,\|_{L^{p+\varepsilon}(\Omega,R^N)} \leq c_2\big(1 + \|Du_h\|_{L^p(\Omega',R^N)}\big) . \tag{3.10}$$

Since u_h is a sequence of minimum points, we have $\|Du_h\|_{L^p(\Omega',R^N)} \leq c_3$ and from (3.10) it follows $\|u_h\|_{H^{1,p+\varepsilon}} \leq c_4$. By Theorem 2.5 in [3], the sequence u_h is converging with respect to the weak topology of $H^{1,p+\varepsilon}_{\text{loc}}(\Omega', R^N)$ to the unique minimum point u (here we use the strict convexity of f) and moreover

$$\lim_h \int_{\Omega'} \Big(g_h(x, Du_h) - \sum_{i,\alpha} g_{\xi^i_\alpha}(x, Du)\cdot D_\alpha u^i_h\Big)\, dx = \int_{\Omega'} \Big(g(x, Du) - \sum_{i,\alpha} g_{\xi^i_\alpha}(x, Du)\, D_\alpha u^i\Big)\, dx. \tag{3.11}$$

Therefore,

$$\lim_h \int_{\Omega'} g_h(x, Du_h)\, dx = \int_{\Omega'} g(x, Du)\, dx;$$

since, a priori

$$\begin{cases} \liminf\limits_h \displaystyle\int_{\Omega'} g_h(x, Du_h)\, dx \geq \int_{\Omega'} g(x, Du)\, dx , \\ \liminf\limits_h \displaystyle\int_{\Omega'-\Omega} |Du_h|^p\, dx \geq \int_{\Omega'-\Omega} |Du|^p\, dx , \end{cases}$$

we deduce also

$$\lim_h \int_{\Omega} f_h(x, Du_h)\, dx = \int_{\Omega} f(x, Du)\, dx . \tag{3.12}$$

Therefore, ii) of Definition 2.1 is checked; clearly i) is true, since the topology is stronger.

4. – Γ-convergence in the topology $L^\infty(\Omega, R^N)$.

We begin this section showing by means of an example that, in the vector case $N > 1$, there is no equivalence between the Γ-convergence in L^p (or in $H^{1,p}$ weakly) and in L^∞ (contrarily to what happens in the case $N = 1$, like DE GIORGI has proved ([8], theorem 1)).

The starting point is the counterexample of Eisen, where he considers the vector case $N = 2$, $n = 1$ and the following integral:

$$\int_0^1 (u^1\cdot(u^2)')^2\,dx$$

and constructs then a vector functions sequence on $(0, 1)$, so that: $u_h^1 \to 1 = u^1$, $u_h^2 \to x = u^2$ in the strong topology of L^p, for any p, and $u_h^1\cdot u_h^2 = 0$ a.e. We have:

$$\liminf_h \int_0^1 (u_h^1(u_h^2)')^2\,dx = 0 < 1 = \int_0^1 (u^1(u^2)')^2\,dx\,.$$

On the contrary, we verify that such an integral is L^∞ lower-semicontinuous. Indeed, since for h large, $u_h^1 > 1-\varepsilon$, we have:

$$\liminf_h \int_0^1 (u_h^1(u_h^2)')^2\,dx \geqslant (1-\varepsilon)^2 \liminf_h \int_0^1 ((u_h^2)')^2\,dx \geqslant (1-\varepsilon)^2 \int_0^1 ((u^2)')^2\,dx\,.$$

Since ε is arbitrary, we have the assertion.

We actually show that the assumptions we have made enable us to prove the following:

THEOREM 4.1. – *For any* $u \in C^1(\overline{\Omega}, R^N)$,

$$F(\Omega, u) = \Gamma(L^\infty(\Omega, R^N)) \lim_h F_h(\Omega, u) = \Gamma(L_0^\infty(\Omega, R^N)) \lim_h F_h(\Omega, u)\,.$$

PROOF. – By Theorem 3.1, we deduce the existence of a sequence of functions u_h converging to u weakly in $H^{1,p+\varepsilon}(\Omega, R^N)$ such that

$$\lim_h \int_\Omega f_h(x, Du_h)\,dx = \int_\Omega f(x, Du)\,dx\,; \tag{4.1}$$

for any h, we set:

$$\int_\Omega |u_h^i - u^i|\,dx = (\varrho_h^i)^2\,, \qquad \varrho_h^i \geqslant 0\,; \tag{4.2}$$

$$\Omega_h^i = \{x \in \Omega\colon |u_h^i - u^i| > \varrho_h^i\}\,, \qquad i = 1, \dots, N\,. \tag{4.3}$$

Clearly, we have for any i:

$$\lim_h \varrho_h^i = 0\,; \qquad |\Omega_h^i| < \varrho_h^i\,. \tag{4.4}$$

Let also be $\beta_h(t) \equiv (\beta_h^i(t))$ a sequence of functions in $C^1(R, R^N)$ verifying the conditions:

$$\left\{\begin{array}{ll} 0 \leqslant \dfrac{d}{dt}\beta_h^i(t) \leqslant 1 & \forall t \in R\,, \\ \beta_h^i(t) = t & |t| \leqslant \varrho_h^i\,, \\ \dfrac{d}{dt}\beta_h^i(t) = 0 & |t| > 2\varrho_h^i\,. \end{array}\right. \tag{4.5}$$

Now, we define a sequence $w_h \equiv (w_h^i)$ by setting:

$$w_h^i = u^i + \beta_h^i(u_h^i - u^i)\,. \tag{4.6}$$

We observe at once that, by the third of (4.5) and (4.6) we have:

$$\|w_h - u\|_{L^\infty} = \sup_{t\in R} |\beta_h(t)| \leqslant 2|\varrho_h| \to 0\,. \tag{4.7}$$

By (4.3), (4.5) and (1.3) we deduce moreover:

$$\begin{aligned} \int_\Omega (f_h(x, Dw_h) - f_h(x, Du_h))\, dx &= \int_{\bigcup_i \Omega_h^i} (f(x, Dw_h) - f(x, Du_h))\, dx \leqslant \int_{\bigcup_i \Omega_h^i} f_h(x, Dw_h)\, dx \leqslant \\ &\leqslant \int_{\bigcup_i \Omega_h^i} \Lambda(1 + |Dw_h|^p)\, dx \leqslant \Lambda\cdot 2^{p-1}\int_{\bigcup_i \Omega_i^h} (1 + |Du|^p + |Du_h|^p)\, dx \leqslant \\ &\leqslant \Lambda\cdot 2^{p-1}\left[\sum_i |\Omega_h^i|\cdot(1 + \|Du\|_{C^0}^p) + \sum_i |\Omega_h^i|^{1/q}\, \|Du_h\|_{L^{p+\varepsilon}}\right], \qquad \left(q = \frac{p+\varepsilon}{p+\varepsilon-1}\right). \end{aligned}$$

Therefore, we have, as $h \to \infty$:

$$\max \lim_h \int_\Omega f_h(x, Dw_h)\, dx \leqslant \lim_h \int_\Omega f_h(x, Du_h)\, dx = F(\Omega, u)\;;$$

since, on the other hand,

$$F(\Omega, u) \leqslant \liminf_h \int_\Omega f_h(x, Dw_h)\, dx$$

we have finally

$$F(\Omega, u) = \lim_h \int_\Omega f_h(x, Dw_h)\, dx\,.$$

5. – Convergence of solutions of obstacle problems.

Let's consider the following closed convex sets in $H_0^{1,p}(\Omega, R^N)$:

$$K_1 = \{v \in H_0^{1,p}(\Omega, R^N):\ v^i \geqslant \psi^i \text{ on } E^i,\ i = 1, \dots, N\},$$
$$K_2 = \{v \in H_0^{1,p}(\Omega, R^N):\ v^i \geqslant \psi^i \text{ on } \Omega,\ i = 1, \dots, N\},$$

where $\psi \equiv (\psi^i)$ is a measurable function such that K_1 and K_2 are not empty and E^i are compact sets of Ω.

Furthermore, we set for any h and $\varphi \in L^{p'}(\Omega, R^N)(1/p + 1/p' = 1)$

$$\Phi_h(\Omega, u) = F_h(\Omega, u) - \int_\Omega \varphi u \, dx\,, \tag{5.1}$$

$$\Phi(\Omega, u) = F(\Omega, u) - \int_\Omega \varphi u \, dx\,, \tag{5.2}$$

from (1.6), it follows that

$$\Phi(\Omega, u) = \Gamma^-(w - H_0^{1,p}(\Omega, R^N)) \lim_h \Phi_h(\Omega, u)\,.$$

We want to prove the following:

THEOREM 5.1. – *If $u_h(\varphi)$ minimizes over K_1 (K_2) the functional* (5.1), *then, as $h \to \infty$, $u_h(\varphi)$ converges in $w - H_0^{1,p}(\Omega, R^N)$ and strongly in $L^p(\Omega, R^N)$ to $u(\varphi)$, where $u(\varphi)$ is the vector which minimizes over K_1 (K_2) the functional* (5.2).

PROOF. – Set $\delta_{K_1}(u) = 0$ if $u \in K_1$ and $\delta_{K_1}(u) = \infty$ if $u \notin K_1$. To prove the assertion, it suffices to show, by Theorem 2.1:

$$\Phi + \delta_{K_1} = \Gamma^-(w - H_0^{1,p}(\Omega, R^N)) \lim (\Phi_h + \delta_{K_1})\,. \tag{5.3}$$

We shall demonstrate (5.3) by using Theorem 2.2. To this aim, we observe that, if u_h, $u \in H_0^{1,p}(\Omega, R^N)$ and $u_h \to u$ in $w - H_0^{1,p}(\Omega, R^N)$, we have:

$$\Phi(\Omega, u) \leqslant \liminf_h \Phi_h(\Omega, u_h) \tag{5.4}$$

and consequently we obtain i).

Besides, we set

$$K_0 = \{u \in C_0^1(\Omega, R^N):\ \exists \varepsilon^i(u^i),\ u^i > \psi^i + \varepsilon^i \text{ on } E^i\}\,.$$

We observe that K_0 is dense in K_1.

For any $u \in K_0$, by using Theorem 4.1, there exists u_h such that

$$\lim_h \|u_h - u\|_{L^\infty} = 0 \,, \qquad \lim \Phi_h(\Omega, u_h) = \Phi(\Omega, u) \,; \tag{5.5}$$

therefore, for h large, $u_h \in K_1$ and so

$$\lim_h \big(\Phi_h(u_h) + \delta_{K_1}(u_h)\big) = \Phi(u) + \delta_{K_1}(u) \,. \tag{5.6}$$

This proves the assertion for K_1, after observing that u_h is bounded on $H_0^{1,p}(\Omega, R^N)$ and so converges in the weak topology of $H_0^{1,p}(\Omega, R^N)$.

For K_2, as in the previous case, we prove that

$$\Phi(\Omega, u) + \delta_{K_2}(u) \leqslant \liminf_h \big(\Phi_h(\Omega, u_h) + \delta_{K_2}(u_h)\big) \,; \tag{5.7}$$

now, we consider the set K_0 dense in K_2:

$$K_0 = \{u \in C_0^1(\Omega, R^N) \colon \forall E^i \subset\subset \Omega,\ \exists \varepsilon^i(u^i, E^i),\ u^i > \psi^i + \varepsilon^i \text{ on } E^i\} \,.$$

If u_h verifies (5.5), by setting $w_h^i = \max\{u_h^i, \psi^i\}$, we have that [3] $w_h \to u$ in $w - H_0^{1,p}(\Omega, R^N)$ and $\Phi(\Omega, u) = \lim_h \Phi_h(\Omega, w_h)$; since $w_h \in K_2$, it follows finally, for any $u \in K_0$

$$\Phi(\Omega, u) + \delta_{K_2}(u) = \lim_h \big(\Phi_h(\Omega, w_h) + \delta_{K_2}(w_h)\big)$$

and this proves the theorem.

REFERENCES

[1] H. ATTOUCH - C. PICARD, *Convergence d'inéquations variationnelles avec obstacles et espaces fonctionnelles en théorie du potentiel,* Applicable An., to appear.

[2] A. BENSOUSSAN - J. L. LIONS - G. PAPANICOLAOU, *Asymptotic methods in periodic structures,* North Holland, 1978.

[3] L. BOCCARDO - P. MARCELLINI, *Sulla convergenza delle soluzioni di disequazioni variazionali,* Ann. Mat. Pura Appl., **110** (1976), pp. 137-159.

[4] L. BOCCARDO - F. MURAT, *Nouveaux resultats de convergence dans des problèmes unilatereaux,* Collège de France, Seminar on Nonlinear Partial Differential Equations 1979-80, ed. by H. BRÉZIS, J. L. LIONS, Research Notes in Math., Pitman, London (1981).

[5] L. CARBONE - C. SBORDONE, *Some properties of Γ-limits of integral functionals,* Ann. Mat. Pura Appl., **122** (1979), pp. 1-60.

[6] G. DAL MASO, *Limiti di soluzioni di problemi variazionali con ostacoli bilaterali,* Atti Accad. Lincei, Rend. Cl. Sci. Fis. Natur., to appear.

[7] G. DAL MASO - P. LONGO, *Γ-limits of obstacles,* Ann. Mat. Pura Appl., (4) **128** (1980), pp. 1-50.

[8] E. DE GIORGI, *Sulla convergenza di alcune successioni d'integrali del tipo dell'area*, Rendiconti Mat., **8** (1975), pp. 277-294.

[9] E. DE GIORGI - T. FRANZONI, *Su un tipo di convergenza variazionale*, Rend. Acc. Naz. Lincei, Roma LVIII, **6** (1975), pp. 842-850.

[10] G. EISEN, *A counterexample for some lower-semicontinuity results*, Math. Z., **162** (1978), 241-243.

[11] M. GIAQUINTA - E. GIUSTI, *On the regularity of the minimum of variational integrals*, Acta Math., **148** (1982), pp. 31-46.

[12] E. GIUSTI, *Equazioni ellittiche del secondo ordine*, Quaderni dell'U.M.I. n. 6, Pitagora Ed. (1978), pp. 174-177.

[13] P. MARCELLINI, *Convergence of second order linear elliptic operators*, Boll. Un. Mat. Ital. (4), **16**-B (1979), pp. 278-290.

[14] P. MARCELLINI - C. SBORDONE, *An approach to the asymptotic behavior of elliptic parabolic operators*, J. Math. Pures Appl., **56** (1977), pp. 157-182.

[15] P. MARCELLINI - C. SBORDONE, *Homogenization of nonuniformly elliptic operators*, Applicable An., **8** (1978), pp. 101-113.

[16] N. MEYERS, *An L^p estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa, (3) **17** (1963), pp. 189-206.

[17] C. SBORDONE, *Su alcune applicazioni di un tipo di convergenza variazionale*, Ann. Scuola Norm. Sup. Pisa, **2** (1975), pp. 617-638.

[18] S. SPAGNOLO, *Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche*, Ann. Scuola Norm. Sup. Pisa, **22** (1968), pp. 577-597.

[19] L. TARTAR, *Convergence d'operateurs différentiels. Analisi convessa e applicazioni*, Quaderno del C.N.R., Roma (1974), pp. 101-104.

[20] V. V. ZHIKOV - S. M. KOZLOV - O. A. OLEINIK - KHA T'EN NGOAN, *Averaging and G-convergence of differential operators*, Uspeki Mat. Nauk, **34**, no. 5 (1979), pp. 65-133; Russian Math. Surveys, **34**, no. 5 (1979), pp. 69-147.