A Classification of Riemannian Manifolds with Structure Group Spin (7) (*).

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Summary. – *Riemannian manifolds with structure group* Spin (7) *are* 8-*dimensional and have a distinguished* 4-*form. In this paper, the covariant derivative of the fundamental* 4-*form is studied, and it is shown that there are precisely four classes of such manifolds.*

1. – Introduction.

An interesting little studied class of manifolds are the 8-dimensional Riemannian manifolds with holonomy group Spin (7) [2].

For such a Riemannian manifold M, BONAN [3] proved that its dimension must be 8 and its Ricci curvature zero. Moreover, ([5], [11], [12], [13], [14]) there exists a representation of Spin (7) on each tangent space of M defined by means of a 3-fold parallel vector cross product.

The 3-fold vector cross products P can be considered as a natural generalization of the almost-complex structures ([4], [8], [13], [14], [15]). Here, in the place of the Kähler form, one has a fundamental 4-form φ which, in special circumstances, for example when P is parallel and M is compact, generates cohomology in dimension 4.

There are no known nonflat examples of manifolds with holonomy group Spin (7). Here we consider instead the class $\mathcal{W}$ of all 8-dimensional Riemannian manifolds M for which the bundle of orthonormal frames with structure group 0(8) can be reduced to Spin (7). The existence of such reduction is equivalent to the existence of a 3-fold vector cross product on M [13]. Then, the class $\mathcal{W}$ contains all parallelizable 8-dimensional manifolds and is analogous to the class of almost Hermitian manifolds [17], and to the class of Riemannian manifolds with structure group G_2 [9]. Within the class $\mathcal{W}$ one can search for analogs to the classes of almost Kähler and locally conformally equivalent to Kähler manifolds as well as analogs to some other special types of almost Hermitian manifolds.

This search, done in a systematic way by using the method in [9] and [17], is the principal subject of this paper. The idea is to study the representation of Spin (7) on the space W of tensors having the same symmetries as the covariant

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derivative $\nabla\varphi$ of the fundamental 4-form φ, next decompose this representation into irreducible components, and then associate a subclass of $\mathcal{W}$ with each invariant subspace of W.

In fact, by using the theory of root systems of semisimple Lie algebras, we show that the representation of Spin (7) on W has only two irreducible components: $W = W_1 \oplus W_2$. Thus, there are a total of 4 invariant subspaces of W, and hence 4 subclasses of $\mathcal{W}$. The manifolds with parallel vector cross products are in the class $\mathcal{T}$ corresponding to $\{0\}$. In contrast to what happens in the case of almost Hermitian manifolds [17] and the Riemannian manifolds with structure group G_2 [9], the following conditions are equivalent:

i) $\nabla P = 0$;

ii) $\nabla_X(P)(X, Y, Z) = 0$, for all $X, Y, Z \in \mathfrak{X}(M)$;

iii) $d\varphi = 0$;

iv) $\delta\varphi = 0$.

Finally, the class of manifolds with locally conformally parallel vector cross products corresponds to W_2.

In section 2, we discuss the algebra of the 3-fold vector cross products P, and extend the definition of P so that P operates on k-vectors and k-forms. Representations of the Lie group Spin (7) and the relevance of the vector cross products into the study of these representations are studied in section 3.

In section 4, we define the space W, and the decomposition $W = W_1 \oplus W_2$ is established. In section 5, we show how each invariant subspace of W corresponds to a subclass of $\mathcal{W}$, and define the 4 classes.

In section 6, we construct a certain tensor field ν which measures the failure of a manifold with vector cross product to be locally conformally related to a manifold with parallel vector cross product. A similar tensor field has been introduced in [17] for almost Hermitian manifolds, and in [9] for Riemannian manifolds with structure group G_2. Using ν we determine which of the 4 classes are preserved under conformal changes of metric. Finally, in section 7, we discuss the strictness of the four inclusion relations between the classes. In fact, we show that three of the four are strict. So for we have been unable to settle the strictness of the inclusion of $\mathcal{T}$ in $\mathcal{W}_1$.

Added in proof. – Recently we have also been able to show the strictness of the inclusion $\mathcal{T} \subset w_1$. The details will appear in a future paper.

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2. – The algebra of 3-fold vector cross products on vector spaces.

In this section, we shall study the algebra of 3-fold vector cross products P and extend the definition of P so that P operates on k-vectors.

DEFINITION ([7]). – Let V be a finite dimensional vector space over $\boldsymbol{R}$ with a (positive definite) inner product $\langle\ ,\ \rangle$. A *3-fold vector cross product* on V is a trilinear map $P\colon V\times V\times V\to V$ satisfying

$$\langle P(x, y, z), x\rangle = \langle P(x, y, z), y\rangle = \langle P(x, y, z), z\rangle = 0\ , \tag{2.1}$$

$$\|P(x, y, z)\|^2 = \det\begin{pmatrix} \|x\|^2 & \langle x, y\rangle & \langle x, z\rangle \\ \langle y, x\rangle & \|y\|^2 & \langle y, z\rangle \\ \langle z, x\rangle & \langle z, y\rangle & \|z\|^2 \end{pmatrix}, \tag{2.2}$$

for $x, y, z \in V$.

Let $\Lambda^k(V)$ denote the k-th Grassmann space over V (i.e., the space generated by the skew-symmetric products $v_1\wedge\ldots\wedge v_k$). It follows from (2.1) that $P(x, y, z)$ is antisymmetric in x, y, z. Hence P may be extended to a linear mapping $P\colon \Lambda^3(V)\to V$. For this reason we shall usually write $P(x\wedge y\wedge z)$ instead of $P(x, y, z)$. Furthermore, the inner product $\langle\ ,\ \rangle$ can be extended to $\Lambda^k(V)$ by the formula

$$\langle v_1\wedge\ldots\wedge v_k, w_1\wedge\ldots\wedge w_k\rangle = \det\left(\langle v_i, w_j\rangle\right)\ ,$$

for $v_1, \ldots, v_k, w_1, \ldots, w_k \in V$. Then, axiom (2.2) becomes

$$\|P(x\wedge y\wedge z)\|^2 = \|x\wedge y\wedge z\|^2\ . \tag{2.3}$$

REMARK. – (2.3) does not mean that P is an isometry, but only an isometry on decomposable vectors.

DEFINITION. – The fundamental 4-form φ of a 3-fold vector cross product P is given by

$$\varphi(x\wedge y\wedge z\wedge w) = \langle P(x\wedge y\wedge z), w\rangle$$

for $x, y, z, w \in V$. (From (2.1) it follows that φ is skew-symmetric.)

ECKMANN [7], WHITEHEAD [23], and BROWN-GRAY [4] have shown that if V has a 3-fold vector cross product then necessarily $\dim V = 4$ or 8. When $\dim V = 4$, the study of P amounts to the study of the volume element of V, namely φ, so we restrict ourselves to the case $\dim V = 8$. In this case, it is well known ([4]) that there are two non-isomorphic 3-fold vector cross products P_+ and P_-, given ([4], [25]) in terms of the Cayley numbers by

$$P_+(x\wedge y\wedge z) = -\,x(\bar{y}z) + \langle x, y\rangle z + \langle y, z\rangle x - \langle x, z\rangle y\ ,$$
$$P_-(x\wedge y\wedge z) = -\,(x\bar{y})z + \langle x, y\rangle z + \langle y, z\rangle x - \langle x, z\rangle y\ ,$$

for $x, y, z \in V$, and where $x\to\bar{x}$ is the conjugation in *Cay* (that is, $\bar{x} = -x + 2\langle x, 1\rangle 1$). The reason why there are two distinct 3-fold vector cross products is that the Cayley numbers are nonassociative. Furthermore, the automorphism groups of P_+ and P_- are both isomorphic to Spin (7). We write $\varphi_\pm$ for the fundamental 4-form of $P_\pm$.

From now on, we shall write P when no distinction between P_+ and P_- is needed.

First, we shall write some elementary consequences of (2.1)-(2.3).

LEMMA 2.1. – We have

$$\langle P(x\wedge y\wedge z), P(x\wedge y\wedge w)\rangle = \langle x\wedge y\wedge z, x\wedge y\wedge w\rangle\,, \tag{2.4}$$

$$P(x\wedge y\wedge P(x\wedge y\wedge z)) = -\|x\wedge y\|^2 z + \langle x\wedge y, x\wedge z\rangle y + \langle y\wedge x, y\wedge z\rangle x\,, \tag{2.5}$$

for $x, y, z, w \in V$.

Furthermore, from (2.5) it follows

$$P\big(x\wedge y\wedge P(x\wedge y\wedge P(x\wedge y\wedge z))\big) = -\|x\wedge y\|^2 P(x\wedge y\wedge z)\,. \tag{2.6}$$

We now introduce a mapping p that will turn out to be the adjoint of P.

DEFINITION. – The linear mapping $p\colon V\to \Lambda^3(V)$ is given by

$$p(x) = -\tfrac{1}{6}\sum_{i,j=0}^{7} e_i\wedge e_j\wedge P(e_i\wedge e_j\wedge x)$$

for $x\in V$, and where $\{e_0, \ldots, e_7\}$ is any orthonormal basis of V.

LEMMA 2.2. – The mapping p has the following properties

(i) p is the adjoint of P; that is for $x\in V$, $\xi\in \Lambda^3(V)$ we have

$$\langle p(x), \xi\rangle = \langle x, P(\xi)\rangle\,,$$

(ii)
$$P\big(p(x)\big) = 7x\,,$$

for $x\in V$.

PROOF. – (i) follows from (2.1), (2.3) and definition of p. From (2.5) we obtain (ii).

Next, we shall define another linear mapping q which will be useful in section 3.

DEFINITION. – The linear mapping $q\colon \Lambda^2(V)\to \Lambda^2(V)$ is given by

$$q(x\wedge y) = \tfrac{1}{2}\sum_{i=0}^{7} e_i\wedge P(e_i\wedge x\wedge y)$$

for $x, y\in V$, and where $\{e_0, \ldots, e_7\}$ is any orthonormal basis of V.

From (2.5) one has the following

LEMMA 2.3. – The mapping q satisfies

(i) $\langle q(x\wedge y), z\wedge w\rangle = \langle P(x\wedge y\wedge z), w\rangle$;

(ii) $\langle q(x\wedge y), x\wedge y\rangle = 0$;

(iii) $\|q(x\wedge y)\|^2 = 3\|x\wedge y\|^2$;

for $x, y, z, w \in V$.

Now, we extend P, p and q to linear maps $P: \Lambda^{k+2}(V) \to \Lambda^k(V)$, $p: \Lambda^k(V) \to \Lambda^{k+2}(V)$ and $q: \Lambda^k(V) \to \Lambda^k(V)$.

DEFINITION. – Let $v_1, \dots, v_{k+2} \in V$. Then

$$P(v_1\wedge\dots\wedge v_{k+2}) = \sum_{1\leqslant i<j<l\leqslant k+2} (-1)^{i+j+1} P(v_i\wedge v_j\wedge v_l)\wedge v_1\wedge\dots\wedge\hat{v}_i\wedge\dots\wedge\hat{v}_j\wedge\dots\wedge\hat{v}_l\wedge\dots\wedge v_{k+2},$$

$$p(v_1\wedge\dots\wedge v_k) = \sum_{i=1}^{k} (-1)^{i+1} p(v_i)\wedge v_1\wedge\dots\wedge\hat{v}_i\wedge\dots\wedge v_k,$$

$$q(v_1\wedge\dots\wedge v_k) = \sum_{1\leqslant i<j\leqslant k} (-1)^{i+j+1} q(v_i\wedge v_j)\wedge v_1\wedge\dots\wedge\hat{v}_i\wedge\dots\wedge\hat{v}_j\wedge\dots\wedge v_k,$$

and $P: \Lambda^{k+2}(V) \to \Lambda^k(V)$, $p: \Lambda^k(V) \to \Lambda^{k+2}(V)$ and $q: \Lambda^k(V) \to \Lambda^k(V)$ are their canonical linear extensions.

Note that the sequences

$$\Lambda^7(V) \underset{p_5}{\overset{P_5}{\rightleftarrows}} \Lambda^5(V) \underset{p_3}{\overset{P_3}{\rightleftarrows}} \Lambda^3(V) \underset{p_1}{\overset{P_1}{\rightleftarrows}} V$$

$$\Lambda^6(V) \underset{p_4}{\overset{P_4}{\rightleftarrows}} \Lambda^4(V) \underset{p_2}{\overset{P_2}{\rightleftarrows}} \Lambda^2(V)$$

are non-exact.

In particular, for $w, x, y, z \in V$ we have

$$p(x\wedge y) = p(x)\wedge y - p(y)\wedge x,$$

$$p(x\wedge y\wedge z) = \underset{xyz}{\mathfrak{S}}\, p(x)\wedge y\wedge z,$$

$$P(w\wedge x\wedge y\wedge z) = \underset{xyz}{\mathfrak{S}}\, \{P(w\wedge x\wedge y)\wedge z\} - P(x\wedge y\wedge z)\wedge w,$$

$$q(x\wedge y\wedge z) = \underset{xyz}{\mathfrak{S}}\, q(x\wedge y)\wedge z,$$

where $\mathfrak{S}$ denotes the cyclic sum.

Let $\Lambda(V) = \bigoplus_{k=0}^{8} \Lambda^k(V)$, and let $*: \Lambda(V) \to \Lambda(V)$ be the Hodge star operator. Then $*\big(\Lambda^k(V)\big) = \Lambda^{8-k}(V)$. Next, we determine the form $*\varphi_\pm$. (Here the 4-form $*\varphi_\pm$ can be defined by

$$(*\varphi_\pm)(x\wedge y\wedge z\wedge w) = \varphi_\pm\big(*(x\wedge y\wedge z\wedge w)\big)\big).$$

LEMMA 2.4.

$$* \varphi_\pm = \pm\varphi_\pm . \tag{2.7}$$

PROOF. – Consider $\{1, e_0, \ldots, e_6\}$ an orthonormal basis for V where 1 is the identity in *Cay*, and such that $e_i e_{i+1} = e_{i+3}$ for $i \in Z_7$; and let $\{\theta_{-1}, \theta_0, \ldots, \theta_6\}$ be the dual basis. Then

$$\begin{aligned}\varphi_\pm = {} & \theta_{-1}\wedge(\theta_0\wedge\theta_1\wedge\theta_3 + \theta_0\wedge\theta_2\wedge\theta_6 + \theta_0\wedge\theta_4\wedge\theta_5 + \theta_1\wedge\theta_2\wedge\theta_4 + \\ & + \theta_1\wedge\theta_5\wedge\theta_6 + \theta_2\wedge\theta_3\wedge\theta_5 + \theta_3\wedge\theta_4\wedge\theta_6) \pm \\ & \pm \theta_0\wedge\theta_1\wedge\theta_2\wedge\theta_5 \mp \theta_0\wedge\theta_1\wedge\theta_4\wedge\theta_6 \mp \theta_0\wedge\theta_2\wedge\theta_3\wedge\theta_4 \pm \\ & \pm \theta_0\wedge\theta_3\wedge\theta_5\wedge\theta_6 \pm \theta_1\wedge\theta_2\wedge\theta_3\wedge\theta_6 \mp \theta_1\wedge\theta_3\wedge\theta_4\wedge\theta_5 \mp \theta_2\wedge\theta_4\wedge\theta_5\wedge\theta_6 .\end{aligned} \tag{2.8}$$

Applying the Hodge star operator to both sides of (2.8), we get (2.7).

LEMMA 2.5. – For $P_\pm\colon \Lambda^4(V) \to \Lambda^2(V)$ we have

$$P_{\pm *} = \pm P_\pm . \tag{2.9}$$

PROOF. – This can be checked by choosing an orthonormal basis for V as that in Lemma 2.4, and computing the maps $P_{\pm *}$ and $P_\pm$.

3. – Some representations of Spin(7).

We shall describe, in this section, some representations of Spin (7) that will be needed in the next section.

A simple method to describe the 8-dimensional representation of Spin (7) is by means of the 3-fold vector cross products P. In fact,

$$\text{Spin}\,(7) = \{g \in 0(8)\,|\,P(gx\wedge gy\wedge gz) = gP(x\wedge y\wedge z) \text{ for all } x, y, z \in V\};$$

and the 21-dimensional irreducible representation of Spin (7) is the adjoint representation.

Next, we shall consider, for convenience, the covariant versions of P, p and q which will be denoted by the same letters. Let V^* denote the dual space of V.

DEFINITION. – The mappings $P\colon \Lambda^k(V^*) \to \Lambda^{k+2}(V^*)$, $p\colon \Lambda^{k+2}(V^*) \to \Lambda^k(V^*)$ and $q\colon \Lambda^k(V^*) \to \Lambda^k(V^*)$ are given by $P(\alpha) = \alpha\circ P$, $p(\beta) = \beta\circ p$ and $q(\alpha) = \alpha\circ q$, for $\alpha \in \Lambda^k(V^*)$ and $\beta \in \Lambda^{k+2}(V^*)$. (To avoid confusion we shall sometimes write P_k, p_k and q_k.)

Thus we have the following non-exact sequences:

$$V^* \underset{p_1}{\overset{P_1}{\rightleftarrows}} \Lambda^3(V^*) \underset{p_3}{\overset{P_3}{\rightleftarrows}} \Lambda^5(V^*) \underset{p_5}{\overset{P_5}{\rightleftarrows}} \Lambda^7(V^*)$$

$$\Lambda^2(V^*) \underset{p_2}{\overset{P_2}{\rightleftarrows}} \Lambda^4(V^*) \underset{p_4}{\overset{P_4}{\rightleftarrows}} \Lambda^6(V^*)\,.$$

We shall now determine the irreducible components of the induced representation of Spin (7) on each space $\Lambda^k(V^*)$. First, let us note that the representations of Spin (7) on $\Lambda^k(V^*)$ and on $\Lambda^{8-k}(V^*)$ are the same because the Hodge star operator $*: \Lambda^k(V^*) \to \Lambda^{8-k}(V^*)$ is an isometry. (We are using on $\Lambda^k(V^*)$ the inner product $\langle\,,\rangle$ given by

$$\langle\alpha, \beta\rangle = \sum_{i_1,\dots,i_k=0}^{7} \alpha(e_{i_1}\wedge\dots\wedge e_{i_k})\beta(e_{i_1}\wedge\dots\wedge e_{i_k})\,,$$

where $\{e_0, \dots, e_7\}$ is an arbitrary basis of V). Hence, it suffices to describe the representations of Spin (7) on V^*, $\Lambda^2(V^*)$, $\Lambda^3(V^*)$, and $\Lambda^4(V^*)$. The representation of Spin (7) on V^* is the irreducible 8-dimensional representation, but the representations of Spin (7) on $\Lambda^2(V^*)$, $\Lambda^3(V^*)$ and $\Lambda^4(V^*)$ are all reducible. In order to describe the irreducible summands of these representations by means of the vector cross product P, we first use Weyl's formula (see for example [21]) to calculate the degrees of the first ten irreducible representations of Spin (7). It can be verified that these degrees are 1, 7, 8, 21, 27, 35, 42, 48, 105, 112. Next, we shall define the following spaces

$$\Lambda^2_1(V^*) = \{\alpha \in \Lambda^2(V^*) | q\alpha = 0\}\,,$$
$$\Lambda^2_2(V^*) = \{\alpha \in \Lambda^2(V^*) |\ \alpha = q\beta \text{ for some } \beta \in \Lambda^2(V^*)\}\,,$$
$$\Lambda^3_1(V^*) = \{\alpha \in \Lambda^3(V^*) | p\alpha = 0\}\,,$$
$$\Lambda^3_2(V^*) = \{\alpha \in \Lambda^3(V^*) | 7\alpha = Pp\alpha\}\,,$$
$$\Lambda^4_{1+}(V^*) = \{\varphi_+\}\,,$$
$$\Lambda^4_{2+}(V^*) = \Big\{\alpha \in \Lambda^4(V^*) | p_+(\alpha) = 0\,, \quad *\alpha = +\alpha \quad \text{and} \quad \sum_{i,j,k=0}^{7} \alpha(e_i\wedge e_j\wedge e_k\wedge P_+(e_i\wedge e_j\wedge e_k)) = 0\Big\}\,,$$
$$\Lambda^4_{3+}(V^*) = \{\alpha \in \Lambda^4(V^*) |\ *\alpha = -\alpha\}\,,$$
$$\Lambda^4_{4+}(V^*) = \{\alpha \in \Lambda^4(V^*) |\ \alpha(x\wedge y\wedge z\wedge P_+(x\wedge y\wedge z)) = 0 \text{ for all } x, y, z \in V\}\,,$$
$$\Lambda^4_{1-}(V^*) = \{\varphi_-\}\,,$$
$$\Lambda^4_{2-}(V^*) = \Big\{\alpha \in \Lambda^4(V^*) |\ p_-(\alpha) = 0\,, \quad *\alpha = -\alpha \quad \text{and} \quad \sum_{i,j,k=0}^{7} \alpha(e_i\wedge e_j\wedge e_k\wedge P_-(e_i\wedge e_j\wedge e_k)) = 0\Big\}\,,$$
$$\Lambda^4_{3-}(V^*) = \{\alpha \in \Lambda^4(V^*) |\ *\alpha = +\alpha\}\,,$$
$$\Lambda^4_{4-}(V^*) = \{\alpha \in V^4(V^*) |\ \alpha(x\wedge y\wedge z\wedge P_-(x\wedge y\wedge z)) = 0 \text{ for all } x, y, z \in V\}\,.$$

In order to check that Spin (7) acts irreducibly on these spaces it will be necessary to make use of the theory of root systems of semisimple Lie algebras. The Dynkin diagram of the simple Lie algebra b_3 of Spin (7) is

$$\underset{\alpha_1}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\underset{\alpha_2}{\circ}\!\!\Longrightarrow\!\!\underset{\alpha_3}{\circ}$$

where $\{\alpha_1, \alpha_2, \alpha_3\}$ is a system of simple roots, and $\alpha_1 = \omega_1 - \omega_2$, $\alpha_2 = \omega_2 - \omega_3$, $\alpha_3 = \omega_3$, $\langle \omega_i, \omega_j \rangle = \delta_{ij}$. Then

$$\frac{\langle \alpha_1, \alpha_2 \rangle}{\|\alpha_1\| \, \|\alpha_2\|} = -\frac{1}{2} = \cos\frac{2}{3}\pi ,$$

$$\frac{\langle \alpha_2, \alpha_3 \rangle}{\|\alpha_2\| \, \|\alpha_3\|} = -\frac{\sqrt{2}}{2} = \cos\frac{3}{4}\pi ,$$

and the angle between α_1, α_3 is equal to $\pi/2$. We shall denote by λ_i $(i = 1, 2, 3)$ the fundamental dominant weights, and by φ_{λ_i} $(i = 1, 2, 3)$ the corresponding irreducible representations. (In general for an extreme weight λ, φ_λ will denote the corresponding irreducible representation.) It can be easily verified that these weights are $\lambda_1 = \omega_1$, $\lambda_2 = \omega_1 + \omega_2$ and $\lambda_3 = \frac{1}{2}(\omega_1 + \omega_2 + \omega_3)$. Moreover, φ_{λ_1} has dimension 7. φ_{λ_2} has dimension 21; it is the adjoint representation. φ_{λ_3} has dimension 8; it can be identified with the representation on V.

Now, a long but not difficult computation shows that:

i) the weights of (irreducibile) representation φ_{λ_1} are

$$\pm\lambda_1 = \pm\omega_1 , \quad \pm\omega_2 , \quad \pm\omega_3 , \quad 0 ;$$

ii) the weights of φ_{λ_2}, i.e. the roots of b_3, are

$$\pm\lambda_2 = \pm(\omega_1 + \omega_2) , \quad \pm(\omega_1 + \omega_3) , \quad \pm(\omega_2 + \omega_3) , \quad \pm\omega_1 , \quad \pm(\omega_1 - \omega_3) ,$$
$$\pm\omega_2 , \quad \pm(\omega_1 - \omega_2) , \quad \pm(\omega_2 - \omega_3) , \quad \pm\omega_3 , \quad 0(3) ;$$

iii) the weights of φ_{λ_3} are

$$\pm\lambda_3 = \pm\tfrac{1}{2}(\omega_1 + \omega_2 + \omega_3) , \quad \pm\tfrac{1}{2}(\omega_1 + \omega_2 - \omega_3) , \quad \pm\tfrac{1}{2}(\omega_1 - \omega_2 + \omega_3) ,$$
$$\pm\tfrac{1}{2}(-\omega_1 + \omega_2 + \omega_3) ;$$

iv) the weights of the 48-dimensional irreducible representation $\varphi_{\lambda_1 + \lambda_3}$ are

$$\pm\tfrac{1}{2}(\omega_1 + \omega_2 + \omega_3)(3) , \quad \pm\tfrac{1}{2}(\omega_1 + \omega_2 - \omega_3)(3) , \quad \pm\tfrac{1}{2}(\omega_1 - \omega_2 + \omega_3)(3) ,$$
$$\pm\tfrac{1}{2}(-\omega_1 + \omega_2 + \omega_3)(3) , \quad \pm\tfrac{1}{2}(3\omega_1 + \omega_2 + \omega_3) , \quad \pm\tfrac{1}{2}(3\omega_1 + \omega_2 - \omega_3) ,$$
$$\pm\tfrac{1}{2}(3\omega_1 - \omega_2 + \omega_3) , \quad \pm\tfrac{1}{2}(3\omega_1 - \omega_2 - \omega_3) , \quad \pm\tfrac{1}{2}(\omega_1 + 3\omega_2 + \omega_3) ,$$

$\pm\frac{1}{2}(\omega_1+3\omega_2-\omega_3)$, $\pm\frac{1}{2}(-\omega_1+3\omega_2+\omega_3)$, $\pm\frac{1}{2}(-\omega_1+3\omega_2-\omega_3)$;

$\pm\frac{1}{2}(\omega_1+\omega_2+3\omega_3)$, $\pm\frac{1}{2}(\omega_1-\omega_2+3\omega_3)$, $\pm\frac{1}{2}(-\omega_1+\omega_2+3\omega_3)$;

$\pm\frac{1}{2}(-\omega_1-\omega_2+3\omega_3)$;

v) the weights of the 27-dimensional irreducible representation $\varphi_{2\lambda_1}$ are

$\pm\omega_1$, $\pm\omega_2$, $\pm\omega_3$, $\pm(\omega_1+\omega_2)$, $\pm(\omega_1+\omega_3)$, $\pm(\omega_2+\omega_3)$,

$\pm(\omega_1-\omega_2)$, $\pm(\omega_1-\omega_3)$, $\pm(\omega_2-\omega_3)$, $\pm2\omega_1$, $\pm2\omega_2$, $\pm2\omega_3$,

$0(3)$;

vi) the weights of the 35-dimensional irreducible representation $\varphi_{2\lambda_3}$ are

$\pm\omega_1(2)$, $\pm\omega_2(2)$, $\pm\omega_3(2)$, $\pm(\omega_1+\omega_2)$, $\pm(\omega_1+\omega_3)$,

$\pm(\omega_2+\omega_3)$, $\pm(\omega_1-\omega_2)$, $\pm(\omega_1-\omega_3)$, $\pm(\omega_2-\omega_3)$,

$\pm(\omega_1+\omega_2+\omega_3)$, $\pm(\omega_1+\omega_2-\omega_3)$, $\pm(\omega_1-\omega_2+\omega_3)$,

$\pm(\omega_1-\omega_2-\omega_3)$, $0(3)$.

We write $\Lambda^k\varphi_{\lambda_3}$ for the induced reducible representation on the k-th Grassmann space $\Lambda^k(V^*)$ over V^*. Then, by computing the weights of $\Lambda^2\varphi_{\lambda_3}$, $\Lambda^3\varphi_{\lambda_3}$ and $\Lambda^4\varphi_{\lambda_3}$, one follows that

$$\Lambda^2\varphi_{\lambda_3}=\varphi_{\lambda_1}\oplus\varphi_{\lambda_2}, \tag{3.1}$$

$$\Lambda^3\varphi_{\lambda_3}=\varphi_{\lambda_1+\lambda_3}\oplus\varphi_{\lambda_3}, \tag{3.2}$$

$$\Lambda^4\varphi_{\lambda_3}=\varphi_{2\lambda_3}\oplus\varphi_{2\lambda_1}\oplus\varphi_{\lambda_1}\oplus\varphi_0, \tag{3.3}$$

where φ_0 is the 1-dimensional irreducible representation.

LEMMA 3.1. – We have

$$\Lambda^2(V^*)=\Lambda^2_1(V^*)\oplus\Lambda^2_2(V^*). \tag{3.4}$$

Also Spin (7) acts irreducibly on $\Lambda^2_i(V^*)$ and

$$\dim\Lambda^2_1(V^*)=21, \quad \dim\Lambda^2_2(V^*)=7.$$

PROOF. – Using definition of q it can be verified that (3.4) holds and that the spaces $\Lambda^2_i(V^*)$ have the stated dimensions. That Spin (7) acts irreducibly on $\Lambda^2_i(V^*)$ is immediate from (3.1).

LEMMA 3.2. – We have the following orthogonal direct sum

$$\Lambda^3(V^*) = \Lambda^3_1(V^*) \oplus \Lambda^3_2(V^*) . \tag{3.5}$$

Also Spin (7) acts irreducibly on $\Lambda^3_i(V^*)$ and

$$\dim \Lambda^3_1(V^*) = 48 , \quad \dim \Lambda^3_2(V^*) = 8 .$$

PROOF. – From Lemma 2.2 (ii) we obtain

$$p_1 \circ P_1 = 7 I_1 , \tag{3.6}$$

where $I_k \colon \Lambda^k(V^*) \to \Lambda^k(V^*)$ denotes the identity map. Using (3.6) it is easy to verify that $\frac{1}{7} P_1 \circ p_1$ and $I_3 - \frac{1}{7} P_1 \circ p_1$ are projections of $\Lambda^3(V^*)$ onto $\Lambda^3_2(V^*)$ and $\Lambda^3_1(V^*)$, respectively. This proves (3.5).

Again using (3.6), one deduces that $P_1 \colon V^* \to \Lambda^3(V^*)$ is injective and $p_1 \colon \Lambda^3(V^*) \to \to V^*$ is surjective. Furthermore, Image $P_1 = \Lambda^3_2(V^*)$. Thus, $\Lambda^3_1(V^*)$ and $\Lambda^3_2(V^*)$ have the stated dimensions. These representations of Spin (7) are irreducible by (3.2).

LEMMA 3.3. – We have the following orthogonal direct sum

$$\begin{aligned} \Lambda^4(V^*) &= \Lambda^4_{1+}(V^*) \oplus \Lambda^4_{2+}(V^*) \oplus \Lambda^4_{3+}(V^*) \oplus \Lambda^4_{4+}(V^*) = \\ &= \Lambda^4_{1-}(V^*) \oplus \Lambda^4_{2-}(V^*) \oplus \Lambda^4_{3-}(V^*) \oplus \Lambda^4_{4-}(V^*) . \end{aligned} \tag{3.7}$$

Also Spin (7) acts irreducibly on each space $\Lambda^4_{i\pm}(V^*)$ and

$$\dim \Lambda^4_{1\pm}(V^*) = 1 , \quad \dim \Lambda^4_{2\pm}(V^*) = 27 , \quad \dim \Lambda^4_{3\pm}(V^*) = 35 , \quad \dim \Lambda^4_{4\pm}(V^*) = 7 .$$

PROOF. – Let P_+ be the 3-fold vector cross product on V. From definition of $P_+ \colon \Lambda^2(V^*) \to \Lambda^4(V^*)$, it is not difficult to prove that Image $P_+ = \Lambda^4_{4+}(V^*)$, and that P_+ maps $\Lambda^2_2(V^*)$ injectively into $\Lambda^4_{4+}(V^*)$. Hence $\dim \Lambda^4_{4+}(V^*) = 7$, and then the subspace U of $\Lambda^4(V)$, annihilated by $\Lambda^4_{4+}(V^*)$ and generated by the elements of the form $x \wedge y \wedge z \wedge P_+(x \wedge y \wedge z)$, $x, y, z \in V$, has $\dim U = 63$.

Now, let us consider the mapping $P_+ \colon \Lambda^4(V) \to \Lambda^2(V)$. Since $\dim U = 63$ and $U \subseteq \operatorname{kernel} P_+$ it follows that $U = \operatorname{kernel} P_+$. Using (2.8) and Lemma 2.5 we obtain $P_+(\Lambda^{4-}(V)) = P_+(\varphi^*_+) = 0$, where φ^*_+ is the dual element of the fundamental 4-form φ_+.

Let C denote the subspace of $\Lambda^4(V)$ given by

$$C = \{\xi \in \Lambda^4(V) |\ * \xi = + \xi \text{ and } \langle \xi, \varphi^*_+ \rangle = P_+ \xi = 0\} .$$

It is easy to verify that kernel $P_+ = \{\varphi^*_+\} \oplus \Lambda^{4-}(V) \oplus C$ and this sum is orthogonal direct.

Considering the dual spaces of $\{\varphi_+^*\}$, $\Lambda^{4-}(V)$ and C (i.e. the spaces $\Lambda^4_{1+}(V^*)$, $\Lambda^4_{3+}(V^*)$ and $\Lambda^4_{2+}(V^*)$, respectively) we get (3.7). Furthermore, all the spaces have the stated dimensions. Using (3.3), it is obvious that Spin (7) acts irreducibly on $\Lambda^4_{i+}(V^*)$. Similarly, we obtain the decomposition

$$\Lambda^4(V^*) = \Lambda^4_{1-}(V^*) \oplus \Lambda^4_{2-}(V^*) \oplus \Lambda^4_{3-}(V^*) \oplus \Lambda^4_{4-}(V^*)$$

if P_- is the vector cross product on V.

4. – The space of covariant derivatives of the fundamental 4-form.

The covariant derivative $\nabla\varphi$ of the fundamental form φ of a 3-fold vector cross product on a 8-dimensional manifold is a covariant tensor of degree 5 which has various symmetry properties. In this section, we shall define a finite dimensional vector space W that will consist of those tensors that posses the same symmetries, and study the decomposition of W into irreducible components under the natural representation of Spin (7).

Let us consider the space $V^* \otimes \Lambda^4(V^*)$, and let W be the subspace of $V^* \otimes \Lambda^4(V^*)$ defined by

$$W = \{\alpha \in V^* \otimes \Lambda^4(V^*) |\; \alpha(w, x\wedge y\wedge z\wedge P(x\wedge y\wedge z)) = 0 \text{ for all } w, x, y, z \in V\} .$$

LEMMA 4.1. – $\dim W = 56$.

PROOF. – Clearly W is naturally isomorphic to $V^* \otimes \Lambda^4_4(V^*)$. Since $\dim V^* = 8$ and $\dim \Lambda^4_4(V^*) = 7$ by Lemma 3.3, the result follows.

We note that there is a natural inner product on W given by

$$\langle \alpha, \beta \rangle = \sum_{h,i,j,k,l=0}^{7} \alpha(e_h, e_i\wedge e_j\wedge e_k\wedge e_l)\beta(e_h, e_i\wedge e_j\wedge e_k\wedge e_l) ,$$

where $\{e_0, \ldots, e_7\}$ is an arbitrary orthonormal basis of V.

It will also be useful to consider linear maps $L_3\colon W \to \Lambda^3(V^*)$, and $L_1\colon W \to V^*$ given by

$$L_3(\alpha)(x\wedge y\wedge z) = \sum_{i=0}^{7} \alpha(e_i, e_i\wedge x\wedge y\wedge z) ,$$

$$L_1(\alpha)(x) = \sum_{i,j,k=0}^{7} \alpha(P(e_i\wedge e_j\wedge e_k), e_i\wedge e_j\wedge e_k\wedge x) ,$$

for $x, y, z \in V$, $\alpha \in W$. First, we shall study the properties of those maps.

LEMMA 4.2. – We have

$$L_1(\alpha)(x) = -\sum_{i,j,k=0}^{7} \alpha(e_i,\, e_j\wedge e_k\wedge P(e_i\wedge e_j\wedge e_k)\wedge x) = 6(p\circ L_3)(\alpha)(x)\,, \tag{4.1}$$

for $x \in V$, $\alpha \in W$.

PROOF. – Substituting $P(e_i\wedge e_j\wedge e_k)$ for e_k in the definition of L_1 and using (2.5), we obtain

$$\begin{aligned} L_1(\alpha)(x) &= \sum_{i,j,k=0}^{7} \alpha\big(P(e_i\wedge e_j\wedge P(e_i\wedge e_j\wedge e_k)),\, e_i\wedge e_j\wedge P(e_i\wedge e_j\wedge e_k)\wedge x\big) = \\ &= \sum_{i,j,k=0}^{7} \{\alpha((-1+\delta_{ij}^2)e_k,\, e_i\wedge e_j\wedge P(e_i\wedge e_j\wedge e_k)\wedge x) + \\ &+ \alpha((\delta_{jk}-\delta_{ik}\delta_{ij})e_j,\, e_i\wedge e_j\wedge P(e_i\wedge e_j\wedge e_k)\wedge x) + \\ &+ \alpha((\delta_{ik}-\delta_{jk}\delta_{ij})e_i,\, e_i\wedge e_j\wedge P(e_i\wedge e_j\wedge e_k)\wedge x)\} = \\ &= -\sum_{i,j,k=0}^{7} \alpha(e_k,\, e_i\wedge e_j\wedge P(e_i\wedge e_j\wedge e_k)\wedge x) = \\ &= -\sum_{i,j,k=0}^{7} \alpha(e_i,\, e_j\wedge e_k\wedge P(e_i\wedge e_j\wedge e_k)\wedge x)\,. \end{aligned} \tag{4.2}$$

On other hand, from definition of $p\colon V \to \Lambda^3(V)$, we see that

$$(p\circ L_3)(\alpha)(x) = -\tfrac{1}{6}\sum_{i,j,k=0}^{7} \alpha(e_i,\, e_i\wedge e_j\wedge e_k\wedge P(e_j\wedge e_k\wedge x)) = \tfrac{1}{6}L_1(\alpha)(x)\,,$$

which proves (4.1).

LEMMA 4.3. – Let be $\alpha \in W$, and suppose there is a constant a such that

$$\begin{aligned} &a(P(x\wedge y\wedge z),\, x\wedge y\wedge z\wedge w) = \\ &\quad = a\{\alpha(x,\, P(x\wedge y\wedge z)\wedge y\wedge z\wedge w) - \alpha(y,\, P(x\wedge y\wedge z)\wedge x\wedge z\wedge w) + \\ &\quad + \alpha(z,\, P(x\wedge y\wedge z)\wedge x\wedge y\wedge w)\}\,, \end{aligned} \tag{4.3}$$

for all $x, y, z, w \in V$. If $a \neq -\frac{1}{3}$ then $(p\circ L_3)(\alpha) = 0$.

PROOF. – From Lemma 4.2 and equation (4.3) we obtain

$$\begin{aligned} (p\circ L_3)(\alpha)(x) &= \frac{1}{6}L_1(\alpha)(x) = \frac{1}{6}\sum_{i,j,k=0}^{7} \alpha(P(e_i\wedge e_j\wedge e_k),\, e_i\wedge e_j\wedge e_k\wedge x) = \\ &= \frac{a}{6}\sum_{i,j,k=0}^{7} \{\alpha(e_i,\, P(e_i\wedge e_j\wedge e_k)\wedge e_j\wedge e_k\wedge x) - \alpha(e_j,\, P(e_i\wedge e_j\wedge e_k)\wedge e_i\wedge e_k\wedge x) + \\ &+ \alpha(e_k,\, P(e_i\wedge e_j\wedge e_k)\wedge e_i\wedge e_j\wedge x)\} = -\frac{a}{2}L_1(\alpha)(x) = -3a(p\circ L_3)(\alpha)(x)\,, \end{aligned}$$

hence the lemma follows.

We now define two subspaces W_1 and W_2 of W by

$$W_1 = \{\alpha \in W \mid L_1(\alpha) = 0 \text{ or } p \circ L_3(\alpha) = 0\},$$

$$\begin{aligned} W_2 = \{\alpha \in W \mid\ & 28\alpha(w, x_1 \wedge x_2 \wedge x_3 \wedge x_4) = \\ & = \sum_{i=1}^{4} (-1)^{i+1} \{ p \circ L_3(\alpha)(x_i) \varphi(w \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_4) + \\ & + 7 \langle w, x_i \rangle L_3(\alpha)(x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_4)\}\,. \end{aligned}$$

The usual representation of Spin (7) on V induces a representation of Spin (7) on W. We shall show that W_1 and W_2 are the two irreducible components of this induced representation. First, we need several lemmas.

LEMMA 4.4. – $W \cap \Lambda^5(V^*) = \{0\}$.

PROOF. – Let be $\alpha \in W \cap \Lambda^5(V^*)$, and $w, x, y, z \in V$. If $z = \lambda P(w \wedge x \wedge y)$, then for any $u \in V$ we have

$$\alpha(w \wedge x \wedge y \wedge z \wedge u) = \lambda \alpha\big(u \wedge w \wedge x \wedge y \wedge P(w \wedge x \wedge y)\big) = 0\,.$$

Therefore w, x, y, z and $P(w \wedge x \wedge y)$ may be assumed to be orthogonal, and then $\{w, x, y, z, P(w \wedge x \wedge y), P(w \wedge x \wedge z), P(w \wedge y \wedge z), P(x \wedge y \wedge z)\}$ forms an orthogonal basis for V. Thus $\alpha(w \wedge x \wedge y \wedge z \wedge u) = 0$ for all $u \in V$. Hence $\alpha = 0$.

LEMMA 4.5. – Suppose $\alpha \in W$ with $L_3(\alpha) \neq 0$ and

$$\begin{aligned} (4.4) \qquad \alpha(w, x_1 \wedge x_2 \wedge x_3 \wedge x_4) = & \sum_{i=1}^{4} (-1)^{i+1} \{ a (p \circ L_3)(\alpha)(x_i) \varphi(w \wedge x_1 \wedge \ldots \hat{x}_i \wedge \ldots \wedge x_4) + \\ & + b \langle w, x_i \rangle L_3(\alpha)(x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_4)\}\,, \end{aligned}$$

for $w, x_1, x_2, x_3, x_4 \in V$. Then $a = 1/28$, $b = 1/4$ and $PpL_3(\alpha) = 7L_3(\alpha)$.

PROOF. – In (4.4) we consider $x_1 = e_i$, $x_2 = e_j$, $x_3 = e_k$, $x_4 = P(e_i \wedge e_j \wedge e_k)$ and take the sum over $i, j, k = 0, \ldots, 7$. Then

$$(4.5) \qquad (168a - 24b) pL_3(\alpha)(w) = 0\,,$$

and thus $7a = b$.

Applying (4.4) and (4.5), we compute $L_3(\alpha)(y \wedge z \wedge u)$ and get

$$(4.6) \qquad (1 - 35a) L_3(\alpha) = -aPpL_3(\alpha)\,.$$

Hence, applying p to both sides of (4.6), we obtain $a = 1/28$, and then $b = 1/4$. Finally, substituting the value of a in (4.6) we find $PpL_3(\alpha) = 7L_3(\alpha)$.

LEMMA 4.6.

$$W_2 \cap \text{kernel } L_3 = \{0\}\,, \tag{4.7}$$

$$W_1 \cap W_2 = \{0\}\,, \tag{4.8}$$

$$\text{kernel } L_3 = \{0\}\,. \tag{4.9}$$

PROOF. – (4.7) is an obvious consequence of the defining condition of the space W_2. In order to prove (4.8) let $\alpha \in W_1 \cap W_2$; then $L_1(\alpha) = 0$. Now, by using (4.1), (4.7) and lemma 4.5, we see that $L_3(\alpha) = 0$. Hence from (4.7) we have $\alpha = 0$. On the other hand, it is easy show that

$$L_3(W_1) \subset \Lambda_1^3(V^*) \quad \text{and} \quad L_3(W_2) \subset \Lambda_2^3(V^*)\,.$$

Since the spaces $L_3(W_1)$ and $L_3(W_2)$ both are invariants under the induced representation of Spin (7) on $\Lambda^3(V^*)$, it follows, from Lemma 3.2, that

$$L_3(W_1) = \Lambda_1^3(V^*) \quad \text{and} \quad L_3(W_2) = \Lambda_2^3(V^*)\,.$$

Hence from (4.7), we get $\dim W_1 \geqslant 48$ and $\dim W_2 = 8$. Now (4.8) and Lemma 4.1 imply (4.9).

THEOREM 4.7. – We have $W = W_1 \oplus W_2$. This direct sum is orthogonal, and it is preserved under the induced representation of Spin (7) on W. The induced representation of Spin (7) on W_i is irreducible and

$$\dim W_1 = 48\,, \qquad \dim W_2 = 8\,.$$

PROOF. – In the Lemma 4.6 the dimensions of W_i have been calculated, and also it has been proved that $W = W_1 \oplus W_2$, where the sum is direct and orthogonal.

Obviously, the representation of Spin (7) on W is $\varphi_{\lambda_1} \oplus \varphi_{\lambda_3}$. Since the weights of this representation are

$\pm\frac{1}{2}(\omega_1 + \omega_2 + \omega_3)(4)$, $\pm\frac{1}{2}(\omega_1 + \omega_2 - \omega_3)(4)$, $\pm\frac{1}{2}(\omega_1 - \omega_2 + \omega_3)(4)$,

$\pm\frac{1}{2}(-\omega_1 + \omega_2 + \omega_3)(4)$, $\pm\frac{1}{2}(3\omega_1 + \omega_2 + \omega_3)$, $\pm\frac{1}{2}(3\omega_1 + \omega_2 - \omega_3)$,

$\pm\frac{1}{2}(3\omega_1 - \omega_2 + \omega_3)$, $\pm\frac{1}{2}(3\omega_1 - \omega_2 - \omega_3)$, $\pm\frac{1}{2}(\omega_1 + 3\omega_2 + \omega_3)$,

$\pm\frac{1}{2}(\omega_1 + 3\omega_2 - \omega_3)$, $\pm\frac{1}{2}(-\omega_1 + 3\omega_2 + \omega_3)$, $\pm\frac{1}{2}(-\omega_1 + 3\omega_2 - \omega_3)$,

$\pm\frac{1}{2}(\omega_1 + \omega_2 + 3\omega_3)$, $\pm\frac{1}{2}(\omega_1 - \omega_2 + 3\omega_3)$, $\pm\frac{1}{2}(-\omega_1 + \omega_2 + 3\omega_3)$,

$\pm\frac{1}{2}(-\omega_1 - \omega_2 + 3\omega_3)$, we have

$$\varphi_{\lambda_1} \otimes \varphi_{\lambda_3} = \varphi_{\lambda_3} \oplus \varphi_{\lambda_1+\lambda_3}\,. \tag{4.10}$$

Therefore, $W = W_1 \oplus W_2$ is the decomposition of W into irreducible components under the natural action of Spin (7). Moreover, φ_{λ_3} and $\varphi_{\lambda_1+\lambda_3}$ are the irreducible representations corresponding to W_2 and W_1, respectively.

5. – The four classes of 8-dimensional Riemannian manifolds with a 3-fold vector cross product.

Using the results obtained in the previous sections, we now establish a classification of 8-dimensional Riemannian manifolds with a 3-fold vector cross product.

Let M be a C^∞ 8-dimensional Riemannian manifold with metric $\langle\, ,\rangle$. Denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M, and by $\mathcal{F}(M)$ the algebra of C^∞ functions on M. For each $m \in M$ the tangent space at m will be denoted by M_m.

DEFINITION. – We say that $(M, \langle\, ,\rangle)$ has a 3-fold vector cross product P if each tangent space M_m has a 3-fold vector cross product $P_m\colon M_m \times M_m \times M_m \to M_m$, and the mapping $m \to P_m$ is C^∞.

It is clear that P gives rise to a tensor field $P\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$ of type (3,1), which satisfies

$$\langle P(X, Y, Z), X\rangle = \langle P(X, Y, Z), Y\rangle = \langle P(X, Y, Z), Z\rangle = 0\,, \tag{5.1}$$

$$\|P(X, Y, Z)\|^2 = \det\begin{pmatrix} \|X\|^2 & \langle X, Y\rangle & \langle X, Z\rangle \\ \langle Y, X\rangle & \|Y\|^2 & \langle Y, Z\rangle \\ \langle Z, X\rangle & \langle Z, Y\rangle & \|Z\|^2 \end{pmatrix}, \tag{5.2}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Furthermore, the algebraic study carried out in the previous sections can obviously be extended to manifolds. We note that the fundamental 4-form φ becomes a differential 4-form on M.

Let ∇ denote the Riemannian connection of $\langle\, ,\rangle$. The covariant derivatives ∇P and $\nabla\varphi$ are given by

$$\begin{aligned}\nabla_W(P)(X, Y, Z) = \nabla_W\big(P(X, Y, Z)\big) - P(\nabla_W X, Y, Z) - P(X, \nabla_W Y, Z) - \\ - P(X, Y, \nabla_W Z)\,,\end{aligned} \tag{5.3}$$

$$\begin{aligned}\nabla_U(\varphi)(W, X, Y, Z) = U\{\varphi(W, X, Y, Z)\} - \varphi(\nabla_U W, X, Y, Z) - \\ - \varphi(W, \nabla_U X, Y, Z) - \varphi(W, X, \nabla_U Y, Z) - \varphi(W, X, Y, \nabla_U Z)\,,\end{aligned} \tag{5.4}$$

for $U, W, X, Y, Z \in \mathfrak{X}(M)$. From (5.3) and (5.4) one has

$$\nabla_U(\varphi)(W, X, Y, Z) = \langle \nabla_U(P)(W, X, Y), Z \rangle \,, \tag{5.5}$$

and so the study of the covariant derivatives of P is equivalent to that of the covariant derivatives of the fundamental 4-form φ.

LEMMA 5.1.

$$\begin{aligned}\nabla_U(\varphi)(W, X, Y, Z) = -\nabla_U(\varphi)(X, W, Y, Z) = -\nabla_U(\varphi)(W, Y, X, Z) = \\ = -\nabla_U(\varphi)(W, X, Z, Y) \,,\end{aligned} \tag{5.6}$$

$$\nabla_W(\varphi)\big(X, Y, Z, P(X, Y, Z)\big) = 0 \,, \tag{5.7}$$

for all $U, W, X, Y, Z \in \mathfrak{X}(M)$.

PROOF. – (5.6) is easily checked from (5.4). (5.7) is proved by applying the vector field W to both sides of (5.2) and using (5.3) and (5.5).

We shall henceforth write $\nabla_W(P)(X \wedge Y \wedge Z)$ for $\nabla_W(P)(X, Y, Z)$, etc.

Now consider the natural 8-dimensional representation of Spin (7) on each tangent space M_m, and let W_m be the space

$$W_m = \{\alpha \in M_m^* \otimes \Lambda^4(M_m^*) |\ \alpha\big(w, x \wedge y \wedge z \wedge P(x \wedge y \wedge z)\big) = 0 \text{ for all } w, x, y, z \in M_m\} \,.$$

Then the induced representation of Spin (7) on W_m has the two components W_{m1}, W_{m2} as described in section 4. It is possible to form from these two a total of four invariant subspaces of W_m (including $\{0\}$ and W_m).

DEFINITION. – Let M be a 8-dimensional Riemannian manifold with a 3-fold vector cross product. For $m \in M$, let U_m denote one of the four invariant subspaces of W_m. Then $\mathcal{U}$ will denote the class of all 8-dimensional Riemannian manifolds with a 3-fold vector cross product such that $(\nabla\varphi)_m \in U_m$ for all $m \in M$.

In order to make this definition meaningful, one must note that for any 8-dimensional Riemannian manifold M with a 3-fold vector cross product, $(\nabla\varphi)_m \in W_m$ for all $m \in M$ by virtue of (5.6) and (5.7).

The class corresponding to W_{mi} will be denoted by $\mathcal{W}_i$. $\mathcal{P}$ will also correspond to $\{0\}$ and $\mathcal{W}$ to W_m.

REMARK. – There are obvious analogies between some of these classes and the corresponding ones for almost Hermitian manifolds [17], and for the 7-dimensional Riemannian manifolds with a 2-fold vector cross product [9]. Nevertheless, Lemma 4.4 implies that if $\nabla_W(\varphi)(W \wedge X \wedge Y \wedge Z) = 0$ for all $W, X, Y, Z \in \mathfrak{X}(M)$, then $\nabla\varphi = 0$. That is, the class $\mathcal{NP}$ of 8-dimensional Riemannian manifolds with a

3-fold nearly parallel vector cross product, defined by analogy with the nearly Kähler manifolds in [17], is actually the class $\mathcal{T}$. This was proved in [13] by a different method.

Let d and δ be the exterior differential and the coderivative of the Riemannian manifold M. If η is a 4-form on M we have the following explicit formulas for $d\eta$ and $\delta\eta$

$$(5.8)\qquad d\eta(U\wedge W\wedge X\wedge Y\wedge Z) = \nabla_U(\eta)(W\wedge X\wedge Y\wedge Z) - \nabla_W(\eta)(U\wedge X\wedge Y\wedge Z) + \\ + \nabla_X(\eta)(U\wedge W\wedge Y\wedge Z) - \nabla_Y(\eta)(U\wedge W\wedge X\wedge Z) + \nabla_Z(\eta)(U\wedge W\wedge X\wedge Y)\,,$$

$$(5.9)\qquad \delta\eta(X\wedge Y\wedge Z) = -\sum_{i=0}^{7}\nabla_{E_i}(\eta)(E_i\wedge X\wedge Y\wedge Z)\,,$$

for $U, W, X, Y, Z \in \mathfrak{X}(M)$. Here $\{E_0, \ldots, E_7\}$ is an arbitrary local frame filed.

Now, assume that M has a vector cross product P with fundamental 4-form φ, then we note that

$$(5.10)\qquad \delta\varphi = -L_3(\nabla\varphi)\,,$$

$$(5.11)\qquad L_1(\nabla\varphi)(X) = \sum_{i,j,k=0}^{7}\nabla_{P(E_i\wedge E_j\wedge E_k)}(\varphi)(E_i\wedge E_j\wedge E_k\wedge X)\,,$$

for $X \in \mathfrak{X}(M)$. Using (5.10), (5.11) and Lemma 4.2 it follows that

$$(5.12)\qquad L_1(\nabla\varphi) = -6p(\delta\varphi)\,.$$

Also, we have

LEMMA 5.2. – $d\varphi = 0$ if and only if $\delta\varphi = 0$.

PROOF. – We write the 3-form $\delta\varphi$ in terms of the exterior differential d and of the Hodge star operator $*$

$$(5.13)\qquad \delta\varphi = - * d * \varphi\,.$$

Then, applying Lemma 2.4 to (5.13), we get the result.

REMARK. – Let $\mathcal{AT}$ and $\mathcal{ST}$ be the classes of 8-dimensional Riemannian manifolds with a 3-fold vector cross product satisfying $d\varphi = 0$ and $\delta\varphi = 0$, respectively. Then from Lemma 5.2 we have $\mathcal{AT} = \mathcal{ST}$. Furthermore, (4.9) and (5.10) imply that if $\delta p = 0$, then $\nabla\varphi = 0$. Hence we get $\mathcal{T} = \mathcal{NT} = \mathcal{AT} = \mathcal{ST}$.

THEOREM 5.3. – The defining relations for each of the four classes are given in table 1 below

TABLE 1.

Class	Defining relations
$\mathcal{T} = \mathcal{N}\mathcal{T} = \mathcal{A}\mathcal{T} = \mathcal{S}\mathcal{T}$	$\nabla\varphi = 0$ (or $d\varphi = 0$, or $\delta\varphi = 0$)
$\mathcal{W}_1$	$L_1(\nabla\varphi) = 0$ (or $p\,\delta\varphi = 0$)
$\mathcal{W}_2 = \mathcal{C}\mathcal{T}$	$28\,\nabla_W(\varphi)(X_1\wedge X_2\wedge X_3\wedge X_4) = -\sum_{i=1}^{4}(-1)^{i+1}\{p\,\delta\varphi(X_i)\varphi(W\wedge X_1\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge X_4) + 7\langle W, X\rangle\,\delta\varphi(X_1\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge X_4)\}$
$\mathcal{W} = \mathcal{W}_1\oplus\mathcal{W}_2$	No relation

6. – Conformal changes of metric.

In this section, we determine which of the 4 classes are preserved under a conformal change of metric. Let M be a 8-dimensional Riemannian manifold with metric $\langle\,,\rangle$; and let $\langle\,,\rangle^0$ be a metric on M conformally related to $\langle\,,\rangle$ via

$$\langle\,,\rangle^0 = e^{2\sigma}\langle\,,\rangle\,, \tag{6.1}$$

where $\sigma\in\mathcal{F}(M)$. It is well known ([11], [12]) that the connections ∇^0 of $\langle\,,\rangle^0$ and ∇ of $\langle\,,\rangle$ are related by

$$\nabla^0_X Y = \nabla_X Y + (X\sigma)Y + (Y\sigma)X - \langle X, Y\rangle\,\mathrm{grad}\,\sigma\,, \tag{6.2}$$

for $X, Y\in\mathfrak{X}(M)$, and where $\mathrm{grad}\,\sigma\in\mathfrak{X}(M)$ is the vector field such that $\langle X, \mathrm{grad}\,\sigma\rangle = X\sigma$ for $X\in\mathfrak{X}(M)$.

Suppose that $(M, \langle\,,\rangle)$ has a 3-fold vector cross product P. Let P^0 be a 3-fold vector cross product on $(M, \langle\,,\rangle^0)$ and $f\in\mathcal{F}(M)$ such that $P^0 = fP$. Then

$$\|P^0(X\wedge Y\wedge Z)\|^{02} = \|X\wedge Y\wedge Z\|^{02} = e^{6\sigma}\|X\wedge Y\wedge Z\|^2 = e^{6\sigma}\|P(X\wedge Y\wedge Z)\|^2 = e^{4\sigma}\|P(X\wedge Y\wedge Z)\|^{02}\,,$$

for $X, Y, Z\in\mathfrak{X}(M)$. Thus we must have $f^2 = e^{4\sigma}$. This leads us to the following

DEFINITION. – Let M be a 8-dimensional Riemannian manifold with metrics $\langle\,,\rangle$, $\langle\,,\rangle^0$ conformally related by (6.1). Let P be a 3-fold vector cross product on $(M, \langle\,,\rangle)$, then

$$P^0 = e^{2\sigma} P\,, \tag{6.3}$$

is a 3-fold vector cross product on $(M, \langle\,,\rangle^0)$. In this case we say that P and P^0 are conformally related.

Let φ, φ^0 denote the fundamental 4-forms corresponding to P and P^0, and let p, p^0 be the corresponding adjoints. Also let δ, δ^0 denote the coderivatives of $\langle\,,\rangle$, $\langle\,,\rangle^0$, respectively

LEMMA 6.1. – We have

$$\varphi^0 = e^{4\sigma}\varphi\,, \tag{6.4}$$

$$p^0 = e^{-2\sigma} p\,, \tag{6.5}$$

$$\begin{aligned} \nabla^0_W(\varphi^0)(X_1\wedge X_2\wedge X_3\wedge X_4) &= e^{4\sigma}\Big\{\nabla_W(\varphi)(X_1\wedge X_2\wedge X_3\wedge X_4) + \\ &+ \sum_{i=1}^{4}(-1)^i\big((X_i\sigma)\varphi(W\wedge X_1\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge X_4) + \langle W, X_i\rangle P(X_1\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge X_4)\sigma\big)\Big\}\,, \end{aligned} \tag{6.6}$$

for $W, X_1, X_2, X_3, X_4 \in \mathfrak{X}(M)$,

$$\delta^0\varphi^0(X\wedge Y\wedge Z) = e^{2\sigma}\{\delta\varphi(X\wedge Y\wedge Z) + 4P(X\wedge Y\wedge Z)\sigma\}\,, \tag{6.7}$$

for $X, Y, Z \in \mathfrak{X}(M)$,

$$p^0\,\delta^0\varphi^0 = p\;\delta\varphi + 28\;d\sigma\,, \tag{6.8}$$

$$d\varphi^0 = e^{4\sigma}\{4\;d\sigma\wedge\varphi + d\varphi\}\,. \tag{6.9}$$

PROOF. – Equation (6.4) is an obvious consequence of (6.1) and (6.3). Taking the exterior derivative of (6.4) we get (6.9). Equation (6.5) follows from (6.1), (6.3) and from the fact that if $\{E_0, \ldots, E_7\}$ is a frame field on an open subset of $(M, \langle\,,\rangle)$, then $\{e^{-\sigma}E_0, \ldots, e^{-\sigma}E_7\}$ is a frame field on an open subset of $(M, \langle\,,\rangle^0)$. (6.6) follows from (5.4), (6.2) and (6.4). From (6.6) and (5.9) we deduce (6.7); and from (6.5), (6.7) and Lemma 2.2 (ii), we obtain (6.8).

Next we shall introduce a tensor field ν that will turn out to be a conformal invariant for 3-fold vector cross products. A similar tensor field has been introduced in [17] for almost Hermitian manifolds, and in [9] for the Riemannian manifolds with structure group G_2.

DEFINITION. – Let M be a 8-dimensional Riemannian manifold with metric $\langle\,,\rangle$ and vector cross product P. Then ν is the covariant tensor field of type $(5, 0)$

given by

$$\nu(W, X_1, X_2, X_3, X_4) = \nabla_W(\varphi)(X_1\wedge X_2\wedge X_3\wedge X_4) + \\ + 1/28 \sum_{i=1}^{4} (-1)^{i+1}\{p\, \delta\varphi(X_i)\varphi(W\wedge X_1\wedge \dots \wedge \hat{X}_i\wedge \dots \wedge X_4) + \\ + 7\langle W, X_i\rangle\, \delta\varphi(X_1\wedge \dots \wedge \hat{X}_i \wedge \dots \wedge X_4)\}, \tag{6.10}$$

for $W, X_1, X_2, X_3, X_4 \in \mathfrak{X}(M)$.

LEMMA 6.2. – Suppose $(P, \langle\, , \rangle)$ and $(P^0, \langle\, , \rangle^0)$ are conformally related. Then the corresponding tensor fields ν and ν^0 satisfy $\nu^0 = e^{4\sigma}\nu$.

PROOF. – This follows from Lemma 6.1 and equation (6.10).

DEFINITION. – Let $\mathcal{U}$ be one of the four classes given in table I. Then $\mathcal{U}^0$ will denote the class of all manifolds locally conformally related to manifolds in $\mathcal{U}$. In other words, $(M, P^0, \langle\, , \rangle^0) \in \mathcal{U}^0$ if and only if for each $m \in M$ there exists an open neighborhood V of m such that $(V, P^0, \langle\, , \rangle^0)$ is conformally related to $(V, P, \langle\, , \rangle) \in \mathcal{U}$.

THEOREM 6.3. – For any $\mathcal{U}$ given in table 1 we have $\mathcal{U}^0 \subseteq \mathcal{W}_2 \oplus \mathcal{U}$. Thus $\mathcal{U}^0 = \mathcal{U}$ if and only if $\mathcal{W}_2 \subseteq \mathcal{U}$. Hence the conformally invariant classes are $\mathcal{W}_2$ and $\mathcal{W}$.

PROOF. – The defining condition for each of the classes mentioned in the statement of the Theorem can be rewritten in terms of ν. From table 1 we have

$$M \in \mathcal{W}_2 \quad \text{if and only if} \quad \nu = 0 .$$

7. – Inclusion relations.

In this section, we establish the strictness of some of the inclusions among the four classes.

First we note that the special unitary group $SU(3) = U(3) \cap Sl(3, \mathbf{C})$ is a parallelizable 8-dimensional manifold, and hence $SU(3) \in \mathcal{W}$.

It is well known (see, for example, [19, p. 515]) that the Lie algebra $su(3)$, of $SU(3)$, is a compact real form of simple Lie algebra $sl(3, \mathbf{C})$. Thus ([19, p. 181]) the killing form B of $su(3)$ is strictly negative definite, being in fact equal to the restriction of Killing form of $sl(3, \mathbf{C})$ to $su(3)\times su(3)$. Therefore, ([19, p. 187]), $B(X, Y) = 6Tr(XY)$ for all $X, Y \in su(3)$. Furthermore, the bilinear symmetric form $-B$ defines a bi-invariant metric on $SU(3)$.

On the other hand, since $su(3)$ is a 8-dimensional vector space over $\mathbf{R}$ with a (positive definite) inner product, it follows that $su(3)$ has a Cayley multiplication, and hence the two 3-fold vector cross products $P_{\pm}$ given as in section 2. We determine $P_{\pm}$ identifying $su(3)$ with the space *Cay* by means of the orthonormal basis $\{1, e_0, \dots, e_6\}$ of $su(3)$ given by

$$1=\frac{1}{\sqrt{12}}\begin{pmatrix} i & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -i\end{pmatrix},\qquad e_0=\frac{1}{\sqrt{12}}\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$e_1=\frac{1}{\sqrt{12}}\begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},\qquad e_2=\frac{1}{\sqrt{12}}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix},$$

$$e_3=\frac{1}{6}\begin{pmatrix} i & 0 & 0\\ 0 & -2i & 0\\ 0 & 0 & i\end{pmatrix},\qquad e_4=\frac{1}{\sqrt{12}}\begin{pmatrix} 0 & i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$e_5=\frac{1}{\sqrt{12}}\begin{pmatrix} 0 & 0 & i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix},\qquad e_6=\frac{1}{\sqrt{12}}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & i\\ 0 & i & 0\end{pmatrix}.$$

Then, it can be verified that the 8-dimensional Riemannian manifold $SU(3)$, with the bi-invariant metric defined by $-B$ and the 3-fold vector cross products $P_{\pm}$, is not in the class $\mathcal{W}_1$ nor in the class $\mathcal{W}_2$. (In fact, the covariant derivative $\nabla\varphi_{\pm}$ of the fundamental form $\varphi_{\pm}$ does not satisfy the defining relations given in table I for these classes.) Furthermore, $SU(3)\notin\mathcal{T}$ because $H^4(SU(3), \boldsymbol{R}) = \{0\}$.

THEOREM 7.1. – The following inclusion relations are strict: $\mathcal{T}\subset\mathcal{W}_2$, $\mathcal{W}_1\cup\mathcal{W}_2\subset$ $\subset\mathcal{W}_1\oplus\mathcal{W}_2$.

PROOF. – Consider $\boldsymbol{R}^8$ with the two 3-fold parallel vector cross products, and $SU(3)$ as before. Let $(\boldsymbol{R}^8)^0$ denote the manifold $\boldsymbol{R}^8$ with a nontrivial change of conformal metric. Then, we have

$$(\boldsymbol{R}^8)^0\in\mathcal{W}_2-\mathcal{T}$$

$$SU(3)\in\mathcal{W}_1\oplus\mathcal{W}_2-\mathcal{W}_1\cup\mathcal{W}_2 .$$

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