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# A Semilinear Parabolic System in a Bounded Domain (*).

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**Summary.** – *Consider the system*

$$(\mathrm{S})\qquad \begin{cases} u_t - \Delta u = v^p, & \text{in } Q = \{(t, x), t > 0, x \in \Omega\}, \\ v_t - \Delta v = u^q, & \text{in } Q, \\ u(0, x) = u_0(x), & v(0, x) = v_0(x) \text{ in } \Omega, \\ u(t, x) = v(t, x) = 0, & \text{when } t \geqslant 0, x \in \partial\Omega, \end{cases}$$

*where $\Omega$ is a bounded open domain in $\mathbb{R}^N$ with smooth boundary, p and q are positive parameters, and functions $u_0(x)$, $v_0(x)$ are continuous, nonnegative and bounded. It is easy to show that (S) has a nonnegative classical solution defined in some cylinder $Q_T = (0, T) \times \Omega$ with $T \leqslant \infty$. We prove here that solutions are actually unique if $pq \geqslant 1$, or if one of the initial functions $u_0$, $v_0$ is different from zero when $0 < pq < 1$. In this last case, we characterize the whole set of solutions emanating from the initial value $(u_0, v_0) = (0, 0)$. Every solution exists for all times if $0 < pq \leqslant 1$, but if $pq > 1$, solutions may be global or blow up in finite time, according to the size of the initial value $(u_0, v_0)$.*

## 1. – Introduction and description of results.

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ ($N \geqslant 1$) with smooth boundary $\partial\Omega$. We shall consider here the following Cauchy-Dirichlet problem

$$(1.1a)\qquad u_t - \Delta u = v^p \quad \text{when } t > 0, \quad x \in \Omega,$$

$$(1.1b)\qquad v_t - \Delta v = u^q \quad \text{when } t > 0, \quad x \in \Omega,$$

$$(1.2)\qquad u = v = 0 \quad \text{if } t \geqslant 0, \quad x \in \partial\Omega,$$

$$(1.3)\qquad u(0, x) = u_0(x); \qquad v(0, x) = v_0(x), \qquad \text{if } x \in \Omega,$$

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where

(1.4) $p > 0$, $q > 0$ and $u_0(x)$, $v_0(x)$ are continuous, nonnegative and bounded real functions.

Equations (1.1) constitute a simple example of a semilinear reaction diffusion system exhibiting a nontrivial coupling on the unknowns $u(t, x)$, $v(t, x)$. These can be thought of as the temperatures of two substances which constitute a combustible mixture, where heat release is described by the power laws in the right-hand side of (1.1). While such a model is rather crude indeed, it represents a first step towards the understanding of more realistic and complex processes, and as such it has been recently considered by several authors: cf. for instance [GKS1], [GKS2] and [FG].

Local (in time) existence of nonnegative classical solutions of (1.1), (1.3) is rather standard, and will be recalled briefly below. These will be denoted alternatively by $(u(t\,x), v(t, x))$ or $(u(t), v(t))$ in what follows. We shall concentrate here in the study of uniqueness and global existence for such solutions. In particular, we show

THEOREM 1. – *Assume that* (1.4) *holds. We then have*

*a) If one of the initial values* $u_0(x)$, $v_0(x)$, *is different from zero, or if* $pq \geqslant 1$, *there exists a unique solution of* (1.1)-(1.3) *which is defined in some time interval* $(0, T)$ *with* $T \leqslant +\infty$.

*b) If* $pq < 1$ *and* $u_0(x) = v_0(x) \equiv 0$, *the set of solutions of* (1.1)-(1.3) *consists of*

*b1) The trivial solution* $u(t, x) = v(t, x) \equiv 0$,

*b2) A solution* $(U(t, x), V(t, x))$ *such that* $U(t, x) > 0$ *and* $V(t, x) > 0$ *for any* $t > 0$ *and* $x \in \Omega$,

*b3) A monoparametric family* $(U_\mu(t, x), V_\mu(t, x))$ *where* $\mu$ *is any positive number,* $U_\mu(t, x) = U((t-\mu)_+, x)$, $V_\mu(t, x) = V((t-\mu)_+, x)$ *and* $\xi_+ = \max\{\xi, 0\}$.

We then consider the question of the life span of solutions. To this end, we shall say that $u(t, x)$ (resp. $v(t, x)$) blows up in a time $T < +\infty$ if

$$\limsup_{t\uparrow T}\,(\max_{x\in\bar\Omega} u(t, x)) = +\infty \qquad (\text{resp. } \limsup_{t\uparrow T}\,(\max_{x\in\bar\Omega} v(t, x)) = +\infty).$$

It is readily seen that if one of the functions $(u(t), v(t))$ blows up at $t = T < +\infty$, so does the other one. To proceed further, let $\lambda_1 > 0$ the first eigenvalue of $(-\Delta)$ in $\Omega$

with homogeneous Dirichlet conditions, and let $\varphi_1$ be such that

$$-\Delta\varphi_1 = \lambda_1\varphi_1 \quad \text{in } \Omega\,, \tag{1.5a}$$

$$\varphi_1 = 0 \quad \text{in } \partial\Omega\,, \tag{1.5b}$$

$$\varphi_1 > 0 \quad \text{in } \Omega\,, \ \int_\Omega \varphi_1(x)\,dx = 1\,. \tag{1.5c}$$

We then have

THEOREM 2. – *Assume that* (1.4) *holds. Then*

*a) If* $pq \leqslant 1$, *every solution of* (1.1)-(1.3) *is global in time*

*b) If* $pq > 1$, *some solutions are global while some others blow up in finite time. More precisely,*

(1.6) *There exists* $C > 0$ *such that, if* $\varphi_1(x)$ *is given in* (1.5) *and*

$$\int_\Omega (u_0(x) + v_0(x))\,\varphi_1(x)\,dx \geqslant C\,,$$

*the corresponding solution* $(u(t), v(t))$ *of* (1.1)-(1.3) *blows up in a finite time* $T$,

(1.7) *There exists* $K > 0$ *such that, if*

$$\|u_0\|_\infty + \|v_0\|_\infty \leqslant K \tag{1.7}$$

*the corresponding solution* $(u(t), v(t))$ *of* (1.1)-(1.3) *exists for all times* $t > 0$.

We now comment briefly on our results. Blow up for nonlinear evolution equations has deserved a great deal of interest ever since the pioneering papers [Fu1], [Fu2]. In particular, it is well known that if $u_0(x)$ is as in (1.4), the Cauchy-Dirichlet problem

$$u_t - \Delta u = u^p \quad \text{when} \quad t > 0,\ x \in \Omega\,, \tag{1.8a}$$

$$u(0, x) = u_0(x) \quad \text{when} \quad x \in \Omega\,, \tag{1.8b}$$

$$u(t, x) = 0 \quad \text{if} \quad t \geqslant 0 \text{ and } x \in \partial\Omega\,, \tag{1.8c}$$

is such that when $0 < p \leqslant 1$ every solution is global in time, although uniqueness fails if $0 < p < 1$ (cf. for instance [FuW]). When $p > 1$, there exist initial values for which solutions blow up in finite time. However, if $u_0(x)$ is small enough, (1.8) has a unique solution which exists for all times: see for instance [F], [MW], [L1].

To our knowledge, no such a complete picture of the situation concerning uniqueness and blow up was available for (1.1)-(1.4) prior to this work. For instance, uniqueness is indeed straightforward if $p \geqslant 1$ and $q \geqslant 1$, and rather easy to prove if $pq \geqslant 1$ (cf. Lemma 2.1 below). However, it requires of some elaboration in the case $0 < pq < 1$. On the other hand, the blow-up result in Theorem 2 can be proved by means of a technique already used in [GKS2] if $p > 1$ and $q > 1$. A somewhat different method is required though when one of the constants $p$, $q$, is allowed to be less than one.

We wish to conclude this Introduction by comparing (1.1)-(1.4) and (1.8) with the corresponding Cauchy problems in the whole space. This last situation is characterized by the onset if critical blow-up parameters, i.e., by the fact that in some range of the parameters $p$ and $q$ (resp. $p$), every nontrivial solution blows up in finite time. In particular, if we replace (1.8) by

$$u_t - \Delta u = u^p \quad \text{if} \quad t > 0, \ x \in \mathbb{R}^N, \ N \geqslant 1, \tag{1.9a}$$

$$u(0, x) = u_0(x) \quad \text{if} \quad x \in \mathbb{R}^N, \tag{1.9b}$$

where $u_0(x)$ is as before, it turns out that every solution of (1.9) with $u_0(x) \neq 0$ blows up in finite time if $1 < p \leqslant 1 + 2/N$, whereas global existsence and blow up coexist if $p > 1 + 2/N$. Indeed, solutions are global if $0 < p \leqslant 1$, but non unique if $u_0(x) \equiv 0$ and $0 < p < 1$; cf for instance [Fu1], [KST], [AW], [AE], .... A similar situation holds for the Cauchy problem (CP) corresponding to (1.1), (1.3), (1.4), where every nontrivial solution blows up in finite time if $pq > 1$ and $\max\{p, q\} \geqslant 2/N(pq-1) - 1$ (see [EH1]). We refer to [L2] for a survey on the role of critical parameters in evolution problems, and to [EL] for recent work on more general systems.

As to the number of solutions of (CP), it has been proved in [EH2] that

i) If $u_0(x) \not\equiv 0$ or $v_0(x) \not\equiv 0$, solutions are unique. Uniqueness also holds in $pq > 1$ when $u_0 = v_0 \equiv 0$.

ii) When $0 < pq < 1$, the set of solutions of (CP) with zero initial value is given by

$$u(t; s) = C_1(t-s)_+^{(p+1)/(1-pq)}, \qquad v(t; s) = C_2(t-s)_+^{(q+1)/(1-pq)},$$

where

$$0 \leqslant s \leqslant t, \qquad C_1^{1-pq} = (1-pq)^{p+1}(p+1)^{-1}(q+1)^{-p}, \qquad C_2 = C_1^q(1-pq)(q+1)^{-1}.$$

Notice the analogy between these results and those in Theorem 1. However, the core of the proof of uniqueness (which consists in the analysis of the situation where $0 < pq < 1$) is different for the cases $\Omega = \mathbb{R}^N$ and $\Omega$ bounded. In particular, our arguments of [EH2] do not carry through here.

Finally, the plan of this paper is as follows. Some preliminary facts (including suitable comparison tools) are gathered in Section 2 below. Uniqueness is then proved in Section 3 in the case $0 < pq < 1$, while global existence and blow up make the content of Section 4.

## 2. - Preliminaries.

In this Section we shall state some basic facts and obtain auxiliary results which will later be used in the main proofs. By a nonnegative classical solution of (1.1)-(1.3) (a nonnegative solution, for short) in some cylinder $Q_T = (0, T) \times \Omega$ with $T \leqslant +\infty$, we shall mean a pair of nonnegative $C^{1,2}$ functions $(u(t, x), v(t, x))$ such that they satisfy (1.1)-(1.3) in $S_T$. Existence and uniqueness of nonnegative solutions is straightforward if $p \geqslant 1$ and $q \geqslant 1$. For instance, consider the associated integral system

$$u(t) = S(t)u_0 + \int_0^t S(t-s)|v(s)|^{p-1}v(s)\,ds \equiv \Phi_1(v)\,, \tag{2.1a}$$

$$v(t) = S(t)v_0 + \int_0^t S(t-s)|u(s)|^{q-1}u(s)\,ds \equiv \Phi_2(u)\,, \tag{2.1b}$$

where for any function $f \in L^1(\Omega)$, $S(t)f$ denotes the solution of

$$\begin{aligned} &u_t - \Delta u = 0 && \text{in } (0, \infty) \times \Omega\,, \\ &u(0, x) = f(x) && \text{in } \Omega\,, \qquad u(t, x) = 0 \text{ in } [0, \infty) \times \Omega\,. \end{aligned}$$

This notation will be retained henceforth. As in [EH1], we now take $T > 0$ fixed, but otherwise arbitrary, and consider the set

$$E_T = \{(u, v): [0, T] \to L^\infty(\Omega) \times L^\infty(\Omega) \text{ such that } |||(u, v)||| < +\infty\}$$

where

$$|||(u, v)||| = \sup_{0 \leqslant t \leqslant T} (\|u(t)\|_\infty + \|v(t)\|_\infty)\,.$$

For simplicity, we shall often write $\| \ \|$ instead of $\| \ \|_\infty$. Clearly, $E$ is a Banach space, and $P_T = \{(u, v) \in E_T: u \geqslant 0, v \geqslant 0\}$ is a closed subset of $E_T$. Let $B_R = \{(u, v) \in E_T: |||(u, v)||| < R\}$. One then readly sees that, if $R > 0$ is large enough and $T > 0$ is sufficiently small, $\psi(u, v) = (\Phi_1(v), \Phi_2(u))$ is a strict contraction of $B_R \cap E_T$ into itself, whence the result.

We now have

LEMMA 2.1. - *Assume that* (1.4) *holds. Then there exists* $T \leqslant +\infty$ *such that* (1.1)-(1.3) *has a nonnegative solution* $(u(t, x), v(t, x))$ *in* $Q_T$. *Moreover, such solution is unique if* $pq \geqslant 1$.

PROOF. – We have just recalled the case $p \geqslant 1$ and $q \geqslant 1$. If one of the parameters $p$, $q$ is less than one, but $pq \geqslant 1$, it is still possible to set up a contraction mapping argument. Indeed, assume for definiteness that $0 < p < 1 < (1/p) \leqslant q$, and for any given $T > 0$, set $X_T = \{u[0, T] \to L^\infty(\Omega)$ such that $|||u||| = \sup_{0 \leqslant t \leqslant T} \|u(t)\| < +\infty\}$, $X_T^+ = \{u \in X_T : u \geqslant 0\}$ and $B_R = \{u \in X_T = |||u||| < R\}$. Recalling (2.1), it will suffice to show that for $R > 0$ large enough and $T > 0$ small enough, the mapping

$$\Phi(u)(t) = S(t)\,u_0 + \int_0^t S(t-s)\left(S(s)\,v_0 + \int_0^s S(s-\sigma)\,u(\sigma)^q\,d\sigma\right)^p ds\,,$$

is a strict contraction from $B_R \cap X_T^+$ into itself. Using mean value theorem, it follows that

$$(2.2)\qquad (\Phi(u) - \Phi(\overline{u}))(t) \leqslant pq\left(\int_0^t S(t-s)\left(\int_0^s S(s-\sigma)(\theta u + (1-\theta)\,\overline{u})^q\,d\sigma\right)^{p-1}\cdot\right.$$
$$\left.\cdot\left(\int_0^s S(s-\sigma)(\theta u + (1-\theta)\,\overline{u})^{q-1}(u-\overline{u})\,d\sigma\right)ds\right)$$

for some $\theta = \theta(s) \in (0, 1)$. We now notice that

(2.3) For any nonnegative and integrable functions $f$, $g$, and any $r \geqslant 1$, there holds

$$S(t)\,fg \leqslant \|f\|_\infty\,(S(t)\,g^r)^{1/r}.$$

Let us assume (2.3) for the moment and continue. It then follows from (2.2) and (2.3) that, if $u, \overline{u} \in B_R \cap X_T^+$,

$$|||\Phi(u) - \Phi(\overline{u})||| \leqslant pq|||u - \overline{u}|||\int_0^t \|S(s-\sigma)(\theta u + (1-\theta)\,\overline{u})^q\,d\sigma\|^{p-1}\cdot$$
$$\cdot\int_0^s \|S(s-\sigma)(\theta u + (1-\theta)\,\overline{u})^q\|^{(q-1)/q}\,d\sigma\,ds \leqslant$$
$$\leqslant pq|||u - \overline{u}|||\int_0^t\left\|\int_0^s S(s-\sigma)(\theta u + (1-\theta)\overline{u})^q\,ds\right\|^{(pq-1)/q}\cdot s^{1/q}\,ds \leqslant$$
$$\leqslant pq|||u - \overline{u}|||\,R^{pq-1}\,C(T)\,,$$

where $C(T) \to 0$ as $T \to 0$, whence the result. To check (2.3), we set $\omega(t) = (S(t)\,fg)$,

$\omega_1(t) = \|f\|_\infty S(t) g^r$, and notice that

$$(\omega^r)_t - \Delta(\omega^r) = -r(r-1)\omega^{r-2}|\nabla\omega|^2 \leq 0,$$

$$(\omega_1)_t - \Delta(\omega_1) = 0,$$

$$\omega^r = \omega_1 = 0 \quad \text{in } [0, \infty) \times \partial\Omega,$$

whereas $\omega^r(0, x) = (fg)^r(x) \leq \|f\|_\infty^r g^r(x) = \omega_1(0, x)$, so that the result follows at once by comparison. Finally, existence is obtained for the case $pq < 1$ by means of an approximation procedure as in [EH1], Theorem 2.1: if, for instance, $0 < p < 1$ and $q \geq 1$, we replace (1.1*a*) by

$$u_t - \Delta u = g_n(v) \quad \text{when } t > 0, \quad x \in \Omega$$

where, for any positive integer $n$, $g_n(r)$ is a nondecreasing function such that $g_n(r) = r^p$ if $r \geq 1/2n$, $|g_n(r_1) - g_n(r_2)| \leq C_n|r_1 - r_2|$ for any $r_1 \geq 0$ and $r_2 \geq 0$, and $g_n(s) \uparrow s^p$ at any $s \geq 0$ as $n \to \infty$. Replacing also $v_0$ by $(v_0 + 1/n)$, we obtain a unique solution $(u_n(t, x), v_n(t, x))$ of the corresponding Cauchy-Dirichlet problem. Furthermore,

$$u_n(t, x) \leq u_m(t, x), \qquad v_n(t, x) \leq v_m(t, x) \qquad \text{if } n \geq m$$

and the conclusion follows by letting $n \to \infty$. We omit further details. ∎

The following comparison result will be useful in the sequel

LEMMA 2.2. – *Assume that* (1.4) *holds, and let* $(u(t, x), v(t, x))$ *be a solution of* (1.1)-(1.3) *in a cylinder* $Q_T = (0, T) \times \Omega$ *with* $T > 0$. *Suppose that* $(u_1(t, x), v_2(t, x))$ *is a solution of* (1.1) *in* $Q_T$ *such that*

$$u_1(0, x) > u_0(x), \quad v_1(0, x) > v_0(x) \quad \text{in } \Omega,$$
$$u_1(t, x) > 0, \qquad v_1(t, x) > 0 \qquad \text{in } [0, T) \times \partial\Omega.$$

*Then*

$$u_1(t, x) > u(t, x) \quad \text{and} \quad v_1(t, x) > v(t, x) \quad \text{in } Q_T.$$

PROOF. – It suffices to consider the case where $u_1(0, x) = u_0(x) + \delta$, $v_1(0, x) = v_0(x) + \delta$ and $u_1(t, x) = v_1(t, x) = \delta$ on $[0, T) \times \partial\Omega$, where $\delta$ is any fixed (but otherwise arbitrary) positive constant. Then, by continuity, there exists $\tau > 0$ such that

$$u_1(t, x) > u(t, x) \quad \text{and} \quad v_1(t, x) > v(t, x) \quad \text{in } Q_\tau = \Omega \times (0, \tau).$$

We now argue by contradiction. Let

$$\sigma_1 = \inf\{t\colon \text{there exists } x \in \Omega \text{ such that } u_1(t, x) \leq u(t, x)\}$$

$$\sigma_2 = \inf\{t\colon \text{there exists } x \in \Omega \text{ such that } v_1(t, x) \leq v(t, x)\}$$

and let $\sigma = \min\{\sigma_1, \sigma_2\}$. Clearly, $\sigma > 0$ and either $u_1(\sigma, y) = u(\sigma, y)$ or $v_1(\sigma, y) = v(\sigma, y)$ at some $y \in \Omega$. Setting $z = u_1 - u$, $\theta = v_1 - v$, one readily checks that

$$z_t - \Delta z > 0 \quad \text{in } Q_\sigma, \qquad \theta_t - \Delta\theta > 0 \quad \text{in } Q_\sigma$$

and since $z$, $\theta$, are positive at the parabolic boundary of $Q_\sigma$, it follows that $z(\sigma, x) > 0$, $\theta(\sigma, x) > 0$ everywhere in $\Omega$. This concludes the proof. ■

We next specialize to the case $0 < pq < 1$, and consider the auxiliary functions

(2.4*a*) $$\bar{u}(t) = (\alpha + C_1^{(1-pq)/(p+1)} \cdot t)^{(p+1)/(1-pq)},$$

(2.4*b*) $$\bar{v}(t) = (\beta + C_2^{(1-pq)/(q+1)} \cdot t)^{(q+1)/(1-pq)},$$

where

(2.4*c*) $$C_1 = (1-pq)^{(1-pq)/(p+1)}(p+1)^{-1/(1-pq)}(q+1)^{-p/(1-pq)},$$

(2.4*d*) $$C_2 = (1-pq)^{(q+1)/(1-pq)}(q+1)^{-1/(1-pq)}(p+1)^{-q/(1-pq)},$$

(2.4*e*) $$\alpha^{(p+1)/(1-pq)} \geqslant \|u_0\|_\infty + 1, \qquad \beta^{(q+1)/(1-pq)} \geqslant \|v_0\|_\infty + 1,$$

(2.4*f*) $$\alpha = \beta C_1^{(1-pq)/(p+1)} \cdot C_2^{(pq-1)/(q+1)}.$$

A routine computation shows then that

(2.5*a*) $$\bar{u}_t - \Delta\bar{u} \equiv \bar{u}_t = \bar{v}^p \quad \text{for any } t > 0,$$

(2.5*b*) $$\bar{v}_t - \Delta\bar{v} \equiv \bar{v}_t = \bar{u}^q \quad \text{for any } t > 0.$$

From Lemma 2.2, we obtain the following comparison result

(2.6) Let $(u(t), v(t))$ be a solution of (1.1)-(1.4) in some cylinder $Q_T$ with $T > 0$, and assume that $0 < pq < 1$. Then

$$u(t) < \bar{u}(t) \quad \text{and} \quad v(t) < \bar{v}(t) \quad \text{in } Q_T.$$

Notice that (2.6) implies that any solution of (1.1)-(1.4) can be continued for all times in the case $0 < pq < 1$. We next show

LEMMA 2.3. – *Assume that $0 < pq < 1$ and $u_0(x) \not\equiv$ or $v_0(x) \not\equiv 0$ in $\Omega$. Then*

(2.7*a*) $$u(t) \geqslant C_1 t^{(p+1)/(1-pq)} S_1(t) \quad \text{in } \Omega \text{ for any } t > 0,$$

(2.7*b*) $$v(t) \geqslant C_2 t^{(q+1)/(1-pq)} S_1(t) \quad \text{in } \Omega \text{ for any } t > 0,$$

*where $C_1$ and $C_2$ are given in (2.14), and $h = S_1(t)$ is the solution of $h_t - \Delta h = 0$ in $(0, \infty) \times \Omega$ which satisfies $h(0, x) = 1$ in $\Omega$ and $h(t, x) = 0$ in$(0, \infty) \times \partial\Omega$.*

PROOF. – Suppose for definiteness that $u_0(x) \not\equiv 0$ and $0 < p < q < 1$. To show *a*), we first observe that

$$\text{If } 0 < q < 1, \qquad (S(t)u_0)^q \geq S(t)u_0^q. \tag{2.8}$$

The proof of (2.8) is similar to that of (2.3) and will be omitted. We then argue as in [EH2], Lemma 2. Since $u(t) \geq S(t)u_0$ in $\Omega$, it follows from (2.1) that

$$v(t) \geq \int_0^t S(t-s)(S(s)u_0^q)\,ds \geq tS(t)u_0^q$$

whence

$$u(t) \geq \int_0^t s^p S(t-s)(S(s)u_0^q)^p\,ds \geq \frac{t^{p+1}}{p+1}S(t)u_0^{pq}$$

which in turn yields

$$v(t) \geq \left(\frac{1}{p+1}\right)^q \left(\frac{1}{q(p+1)+1}\right) t^{q(p+1)+1} S(t)u_0^{pq^2}.$$

Iterating the previous procedure, we obtain

$$u(t) \geq A_{k+1}B_{k+1}t^{\gamma_{k+1}}S(t)(u_0)^{pq^{k+1}} \tag{2.9a}$$

where

$$\gamma_k = (p+1)((pq)^k + (pq)^{k-1} + \ldots + pq + 1) \tag{2.9b}$$

$$A_{k+1} = \left(\prod_{j=0}^{k-1}(p+1)((pq)^j + (pq)^{j-1} + \ldots + pq + 1)^{-(pq)^{k-1-j}}\right)^p, \tag{2.9c}$$

$$B_{k+1} = (p+1)^{-(1-(pq)^{k+1})/(1-pq)}(1+pq)^{-(pq)^{k-1}}((pq)^k + (pq)^{k-1} + \ldots + 1)^{-1}. \tag{2.9d}$$

Since

$$A_{k+1} \geq \left(\frac{1-pq}{q+1}\right)^{p/(1-pq)}, \qquad B_{k+1} \geq (p+1)^{-1/(1-pq)}(1-pq)^{1/(1-pq)},$$

letting $k \to \infty$, it follows from (2.9) that

$$u(t) \geq C_1 t^{(p+1)/(1-pq)} S(t)\chi(u_0) \quad \text{in } \Omega \text{ for any } t > 0,$$

where $C_1$ is given in (2.4) and $\chi(u_0) = 1$ where $u_0 > 0$, and zero otherwise. If $u_0(x)$ vanishes somewhere, we still have that $u(t) > 0$ and $v(t) > 0$, for any $t > 0$. In particular, for any sequence $\{t_n\}$ of positive numbers such that $\lim_{n\to\infty} t_n = 0$, we see that

$$u(t+t_n) \geq C_1 t^{(p+1)/(1-pq)} S_1(t) \quad \text{in } \Omega \text{ for any } t>0,$$

so that (2.7*a*) follows by letting $n \to \infty$. The proof of (2.7*b*) is similar. ∎

We next show that (1.1)-(1.4) has a maximal solution if $0 < pq < 1$.

LEMMA 2.4. – *Assume that* $0 < pq < 1$. *Then there exists a solution* $(u_M(t), v_M(t))$ *of* (1.1)-(1.4) *such that, if* $(u(t), v(t))$ *is any other solution there holds*

$$u(t) \leq u_M(t) \quad \textit{and} \quad v(t) \leq v_M(t) \quad \textit{in } Q=(0,\infty)\times\Omega.$$

PROOF. – Let $\bar{u}(t)$, $\bar{v}(t)$ be the functions given in (2.4). We now define the sequences $\{u_n\}$, $\{v_n\}$ as follows. To start with, $u_1$ and $v_1$ solve respectively

$$\begin{array}{llll} u_t - \Delta u = \bar{v}^p & \text{in } Q, & v_t - \Delta v = \bar{u}^q & \text{in } Q, \\ u(0,x) = u_0(x) & \text{in } \Omega, & v(0,x) = v_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{in } [0,\infty)\times\Omega & v(t,x)=0 & \text{in } [0,\infty)\times\Omega \end{array}$$

whereas, for $j \geq 2$, $u_j$ and $v_j$ are the respective solutions of

$$\begin{array}{llll} u_t - \Delta u = v_{j-1}^p & \text{in } Q, & v_t - \Delta v = u_{j-1}^q & \text{in } Q, \\ u(0,x) = u_0(x) & \text{in } \Omega, & v(0,x) = v_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{in } [0,\infty)\times\partial\Omega & v(t,x)=0 & \text{in } [0,\infty)\times\partial\Omega. \end{array}$$

It is then readily seen that, for any $n \geq 3$

$$\bar{u} \geq u_1 \geq u_2 \geq \ldots u_n \geq \ldots$$

$$\bar{v} \geq v_1 \geq v_2 \geq \ldots v_n \geq \ldots$$

and that for any $T>0$, we may select $C = C(T, u_0, v_0) > 0$ such that

$$\sup_{0\leq t\leq T} (\|u_n(t)\|_{H^1(\Omega)} + \|v_n(t)\|_{H^1(\Omega)}) \leq C < +\infty.$$

Therefore there exist functions $(u_M, v_M)$ such that

$$u_M(t) = \lim_{n\to\infty} u_n(t), \qquad v_M(t) = \lim_{n\to\infty} v_n(t),$$

$$(u_M(t,x), v_M(t,x)) \quad \text{solve (1.1)-(1.4)}.$$

Let now $(u,v)$ be any other solution of (1.1)-(1.4). By Lemma 2.2, we certainly have

$$u(t) < \bar{u}(t), \qquad v(t) < \bar{v}(t) \quad \text{in } Q.$$

Assume now that $u \leqslant u_j$, $v \leqslant v_j$ in $Q$ for some $j \geqslant 1$. Then

$$(u_{j+1} - u)_t - \Delta(u_{j+1} - u) = v_j^p - v^p \geqslant 0 \quad \text{in } Q,$$

$$u_{j+1} - u = 0 \quad \text{if } x \in \Omega \text{ and } t = 0 \text{ or } x \in \partial\Omega \text{ and } t > 0,$$

where upon $u_{j+1} \geqslant u$ in $Q$, and similarly $v_{j+1} \geqslant v$ in $Q$. Letting $j \to \infty$, the last statement in the Lemma follows. ∎

Let now $a(t, x)$, $b(t, x)$ be smooth functions such that, for some $\tau > 0$,

(2.10*a*) $$-1 \leqslant a(t, x) \leqslant 1 \quad \text{in } \overline{Q}_\tau = [0, \tau] \times \Omega,$$

(2.10*b*) $$-1 \leqslant b(t, x) \leqslant 1 \quad \text{in } \overline{Q}_\tau$$

and consider the system

(2.11*a*) $$u_t - \Delta u = a(t, x) v^p \quad \text{in } Q_\tau = (0, \tau) \times \Omega,$$

(2.11*b*) $$v_t - \Delta u = b(t, x) u^q \quad \text{in } Q_\tau,$$

(2.11*c*) initial and boundary conditions (1.2), (1.3), under assumptions (1.4).

We then have

LEMMA 2.5. – *Assume that conditions* (2.10) *hold, and let* $(u_M(t), v_M(t))$ *be the maximal solution of* (1.1)-(1.4) *obtained in Lemma* 2.4. *Then, if* $(u(t), v(t))$ *is a non-negative solution of* (2.11), *we have*

$$u(t) \leqslant u_M(t), \qquad v(t) \leqslant v_M(t) \quad \text{in } Q_\tau.$$

PROOF. – Let $(u_n, v_n)$ be the sequence leading to $(u_M, v_M)$ in the proof of Lemma 2.4. Set $u_1 = \overline{u}$, $v_1 = \overline{v}$. We first observe that

(2.12) $$u < \overline{u} \quad \text{and} \quad v < \overline{v} \quad \text{in } Q_\tau.$$

Indeed, $u < \overline{u}$ (resp. $v < \overline{v}$) holds in $\Omega$ at $t = 0$, and is satisfied at $\partial\Omega$ for any $t > 0$. Then (2.12) follows by means of a contradiction argument as in the proof of Lemma 2.2. We then proceed by induction as in the comparison result in Lemma 2.4. ∎

Let $\lambda_1 > 0$ be the first eigenvalue of $(-\Delta)$ with homogeneous Dirichlet conditions, and let $\psi_1(x)$ be a function such that

(2.13*a*) $$-\Delta\psi_1 = \lambda_1 \psi_1 \quad \text{in } \Omega,$$

(2.13*b*) $$\psi_1 = 0 \quad \text{in } \partial\Omega,$$

(2.13*c*) $$0 \leqslant \psi_1 \leqslant 1, \qquad |\nabla\psi_1| \text{ is bounded in } \Omega.$$

We are now prepared to show that maximal solutions are positive, thus extending the corresponding result obtained in [FuW] for the scalar problem (1.8).

LEMMA 2.6. – *Suppose that* $0 < pq < 1$, *and let* $(u_M, v_M)$ *be the solution obtained in Lemma* 2.4. *Then*

$$u_M(t) > 0 \quad \text{and} \quad v_M(t) > 0 \quad \text{in } Q = (0, \infty) \times \Omega . \tag{2.14}$$

*Moreover, there exist* $\varepsilon > 0$ *and* $\sigma > 0$ *such that*

$$u_M(t, x) \geqslant (\varepsilon t \psi_1(x)^2)^{(p+1)/(1-pq)} \quad \text{in } Q_\sigma = (0, \sigma) \times \Omega , \tag{2.15a}$$

$$v_M(t, x) \geqslant (\varepsilon t \psi_1(x)^2)^{(q+1)/(1-pq)} \quad \text{in } Q_\sigma . \tag{2.15b}$$

PROOF. – It suffices to examine the case where $u_0 = v_0 \equiv 0$. Consider the auxiliary functions

$$u_\varepsilon(t, x) = (\varepsilon t \psi_1(x)^2)^{(p+1)/(1-pq)} ,$$

$$v_\varepsilon(t, x) = (\varepsilon t \psi_1(x)^2)^{(q+1)/(1-pq)} .$$

Clearly, $u_\varepsilon = v_\varepsilon = 0$ in $\Omega$ if $t = 0$, and in $\partial\Omega$ if $t > 0$. Furthermore,

$$(u_\varepsilon)_t - \Delta u_\varepsilon = a_\varepsilon(t, x) v_\varepsilon^p \quad \text{in } Q ,$$

$$(v_\varepsilon)_t - \Delta v_\varepsilon = b_\varepsilon(t, x) u_\varepsilon^q \quad \text{in } Q ,$$

where

$$a_\varepsilon(t, x) = \left( \left( \frac{p+1}{1-pq} \right) \psi_1^2 + \left( \frac{2(p+1)}{1-pq} \right) \lambda_1 t \psi_1^2 - \left( \frac{2(p+1)-1}{1-pq} \right) \left( \frac{\partial \psi_1}{\partial x} \right)^2 t \right) \varepsilon ,$$

$$b_\varepsilon(t, x) = \left( \left( \frac{q+1}{1-pq} \right) \psi_1^2 + \left( \frac{2(q+1)}{1-pq} \right) \lambda_1 t \psi_1^2 - \left( \frac{2(q+1)-1}{1-pq} \right) \left( \frac{\partial \psi_1}{\partial x} \right)^2 t \right) \varepsilon ,$$

so that (2.11) holds provided that $\varepsilon$ and $t$ are small enough. It then follows from Lemma 2.5 that

$$u_M(t, x) \geqslant u_\varepsilon(t, x), \qquad v_M(t, x) \geqslant v_\varepsilon(t, x) \quad \text{in } Q_\sigma$$

for some $\sigma > 0$ and $\varepsilon > 0$ small enough, and this in turn implies (2.14). ∎

## 3. – End of the proof of Theorem 1.

In this Section we shall conclude our analysis of uniqueness of solutions of (1.1)-(1.4). In view of Lemma 2.1, it only remains to consider the case where $0 < pq < 1$, an assumption to be retained henceforth. Moreover, it will suffice to show that positive solutions are unique. This is done in our next result

LEMMA 3.1. – *There exists a unique solution* $(u(t), v(t))$ *of* (1.1)-(1.4) *such that* $u(t) > 0$ *and* $v(t) > 0$ *in* $Q = (0, \infty) \times \Omega$.

PROOF. – The existence statement has been already proved in Lemma 2.4. As to the uniqueness we shall proceed by contradiction, and assume that $(u_1, v_1)$ and $(u_2, v_2)$ are two such solutions. We shall distinguish two cases

*Case I*: $u_0 = v_0 \equiv 0$.

By (2.6), we have that, for $i = 1, 2$

$$v_i(t) < (\beta + C_2^{(1-pq)/(q+1)}t)^{(q+1)/(1-pq)} \quad \text{in } Q\,,$$

where $\beta$, $C_2$, are given in (2.4). Using the integral equation (2.1$a$), we obtain

$$u_i(t) < \int_0^t S(t-r)(\beta + C_2^{(1-pq)/(q+1)}r)^{p(q+1)/(1-pq)}\,dr \leqslant$$

$$\leqslant (\beta + C_2^{(1-pq)/(q+1)}t)^{p(q+1)/(1-pq)}\int_0^1 S_1(t-r)\,dr = (\beta + C_2^{(1-pq)/(q+1)}t)^{p(q+1)/(1-pq)}\int_0^t S_1(r)\,dr$$

with $S_1(t)$ defined in (2.7), and $i = 1, 2$. Since

$$\Delta\int_0^t S_1(r)\,dr = S_1(t) - 1\,,$$

it follows that, for any given $T > 0$ and any $m > 1$,

$$\int_0^t S_1(r)\,dr \in L^\infty((0, T); W^{2m}(\Omega))$$

whence

$$\left\|(\beta + C_2^{(1-pq)/(q+1)}t)^{(q+1)/(1-pq)}\int_0^t S_1(r)\,dr\right\|_{C^1(\bar{\Omega})} \to 0 \quad \text{as } t \to 0\,, \tag{3.1$a$}$$

and analogously,

$$\left\|(\alpha + C_1^{(1-pq)/(q+1)}t)^{(q+1)/(1-pq)}\int_0^t S_1(r)\,dr\right\|_{C^1(\bar{\Omega})} \to 0 \quad \text{as } t \to 0\,. \tag{3.1$b$}$$

Let $\psi_1(x)$ be a function satisfying (2.13). It then follows from (3.1) that

For any $C > 0$, there exists $t_0$ such that, if $t \leqslant t_0$

$$(\beta + C_2^{(1-pq)/(q+1)} t)^{(q+1)/(1-pq)} \int_0^t S_1(s)\,ds \leqslant C\psi_1 \quad \text{in } \Omega, \tag{3.2a}$$

$$(\alpha + C_1^{(1-pq)/(q+1)} t)^{(q+1)/(1-pq)} \int_0^t S_1(s)\,ds \leqslant C\psi_1 \quad \text{in } \Omega. \tag{3.2b}$$

On the other hand, by comparison

$$S_1(t) \geqslant e^{-\lambda_1 t}\psi_1(x) \quad \text{in } Q. \tag{3.3}$$

Therefore, taking into account (2.7), (3.2) and (3.3), it follows that

(3.4) For any fixed $s \in (0,1)$, there exist $\tau = \tau(s)$ such that $\lim_{s\to 0} \tau(s) = 0$ and, for $t \leqslant \tau$,

$$u_1(t+s) - u_2(t) \geqslant 0, \qquad u_2(t+s) - u_1(t) \geqslant 0,$$

$$v_1(t+s) - v_2(t) \geqslant 0, \qquad v_2(t+s) - v_1(t) \geqslant 0.$$

We now fix $T > 0$, and for $t \in [0, T]$ we set

$$\underline{w}(t) = u_1(t + s + \tau(s)), \qquad \underline{\omega}(t) = v_1(t + s + \tau(s))$$

$$\bar{w}(t) = u_2(t + \tau(s)), \qquad \overline{\omega}(t) = v_2(t + \tau(s)).$$

Recalling (3.4), we have that

$$\underline{w}(0) \geqslant \bar{w}(0), \qquad \underline{\omega}(0) \geqslant \overline{\omega}(0). \tag{3.5}$$

Suppose now that $0 < p$, $q < 1$. We then consider the equation for $(\bar{w} - \underline{w})$, multiply both sides there by $(\bar{w} - \underline{w})_+$ and integrate over $\Omega$. Using mean value theorem as well as (2.7) and (3.3), we arrive at

$$\frac{d}{dt}\int_\Omega (\bar{w} - \underline{w})_+^2\,dx + \int_\Omega |D(\bar{w} - \underline{w})_+|^2\,dx = \int_\Omega (\overline{\omega}^p - \underline{\omega}^p)(\bar{w} - \underline{w})_+\,dx \leqslant \tag{3.6a}$$

$$\leqslant p\int_\Omega (\theta\overline{\omega} + (1-\theta)\underline{\omega})^{p-1}(\overline{\omega} - \underline{\omega})_+(\bar{w} - \underline{w})_+\,dx \leqslant C\int_\Omega \psi_1^{p-1}(\overline{\omega} - \underline{\omega})_+(\bar{w} - \underline{w})_+\,dx \leqslant$$

$$\leqslant \frac{C}{\varepsilon}\int_\Omega (\overline{\omega} - \underline{\omega})_+^2\,dx + C\varepsilon\int_\Omega \psi_1^{2(p-1)}(\bar{w} - \underline{w})_+^2\,dx,$$

where $\varepsilon > 0$, $\theta \in (0,1)$, and here and henceforth $C$ will denote a generic constant depending only on $s$, $p$ and $q$. Arguing in a similar way, we obtain

$$\text{(3.6}b\text{)} \qquad \frac{d}{dt}\int_\Omega (\overline{\omega} - \underline{\omega})_+^2\,dx + \int_\Omega |D(\overline{\omega} - \underline{\omega})_+|^2\,dx \leqslant$$

$$\leqslant \frac{C}{\varepsilon}\int_\Omega (\bar{w} - \underline{w})_+^2\,dx + C\varepsilon\int_\Omega \psi_1^{2(q-1)}(\overline{\omega} - \underline{\omega})_+^2\,dx\,.$$

We now observe that, by Hardy's inequality (cf. for instance [LM], p. 76) and standard properties of the eigenfunction $\psi_1$, we have that, if $h \in H_0^1(\Omega)$,

$$\int_\Omega \left|\frac{h}{\psi_1}\right|^2 dx \leqslant K\int_\Omega |Dh|^2\,dx\,,$$

for some $K = K(\Omega) > 0$. Since $0 < p,\ q < 1$, we may select $\varepsilon > 0$ small enough such that adding up (3.6$a$) and (3.6$b$) yields that $E(t) = \int_\Omega (\overline{\omega} - \underline{\omega})_+^2\,dx$ satisfies

$$E'(t) \leqslant CE(t)$$

whereas, by (3.5), $E(0) = 0$. Therefore $E(t) = 0$ for $t \in [0, T]$, whence $\bar{w}(t) \leqslant \underline{w}(t)$ and $\overline{\omega}(t) \leqslant \underline{\omega}(t)$ in $[0, T]$. Letting $s \downarrow 0$ we finally obtain

$$u_2(t) \leqslant u_1(t)\,, \qquad v_2(t) \leqslant v_1(t) \quad \text{ in } [0, T]$$

and, since the roles of $u_1$ and $u_2$ (resp., $v_1$ and $v_2$) can be exchanged, we also have that $u_1(t) \leqslant u_2(t)$ and $v_1(t) \leqslant v_2(t)$ in $[0, T]$ under our current assumptions. The case where, say, $p < 1 < q$, is similar.

*Case II*: $(u_0, v_0) \not\equiv (0, 0)$.

We argue again by contradiction, and assume that (1.1)-(1.4) has two different solutions, namely $(u(t), v(t))$ and the maximal solution $(u_M(t), v_M(t))$ constructed in Lemma 2.4. Clearly, it will suffice to show that

$$\text{(3.7)} \qquad u_M(t) \leqslant u(t)\,, \qquad v_M(t) \leqslant v(t) \quad \text{ in } Q\,.$$

To this end, we set

$$w_1(t) = u_M(t) - u(t)\,, \qquad w_2(t) = v_M(t) - v(t)$$

and suppose to start with that $0 < p,\ q < 1$. Then $w_1, w_2$ satisfy

$$\text{(3.8}a\text{)} \qquad (w_1)_t - \Delta w_1 \leqslant w_2^p \quad \text{ in } Q\,,$$

$$\text{(3.8}b\text{)} \qquad (w_2)_t - \Delta w_2 \leqslant w_1^q \quad \text{ in } Q\,,$$

$$\text{(3.8}c\text{)} \quad w_1(t, x) = w_2(t, x) = 0 \quad \text{ if } t = 0 \text{ and } x \in \Omega, \text{ or if } t > 0 \text{ and } x \in \partial\Omega\,.$$

Let $(\omega_1, \omega_2)$ be the unique positive solution of

$$(\omega_1)_t - \Delta\omega_1 = \omega_2^p \quad \text{in } Q, \tag{3.9a}$$

$$(\omega_2)_t - \Delta\omega_2 = \omega_1^q \quad \text{in } Q, \tag{3.9b}$$

$$\omega_1(t, x) = \omega_2(t, x) = 0 \quad \text{if } t = 0 \text{ and } x \in \Omega, \text{ or if } t > 0 \text{ and } x \in \partial\Omega. \tag{3.9c}$$

By Lemma 2.5, we certainly have that

$$w_1 \leqslant \omega_1, \qquad w_2 \leqslant \omega_2 \quad \text{in } Q.$$

We then define functions $f$ and $g$ as follows

$$f = \omega_1 - w_1 = \omega_1 - u_M(t) + u(t) \geqslant 0,$$

$$g = \omega_2 - w_2 = \omega_2 - v_M(t) + v(t) \geqslant 0$$

and claim that $f$ and $g$ satisfy

$$f_t - \Delta f \geqslant g^p \quad \text{in } Q, \tag{3.10a}$$

$$g_t - \Delta g \geqslant f^p \quad \text{in } Q, \tag{3.10b}$$

$$f(t, x) = g(t, x) = 0 \quad \text{if } t = 0 \text{ and } x \in \Omega, \text{ or if } t > 0 \text{ and } x \in \Omega. \tag{3.10c}$$

To check (3.10*a*), we notice that $f_t - \Delta f = \omega_2^p - v_M^p + v^p$. We then take advantage of the following elementary inequality, which is recalled for instance in [AE], Corollary 2.20

(3.11) Let $\omega_2$, $v_M$ and $v$ nonnegative quantities such that $\omega_2 + v \geqslant v_M$ $\omega_2 \leqslant v_M$ and $v \leqslant v_M$. Then if $0 < p < 1$, $\omega_2^p - v_M^p + v^p \geqslant (\omega_2 - v_M + v)^p$.

Obviously, the same argument yields (3.10*b*) at once. We now remark that (3.9) and (3.10) strongly suggest that

$$f \geqslant \omega_1 \quad \text{and} \quad g \geqslant \omega_2 \quad \text{in } Q, \tag{3.12}$$

which in turn implies (3.7) by the very definition of $f$ and $g$. However, to derive (3.12) some care is needed. The crucial point consists in showing that

$$f > 0 \quad \text{and} \quad g > 0 \quad \text{in } Q. \tag{3.13}$$

Actually, if (3.13) fails, we should have $f = g = 0$ in $Q_\tau = (0, \tau) \times \Omega$ for some $\tau > 0$. But then, recalling the definition of $f$ and $g$, we would obtain

$$(\omega_1 + u)^q = u_M^q = (v_M)_t - \Delta v_M = (\omega_2 + v)_t - \Delta(\omega_2 + v) = \omega_1^q + u^q \quad \text{in } Q_\tau,$$

$$(\omega_2 + v)^p = u_M^p = (u_M)_t - \Delta u_M = (\omega_1 + u)_t - \Delta(\omega_1 + u) = \omega_2^p + u^p \quad \text{in } Q_\tau,$$

which is impossible, since by assumption $\omega_1$, $\omega_2$, $u$ and $v$ are strictly positive in $Q_\tau$.

We then may repeat the arguments in Lemma 2.3 to show that

(3.14*a*) $$f \geq C_1 t^{(p+1)/(1-pq)} S_1(t) \quad \text{in } \Omega \quad \text{for any } t > 0,$$

(3.14*b*) $$g \geq C_2 t^{(q+1)/(1-pq)} S_1(t) \quad \text{in } \Omega \quad \text{for any } t > 0,$$

(cf. (2.7)). As next step, we proceed by showing that

(3.15) There exists a solution of (3.9), $(\overline{\omega}_1, \overline{\omega}_2)$, such that

$$C_1 t^{(p+1)/(1-pq)} S_1(t) \leq \overline{\omega}_1 \leq f \quad \text{in } Q,$$

$$C_2 t^{(q+1)/(1-pq)} S_1(t) \leq \overline{\omega}_2 \leq g \quad \text{in } Q.$$

Once (3.15) is obtained, the proof is concluded under our current assumptions. Indeed, by the uniqueness result established in Step 1, we have that $\overline{\omega}_1 = \omega_1$, $\overline{\omega}_2 = \omega_2$, so that (3.12) follows. To show (3.15), we notice that, if we set $h_1(t) = C_1 t^{(p+1)/(1-pq)} S_1(t)$, $h_2(t) = C_2 t^{(q+1)/(1-pq)} S_1(t)$, there holds

$$(h_1)_t - \Delta h_1 = \frac{C_1(p+1)}{1-pq} t^{p(1+q)/(1-pq)} S_1(t) \leq (C_2 t^{(q+1)/(1-pq)} S_1(t))^p = h_2^p$$

and, in a similar way

$$(h_2)_t - \Delta h_2 \leq h_1^q.$$

We now define sequences $\{\omega_1^j\}, \{\omega_2^j\}$ as follows. To begin with, $\omega_1^1$ solves

(3.16*a*) $$u_t - \Delta u = g^p \quad \text{in } Q,$$

(3.16*b*) $$u = 0 \qquad t = 0,\ x \in \Omega \text{ and if } t > 0 \textit{ and } x \in \partial\Omega.$$

In view of (3.15), we readily see that

(3.17) $$h_1(t) \leq \omega_1^1 \leq f.$$

As to $\omega_2^1$, it solves

$$v_t - \Delta v = f^q \quad \text{in } Q$$

together with (3.16*b*), so that

(3.18) $$h_2(t) \leq \omega_2^1 \leq g.$$

The sequence $\{\omega_1^j\}$ (resp. $\{\omega_2^j\}$) is then defined by induction as in the proof of Lemma 2.4. Namely, for $j \geq 2$, $\omega_1^j$ solves (3.16*a*) (with $g^p$ replaced by $\omega_2^{j-1}$) and (3.16*b*). In this way, (3.17) holds for any $\omega_1^j$, and since (3.18) is satisfied for any $\omega_2^j$, the result follows by letting $j \to \infty$. This concludes the proof in the case where $0 < p$, $q < 1$.

It then remains to consider the situation where, for instance, $0 < p < 1 \leq q$. Some modifications are required for the previous argument to work in such case, and we now proceed to sketch them briefly. Let $(u(t), v(t))$ and $(u_M(t), v_M(t))$ be as before.

We first claim that we may assume without loss of generality that there exists $T > 0$ such that

$$\frac{u_M(t)^q - u(t)^q}{u_M(t) - u(t)} \leqslant 1 \quad \text{if } t \leqslant T. \tag{3.19}$$

Indeed, set $\bar{w} = \lambda u_M$, $\bar{\omega} = \beta v_M$, $w = \lambda u$ and $\omega = \beta v$, where $\lambda$ and $\beta$ are real parameters to be determined presently. We have that

$$(\bar{w} - w)_t - \Delta(\bar{w} - w) = \lambda\beta^{-p}(\bar{\omega}^p - \omega^p) \leqslant \lambda\beta^{-p}(\bar{\omega} - \omega)^p,$$

$$(\bar{\omega} - \omega)_t - \Delta(\bar{\omega} - \omega) = \beta\lambda^{-q}\left(\frac{\bar{w}^q - w^q}{\bar{w} - w}\right)(\bar{w} - w).$$

Furthermore

$$\beta\lambda^{-q}\left(\frac{\bar{w}^q - w^q}{\bar{w} - w}\right) = \beta\lambda^{-q} q\int_0^1 (\theta\bar{w} + (1-\theta)w)^{q-1}\,d\theta \leqslant q\beta\lambda^{-1}M,$$

for some $M > 0$, which depends only on the bound for $u_M$ in $Q_T$. It therefore suffices to select $\lambda$ and $\beta$ such that

$$\lambda\beta^{-1} < 1, \qquad q\beta\lambda^{-1}M < 1$$

to obtain that functions $w_1(t)$, $w_2(t)$ defined just as before satisfy (3.8) with $q = 1$ there. We then set $q = 1$ from (3.8) on, and notice that the previous approach continues to work under the present circumstances. For instance, one still has that

$$\omega_2 \leqslant v_M \tag{3.20}$$

in (3.11), although $(\omega_1, \omega_2)$ and $(u_M, v_M)$ satisfy now different systems of equations. Indeed, if $\{\omega_2^j\}$ and $\{v_M^j\}$ are the sequences leading to $\omega_2$ and $v_M$ in the proof of Lemma 2.4, we see that, since $C_2 < 1$,

$$v_M^1 \geqslant (1 + C_2^{(1-pq)/(q+1)}t)^{(q+1)/(1-pq)} \geqslant (1 + C_2^{(1-p)/2}t)^{2/(1-p)} = \omega_2^1,$$

and this first inequality provides the foothole whereupon a suitable iteration argument leading to (3.20) can start. We omit further details. ∎

We conclude this Section by stating the following description of solutions of (1.8) in the sublinear case $p < 1$, which can be obtained by means of a simplified version of our previous arguments.

COROLLARY 3.2. – *Assume that* $0 < p < 1$. *Then, if* $u_0(x) \not\equiv 0$, *there exists a unique solution of* (1.8). *When* $u_0 = 0$, *the set of solutions of* (1.8) *consists of*

1) *the trivial function* $u(t, x) = 0$,

2) *a solution* $U(t, x)$ *which is positive in* $Q = (0, \infty) \times \Omega$,

3) *a monoparametric family* $\{U_\mu(t, x)\}$, *where* $\mu$ *is any positive number and* $U_\mu(t, x) = U((t-\mu)_+, x)$.

## 4. - Global existence and blow up.

We have already seen that every solution is global if $0 < pq < 1$ (cf. the remark following (2.6)). If, say, $p < 1$, there holds

(4.1) If $(u(t), v(t))$ is a solution of (1.1)-(1.3), and $0 < pq < 1$, the function $z(t) = v(t) + S(t) u_0^{1/p}$ is such that

$$z_t - \Delta z \leq 2^{q(1-p)}(1+t)^q z^{pq} \leq 2^{(1-p)}(1+t)^q(1+z).$$

The proof of (4.1) follows from the representation formulae (2.1) and the concavity estimate (2.8); see [EH1], Lemma 3.1 for details.

When $pq > 1$, global existence of solutions spreading from small initial values follows from a slight modification of a well known argument for the scalar equation (1.8). Let $\varphi_0(x)$ be the unique solution of

$$-\Delta\varphi_0 = 1 \quad \text{in } \Omega,$$

$$\varphi_0 = 0 \quad \text{in } \partial\Omega.$$

Clearly, $0 \leq \varphi_0 \leq C$ for some $C > 0$. If we define now

$$\tilde{u}(x) = a(1 + \varphi_0(x)), \qquad \tilde{v}(x) = b(1 + \varphi_0(x)),$$

with $a > 0$, $b > 0$, we readily check that

$$\tilde{u}_t - \Delta\tilde{u} - \tilde{v}^p \geq 0, \qquad \tilde{v}_t - \Delta\tilde{v} - \tilde{u}^q \geq 0 \quad \text{in } Q$$

provided that $0 < a^{pq-1} \leq (1+C)^{-(1+pq)}$ and $b^p \leq (1+C)^{-(1+p+pq)}$. Then if $\|u_0\|_\infty \leq a/2$ and $\|v_0\|_\infty \leq b/2$, the corresponding solution $(u(t), v(t))$ exists for all times, as can be seen by means of the argument in the proof of Lemma 2.2.

We finally consider the case of large initial values, and prove

LEMMA 4.1. - *Assume that* $pq > 1$, *and let* $\varphi_1(x)$ *be the function defined in* (1.5). *Then there exists* $C > 0$ *such that, if*

$$\int_\Omega (u_0(x) + v_0(x))\varphi_1(x)\, dx \geq C,$$

*the corresponding solution* $(u(t), v(t))$ *of* (1.1)-(1.4) *blows up in finite time.*

PROOF. - The case where $p > 1$ and $q > 1$ can be obtained as in [GKS2]. To deal with the general situation, we suppose without loss of generality that $0 < p < 1 < q$

and $u_0(x) \not\equiv 0$. We then make use of (2.1), (2.8) and Hölder's inequality, to obtain

$$(4.2) \qquad u(t) \geqslant S(t)u_0 + \int_0^t S(t-s)\left(\int_0^s S(s-\sigma)\,u(\sigma)^q\,d\sigma\right)^p ds \geqslant$$

$$\geqslant S(t)u_0 + \int_0^t s^{p-1} S(t-s)\left(\int_0^s S(s-\sigma)\,u(\sigma)^{qp}\,d\sigma\right) ds \geqslant$$

$$\geqslant S(t)u_0 + \int_0^t s^{p-1} S(t-s)\left(\int_0^s S(s-\sigma)\,u(\sigma)^{pq}\,d\sigma\right) ds =$$

$$= S(t)u_0 + \int_0^t\int_0^s s^{p-1} S(t-\sigma)\,u(\sigma)^{pq}\,d\sigma\,ds\,.$$

Write now

$$(u(t), \varphi_1) = \int_\Omega u(t,x)\,\varphi_1(x)\,dx\,.$$

Multiplying both sides in (1.1$a$) by $\varphi_1(x)$, and integrating in space and time yields

$$(u(t), \varphi_1) \geqslant e^{-\lambda_1 t}(u_0, \varphi_1)\,.$$

We then deduce from (4.2) that

$$(u(t), \varphi_1) \geqslant e^{-\lambda_1 t}(u_0, \varphi_1) + \int_0^t\int_0^s s^{p-1} e^{-\lambda_1(t-\sigma)}(u(\sigma)^{pq}, \varphi_1)\,d\sigma\,ds \geqslant$$

$$\geqslant e^{-\lambda_1 t}(u_0, \varphi_1) + \int_0^t\int_0^s s^{p-1} e^{-\lambda_1(t-\sigma)}(u(\sigma), \varphi_1)^{pq}\,d\sigma\,ds$$

whence

$$(4.3) \qquad e^{-\lambda_1 t}(u(t), \varphi_1) \geqslant (u_0, \varphi_1) + \int_0^t\int_0^s s^{p-1} e^{-\lambda_1(1-pq)}(e^{-\lambda_1\sigma}u(\sigma), \varphi_1)^{pq}\,d\sigma\,ds\,.$$

Se not

$$h(t) = \int_0^t\int_0^s s^{p-1} e^{\lambda_1(1-pq)\sigma}(e^{\lambda_1\sigma}u(\sigma), \varphi_1)^{pq}\,d\sigma\,ds\,,$$

$$\chi(t) = e^{\lambda_1(1-pq)t}, \qquad f(t) = (e^{\lambda_1 t}u(t), \varphi_1)\,.$$

We readily check that

$$h'(t) = t^{p-1}\int_0^t \chi(s)\, f(s)^{pq}\, ds\,,$$

$$h''(t) = t^{p-1} f(t)^{pq}\chi(t) - \left(\frac{1-p}{t}\right) h'(t)$$

whence

$$(f^{pq}\chi)(t) = (t^{1-p}h')'\,,$$

that is to say

$$\chi^{-1/pq}((t^{1-p}h')') = f \geqslant (u_0, \varphi_1) + h$$

i.e.,

$$(t^{1-p}h')' \geqslant \chi((u_0, \varphi_1) + h)^{pq}.$$

If we write now

$$a_0 = (u_0, \varphi_1), \qquad g(t) = (u_0, \varphi_1) + h(t).$$

We are finally led to

$$(t^{1-p}g'(t))' \geqslant \chi(t)\, g(t)^{pq}. \tag{4.4}$$

We now conclude as in [K], Thm. 1. Suppose that $u(t, x)$ can be continued for all times, and in particular is well defined for $t \in [1, 2]$. From (4.4), we deduce that there exists $m > 0$ such that

$$(t^{2(1-p)}g'(t))^2 = g'(1)^2 + 2\int_1^t (s^{1-p}g'(s))(s^{1-p}g'(s))'\, ds \geqslant$$

$$\geqslant g'(1)^2 + 2\int_1^t s^{1-p}\chi(s)\, g(s)^{pq} g'(s)\, ds \geqslant g'(1)^2 + 2m\int_1^t g(s)^{pq} g'(s)\, ds =$$

$$= g'(1)^2 + \frac{2m}{1+pq}(g(t)^{pq+1} - g(1)^{pq+1})$$

whence, setting $t = 2$

$$\int_{g(1)}^{g(2)} \left(g'(1) + \frac{2m}{1+pq}(r^{pq+1} - g(1)^{pq+1})\right)^{-1/2} dr \geqslant \int_1^2 s^{p-1}\, ds > 0\,. \tag{4.5}$$

Since $\int_0^\infty (1 + r^{pq+1})^{-1/2}\, dr < +\infty$, we see at once that (4.5) cannot possibly hold if $g'(1)$

is large enough, which in turn holds if $(u_0, \varphi_1)$ is sufficiently large. ■

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REFERENCES

[AE] J. AGUIRRE - M. ESCOBEDO, *A Cauchy problem for $u_t - \Delta u = u^p$: Asymptotic behaviour of solutions,* Ann. Fac. Sci. Toulouse, 8 (1986-87), pp. 175-203.

[AW] D. G. ARONSON - H. WEINBERG, *Multidimensional nonlinear diffusion arising in population genetics,* Advances Math., **30** (1978), pp. 33-76.

[EH1] M. ESCOBEDO - M. A. HERRERO, *Boundedness and blow up for a semilinear reaction diffusion system,* J. Diff. Eq., **89**, 1 (1991), pp. 176-202.

[EH2] M. ESCOBEDO - M. A. HERRERO, *A uniqueness result for a semilinear reaction diffusion system,* Proc. Amer. Math. Soc., to appear.

[EL] M. ESCOBEDO - H. A. LEVINE, *Critical blow-up and global existence numbers for a weakly coupled system of reaction-diffusion equations,* to appear.

[F] A. FRIEDMAN, *Blow-up of solutions of nonlinear parabolic equations,* in *Nonlinear Diffusion Equations and Their Equilibrium States,* Vol. I; edited by W. M. NI *et al.*, Springer-Verlag (1988), pp. 301-318.

[FG] A. FRIEDMAN - Y. GIGA, *A single point blow up for solutions of nonlinear parabolic systems,* J. Fac. Sci. Univ. Tokyo, Sect. I, **34** (1987), pp. 65-79.

[Fu1] H. FUJITA, *On the blowing up of solutions of the Cauchy problem for $u_t - \Delta u = u^{1+\alpha}$,* J. Fac. Sci. Univ. Tokyo, Sect. I, **13** (1960), pp. 109-124.

[Fu2] H. FUJITA, *On nonlinear equations $\Delta u + e^u = 0$ and $\partial u/\partial t = \Delta v + e^v$,* Bull. Amer. Math. Soc., **75** (1969), pp. 132-135.

[FuW] H. FUJITA - S. WATANABE, *On the uniqueness and non-uniqueness of solutions of initial value problems for some quasi-linear parabolic equations,* Comm. Pure Appl. Math., **21** (1968), pp. 631-652.

[GKS1] V. A. GALAKTIONOV - S. P. KURDYUMOV - A. A. SAMARSKII, *A parabolic system of quasilinear equations - I,* Diff. Eq., **19**, 12 (1983), pp. 2123-2143.

[GKS2] V. A. GALAKTIONOV - S. P. KURDYUMOV - A. A. SAMARSKII, *A parabolic system of quasilinear equations - II,* Diff. Eq., **21**, 9 (1985), pp. 1544-1559.

[K] V. KAMKOV, *Continuability and estimates of solutions of $(a(t)\varphi(x)x') + c(t)f(x) = 0$,* Annales Polinici. Math., **30** (1974), pp. 125-137.

[KST] K. KOBAYASHI - T. SIRAO - H. TANAKA, *On the growing up problem for semilinear heat equations,* J. Math. Soc. Japan, **29** (1977), pp. 407-429.

[L1] H. A. LEVINE, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$,* Arch. Rat. Mech. Anal., **51**, 5 (1973), pp. 371- 385.

[L2] H. A. LEVINE, *The role of critical exponents in blow-up theorems,* SIAM Review, **32**, 2 (1990), pp. 262-288.

[LM] J. L. LIONS - E. MAGENES, *Problèmes aux limites non homogènes et applications,* Vol. I, Dunod, Paris (1968).

[MW] C. E. MÜLLER - F. B. WEISSLER, *Single point blow-up for a general semilinear heat equation,* Indiana Univ Math. J., **34** (1983), pp. 881-913.